BIOGRAPHICAL MEMOIRS

Martin David Kruskal. 28 September 1925 –– 26 December 2006

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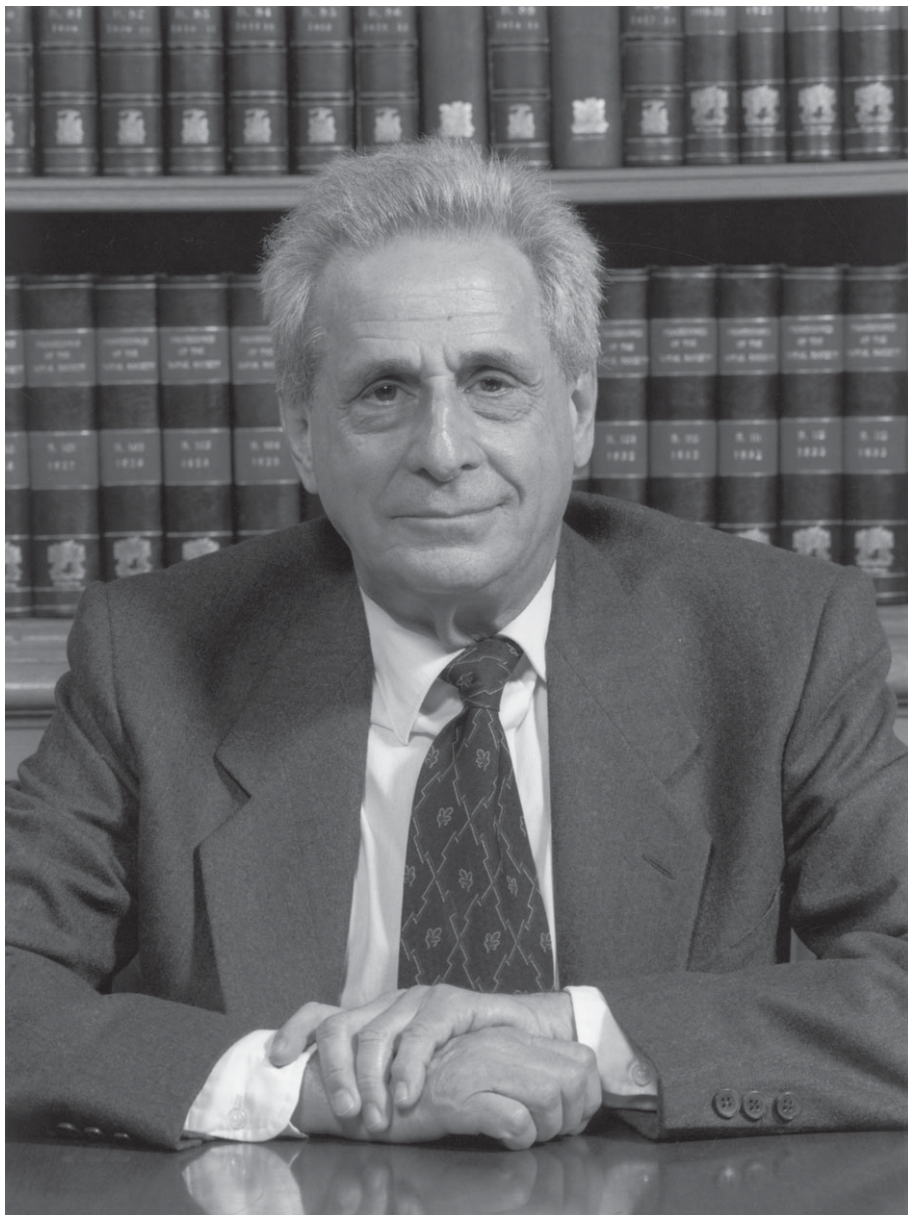
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Elected ForMemRS 1997

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Martin David Kruskal was one of the most versatile theoretical physicists of his generation and is distinguished for his enduring work in several different areas, most notably plasma physics, a memorable detour into relativity and his pioneering work in nonlinear waves. In the latter, together with Norman Zabusky, he invented the concept of the soliton and, with others, developed its application to classes of partial differential equations of physical significance.

INTRODUCTION

Imagine you are at a conference at a European venue enjoying a leisurely, silent breakfast with a few other participants. Suddenly the door opens and an older, senior man joins the group. He picks up a jar of jam and starts to interrogate it energetically. 'Why is it round and not square?', he asks suddenly, addressing the jar and its constituent parts in the third person. 'Why is it not hexagonal or even octagonal? Why is the lid not thicker? Why is the label pink and not green?' The questions come thick and fast: Why, why, …? The jar, we might add, remained unruffled and refused to answer. This interrogation upset the equilibrium of the

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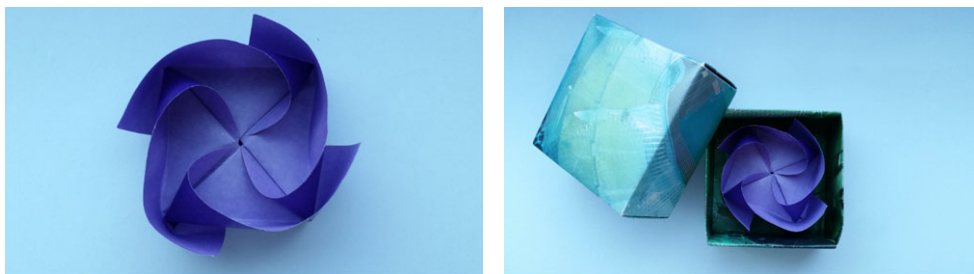§m.a.h.maccallum@qml.ac.uk

Figure 1. An origami flower and its box, both made by Martin Kruskal and presented to Sheila Gibbon about 30 years ago.

other occupants of the table to such a degree that one lady retorted 'Oh, I don't *know*, Martin' and left, at which point the occupants of the table returned to their original demeanour. Later that morning, with a shy smile of silent apology, Martin presented the lady with a beautifully constructed origami flower in an origami box, which has been kept and treasured to this day (figure 1). This anecdote is an illustration of Martin Kruskal's relentless desire to interrogate any problem or issue that caught his interest, turning it inside out and upside down, until he had thoroughly understood it, yet with a disarming charm that gained the affection of those who knew and worked with him. He was recognized by all as a kind and generous man who had a penetrating, hyperactive and insatiably curious intellect. Fundamentally, he just wanted to understand whatever was in front of him.

