Estimates for the LANS-α, Leray-α and Bardina Models in Terms of a Navier-Stokes Reynolds Number

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In honor of Ciprian Foias on his 75th birthday:
Wisdom is vindicated by all her children (Luke, 7:35)[1]

ABSTRACT. Estimates for the three α-models known as the LANS-α, Leray-α and Bardina models are found in terms a Reynolds number associated with a Navier-Stokes velocity field. They are tabulated for comparative purposes and show clearly that all estimates for the Leray-α model are smaller than those for the LANS-α and Bardina models.

1. INTRODUCTION

1.1. Opening remarks. Ciprian Foias has been a wise and gentle inspiration and guide to a younger generation of applied mathematicians who have followed in his footsteps by devoting considerable portions of their careers to studying the Navier-Stokes equations and the various problems associated with them. Ciprian has taught us respect for the severe difficulties encountered when addressing three-dimensional Navier-Stokes regularity properties [2–10], but he has also been the leader these last few years in a program that has seen the development of a set of three-dimensional models that regularize the Navier-Stokes equations. Known more commonly as α-models, the most prominent of these are the LANS-α model (LANS stands for "Lagrangian-averaged Navier-Stokes") [11–13], the Leray-α model [14] and the Bardina model [15]. At various levels

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these share properties of the Navier-Stokes equations but, unlike their parent, they possess regular solutions. These comforting regularity properties are the motivation for their practical use in turbulence modelling [16–18].

Proofs of the regularity of solutions and estimates on the attractor dimension for all three α-models have been found in terms of the Grashof number Gr, which is a measure of the forcing [12, 14, 15], but comparisons between estimates make better sense if they are made in terms of the same parameter that is also intrinsically associated with the Navier-Stokes equations. The obvious choice for this is the Reynolds number Re based on a velocity field U, the space-time average of the Navier-Stokes velocity field $\boldsymbol{u}(\mathbf{x}, t)$. Although technically not a control parameter but a measure of the fluid response, Re is a better choice than the Grashof number Gr because of its standard use in computational fluid calculations and scaling methods in statistical physics [19, 20]. Comparisons between estimates for the three α-models and the Navier-Stokes equations are tabulated at the end of this main section. These extend the comparison made between LANS-α and Navier-Stokes by the present authors [21].

1.2. α-models. The idea is to introduce a regularized velocity field $\boldsymbol{v}(\mathbf{x}, t)$ defined in terms of the Navier-Stokes velocity field $\boldsymbol{u}(\mathbf{x}, t)$ as

$$\boldsymbol{v} = (1 - \alpha^2\Delta)\boldsymbol{u} \tag{1.1}$$

where α is the coherence length of the Lagrangian statistics: clearly $\boldsymbol{v} \to \boldsymbol{u}$ in the limit $\alpha \to 0$. The four partial differential equations in question are [12, 14, 15]

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \nu\Delta\boldsymbol{u} + \boldsymbol{f}(\mathbf{x}) \qquad \text{NS} \tag{1.2}$$

$$\left.\begin{array}{l} \boldsymbol{v}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{v} + \nabla\boldsymbol{u}^T\cdot\boldsymbol{v} + \nabla p \\ \boldsymbol{v}_t - \boldsymbol{u}\times\operatorname{curl}\boldsymbol{v} + \nabla\tilde{p} \end{array}\right\} = \nu\Delta\boldsymbol{v} + \boldsymbol{f}(\mathbf{x}) \qquad \text{LANS-}\alpha \tag{1.3}$$

$$\boldsymbol{v}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{v} + \nabla p = \nu\Delta\boldsymbol{v} + \boldsymbol{f}(\mathbf{x}) \qquad \text{Leray-}\alpha \tag{1.4}$$

$$\boldsymbol{v}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \nu\Delta\boldsymbol{v} + \boldsymbol{f}(\mathbf{x}) \qquad \text{Bardina} \tag{1.5}$$

taken on a three-dimensional periodic domain $[0, L]^3$ with $\operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{v} = 0$. In the two alternative versions of LANS-α, the two pressures $\tilde{p}$ and p are related by $\tilde{p} = p + \boldsymbol{u}\cdot\boldsymbol{v}$.

