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Enstrophy bounds and the range of space-time scales in the hydrostatic primitive equations

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The hydrostatic primitive equations (HPEs) form the basis of most numerical weather, climate, and global ocean circulation models. Analytical (not statistical) methods are used to find a scaling proportional to $(\mathrm{Nu\,Ra\,Re})^{1/4}$ for the range of horizontal spatial sizes in HPE solutions, which is much broader than is currently achievable computationally. The range of scales for the HPE is determined from an analytical bound on the time-averaged enstrophy of the horizontal circulation. This bound allows the formation of very small spatial scales, whose existence would excite unphysically large linear oscillation frequencies and gravity wave speeds.

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The hydrostatic primitive equations (HPEs) have been the foundation of most numerical weather, climate, and global ocean circulation calculations for many decades [1–5]. In practice, modern computational power can handle integrations of these on global horizontal grids ranging in size between 15 and 60 km, which correspond respectively to one-eighth degree and one-half degree in latitude and longitude at the equator. This limitation raises the longstanding question, "can numerical simulations at these grid sizes adequately predict climate and other natural phenomena that occur on the much wider range of scales observed in nature?" See Fig. 1.

Important as it may be, this longstanding question is not addressed here. Rather, two questions are addressed associated with the HPE model itself , namely, "what range of scales is available for solutions of the HPE?," and "what scaling law governs the size of horizontal HPE excitations in terms of the system parameters?" These dimensionless parameters are Nu, Ra, Re, ϵ, and σ associated with the names of Nusselt, Rayleigh, Reynolds, Rossby, and Prandtl, respectively, as well as the domain aspect ratio α_a.

FIG. 1. (Color online) A NASA image [6] illustrates the large range of fluid scales that exist in atmospheric circulation. The oceanic range of scales is similar, but is not so easily observed.

The HPEs differ from the three-dimensional (3D) Navier-Stokes equations in that they incorporate both rotation and stratification, and in the imposition of vertical hydrostatic balance. The latter is often regarded as the most accurate of the various assumptions used in large-scale computations of the climate, weather, and ocean circulation. The hydrostatic assumption determines the pressure from the weight of the fluid above a given point, independently of its state of motion. This changes the nature of the dynamics, because the vertical velocity is determined from incompressibility, rather than from its own evolution equation.

Unlike the Navier-Stokes equations, solutions of the HPEs have been proved to be regular by Cao and Titi [7]. Moreover, the HPEs have also been shown to possess a global attractor [8]. Although its solutions are regular, the HPE system may potentially possess a vast range of sizes of excitations [9]. While Kolmogorov introduced the $\mathrm{Re}^{3/4}$ scaling law for the range of spatial sizes of excitations in incompressible fluid flows by using statistical methods [10], the present Rapid Communication will use analytical methods to show that a scaling law exists, proportional to $(\mathrm{Nu\,Ra\,Re})^{1/4}$, for the range of *horizontal* spatial sizes in solutions of the HPEs, with boundary conditions similar to those of Cao and Titi [7]. This result demonstrates that HPE excitations are possible at scales that are smaller than can be captured at present in numerical resolutions.

A dimensionless version of the HPEs may be expressed in terms of two sets of velocity vectors involving the horizontal velocities u,v and the vertical velocity w [11],

$$\boldsymbol{V}(x,y,z,t) = (u,v,\varepsilon w), \quad \boldsymbol{v} = (u,v,0). \tag{1}$$

Under the constraint of incompressibility, $\operatorname{div}\boldsymbol{V} = 0$, these satisfy

$$\varepsilon(\partial_t + \boldsymbol{V}\cdot\nabla)u - v = \varepsilon\,\mathrm{Re}^{-1}\,\Delta u - \partial_x P, \tag{2}$$

$$\varepsilon(\partial_t + \boldsymbol{V}\cdot\nabla)v + u = \varepsilon\,\mathrm{Re}^{-1}\,\Delta v - \partial_y P. \tag{3}$$

Here $\varepsilon = U_0/(fL)$ is the Rossby number, $\mathrm{Re} = U_0L/\nu$ is the Reynolds number, and P the dimensionless pressure for a typical velocity U_0, a domain size L in the horizontal direction, a rotation frequency f, and a viscosity ν.