A string of obituaries appeared on Martin David Kruskal's death in 2006 which contain a variety of facts about his early career: see Eilbeck (2007), the *Los Angeles Times* (2006), the *New York Times* (2007), O'Connor & Robertson (2006), SIAM (2007) and a more recently written memoir for the US National Academy of Sciences by Deift (2016). Martin was born in New York City on 28 September 1925 and grew up in New Rochelle, Westchester County, New York. He attended Fieldston High School in Riverdale, New York, and entered the University of Chicago, from which he obtained his BS in 1945. Richard Courant persuaded Kruskal that he should undertake research at his new Institute, now the famed Courant Institute of Mathematical Sciences of New York University. Kruskal became an assistant instructor there in 1946 and, after studying for his MS, was awarded the degree in 1948. He then undertook research advised by Richard Courant and Bernard Friedman, during which time he married Laura (Lashinsky) in 1950. They subsequently had three children: Karen, Kerry and Clyde. He submitted his thesis, 'The bridge theorem for minimal surfaces', and was awarded his doctorate in 1952. In 1951 he moved to Princeton where he took up a post in Project Matterhorn, which was re-named Princeton Plasma Physics Laboratory after declassification in 1961. In 1961 he became a professor of astronomy, then a founder and chair of the Program in Applied and Computational Mathematics (1968) and then, following that, he became a professor of mathematics (1979). He retired from Princeton in 1989 and joined the mathematics department of Rutgers University, holding the David Hilbert Chair of Mathematics. Professionally, he was always known as Martin, but was always called David by his wife, Laura, and his family. His father, Joseph B. Kruskal Sr, was a fur wholesaler and his mother, Lillian Kruskal Oppenheimer, founded the Origami Center of America in New York City (Origami USA). He was one of five children. His two brothers were Joseph Kruskal (1928–2010), discoverer of multi-dimensional scaling, the Kruskal tree theorem and Kruskal's

Figure 2. Cartoon of Kruskal drawn by Leo Cullum, a longtime cartoonist for *The New Yorker*. Used with permission from the Cullum estate.

algorithm in computer science, and William Kruskal (1919–2005), a statistician known for the Kruskal–Wallis test.

Following his mother, Martin had a great love of origami, games, puzzles and symmetries. The web has many entries regarding his card trick called the Kruskal count (AMS 2016). Laura once said that they originally met at his mother's Origami Center in NYC, and Laura herself was equally well known as a lecturer and writer on that subject. In later years they travelled widely together to many scientific meetings at which Laura would organize anyone in reach into an origami group, whether they wished it or not! They were both much loved characters, larger than life, who made things happen and made any gathering or dinner a fun place to be. One story is that on a visit to a meeting in Rome, Laura sat on a bench in a public garden. As was her habit, she struck up a conversation with another inhabitant of the bench. She soon discovered that he was a Vatican official, who was persuaded to get her an audience with the Pope. We have no corroboration of this story, but if it isn't true it ought to be, because it entirely reflected her character. Whether she managed to teach origami to the Pope is unknown, but she no doubt tried.

The scale of the modern sciences is so huge that results that have taken so much of our own time and labour seem a mere grain of sand on a vast beach. Very few of us achieve results that leave marks on the coastline or even a few short-lived ripples on the sand. Martin was a rare example of one whose talents across a broad range of subjects have left the scientific coastline permanently changed (figure 2). He was an old-school scientist who pursued a career 'lifestyle' that deliberately rejected the devotion of so much time to the writing of grant proposals and the pursuit of influence through service on committees. His career ran against the modern trend in which research has increasingly become an industry, where goals, pathways to objectives and ultimate outcomes dominate a scientist's life and the acquisition

of grant money is the main concern. Instead, he spent his time in the pursuit of ideas and in the encouragement of others, particularly young scientists. Although it is reputed that he never had a grant, he was so successful that the absence of grant money was no barrier to travel because he was in such demand: conferences tended to come alive when he was in attendance. From the 1970s onwards he travelled the world extensively, often accompanied by Laura. In his capacious bag he carried a vast array of 'just in case I might need them' articles, including a large range of materials and implements for his origami. Darryl Holm, now of Imperial College London, tells the story that, in the Australian house of one of the co-authors of this memoir (Nalini Joshi), he used a word the meaning of which Martin felt was contextually wrong. After some discussion, Nalini supplied them with two dictionaries, one of which supported Darryl and the other Martin. 'OK', said Martin, 'In my bag I always carry three dictionaries, so let's check in those.' In the end Martin won 3–2. As much as anything, the story illustrates his enduring passion for precise definitions. He also insisted on logical thought. Whenever a graduate student was stuck on a technical point, his exhortation to them was always 'Follow the logic!'.

He was elected a Member of the US National Academy of Sciences in 1980, the American Academy of Arts and Sciences in 1983, a Foreign Member of the Royal Society (ForMemRS) in 1997, a Foreign member of the Russian Academy of Arts and Sciences in 2000 and an Honorary Fellow of the Royal Society of Edinburgh (FRSE) in 2001. In 2000, he was also awarded an honorary doctorate by Heriot-Watt University, Edinburgh. Among his many prizes, he was awarded the National Medal of Science in 1993 and, together with Gardner, Greene and Miura, the Steele Prize by the American Mathematical Society in 2006. Listed alphabetically among his PhD students are Ovidiu and Rodica Costin, Jishan Hu, Nalini Joshi, Robert Mackay (FRS, 2000) (jointly with John Greene), Steven Orszag and G. V. Ramanathan. His work spanned many fields, including major contributions to plasma physics, relativity, what are now called 'integrable systems' and a lifelong interest in both asymptotology (his label) and surreal numbers.

Contributions to plasma physics and fusion

In 1951 Martin was recruited by Lyman Spitzer[1] to join the, then classified, controlled fusion project at Princeton called Project Matterhorn[2]. This project pursued the idea of magnetic confinement of a fusion plasma by three-dimensional magnetic fields, a concept named the Stellarator by Spitzer. Martin was Spitzer's first employee and an inspired choice, as it turned out. Almost nothing was known about plasma physics in 1951 and certainly magnetic confinement theory was nonexistent. It was, therefore, a perfect time to enter the field and Martin took full advantage of the opportunity to help define and develop a new area of physics. Spitzer asked Martin to look at a mathematical problem while waiting for his security clearance to come through. Specifically, he asked whether they could make

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[1] Lyman Spitzer (1914–1997) was a reknowned astro-physicist who made major contributions to interstellar and plasma physics, space astronomy and nuclear fusion. After World War II he became the director of Princeton's observatory and the director of the classified Matterhorn project. He was the first to suggest the placing of a telescope in space and was a major force behind the development of the Hubble Space Telescope. In 2003 NASA launched an infra-red space observatory and named it the Spitzer Space Telescope in his honour.

[2] This section was written by Steven Cowley FRS.

integrable magnetic fields that lie on surfaces in three dimensions. Spitzer wanted to make sure that the field lines in the Stellarator would stay confined, since the hot plasma ions and electrons would follow the field lines—it is not clear how much of this motivation Spitzer could tell Martin until he was cleared. In an unpublished report $(1)^3$, Martin proved that for small rotational transform, $\iota \ll 1$, the deviation from perfect magnetic surfaces was beyond all orders in powers of ι. Clearly Martin was already thinking deeply about *integrability* and *asymptotics beyond-all-orders* in 1951 (although admittedly in a simplified system), well before, for example, Kolmogorov's famous 1958 paper on the existence of surfaces in Hamiltonian systems (Kolmogorov 1958). Spitzer's question remains central to Stellarator research. Indeed, modern Stellarators are designed by iteratively searching for integrable fields (see e.g. Helander 2014).