The idea of creating a turbulence closure model without enhancing viscous dissipation came originally from Leray [22] who showed how to regularize the Navier-Stokes equations (1.2) by modifying their nonlinearity into the form (1.4) with $\boldsymbol{v} = 0$ on the boundary. The two velocities $\boldsymbol{u}$ and $\boldsymbol{v}$ were related by $\boldsymbol{u} = G_\delta * \boldsymbol{v}$ with the filtering operation defined by $G_\delta * \boldsymbol{v} = \int G_\delta(\mathbf{x}, \mathbf{y})\boldsymbol{v}(\mathbf{y})\, d^3y$ for a symmetric kernel $G_\delta(\mathbf{x}, \mathbf{y})$ of characteristic width δ. The Navier-Stokes equations are recovered in the limit as $\delta \to 0$, so that $\boldsymbol{u} \to \boldsymbol{v}$. The Leray regularization of the Navier-Stokes equations has been reviewed by Gallavotti [23].

The three regularizations given in (1.3), (1.4) and (1.5) with $\boldsymbol{u}$ and $\boldsymbol{v}$ related by (1.1) have been shown to have regular solutions in [12–15]. Generally the most important of the estimates have been found in terms of the Grashof number Gr defined below in terms of the forcing. The most suitable quantity, however, is the Reynolds number Re based on the Navier-Stokes velocity field $\boldsymbol{u}$ which needs to be related to the forcing function $\boldsymbol{f}(\mathbf{x})$ on the right hand sides of (1.2) – (1.5). The forcing is taken to be of narrow-band type such that

$$\|\nabla^n \boldsymbol{f}\|_2 \approx \ell^{-n}\|\boldsymbol{f}\|_2 \,. \tag{1.6}$$

With $f_{rms} = L^{-d/2}\|\boldsymbol{f}\|_2$, where and $\|\boldsymbol{f}\|_2^2 = \int_\Omega |\boldsymbol{f}|^2\,dV$, the standard definition of the Grashof number in d-dimensions is

$$Gr = \frac{\ell^3 f_{rms}}{\nu^2} \,. \tag{1.7}$$

Define the Reynolds number as

$$\mathrm{Re} = \frac{U\ell}{\nu} \qquad U^2 = L^{-d}\langle\|\boldsymbol{u}\|_2^2\rangle \tag{1.8}$$

where $\langle\cdot\rangle$ is the long-time-average

$$\langle g(\cdot)\rangle = \overline{\lim}_{t\to\infty}\frac{1}{t}\int_0^t g(\tau)\,d\tau \,. \tag{1.9}$$

Doering and Foias [10] have addressed the problem of how to relate Gr and Re and have shown that in the limit $Gr \to \infty$, solutions of the d-dimensional Navier-Stokes equations must satisfy[1]

$$Gr \le c\,(\mathrm{Re}^2 + \mathrm{Re}) \,. \tag{1.10}$$

While this relation is gratifying, finding estimates for all three α-models is not so simple as substituting Re^2 for Gr. The time average $\langle\,\cdot\,\rangle$ within U and hence within Re suggests that sharper estimates can be found.

1.3. Comparisons between models. Comparisons between the models can be made at different levels but those for $\langle H_1\rangle$ are particularly instructive as this is one of the few Navier-Stokes quantities known to be bounded with an upper bound proportional to Re^3. The corresponding upper bound for LANS-α and Bardina of $\mathrm{Re}^{5/2}$ is just beaten by the Leray-model with $\mathrm{Re}^{7/3}$. These three models all have the property that the H_1-norm is bounded above point-wise in

[1] In [21] it has been shown that this property holds for the LANS-α equations; the same methods can be used to show this also holds for the Leray-α and Bardina models although these calculations won't be displayed here.

time, which still remains an open problem for the Navier-Stokes equations. The estimate for the attractor dimension for Leray-α model at Re $^{9/7}$ is by far the best. While the equivalent for LANS-α is Re $^{9/4}$ – which appears consistent with Landau's heuristic ideas – this may not actually be sharp [21]. Moreover, while Foias, Holm and Titi obtained an attractor dimension estimate of Re $^{3/2}$ in [11], their definition of Re was different from that used here.