As mentioned earlier, the HPE has no evolution equations for the vertical velocity component w. Instead, this variable is determined (diagnosed) from the incompressibility condition, $\operatorname{div} \boldsymbol{V} = 0$. The z derivative of the pressure field $P_z = \hat{\boldsymbol{k}} \cdot \nabla P$ and the dimensionless temperature Θ enter through the equation for hydrostatic balance,

$$a_0\Theta + \partial_z P = 0. \tag{4}$$

The coefficient $a_0 = (\varepsilon\sigma^{-1}\alpha_a^{-2})\mathrm{Ra}\,\mathrm{Re}^{-2}$ arises from nondimensionalization of the equations. Here $\sigma = \nu/\kappa$ is the Prandtl number (the ratio of viscosity ν and thermal diffusivity κ), Ra is the Rayleigh number defined by $\mathrm{Ra} = g\alpha T_0 H^3(\nu\kappa)^{-1}$, g is the acceleration due to gravity, α is the volumetric expansion coefficient, T_0 is a typical temperature difference, and $\alpha_a = H/L$ is the aspect ratio of the cylindrical domain. When (2)–(4) are combined, an evolution equation for the hydrostatic velocity field $\boldsymbol{v} = (u, v, 0)$ results,

$$\varepsilon(\partial_t + \boldsymbol{V}\cdot\nabla)\boldsymbol{v} + \hat{\boldsymbol{k}}\times\boldsymbol{v} + a_0\hat{\boldsymbol{k}}\Theta = \varepsilon\,\mathrm{Re}^{-1}\,\Delta\boldsymbol{v} - \nabla P, \tag{5}$$

which is taken in tandem with the incompressibility condition $\operatorname{div}\boldsymbol{V} = 0$. The dimensionless temperature Θ (the source of buoyancy) evolves according to

$$(\partial_t + \boldsymbol{V}\cdot\nabla)\Theta = (\sigma\,\mathrm{Re})^{-1}\Delta\Theta + q, \tag{6}$$

in which the nondimensional parameter q specifies heat sources or sinks. The domain Ω is taken to be a cylinder of radius L and height H. The vertical velocity and vertical flux of horizontal momentum both vanish on its flat upper and lower cylinder surfaces ($z = 0, H$): That is, $w = 0$ and $u_z = v_z = 0$ on the boundary. The variables are all taken to be periodic in the azimuthal angle on the vertical sidewall of the cylinder.

The approximations that lead to the HPEs will continue to be valid as long as the vertical acceleration remains negligible. However, even their linearized equations indicate that the HPE solutions may not always respect the approximations under which they were derived. Linearizing the HPEs in (5) and (6), and comparing with their nonhydrostatic equivalent (which has the dynamics of w restored), leads to familiar dispersion relations [12], which are illustrated in Fig. 2. The significance of the comparison of these dispersion curves is that without the frequency cutoff that is enforced by the buoyancy terms in the nonhydrostatic equations, the HPEs admit unphysically high gravity wave frequencies at small horizontal scales. Hence, if the HPE solutions acquire high horizontal wave numbers, then they may leave their range of validity. The result will be HPE gravity waves propagating at a fixed phase speed in the limit of small scales, while in reality gravity waves at these scales cease to propagate at all.

An estimate for the resolution length. Taking the inner product of the divergence-free velocity $\boldsymbol{V}$ with the motion equation (5) gives an equation for the rate of change of the kinetic energy of horizontal motion,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{v}|^2\,d\mathcal{V} = \int_\Omega (\mathrm{Re}^{-1}\,\boldsymbol{V}\cdot\Delta\boldsymbol{v} - a_0 w\Theta)d\mathcal{V}, \tag{7}$$

in which $d\mathcal{V}$ is the volume element and surface terms integrate to zero under the present boundary conditions. For the Navier-Stokes equations it is normal practice to use the energy dissipation rate $\nu\langle\int_\Omega|\boldsymbol{\omega}|^2\,d\mathcal{V}\rangle$ based on the full vorticity $\boldsymbol{\omega} = \operatorname{curl}\boldsymbol{V}$ to define a length scale called the Kolmogorov length [14]. The quantity $\int_\Omega|\boldsymbol{\omega}|^2\,d\mathcal{V}$ is called the *enstrophy* and the angle brackets $\langle\cdot\rangle$ denote the time average over the interval $[0,T]$,

$$\langle\cdot\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T(\cdot)dt. \tag{8}$$