A necessary condition for a practical fusion system is hydromagnetic stability. Martin made many major contributions to the understanding and formulation of plasma stability. In 1954 Kruskal and Martin Schwarzschild published one of the earliest calculations of plasma stability (2). They treated the instability of a plasma supported against gravity by a magnetic field, now commonly called the *Kruskal–Schwarzschild instability*, and the *kink* stability of a current-carrying plasma column. The largest stable plasma current in a cylinder is set by the *Kruskal–Shafranov limit*. Martin calculated this limit in an initially classified report (3) and the result was finally published in the journal *Physics of Fluids* in 1958 (5).

By the mid 1950s, Spitzer had assembled a team of highly talented young theoreticians that produced a string of classic papers that have huge influence to this day. Martin was very much at the centre of this effort—we can only highlight the most important of his contributions. With Russell Kulsrud, he formulated the problem of the equilibrium of a magnetically confined plasma and showed that it can be obtained as a stationary variation of the energy (6). In the famous 'Energy principle' paper, Martin and three others of the Princeton group, Ira Bernstein, Ed Frieman and Russell Kulsrud, showed that positivity of the second variation of the magnetohydrodynamic (MHD) energy is a necessary and sufficient condition for MHD stability (7). They also showed how the principle could be used to calculate stability in complicated geometries. The *Energy principle* is the basis of most modern MHD stability calculations. Particle collisions in fusion plasmas are rare and therefore the MHD fluid equations are an inaccurate description of the plasma behaviour. Martin and Carl Oberman developed the first kinetic energy principle (8) based on the collisionless guiding centre description of the plasma—this paper, too, is the starting point of many modern calculations.

The adiabatic invariants of the guiding centre approximation for charged particles in a magnetic field are represented by an asymptotic series in the small parameter ϵ where ϵ equals the Larmor radius divided by the scale length of the magnetic field. Martin became fascinated by the structure and generality of this problem, in particular showing that invariance could be proved to all orders in ϵ. Russell Kulsrud proved first that the adiabatic invariant of the harmonic oscillator is invariant to all orders in $\dot{\omega}/\omega^2 \ll 0$ where $\omega(t)$ is the instantaneous oscillation frequency, with the dot representing a time derivative (Kulsrud 1957). Martin demonstrated the same result for the first adiabatic invariant of a particle in a magnetic field, then in greater generality for an autonomous system of differential equations with all solutions nearly periodic (10). When Martin presented the results of this paper to the Princeton group, the seminar lasted for two days—most of the audience stayed for it all!

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[3] Numbers in this form refer to the bibliography at the end of the text.

Plasmas support nonlinear electrostatic waves that do not undergo Landau damping. These were discovered by Ira Bernstein, John Greene and Martin (4) and are usually referred to as BGK modes. They showed that self-consistent solutions can be found in one dimension by adding the appropriate distribution of particles trapped in the electrostatic potential. The dynamics of nonlinear collisionless waves in plasmas remains a vibrant area of research and BGK waves play a central role in our current understanding.

Martin's interest in plasma physics and fusion research waned in the 1960s as he became more consumed by the study of nonlinear PDEs and in particular the Korteweg de Vries (KdV) equation. Nonetheless, the influence of Martin's mathematical style is imprinted on modern plasma physics—perhaps most obviously on those of us brought up on his notions of *asymptotology* (12)—but what is perhaps more surprising is that Martin could, and often did, think like a physicist, especially when he worked with experimentalists. He was not afraid to use intuition when rigour was unavailable and he delighted in cartoon explanations of physical processes. It is rare indeed that someone looks like a physicist to physicists and a mathematician to mathematicians!

KRUSKAL'S CONTRIBUTION TO RELATIVITY

In 1960, papers by Kruskal (9), and independently by Szekeres (1960), found the maximal analytic extension of Karl Schwarzschild's[4] vacuum solution, and coordinates for it.[5] The metric is now well understood to represent a spherically symmetric black hole. Its structure, horizon and singularities are explained in introductory courses and texts by making use of the Kruskal–Szekeres form, which is known thus to all students of the theory.

Kruskal's paper was rather unusual in that it was actually written by Wheeler (Wheeler & Ford 1998, pp. 295–296). Kruskal had shown Wheeler his results (allegedly on a napkin in a lunchroom) some time in 1956–7: Charles Misner recalls being told about them in 1958 and Kruskal's paper says Wheeler described them at a 1959 conference. Prompted (as described in Wheeler & Ford 1998) by work of Misner and others, Wheeler wrote the results up and submitted the paper without telling Kruskal (though over his name); the first Kruskal knew of it was when he received the galley proofs. Wheeler records that Kruskal 'was mystified only briefly' and suggested it be published as joint work, but Wheeler demurred on the grounds that all the important ideas belonged to Kruskal.

The starting point was the metric given by Karl Schwarzschild (Schwarzschild 1916), only seven weeks after Einstein's paper presenting the final form of his theory. In what are now called Schwarzschild coordinates, it reads

$$\mathrm{d}s^2 = -(1 - 2m/r)\mathrm{d}t^2 + \mathrm{d}r^2/(1 - 2m/r) + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \qquad [1]$$

where m is the gravitational mass that would be measured by a far away observer, in geometrized units, and $4\pi r^2$ gives the areas of the spheres of symmetry.

The form [1] is valid in $r > 2m$, or in $r < 2m$, but not at $r = 2m$. That surface is now understood to be the black hole horizon, a light-like surface bounding the black hole region from which light cannot escape: the coordinates of [1] are singular there. An imperfect

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[4] Karl Schwarzschild (1873–1916) was the father of Martin Schwarzschild (1912–1997) who was himself a co-author with Martin Kruskal (2).

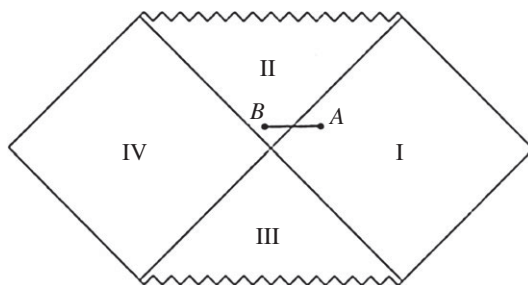[5] This section was written by Malcolm MacCallum.

Figure 3. Conformal diagram of the Kruskal–Szekeres maximal extension of Schwarzschild's solution.

understanding of such coordinate singularities (still sometimes found) be-devilled early interpretations (Eisenstaedt 1982), but Lemaître (1933) wrote that 'the singularity of the field is not real and arises simply because one wanted to use coordinates for which the field is static'. Others had found the same. The Eddington–Finkelstein coordinates, which go smoothly across the horizon (Eddington 1924; Finkelstein 1958), clarified the matter by using a null (light-like) coordinate like those used by Kruskal. This work still did not provide the maximal extension.