The Table compares estimates of various solution properties. These estimates improve as one passes from the Navier-Stokes equations to LANS-α, to Bardina and then to Leray-α, with the Leray-α model showing the most improvement. The milder activity shown by Leray is illustrated by the much tighter estimates for variables $\langle\kappa_{n,0}^2\rangle$ in the penultimate row. These quantities involve higher derivatives, as explained in Section 3 and the appendix. Based on a definition for the Navier-Stokes equations

$$F_n = H_n + \tau^2\|\nabla^n \boldsymbol{f}\|_2 \tag{1.11}$$

where the characteristic time τ defined in Section 3, the $\kappa_{n,0}$ are defined as

$$\kappa_{n,0}^{2n} = \frac{F_n}{F_0} = \frac{\int_\Omega k^{2n}\left(\hat{\boldsymbol{u}}^2 + \tau^2\hat{\boldsymbol{f}}^2\right)\mathrm{d}V}{\int_\Omega \left(\hat{\boldsymbol{u}}^2 + \tau^2\hat{\boldsymbol{f}}^2\right)\mathrm{d}V}\,. \tag{1.12}$$

These are the $2n^{th}$ Fourier-moments of the velocity field. Being squares of inverse lengths, the time-averages $\langle\kappa_{n,0}^2\rangle$ indicate the expected activity as a function of length-scale, with emphasis on activity in the higher wave-numbers at higher values of n. The Table shows that the asymptotic exponent of 17/12 in $\langle\kappa_{n,0}^2\rangle$ as $n \to \infty$ for Leray-α is a great improvement over the 11/4 for LANS-α and Bardina. It should be noted, however, as explained in Section 3, that the definitions of $\kappa_{n,0}$ are different for each model, although they play the same physical role.

2. WHY DO GENERAL α-MODEL ESTIMATES DIFFER FROM NAVIER-STOKES ESTIMATES?

What is it about the filtering that makes the α-models different from the Navier-Stokes equations? This can be illustrated by looking at Leray's energy inequality for the Navier-Stokes equations [2–10]. The semi-norms H_n are defined on a periodic domain $\Omega = [0, L]^3$

$$H_n = \int_\Omega |\nabla^n \boldsymbol{u}|^2\,\mathrm{d}V\,. \tag{2.1}$$

The energy $H_0 = \|\boldsymbol{u}\|_2^2$ satisfies

$$\frac{1}{2}\frac{dH_0}{dt} \leq -\nu H_1 + \|\boldsymbol{f}\|_2 H_0^{1/2}\,. \tag{2.2}$$

	NS	LANS-α	Leray	Bardina
Bounded $H_1(t)$?	No	Yes	Yes	Yes
$\langle H_1 \rangle$?	Re^{3}	$\mathrm{Re}^{5/2}$	$\mathrm{Re}^{7/3}$	$\mathrm{Re}^{5/2}$
$\ell \lambda_k^{-1}$	$\mathrm{Re}^{3/4}$	$\mathrm{Re}^{5/8}$	$\mathrm{Re}^{7/12}$	$\mathrm{Re}^{5/8}$
$\langle H_2 \rangle$	–	Re^{3}	$\mathrm{Re}^{8/3}$	Re^{3}
$\langle H_3 \rangle$	–		Re^{3}	
$d_F(\mathcal{A})$	–	$\mathrm{Re}^{9/4}$	$\mathrm{Re}^{9/7}$	$\mathrm{Re}^{9/5}$
$\langle \\|\boldsymbol{u}\\|_\infty^2 \rangle$	–	$\mathrm{Re}^{11/4}$	$\mathrm{Re}^{5/2}$	$\mathrm{Re}^{11/4}$
$\langle \\|\nabla \boldsymbol{u}\\|_\infty \rangle$	–	$\mathrm{Re}^{35/16}$	$\mathrm{Re}^{17/12}$	$\mathrm{Re}^{35/16}$
$\ell^2 \langle \kappa_{n,0}^2 \rangle$	–	$\mathrm{Re}^{11/4-7/(4n)} (\ln \mathrm{Re})^{1/n}$	$\mathrm{Re}^{17/12-5/(12n)} (\ln \mathrm{Re})^{1/n}$	$\mathrm{Re}^{11/4-7/(4n)} (\ln \mathrm{Re})^{1/n}$
$\ell^2 \langle \kappa_{1,0}^2 \rangle$	$\mathrm{Re} (\ln \mathrm{Re})$	$\mathrm{Re} (\ln \mathrm{Re})$	$\mathrm{Re} (\ln \mathrm{Re})$	$\mathrm{Re} (\ln \mathrm{Re})$