However, it is more appropriate in the hydrostatic approximation to use three-dimensional $\boldsymbol{\zeta} = \operatorname{curl}\boldsymbol{v}$ and base a *horizontal* length scale on $\langle\int_\Omega|\boldsymbol{\zeta}|^2\,d\mathcal{V}\rangle$, since the vertical velocity w is diagnosed from the horizontal velocity dynamics. To determine this horizontal length scale from the evolution of the horizontal kinetic energy in (7), let us examine the Laplacian term

$$\int_\Omega \boldsymbol{V}\cdot\Delta\boldsymbol{v}\,d\mathcal{V} = -\int_\Omega \boldsymbol{\omega}\cdot\boldsymbol{\zeta}\,d\mathcal{V}, \tag{9}$$

where the surface terms again vanish for our choice of boundary conditions. Note that $\boldsymbol{\zeta}$ is fully three dimensional, but its horizontal components vanish at the top and bottom of the cylinder. Two more integrations by parts give

$$\int_\Omega \boldsymbol{\omega}\cdot\boldsymbol{\zeta}\,d\mathcal{V} = \int_\Omega [|\boldsymbol{\zeta}|^2 + (\operatorname{div}\boldsymbol{v})^2]d\mathcal{V} \geqslant \int_\Omega |\boldsymbol{\zeta}|^2\,d\mathcal{V}, \tag{10}$$

and substituting into (7) implies

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{v}|^2\,d\mathcal{V} \leqslant -\mathrm{Re}^{-1}\int_\Omega |\boldsymbol{\zeta}|^2\,d\mathcal{V} - a_0\int_\Omega w\Theta\,d\mathcal{V}. \tag{11}$$

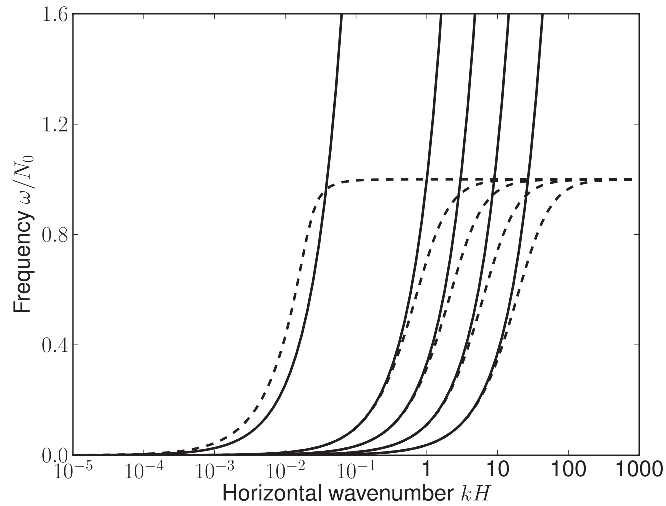

FIG. 2. This comparison of linear mode dispersion relations for the hydrostatic primitive equations (solid curves) with those of the exact nonhydrostatic equations (dashed curves) for oceanic conditions shows that the primitive equations admit very high fluctuation frequencies, especially at high horizontal wave numbers. In contrast, the dispersion relation for the nonhydrostatic equations limits properly to the buoyancy frequency, regardless of how high the horizontal wave number becomes. The oceanic parameters used here are sound speed $c_s = 1500\ \mathrm{ms}^{-1}$, mean depth $H = 5$ km, and buoyancy frequency $N_0 = 0.01\ \mathrm{s}^{-1}$, where the appropriate normalizing length scale H is the mean ocean depth. The multiple curves correspond to different choices of vertical wave number $mH \in [0,1,3,9,27]$, increasing from the left. The value $m = 0$ is the barotropic mode and the others are baroclinic. Figure courtesy of Dukowicz [12]. For additional details and more explanation, see Ref. [13].