That extension can be illustrated by its conformal *Penrose diagram* (see figure 3). Such diagrams depict the metric transformed by a conformal factor Ω, i.e. they show Ω^2 times the original metric where the function $\Omega \to 0$ at infinity in such a way that the original space is mapped to a finite region. In this figure, each point represents a sphere, the coordinates θ and ϕ being suppressed, and light travels on lines at $45°$ to the axes. Time directions are *vertical* (i.e. at more than $45°$) and space directions horizontal in the picture. The region labelled I is the region $r > 2m$ in [1]. Region II ($r < 2m$) is the interior of the black hole, the horizon being the boundary between I and II. The jagged lines represent the future and past (true, not coordinate) singularities at $r = 0$ and the left and right hand edges of the figure consist of points at infinity. The region III is a white hole, which light can emerge from but not travel into. Intriguingly, region IV is a second exterior region. The line AB in the figure represents a three-dimensional surface composed of spheres whose size decreases to a minimum as one moves from A towards B. If one continues the line into region IV, the spheres' sizes increase again. Representing each sphere by a circle, the wormhole can be drawn as in figure 4. One might then imagine, as described in Kruskal's (or Wheeler's) paper, that such a wormhole could join two areas in the same space–time, although this is impossible in the Schwarzschild maximal extension itself. Sadly for science fiction writers, such wormholes in the Schwarzschild space–time, being space-like, can only be traversed if travelling faster than light. However, there are other wormhole solutions that, although perhaps not present in nature owing to the need for exotic forms of matter to sustain them, give very interesting possibilities, including causality violations. An excellent semi-popular account of these, which describes results from later technical papers, appears in chapter 14 of Thorne (1994).

One can arrive at figure 3 as follows. Kruskal & Szekeres used the coordinates

$$u = (r/2m - 1)^{1/2} e^{r/4m} \cosh(t/4m), \qquad [2]$$

$$v = (r/2m - 1)^{1/2} e^{r/4m} \sinh(t/4m), \qquad [3]$$

in terms of which the metric becomes

$$ds^2 = 32m^3 (du^2 - dv^2)/r e^{r/2m} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2), \qquad [4]$$
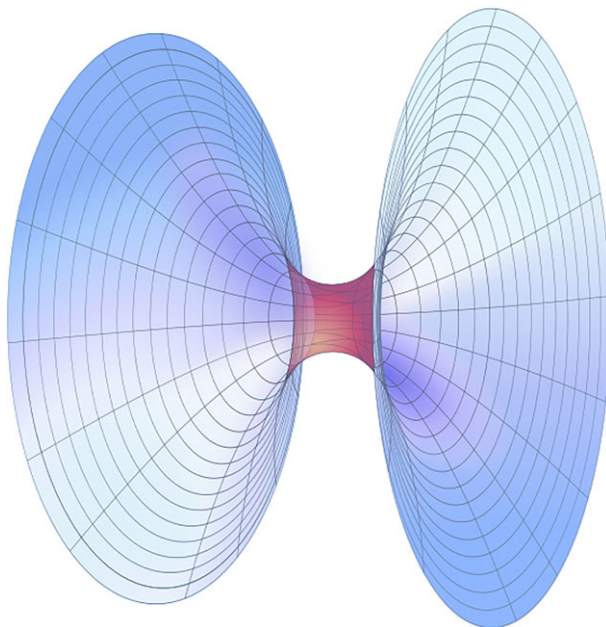
Figure 4. Diagram of a Schwarzschild solution wormhole (from Wikipedia, contributor Kes47).

with r given in terms of the new coordinates implicitly by

$$u^2 - v^2 = e^{r/2m}(r/2m - 1). \tag{5}$$

Kruskal's paper includes a nice diagram showing the four regions. The coordinates of [4] and coordinates covering figure 3 showing the conformally transformed metric, are related by

$$u' = \arctan(u/\sqrt{2m}), \qquad v' = \arctan(v/\sqrt{2m}). \tag{6}$$

In figure 3, $-\frac{1}{2}\pi < u' < \frac{1}{2}\pi$, $-\frac{1}{2}\pi < v' < \frac{1}{2}\pi$ and $-\frac{1}{2}\pi < u' + v' < \frac{1}{2}\pi$.

As well as prompting the intriguing work on wormholes, the results of Kruskal & Szekeres stimulated comprehensive investigations of the structures of other black holes (see Carter 1973) and helped promote the uses of global analysis which has led to work on singularity theorems and other issues (Hawking & Ellis 1973) that still continues.

KDV EQUATION AND INTEGRABLE NONLINEAR SYSTEMS

It is hard for younger scientists to imagine the scientific world of the 1950s, with prehistoric computational facilities, where *linear* thinking still dominated.[6] The large-scale PDE systems of applied mathematics and theoretical physics appeared so intractable that the reflex response was to ask what the linear approximation gave. Physics and applied mathematics have always abounded with special solutions of systems that are intrinsically nonlinear, but the effect that

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[6] This section was written by John Gibbon.

even small nonlinearities could have on a predominantly linear system was not at all well understood, nor did the question necessarily spring to mind. An argument has often been made that the emergence of quantum mechanics in the 1920s, an essentially linear science in that decade, side-lined the knowledge accumulated on nineteenth-century nonlinear differential equations, which had dropped out of fashion. There may be something in this view, but the massive disruptions of the two world wars would suggest that there were many more factors involved than just this. There also existed people whose thinking was ahead of their time. For instance, T. H. R. Skyrme came up with the idea of the sine-Gordon equation[7] (Skyrme 1958)

$$\phi_{xx} - \phi_{tt} = m^2 \sin \phi \qquad [7]$$

as a model nonlinear field theory for strong interactions, a step beyond the linear Klein–Gordon equation. The idea was to write down a fully *nonlinear field equation* which has local solutions of finite energy that cannot be reached by perturbation theory. Equation [7] has a travelling-wave solution, called a kink, of the form

$$\phi(x,\, t) = 4 \arctan \exp \{m\gamma (x - vt) + \delta\} \qquad \gamma^2 = (1 - v^2)^{-1} \qquad [8]$$

where δ is an arbitrary phase shift and v the kink velocity.[8] Perring & Skyrme (1962) actually constructed an exact double kink solution of [7] with equal but opposite velocities fired at one another as particles from $-\infty$ and ∞. These merged and then emerged intact. Neither the importance of the model nor the significance of their solution was recognized at the time.

A set of methods began to be developed in the early 1960s that sought to determine how dispersion or dissipation balanced the nonlinear terms in PDEs on some set of stretched time and space scales determined by a small amplitude parameter ε. Different names have been used, depending on the circumstances, but the methods of reductive perturbation theory, stretched coordinates and multiple scales are names that will be familiar to those who have worked on weakly nonlinear systems. For instance, Stuart (1960) was the first to develop these ideas for plane Poiseuille flow at the point of critical instability of the linear system on a time scale $T = \varepsilon^2 t$. With the inclusion of a space variable $X = \varepsilon x$, these ideas were developed further in a series of papers by Benney & Newell (1967), Newell & Whitehead (1969) and Newell (1974) for fluid convection, and Stewartson & Stuart (1971) for plane Poiseuille flow. For predominantly dispersive systems, such as those found in plasmas, nonlinear chains and surface water waves, reductive perturbation methods that use an amplitude $\varepsilon u(x, t)$ on time and space scales $\xi = \varepsilon^{q/2}(x - c_p t)$ and $\tau = \varepsilon^{3q/2} t$ give PDEs of the type (see Dodd *et al.* 1982)

$$u_{\xi\xi\xi} + 6u^q u_\xi + u_\tau = 0. \qquad [9]$$

$q = 1$ is the KdV equation[9] while $q = 2$ is the modified KdV (mKdV) equation. The KdV equation was already known, and received its name, from a paper by Korteweg and de Vries (1895) on the dynamics of small surface waves in shallow water. The solitary wave, observed

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[7] Subscripts denote partial derivatives.