TABLE 1.1. Upper bounds for the Navier-Stokes, LANS-α , Leray-α, Bardina models with coefficients omitted. In the α-model cases, these coefficients diverge as $\alpha \to 0$. The variables $\kappa_{n,r}$ in the last two rows are defined in Section 3.

Time-averaging (2.2) and using (1.8) and (1.10) yields

$$\langle H_1\rangle \leq \nu^2 L^3 \ell^{-4} Gr\,\mathrm{Re} \;\leq c\,\nu^2 L^3 \ell^{-4}\left(\mathrm{Re}^{\,3} + \mathrm{Re}^{\,2}\right). \tag{2.3}$$

The energy dissipation rate $\varepsilon = \nu L^{-3}\langle H_1\rangle$ is bounded by

$$\varepsilon \leq c\,\nu^3 \ell^{-4}\left(\mathrm{Re}^{\,3} + \mathrm{Re}^{\,2}\right).$$

To leading order the inverse Kolmogorov length $\lambda_k^{-1} = (\varepsilon/\nu^3)^{1/4}$ is then bounded above by

$$\ell\lambda_k^{-1} \leq c\,\mathrm{Re}^{\,3/4}. \tag{2.4}$$

This upper bound conforms with the generally accepted scaling law for the inverse Kolmogorov length with the Reynolds number [19, 20]. Now we turn to improvements on this for the three α-models.

In what follows, the two dimensionless volumes V_ℓ and V_α are defined by

$$V_\ell = \left(\frac{L}{\ell}\right)^3, \quad V_\alpha = \left(\frac{L}{(\ell\alpha)^{1/2}}\right)^3, \tag{2.5}$$

and $\lambda_1 > 0$ is smallest eigenvalue of the Stokes operator.

2.1. The LANS-α model. The key to the improved results for the LANS-α equations is due to Foias, Holm and Titi [12] who showed that the integral of the product $\boldsymbol{u}\cdot\boldsymbol{v}$ has two properties. $\boldsymbol{v}$ is defined in (1.1). The first property is

$$\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}V = \int_\Omega \left\{|\boldsymbol{u}|^2 + \alpha^2|\nabla\boldsymbol{u}|^2\right\} dV \tag{2.6}$$

while the second is

$$\begin{aligned}\frac{d}{dt}\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}V &= \int_\Omega (\boldsymbol{u}_t\cdot(1-\alpha^2\Delta)\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{v}_t)\,\mathrm{d}V\\ &= \int_\Omega \left\{\boldsymbol{u}\cdot\left[1-\alpha^2\Delta)\boldsymbol{u}_t\right] + \boldsymbol{u}\cdot\boldsymbol{v}_t\right\}\,\mathrm{d}V\\ &= 2\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}_t\,\mathrm{d}V,\end{aligned} \tag{2.7}$$

where two integrations by parts have occurred between the first and second lines. From (2.6) we clearly we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(H_0 + \alpha^2 H_1) &= -\nu(H_1 + \alpha^2 H_2) + \int_\Omega \boldsymbol{u}\cdot\boldsymbol{f}\,\mathrm{d}V\\ &\leq -\nu(H_1 + \alpha^2 H_2) + \|\boldsymbol{u}\|_2\|\boldsymbol{f}\|_2.\end{aligned} \tag{2.8}$$