Upon defining the vertical Nusselt number Nu as

$$\mathrm{Nu} := -\left\langle \int_{\Omega} w\Theta \, d\mathcal{V} \right\rangle, \tag{12}$$

the time average of (11) may be written as

$$\left\langle \int_{\Omega} |\boldsymbol{\zeta}|^2 \, d\mathcal{V} \right\rangle \leqslant \left(\varepsilon\sigma^{-1}\alpha_a^{-2}\right) \mathrm{Ra}\, \mathrm{Re}^{-1}\, \mathrm{Nu}, \tag{13}$$

since the horizontal kinetic energy term vanishes in the limit as $T \to \infty$. This bound on the time-averaged enstrophy of the horizontal circulation $\langle \int_{\Omega} |\boldsymbol{\zeta}|^2 \, d\mathcal{V} \rangle$ yields a *horizontal resolution length scale* which emerges upon switching back into dimensional variables. Let $\boldsymbol{\zeta}_{\mathrm{dim}}$ be the dimensional version of $\boldsymbol{\zeta}$; that is, $\boldsymbol{\zeta}_{\mathrm{dim}} = L^{-1} U_0 \boldsymbol{\zeta}$ for a typical horizontal velocity scale U_0. Then a *resolution scale* λ_{res} may be defined using the same approach as that used to find an analytical estimate of the inverse Kolmogorov scale for the Navier-Stokes equations:

$$\begin{aligned} L^4 \lambda_{\mathrm{res}}^{-4} &:= L^4 \left\langle \left(\nu^{-2} L^{-3} \int_{\Omega} |\boldsymbol{\zeta}_{\mathrm{dim}}|^2 \, d^3x \right) \right\rangle \\ &= L^4 (L^{-1} U_0)^2 \nu^{-2} \left\langle \int_{\Omega} |\boldsymbol{\zeta}|^2 \, d\mathcal{V} \right\rangle \\ &= \mathrm{Re}^2 \left\langle \int_{\Omega} |\boldsymbol{\zeta}|^2 \, d\mathcal{V} \right\rangle. \end{aligned} \tag{14}$$

Thus, the main result obtained from Eqs. (13) and (14) is an estimate for the *range* of horizontal scales, defined by the ratio $L\lambda_{\mathrm{res}}^{-1}$, as

$$L\lambda_{\mathrm{res}}^{-1} \leqslant \left(\varepsilon\sigma^{-1}\alpha_a^{-2}\, \mathrm{Nu}\, \mathrm{Ra}\, \mathrm{Re}\right)^{1/4}. \tag{15}$$

This bound incorporates all physical processes in their nondimensional forms. Estimated from the time-averaged enstrophy of the horizontal circulation, the ratio $L\lambda_{\mathrm{res}}^{-1}$ of the domain size to the resolution scale provides an upper bound for the range of *horizontal* (not vertical) length scales available as solutions of the HPEs.

Conclusion. It is now time to put some numbers into the estimate in (15). For example, in regional flows in the ocean of depth $H \approx 10^{0.5}$ km, aspect ratio $\alpha_a = 10^{-2}$, Prandtl number $\sigma \approx 10$, and Rossby number $\varepsilon = 10^{-2}$, one has $\varepsilon\sigma^{-1}\alpha_a^{-2} \approx 10^1$. Thus, the range of scales (15) in this case may be written as

$$L\lambda_{\mathrm{res}}^{-1} \lesssim (10\, \mathrm{Nu}\, \mathrm{Ra}\, \mathrm{Re})^{1/4}. \tag{16}$$

The Rayleigh, Prandtl, and Nusselt numbers usually appear in Rayleigh-Bénard convection in which Nu is observed to scale with Ra such that $\mathrm{Nu} \sim \mathrm{Ra}^{\beta}$ with variations around $\beta = 1/3$: See Ref. [15] for a discussion of the state of the art for heat transfer and large scale dynamics in turbulent Rayleigh-Bénard convection. However, the hydrostatic approximation excludes deep convective processes, in which case $\mathrm{Nu} \approx 1$ [16]. The Rayleigh-Bénard β scaling for Nu would apply only at small vertical turbulence scales where the hydrostatic approximation would be invalid. An important issue in oceanic simulations is to differentiate between mass flux and heat flux. Numerical simulations of ocean circulation must typically be corrected to prevent overestimating the heat flux [17]. The need for this correction is another indication that the Nusselt number tends to be small in oceanic flows.