[8] Kinks (anti-kinks) are solutions that move ϕ from 0 to 2π (and vice-versa). It is the profiles of the partial derivatives ϕ_x and ϕ_t that are the solitons. Thus, Perring & Skyrme (1962) had found the first analytical multiple soliton solution, but there was no hint in the paper of deeper properties that underlie this result.

[9] The coefficient of the uu_ξ-term can be altered at will by a re-scaling of u, so in this memoir we will freely use different values appropriate to the occasion.

and named by Scott Russell (1844) (but not the solution [10]), shows up in the solution

$$u = \frac{1}{2}a^2 \mathrm{sech}^2(ax - a^3t + \delta). \qquad [10]$$

Among many examples, waves in cold ion plasmas obey the KdV equation—see Washimi & Taniuiti (1966). It was this logical development of thought that ultimately led to the KdV or mKdV equations as PDEs that described weakly nonlinear behaviour of what appeared to be intractable bigger systems.

The work of Zabusky and Kruskal and the FPU problem

A very early example of the investigation of the effects of nonlinearity was made at Los Alamos National Laboratory by Fermi, Pasta and Ulam (1955), who were interested in the behaviour of systems that were primarily linear but into which nonlinearity was introduced as a perturbation. In the absence of such perturbations, the energy in each of the normal modes of the linear system would be constant. It was expected that the nonlinear interactions between the modes would lead to the energy of the system being evenly distributed throughout all of the modes: a result that would be in accordance with the equipartition theorem. The results they obtained contradicted this idea. The importance of what is now called the FPU problem is that the unexpected nature of their results stimulated work on these types of nonlinear systems and some of the modern work on solitons stemmed directly from it. A brief résumé of the their original report is this: consider a dynamical system of N identical particles of unit mass (FPU had $N = 64$) on a line with fixed end points with forces acting between nearest neighbours. If $Q_n(t)$ denotes the displacement from equilibrium of the nth particle, then the equation of motion for this particle can be written as

$$\ddot{Q}_n = f(Q_{n+1} - Q_n) - f(Q_n - Q_{n-1}). \qquad [11]$$

Two examples of the choice FPU made for f were either $f = \gamma Q + \alpha Q^2$ or $f = \gamma Q + \beta Q^3$, where γ denoted the linear chain constant and the constants α and β were chosen such that the maximum displacement of Q_n was small. Using these two nonlinearities, FPU integrated equation [11] numerically[10] on one of the earliest valve computers called MANIAC 1. Using initial data in the form of a sine-wave, they found that the energy did not spread throughout all the normal modes, but remained in the initial mode and a few nearby modes. Furthermore, the energy density of those nearby modes had an almost periodic behaviour in time. Over a large number of oscillations, the energy in each normal mode was seen to be almost periodic in time, with no loss of energy to higher modes as time increased. The precise explanation of this periodicity, which they called 'recurrence', stimulated a deeper study of equations such as [11]. In the continuum limit, equation [11] can be transformed into the KdV equation by using the method of stretched coordinates mentioned earlier. The choice of $f(Q) = \exp(-Q)$ makes [11] into the Toda lattice (Toda 1967).

The word 'soliton', after John Scott Russell's 'solitary wave' (Scott Russell 1844), first appeared in the paper by Zabusky & Kruskal (13). Kruskal had been interested in the FPU problem for some time, particularly in the explanation for why recurrence occurred: see his earlier paper on recurrence with respect to his plasma work (10). Together with Norman

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[10] It would appear that these computations were actually performed by a young woman named Mary Tsingou (Dauxois 2008).

Zabusky of Bell Laboratories, a very experienced computational physicist, Kruskal described a numerical study of the KdV equation (13) in a form where we revert to (x, t) coordinates

$$u_t + uu_x + \delta^2 u_{xxx} = 0. \tag{12}$$

It is noteworthy that when $\delta = 0$, the simple PDE $u_t + uu_x = 0$ causes waves to steepen in regions of negative gradient, ultimately causing a shock. Adding the smoothing dissipative term $\delta^2 u_{xx}$ to the right hand side gives Burgers' equation. The addition of $\delta^2 u_{xxx}$ to the left hand side disperses waves. In (12) they chose $\delta = 0.022$ with boundary conditions that were periodic such that $u(x, t) = u(x + 2, t)$, with an initial condition $u(x, 0) = \cos x$. They noticed that initially the wave steepened when it had a negative slope, which was a consequence of the dominance of the nonlinearity over the very small dispersive term, but once the wave had steepened the $\delta^2 u_{xxx}$-term became important and balanced the nonlinearity. On the left of the steepened region oscillations developed, each of which grew and reached a steady but different amplitude with each becoming a solitary wave in shape like (10). *The remarkable property of these was that they passed through one another with only a change of phase* as they went through the cycles of evolution forced by the periodic boundary conditions. This phase shift ensured that the initial state did not quite recur, but nevertheless it came close to recurrence, as in the FPU problem. Zabusky and Kruskal called these solitary waves 'solitons' because of their particle-like property. Subsequent work showed that the particle-like property of solitions is robust with a change of boundary conditions from periodic to the whole line. The particle-like behaviour intrigued Kruskal because the elastic collisional properties (with phase-shifting) reminded him of quantum mechanical scattering. N. J. Zabusky wrote a retrospective account of this work (Zabusky 2005) in the year before Kruskal's death in 2006.

The seven papers of Gardner, Greene, Kruskal, Miura and other collaborators

In the period 1965 to 1974 a series of seven papers were written by Kruskal and his associates. Kruskal was a co-author on four of them and not all four named authors appear on every paper. The first paper by Gardner *et al.* (14) is the key paper, but no numerical labelling appears in the title: the next six were entitled 'Korteweg de Vries equation and generalizations I–VI: ....'. Number I is authored by Miura alone (Miura 1968), number III was authored by Su & Gardner (1969), and concerned the derivation of [13] and [14], and number IV was authored by Gardner alone (Gardner 1971). Number V has Miura, Gardner and Zabusky as co-authors with Kruskal (16). All of the papers explored the properties of the KdV equation.