An absorbing ball for H_1 can then be calculated (see [12]): this is the key result that is missing for the Navier-Stokes equations. It is also possible to estimate the time averages of $\langle H_1\rangle$ and $\langle H_2\rangle$ which can be found, as in (2.2), to satisfy

$$\nu L^{-3}\langle H_1 + \alpha^2 H_2\rangle \le \nu^3\ell^{-4}\mathrm{Re}\; Gr \le c\,\nu^3\ell^{-4}\,\mathrm{Re}^3 . \tag{2.9}$$

The upper bound on $\langle H_2\rangle$, written as

$$\alpha^2\ell\nu^{-2}\langle H_2\rangle \le c\,V_\ell\,\mathrm{Re}^3 , \tag{2.10}$$

can then be used to improve the estimate for $\langle H_1\rangle$ by using both the simple inequality $\langle H_1\rangle \le \langle H_0\rangle^{1/2}\langle H_2\rangle^{1/2}$ together with the velocity U defined by $U^2 = L^{-3}\langle H_0\rangle$. This improvement is

$$\langle H_1\rangle \le c\,\nu^2L^3\ell^{-3}\alpha^{-1}\,\mathrm{Re}^{5/2} . \tag{2.11}$$

This improves the Navier-Stokes result in (2.4) to

$$\ell\lambda_k^{-1} \le c\left(\frac{\ell}{\alpha}\right)^{1/4}\mathrm{Re}^{5/8} . \tag{2.12}$$

Hence the energy dissipation rate ε is also bounded above by $\mathrm{Re}^{5/2}$ but the improved estimate blows up when $\alpha \to 0$; no equivalent result is implied for the $3D$ Navier-Stokes equations.

Foias, Holm and Titi [12] have made two estimates of the fractal dimension $d_F(\mathcal{A})$ of the global attractor $\mathcal{A}$, the first in terms of the Grashof number Gr but the second in terms of $\overline{\varepsilon}$ which includes the H_2-norm. Their definition of $\overline{\varepsilon}$ is

$$\overline{\varepsilon} = \lambda_1^{3/2}\nu\langle H_1 + \alpha^2 H_2\rangle \tag{2.13}$$

where λ_1 is the smallest eigenvalue of the Stokes operator. Their result is [12]

$$d_F(\mathcal{A}) \le c\,\frac{\lambda_1^{-3/2}}{(\alpha^2\lambda_1)^{3/4}}\left(\frac{\overline{\varepsilon}}{\nu^3}\right)^{3/4} . \tag{2.14}$$

We now use the estimate for $\langle H_1 + \alpha^2 H_2\rangle$ from (2.9). Thus

$$ad5\overline{\varepsilon} \le c\,(L\lambda_1^{1/2})^3\nu^3\ell^{-4}\,\mathrm{Re}^3 , \tag{2.15}$$

which turns the result of [12] into

$$d_F(\mathcal{A}) \le c\,\frac{V_\alpha V_\ell^{1/2}}{(L^2\lambda_1)^{9/8}}\,\mathrm{Re}^{9/4} , \tag{2.16}$$

where $L^2\lambda_1 = 4\pi^2$. The right hand side blows up as $\alpha \to 0$ through V_α. This re-working of the Foias, Holm and Titi estimate [12] can be found in [21].