The sizes of Ra and Re for typical flows in the ocean are very large, when based on regional domain size and molecular values of viscosity and diffusivity of heat. For example, with $H \approx 5$ km and $\mathrm{Nu} \approx 1$,

$$\mathrm{Ra} = g\alpha T_0 H^3 (\nu\kappa)^{-1} \approx 10^1 10^{-4} 10^0 10^{11} (10^6 10^7) \approx 10^{21}, \tag{17}$$

and $\mathrm{Re} = U_0 H/(\nu\alpha_a) \approx 10^{-1}(5 \times 10^3)(10^6 10^2) \approx 5 \times 10^{10}$. According to these estimates, $\mathrm{Ra}\, \mathrm{Re}^{-2} = O(1)$, and the quantity $a_0 = (\varepsilon\sigma^{-1}\alpha_a^{-2}) \mathrm{Ra}\, \mathrm{Re}^{-2} \approx 10^1$; so the range of scales is bounded by about *eight* orders of magnitude, since $\frac{1}{4}\log_{10}(10\, \mathrm{Ra}\, \mathrm{Re}) = 8$. That is, in this case, $L\lambda_{\mathrm{res}}^{-1} \lesssim 10^8$. This means that for a domain size of 400 km at a depth of about 4 km, *the horizontal excitation scales could be as small as a few millimeters.* In particular, the estimate (15) with $\mathrm{Nu} \approx 1$ and $\mathrm{Ra} \sim \mathrm{Re}^2$ yields an estimate of Ra, so that

$$L\lambda_{\mathrm{res}}^{-1} \leqslant \left(\varepsilon\sigma^{-1}\alpha_a^{-2}\right)^{1/4} \mathrm{Re}^{3/4}, \tag{18}$$

which is close to the Kolmogorov range of scales in 3D. The very high linear wave frequencies associated with such small horizontal scales would preclude both the physical relevance and the computability of the HPEs. The conclusion is that improving the resolution of HPE numerical solutions may tend to make their results less accurate and much more expensive to perform, because the nonlinear tendency toward much smaller spatial scales produces wave excitations of rapidly increasing linear frequency (as in Fig. 2) that would require reducing the time step beyond the present limits of computability. This fact has already been recognized in practice, since the HPEs are generally applied to climate simulations, but not to regional simulations. What this Rapid Communication shows and emphasizes is that unphysically small spatial scales can potentially be generated in HPEs when molecular values for transport coefficients are used. In fact, modulo appropriate adaptations, the same range of scales would be found to hold for the nonhydrostatic equations, although we do not discuss it here because no proof of existence is available for them [18].

Two caveats about the range of scales estimated here should be mentioned. First, the system of equations explored in this Rapid Communication are used for oceanic general circulation models, but typical atmospheric models must consider the effects of compressibility [18]. In this sense, the results of this Rapid Communication, while indicative of the range of scales for atmospheric models, can only be applied directly to the oceans. Second, the bound provided in this article may not be sharp as the $(\mathrm{div}\, \boldsymbol{v})^2$ term in inequality (10) has been dropped.

Of course, numerical simulations of large-scale circulations in the ocean and atmosphere do not use the molecular values of viscosity and diffusivity. Instead, they introduce effective values for these quantities due to unresolved scales, associated with turbulent "eddies." These effective values are chosen essentially to make the Reynolds number at the horizontal grid scale $\mathrm{Re}(\Delta x)$ equal to unity. If the scaling $\mathrm{Ra} \sim \mathrm{Re}^2$ persists for these simulations and the Nusselt number at the grid scale is of order unity, then the numerical procedure of setting $\mathrm{Re}(\Delta x) = 1$ might tend to properly resolve the hydrostatic excitations of the HPEs. However, it may also be good practice in numerical simulations using the HPEs to evaluate the dimensionless numbers at the vertical grid scale Nu(Δz) and Ra(Δz) corresponding to the other physical aspects of the HPEs. Further study of the scaling law Ra $\sim$ Re^2 for various regimes of ocean and atmosphere circulation might also be fruitful in determining local values of the ranges of scales.

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