Let the Korteweg de Vries (KdV) and modified Korteweg de Vries (mKdV) equations be written in the form

$$u_t - 6uu_x + u_{xxx} = 0, \tag{13}$$

$$v_t - 6v^2 v_x + v_{xxx} = 0. \tag{14}$$

The coefficients of -6 are adjustable by scaling. Robert Miura (Miura 1968) discovered that if

$$u = v_x + v^2 \tag{15}$$

then

$$u_t - 6uu_x + u_{xxx} = \left(\frac{\partial}{\partial x} + 2v \right) (v_t - 6v^2 v_x + v_{xxx}). \tag{16}$$

Clearly, if v satisfies [14] then u satisfies [13], but not necessarily vice-versa. In Gardner *et al.* (14) it was shown that the Riccati equation [15], now known as the Miura transformation, is

exactly linearizable: choose $v = \psi_x/\psi$, and also note that the KdV equation is invariant under a Galilean transformation $x = x' + 6\lambda t'$; $t = t'$ and $u = u' - \lambda$. These turn [15] into (dropping the primes)

$$\left\{ -\frac{\partial^2}{\partial x^2} + u(x, t) \right\} \psi = \lambda \psi. \qquad [17]$$

How does ψ evolve in time? This is found by using $v = \psi_x/\psi$ in [14] and then [17]

$$\left\{ \frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3} - 3(u - \lambda)\frac{\partial}{\partial x} \right\} \psi = f\psi, \qquad [18]$$

with f taken as a constant. Readers will immediately recognize [17] as the Schrödinger equation of quantum mechanics with the KdV variable $u(x, t)$ as the potential and λ as a *constant energy eigenvalue*. In elementary quantum mechanics, one is normally given a potential and asked to solve for ψ together with the corresponding energy spectrum—in fact, for initial data $u(x, 0)$, this is exactly what one does. However, for $t > 0$, one has an *inverse problem*: given asymptotic properties of ψ at $x = \pm\infty$, with a constant spectrum λ and scattering data, namely reflection and transmission coefficients $a(k, 0)$ and $b(k, 0)$, can we reconstruct the potential $u(x, t)$ for all $t > 0$? In the entirely different subject of scattering theory, the solution to this inverse problem had already been answered by Gel'fand & Levitan (1955) and Marchenko (1955). Gardner *et al.* (14) showed how to use this scattering machinery to reconstruct the potential $u(x, t)$ for all $t > 0$.

The Lax pair formulation

In a classic paper, Peter Lax (1968) re-formulated the idea in a wider context and showed how to write down scattering problems for a much wider set of PDEs. Consider a spectral problem (in one-dimensional space x) with differential operator denoted by L together with a time dependence denoted by the operator P

$$L\psi = \lambda(t)\psi, \qquad\qquad \psi_t = P\psi. \qquad [19]$$

What is the condition on P and L such that λ is constant in time? Simple differentiation with respect to t and re-substitution shows that

$$L_t = PL - LP = [P, L]. \qquad [20]$$

We have considerable freedom to choose L and P, but let us begin with the symmetric Schrödinger operator in [17]

$$L = -\frac{\partial^2}{\partial x^2} + u(x, t), \qquad [21]$$

then, writing P as a third-order anti-symmetric operator taken from [18], we find the potential $u(x, t)$ evolves according to the KdV equation [13], as it should. This highlights the fact that while $u(x, t)$ is deforming with time, the spectrum λ remains constant. This is called an *isospectral deformation*. It can now easily be seen that one can play a game: for a fixed L, such as [21], one can choose a hierarchy of anti-symmetric operators P, which yield a corresponding hierarchy of PDEs. Likewise, one can vary L and make it a third-order operator or endow it with a matrix formulation.

In the paper by Miura *et al.* (15) they also recorded a series of higher conservation laws: the first three are the standard mass, energy and momentum while the fourth was found by

Whitham (1965). The Hamiltonian structure of the KdV equation appeared in the context of a hierarchy of PDEs (see Gardner 1971 and (17)). They showed that a recursion relation exists between the Q_k in the Hamiltonian formulation $\partial_t u = -\partial Q_k / \partial x$ and $\partial_t P_k = \delta H_k / \delta u$. Beginning with $Q_1 = u$, one can generate the infinite sequence Q_k. The KdV equation is the second in the hierarchy, with an infinite sequence of conserved quantities. A full Hamiltonian analysis can be found in Zakharov & Faddeev (1971).

Zakharov and Shabat and the AKNS formulation

A further boost to the subject came when the distinguished Russian plasma physicist V. E. Zakharov and co-workers began to work in this area. Zakharov & Shabat (1972) extended these ideas by finding a Lax pair for the so-called nonlinear Schrödinger (NLS) equation

$$i q_t + q_{xx} \pm 2q|q|^2 = 0. \tag{22}$$

The effect of this work opened the possibilities that there could be many more nonlinear systems whose solutions have the same particle-like properties as the KdV and mKdV equations.[11] Ablowitz *et al.* (1973) came up with an elegant formulation of a more general scattering problem (see also Ablowitz & Segur 1981)

$$\frac{\partial \psi_1}{\partial x} + i\lambda \psi_1 = q\psi_2, \qquad \frac{\partial \psi_2}{\partial x} - i\lambda \psi_2 = r\psi_1, \tag{23}$$

with a time dependence on the (ψ_1, ψ_2)

$$\frac{\partial \psi_1}{\partial t} = A\psi_1 + B\psi_2, \qquad \frac{\partial \psi_2}{\partial t} = C\psi_1 - A\psi_2. \tag{24}$$

It is easy to compute the compatibility conditions between [23] and [24], which are

$$A_x = qC - rB \qquad B_x + 2i\lambda B = q_t - 2Aq \qquad C_x - 2i\lambda C = r_t + 2Ar. \tag{25}$$

Several equations (and their variants) fit into this scheme: see table 1 for a list of q, r, A, B and C. In a separate development, it was shown independently by Flaschka (1974) and Manakov (1975) that the Lax pair formulation can be extended to discrete systems when they came up with a Lax pair for what is known as the Toda lattice (Toda 1967). Fordy & Gibbons (1980) also showed that there exists an integrable Toda-like extension of the Klein–Gordon equation. Over the decades these developments provoked the writing of literally thousands of papers on the properties of solutions of this class of PDEs, including a series of textbooks: see, for example, Ablowitz & Segur (1981), Dodd *et al.* (1982), Novikov *et al.* (1984) and Newell (1985).

THE PAINLEVÉ PROPERTY

Martin had an enduring interest in the six Painlevé second-order ordinary differential equations, designated as PI–PVI (see Ince 1956).[12] These are stated in table 2. His interest began when they appeared as symmetry reductions of soliton equations, but when his

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[11] In the meantime, Wadati (1972) had shown how to find the scattering problem for the mKdV equation.
[12] This section was written by Nalini Joshi.

Table 1. For the SIT and sine-Gordon equations (SGE) see Gibbon *et al.* (1979) and references therein ; for the equations of second harmonic resonance (2nd HR) see Kaup (1978).