2.2. The Leray-α model. Below we find some estimates for the Leray-α equations in the same manner as for the Navier-Stokes and LANS-α equations. The results are given in the table.

$$\begin{aligned} \int_\Omega |\boldsymbol{v}|^2\,\mathrm{d}V &= \int_\Omega (1-\alpha^2\Delta)\boldsymbol{u}\cdot(1-\alpha^2\Delta)\boldsymbol{u}\,\mathrm{d}V \\ &= \int_\Omega \left[|\boldsymbol{u}|^2 + 2\alpha^2|\nabla\boldsymbol{u}|^2 + \alpha^4|\Delta\boldsymbol{u}|^2\right]\mathrm{d}V \\ &= H_0 + 2\alpha^2 H_1 + \alpha^4 H_2\,. \end{aligned} \tag{2.17}$$

Now consider the Leray-α equations (1.4) which gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{v}|^2\,\mathrm{d}V = -\nu\int_\Omega |\nabla\boldsymbol{v}|^2\,\mathrm{d}V + \int_\Omega \boldsymbol{v}\cdot\boldsymbol{f}\,\mathrm{d}V\,, \tag{2.18}$$

leading to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(H_0 + 2\alpha^2H_1 + \alpha^4H_2) \\ &\qquad \le -\nu(H_1 + 2\alpha^2H_2 + \alpha^4H_3) + (1+\alpha^2\ell^{-2})\|\boldsymbol{f}\|_2\|\boldsymbol{u}\|_2\,. \end{aligned} \tag{2.19}$$

Time-averaging we obtain

$$\begin{aligned} \nu\langle H_1 + 2\alpha^2H_2 + \alpha^4H_3\rangle &\le (1+\alpha^2\ell^{-2})L^3 f_{rms}U \\ &\le (1+\alpha^2\ell^{-2})\nu^3L^3\ell^{-4}Gr\,\mathrm{Re} \\ &\le c\,(1+\alpha^2\ell^{-2})\nu^3V_\ell\ell^{-1}(\mathrm{Re}^3 + \mathrm{Re}^2)\,. \end{aligned} \tag{2.20}$$

Thus we can write

$$\langle H_3\rangle \le c\,\alpha^{-4}(1+\alpha^2\ell^{-2})\nu^2V_\ell\ell^{-1}(\,\mathrm{Re}^3 + \,\mathrm{Re}^2)\,. \tag{2.21}$$

This can be exploited to bring down the estimate for $\langle H_1\rangle$ from Re ^{3}. In fact we know that $H_1 \le H_3^{1/3}H_0^{2/3}$ and $H_2 \le H_3^{2/3}H_0^{1/3}$. Thus

$$\begin{aligned} \langle H_1\rangle &\le \langle H_3\rangle^{1/3}\langle H_0\rangle^{2/3} \\ &= \langle H_3\rangle^{1/3}L^2(\nu\ell^{-1}\mathrm{Re}\,)^{4/3} \\ &\le c\,\nu^2(1+\ell^{-2}\alpha^2)^{1/3}V_\ell\ell^{1/3}\alpha^{-4/3}\,\mathrm{Re}^{7/3} \\ &\le c\,\nu^2(1+\ell^{-2}\alpha^2)^{1/3}V_\alpha^{8/9}V_\ell^{4/9}L^{-1}\,\mathrm{Re}^{7/3}\,. \end{aligned} \tag{2.22}$$

This is an improvement on the $\mathrm{Re}^{5/2}$ estimate for LANS-α which, in turn, is an improvement on the Re^3 for Navier-Stokes. Moreover,

$$\begin{aligned}\langle H_2\rangle &\le c\,\nu^2(1+\alpha^2\ell^{-2})^{2/3}V_\ell\ell^{-1/3}\alpha^{-8/3}\,\mathrm{Re}^{8/3}\\ &\le c\,\nu^2(1+\ell^{-2}\alpha^2)^{2/3}V_\alpha^{16/9}V_\ell^{2/9}L^{-3}\,\mathrm{Re}^{8/3}.\end{aligned} \tag{2.23}$$

which is an improvement on the Re^3 for LANS-α .