Name	PDE	q	r	A	B	C
NLS	$iq_t + q_{xx} \mp 2q\lvert q\rvert^2 = 0$	q	$\pm q^*$	$-2i\lambda^2 \mp i\lvert q\rvert^2$	$2\lambda q + iq_x$	$\pm(2\lambda q^* - iq_x^*)$
mKdV	$q_t \mp 6q^2 q_x + q_{xxx} = 0$	q	$\pm q$	$-4i\lambda^3 - 2i\lambda qr + rq_x - qr_x$	$-q_{xx} + 2i\lambda q_x + 2q^2 r + 4q\lambda^2$	$-r_{xx} - 2i\lambda r_x + 2qr^2 + 4r\lambda^2$
SIT	$E_{xt} = EN$; $2N_x = -(\lvert E\rvert^2)_t$	$E/2$	$-E^*/2$	$-N/4i\lambda$	$E_t/4i\lambda$	$E_t^*/4i\lambda$
SGE	$\phi_{xt} = \sin\phi$	$\phi_x/2$	$-\phi_x/2$	$-\cos\phi/4i\lambda$	$\phi_{xt}/4i\lambda$	$\phi_{xt}/4i\lambda$
2nd HR	$P_x = QP^*$; $Q_t = P^2$	Q	Q^*	$-\lvert P\rvert^2/2i\lambda$	$P^2/2i\lambda$	$-P^{*2}/2i\lambda$

Table 2. The six Painlevé transcendents in terms of the independent variable t. The primes represent derivatives with respect to t while α, β, γ and δ are constants.

PI	$w'' = 6w^2 + t$
PII	$w'' = 2w^3 + tw + \alpha$
PIII	$w'' = w'^2/w - w/t + (\alpha w^2 + \beta)/t + \gamma w^3 + \delta/w$
PIV	$w'' = w'^2/2w + 3w^3/2 + 4tw^2 + 2w(t^2 - \alpha) + \beta/w$
PV	$w'' = \left(\dfrac{1}{2w} + \dfrac{1}{w-1}\right)w'^2 - \dfrac{w'}{t} + \dfrac{(w-1)^2}{t^2 w}\left(\alpha w^2 + \beta\right) + \dfrac{\gamma w}{t} + \dfrac{\delta w(w+1)}{w-1}$
PVI	$w'' = \dfrac{1}{2}\left(\dfrac{1}{w} + \dfrac{1}{w-1} + \dfrac{1}{w-t}\right)w'^2 - \left(\dfrac{1}{t} + \dfrac{1}{t-1} + \dfrac{1}{w-t}\right)w' + \dfrac{w(w-1)(w-t)}{t^2(t-1)^2}$
	$\times\left(\alpha + \dfrac{\beta t}{w^2} + \dfrac{\gamma(t-1)}{(w-1)^2} + \dfrac{\delta t(t-1)}{(w-t)^2}\right)$

relentless questions about their properties were unanswered, he dissected and broke down each question into natural components and developed new methods to find answers.

Consistent with much of Martin's other work, each approach he developed had a major influence on other progress in the field. His strategy was loosely grouped in three directions: (i) describing singular behaviours; (ii) finding analytic properties of solutions; and (iii) asymptotic analysis. The process was always the same: focus on an interesting question, try an idea, resolve any paradox and then follow the logic until a destination appears. At each step, examples and simpler models shaped the search for answers. In the standard analysis of linear ordinary differential equations (ODEs), if we solve for the highest derivative, the problematic places for defining solutions become clear: these are the places where the coefficient functions are singular. Such singularities are called *fixed* singularities in the sense that their locations are determined for all time by the equation. However, these are not the only possibilities for nonlinear ODEs. Consider, for example, the Riccati equation $w' = t - w^2$, which is linearized by

$$w = y'(t)/y(t) \quad \Rightarrow \quad y'' = ty. \qquad [26]$$

The linear ODE governing $y(t)$ is the classical Airy equation, with a general solution $y(t) = aAi(t) + bBi(t)$, where a and b are arbitrary constants. The solution $w(t)$ of the Riccati

equation becomes singular where $y(t)$ vanishes, but the locations of these zeros are determined by initial conditions. In turn, these determine the constants a and b, which are not visible in the ODE. These are movable singularities; i.e. ones whose locations change or 'move' with initial conditions. In the above example, the solution $w(t)$ has an infinite number of movable poles but, in general, the solution may be multi-valued around movable singularities.

This property was well known to mathematicians in the nineteenth century, who, by 1905, had classified second-order nonlinear ODEs (under some mild conditions) with the property that all solutions should be single-valued around all movable singularities (Ince 1956). Of the resulting class of 50 second-order ODEs, only the six Painlevé equations were found to define new higher transcendental functions as solutions. For these equations, all movable singularities of all solutions turn out to be poles. This is now called the *Painlevé property*.

It created enormous excitement when it was discovered in the late 1970s that all known ODE reductions of soliton equations have this property.[13] Ablowitz *et al*. (1978, 1980) conjectured that this would always be the case and thereby provided a widely used test for integrability. Weiss *et al*. (1983) showed how to apply the test directly to PDEs. Martin provided a simplification of the procedure, called a *reduced ansatz*, from the very beginning (listed as a private communication in the most cited paper of the time), creating clarity around these concepts in ways that are typical of his generous fundamental contributions to the field. He did not seek authorship in these cases, but others included him anyway: during a visit to the USA in 1982, Jimbo and Miwa visited Martin in Princeton and asked him questions about this test, resulting in its first application to the self-dual Yang–Mills system (18); see Mason & Woodhouse (1996) for further results on this topic.

Thousands of papers on the Painlevé property followed these developments. Whenever Martin was approached, he provided ideas on and resolutions to the most common paradoxes and contradictions in the applications of the test. An application to the one-dimensional anisotropic Heisenberg spin chain in a transverse magnetic field is noteworthy because Martin pointed out that singularities occur not only where solutions become unbounded but also where terms multiplying the highest derivative may vanish, leading to a loss of information in the equation (23).

These examples led to Martin's broad understanding of singularity analysis, perhaps the deepest of anyone in the field, but he still felt that fundamental questions remained unanswered. The test for the Painlevé property only gives necessary conditions. How do we find a sufficient proof that an example of interest does have the Painlevé property? There were integrable examples in which solutions were multivalued around movable singularities. How do we test for those? If singularities provide a good enough characterization of integrability, how do we find their other properties, such as Hirota's bilinear forms (Hirota 1971)? How can we test higher order ODEs, such as Chazy's third-order ODE, which has a movable natural barrier, rather than a localized singularity (Clarkson & Olver 1996)?

Martin always had an affectionate view of the Harry Dym equation, as an integrable equation related to the KdV equation, but its solutions have branched movable singularities and, therefore, it fails the test for the Painlevé property at the first step. Martin also had a model in his mind of a distinction between integrability and non-integrability based on whether solutions existed with a dense set of values at any point in the phase plane. Combining these

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[13] The easiest connection is to consider the mKdV equation [16] in similarity variable form $v(x, t) = t^{-1/3} w(\tau)$ with $\tau = x t^{-1/3}$ and then integrate. This turns into a scaled form of PII with τ standing for t in the table.

two ideas, Martin was led to a major extension of the Painlevé property, in joint work with Clarkson, that he called the *poly-Painlevé* test for integrability (24). As usual, his key ideas relied on asymptotic analysis, in this case covering many singularities concurrently. Martin's insights in these directions also led to singularity analysis of nonlinear ODEs with extended movable singularities, such as natural barriers (28).

Martin's questions were often based on a radical premise: it is never acceptable to have an adulatory reference to a celebrated proof without knowing how to prove it yourself. He worked on providing a sufficient proof of the Painlevé property that was simpler than the ones provided by classical mathematicians so that it could be extended to all modern integrable equations. With Joshi, he provided a direct proof of the Painlevé property of the Painlevé equations based on first principles (29). Martin constantly worked on improving and simplifying this proof until he passed away in 2006.

At an August meeting in Potsdam (NY) in 1979, Jim Corones, a materials scientist from Iowa, suggested one evening in jest that the test for integrability required only a postcard: write the equation of interest on a postcard, send it to Ryogo Hirota in Japan and if, in time, he sent back a very long formula then the equation must be integrable! Hirota's insights relied on finding a bilinear form of the equation, and has been used extensively to find specific soliton solutions more easily (Hirota 1971). Martin always wanted to know how to find the bilinear form directly, without relying on Hirota's intuition, and realized that he could do so by converting all movable singularities to movable regular zeroes of solutions. He used this idea in joint work with Hietarinta (25) to find Hirota forms of the Painlevé equations.

In the summer of 1982, Martin was inspired by the idea that the highly transcendental solutions of Painlevé equations should be described in a similar way to the traditional classical special functions. In particular, he heard a talk by Bryce McLeod who suggested that the connection problem for Painlevé transcendents should be tackled in a similar way to that for Airy functions, by following a large semi-circular path in the complex plane (Hastings & McLeod 1980). Carrying this out turned out to be no mean feat, because standard averaging and multiple-scales methods had to be extended in counter-intuitive ways. These methodological extensions were achieved in Joshi's PhD thesis, supervised by Martin, and connection results in the complex plane were obtained for the first and second Painlevé equations (21, 26, 27).

Near infinity, the solutions of these Painlevé equations are asymptotic to (scaled) elliptic functions, which reduce to power series expansions for certain initial conditions in some sectors of the complex plane. These power series are divergent and hide a small free parameter *beyond all orders* of the expansion. This was familiar territory for Martin, who had already encountered asymptotics-beyond-all-orders in the calculation of adiabatic invariants in plasma confinement. At the Santa Barbara conference celebrating Martin's sixtieth birthday (19) and in the programme that followed, he worked with Segur to resolve a similar problem that arose in the study of crystal growth in two dimensions (22) and in the study of breathers in a field model approximating the sine-Gordon equation (20). These papers have had a lasting influence on the field. Later, he applied ideas from surreal number theory with Costin to revisit this problem for classes of nonlinear ODEs (30).

In the 1990s, discrete integrable versions of the Painlevé equations were proposed. Although Martin published only one paper on the subject (31), his influence on this fledgling field was evident. At a conference in Esterel, Quebec, in 1994, at a time when the popular but unsettled test for the singularity confinement property was being proposed as a discrete

Painlevé property, Martin pointed out that it was actually a test for *well-posedness* in the discrete equations. This observation alone sharpened the thinking at the time and allowed the field to develop and grow to the exciting, mature field it is today.

Some final remarks

It is in the nature of science that most of its participants tend to specialize to such a degree that they forever sit in their own valleys and pan for specks of gold in the local river. Many never even raise their heads to observe the hills that flank their particular valley. Martin was one of the few who climbed those hills, explored other valleys and realized that the scientific disciplines form an inter-locking jigsaw whose picture tells a much bigger story. Not only did he make major contributions to the areas sketched in this memoir, he also had a lifelong interest in what he called 'asymptotology', which he defined as 'as the art of dealing with applied mathematical systems in limiting cases' (12). He referred to asymptotology as 'an art, at best a quasi-science, but not a science'.

Those who attended any of the early soliton conferences will recall a strongly mixed set of participants: not only regular PDE applied mathematicians, but plasma and optical physicists, fluid dynamicists, gauge theorists, geometric analysts, meteorologists, water wave theoreticians and experimentalists, and algebraic geometers. It was a heady rainbow mix of scientific cultures whose members participated because they felt something new was happening. It is ironic that in a day when many funding agencies across the world now require evidence of 'interdisciplinarity' in their proposals, a grant-less Martin was an early founder of this style of interdisciplinary science. In his eyes, interdisciplinarity was neither an institutionalized posture nor a box-ticking exercise, but simply the way he worked. The forward-thinking amiability of many of those early meetings could largely be attributed to Martin's friendly, robust and generous personality. He was always ready to give more credit to others and take less himself. He would also pepper speakers with endless questions and ideas although, at times, it could be an un-nerving experience to have one's mind turned inside out under the glare of such a penetrating intellect.

Our discussions above make it clear that Martin's early work is stamped indelibly all over modern plasma physics and relativity. Fifty years after the Gardner *et al.* paper (14), it is pertinent to ask: how successful has the search for integrable systems been? As ever, one can take both a narrow and a wide view. The narrow view is that there appears to be only a handful of integrable systems of physical significance, and that these are mainly restricted to one spatial dimension, although, within this small subset, it ought to be acknowledged that the properties of the NLS equation in fibre optics have had profound consequences in that science (Agrawal 2011). In the early days it was hoped that the local particle properties of soliton solutions might be found in fully 3D systems, but that appears, so far, to have been a vain hope. The 2D systems that are integrable, such as the Davey–Stewartson and Kadomtsev–Petviashvili equations (Benney & Roskes 1969; Kadomtsev & Petviashvili 1970; Davey & Stewartson 1974), yield solutions that are more like wave-fronts than localized humps of finite energy. Associated with many members of the finite set of integrable systems, there are infinite hierarchies of PDEs with no apparent physical significance, although the history of science shows that we should keep an open mind. The wider and more generous view is that Martin was a leader in teaching us that the physical world should not be seen through linear eyes,

with a few special nonlinear solutions tacked on, but it should not only be recognized as inherently nonlinear but also be explored with confidence. Indeed, many rich mathematical structures associated with integrability have been discovered that were undreamed of before. Much of the early work on algebraic geometry associated with the KdV equation on periodic boundary conditions (Dubrovin & Novikov 1974; Lax 1975; Krichever 1977; Segal & Wilson 1985) flowed out and merged with the 'geometry and physics' revolution that was occurring in parallel.

In modern academic and corporate circles, much is made of that ephemeral quality called 'leadership'. Unfortunately, it is increasingly viewed in narrow terms, such as an individual's ability to raise money or lead a large group. Kruskal did neither, yet his very obvious leadership qualities lay in the realm of ideas and shone through to all who knew him. He belonged to that fading generation of scientists, educated just after World War II, who founded the international science research system we know today. The continuing financial support required by this system obviously needs those who excel in managing the processes that are necessary in this sphere, but it also requires individuals who not only have the ability and the vision to conjure major new ideas, but also have the inspirational qualities to disseminate them across far-flung boundaries. Martin Kruskal excelled at this. Seen in this light, his far-seeing contribution to the understanding of modern physics and applied mathematics has been immense.

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