Cheskidov, Holm, Olson and Titi [14] have proved that the Hausdorff and fractal dimensions of the global attractor of the Leray-α model are bounded by

$$d_H(\mathcal{A}) \le d_F(\mathcal{A}) \le \left(\frac{L}{\ell_d}\right)^{12/7}\left(1+\frac{L}{\alpha}\right)^{9/14} \tag{2.24}$$

where

$$\ell_d^{-4} = \varepsilon_{\mathsf{Leray}}\nu^{-3} \tag{2.25}$$

and where

$$\begin{aligned}\varepsilon_{\mathsf{Leray}} &= L^{-3}\nu\langle H_1 + 2\alpha^2 H_2 + \alpha^4 H_3\rangle\\ &\le c\,L^{-3}\left(1+\alpha^2\ell^{-2}\right)\nu^3 V_\ell\ell^{-1}\,\mathrm{Re}^3\end{aligned} \tag{2.26}$$

Thus

$$\ell_d^{-4} = \varepsilon_{\mathsf{Leray}}\nu^{-3} \le c\,(1+\alpha^2\ell^{-2})\ell^{-4}\,\mathrm{Re}^3. \tag{2.27}$$

Thus we have

$$d_H(\mathcal{A}) \le d_F(\mathcal{A}) \le V_\ell^{4/7}(1+\alpha^2\ell^{-2})^{3/7}\left(1+\frac{L}{\alpha}\right)^{9/14}\mathrm{Re}^{9/7}, \tag{2.28}$$

which is entered in the table.

2.3. The Bardina model. Now consider the Bardina model [15] given in (1.5)

$$\boldsymbol{v}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nu\Delta\boldsymbol{v} - \nabla p + \boldsymbol{f}\,, \quad \boldsymbol{v} = \boldsymbol{u} - \alpha^2\Delta\boldsymbol{u} \tag{2.29}$$

with $\operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{v} = 0$. Now we know that

$$\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}V = \int_\Omega\left\{|\boldsymbol{u}|^2 + \alpha^2|\nabla\boldsymbol{u}|^2\right\}\mathrm{d}V = H_0 + \alpha^2 H_1 \tag{2.30}$$

and

$$\frac{d}{dt}\int_{\Omega} \boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}V = \int_{\Omega}(\boldsymbol{u}_t\cdot\boldsymbol{v}+\boldsymbol{u}\cdot\boldsymbol{v}_t)\,\mathrm{d}V = 2\int_{\Omega}\boldsymbol{u}\cdot\boldsymbol{v}_t\,\mathrm{d}V\,. \tag{2.31}$$

Therefore $\langle H_1 + \alpha^2 H_2\rangle$ is found to satisfy the same estimates as the LANS-α model

$$\nu L^{-3}\langle H_1+\alpha^2H_2\rangle \le \nu^3\ell^{-4}\mathrm{Re}\ Gr \le c\,\nu^3\ell^{-4}\,\mathrm{Re}^3\,, \tag{2.32}$$

with

$$\ell\lambda_k^{-1} \le c\left(\frac{\ell}{\alpha}\right)^{1/4}\ \mathrm{Re}^{5/8}\,. \tag{2.33}$$

These latter results are exactly as in LANS-α. The estimate for the dimension $d_F(\mathcal{A})$ of the global attractor $\mathcal{A}$ given in [15] is proportional to Gr^2. This, however, can be improved by noting that their estimate is dependent upon $\langle H_2\rangle$ whose upper bound can be improved to Re^3 as opposed to $Gr^2 \le c\,\mathrm{Re}^4$. With this improvement ait is found that the estimate for $d_{F,\mathsf{Bard}}(\mathcal{A})$ in [15] converts to

$$d_{F,\mathsf{Bard}}(\mathcal{A}) \le \left(\frac{L}{\alpha}\right)^{18/5}\ \mathrm{Re}^{9/5}\,. \tag{2.34}$$

3. ESTIMATES FOR $\langle\kappa_{n,r}^2\rangle$ FOR ALL THREE MODELS

We begin by forming the combination

$$F_n = H_n + \tau^2\|\nabla^n\boldsymbol{f}\|_2^2\,, \tag{3.1}$$

where the quantity τ

$$\tau = \ell^2\nu^{-1}(Gr\ln Gr)^{-1/2}\,. \tag{3.2}$$

For the LANS-α and Bardina models we define the combination

$$J_n = F_n + 2\alpha^2F_{n+1} \tag{3.3}$$

and for the Leray-α-model the combination

$$L_n = F_n + 2\alpha^2F_{n+1} + \alpha^4F_{n+2} \tag{3.4}$$

Theorem 3.1. *As $Gr \to \infty$, for $n \geq 1$, $1 \leq p \leq n$, J_n and L_n satisfy*

$$\frac{dJ_n}{dt} = -\frac{1}{4}\nu\frac{J_n^{1+1/p}}{J_{n-p}^{1/p}} + c_{n,\alpha}\nu^{-1}\|\boldsymbol{u}\|_\infty^2 J_n + c_1\nu\ell^{-2}\mathrm{Re}\,(\ln \mathrm{Re}\,)J_n \tag{3.5}$$

$$\frac{dL_n}{dt} = -\frac{1}{3}\nu\frac{L_n^{1+1/p}}{L_{n-p}^{1/p}} + c_{n,\alpha}\|\nabla\boldsymbol{u}\|_\infty L_n + c_1\nu\ell^{-2}\mathrm{Re}\,(\ln \mathrm{Re}\,)L_n \tag{3.6}$$

and, for $n = 0$,

$$\begin{aligned}\frac{1}{2}\frac{dJ_0}{dt} &\leq -\nu J_1 + c_1\nu\ell^{-2}\mathrm{Re}\,(\ln \mathrm{Re}\,)J_0\,,\\ \frac{1}{2}\frac{dL_0}{dt} &\leq -\nu L_1 + c_1\nu\ell^{-2}\mathrm{Re}\,(\ln \mathrm{Re}\,)L_0\,.\end{aligned} \tag{3.7}$$

Proof. The proof of these follows closely to that for LANS-α in [21] and will not be repeated here. ◻

Important Remark. The $\|\nabla\boldsymbol{u}\|_\infty L_n$ in the middle term in (3.6) is neither valid for LANS-α nor Bardina but must be replaced by $\nu^{-1}\|\boldsymbol{u}\|_\infty^2 J_n$ (see [21]). Estimates for LANS-α can be found in that paper while those for Bardina follow in a similar manner.

However, estimates for Leray-α come out to be much sharper than those for LANS-α and Bardina because of the $\|\nabla\boldsymbol{u}\|_\infty$-term in (3.6) as opposed to the $\nu^{-1}\|\boldsymbol{u}\|_\infty^2$-term in (3.5). To show this define

$$\kappa_{n,r} = \left(\frac{L_n}{L_r}\right)^{1/(2(n-r))}. \tag{3.8}$$

Then from (3.6)

$$\langle\kappa_{n,r}^2\rangle \leq c_{n,r}\nu^{-1}\langle\|\nabla\boldsymbol{u}\|_\infty\rangle + c_1\ell^{-2}\mathrm{Re}\,(\ln \mathrm{Re}\,)\,. \tag{3.9}$$

To estimate the right hand side of (3.9), Agmon's inequality gives

$$\begin{aligned}\langle\|\nabla\boldsymbol{u}\|_\infty\rangle &\leq \langle H_2\rangle^{1/4}\langle H_3\rangle^{1/4}\\ &\leq c\,L^{-2}\nu(1+\alpha^2\ell^{-2})^{5/12}V_\ell^{1/18}V_\alpha^{35/36}\,\mathrm{Re}^{17/12}\,,\end{aligned} \tag{3.10}$$

Thus

$$L^2\langle\kappa_{n,r}^2\rangle \leq c\,(1+\alpha^2\ell^{-2})^{5/12}V_\ell^{1/18}V_\alpha^{35/36}\,\mathrm{Re}^{17/12}\,, \tag{3.11}$$

or

$$\ell^2\langle\kappa_{n,r}^2\rangle \leq c\,(1+\alpha^2\ell^{-2})^{5/12}V_\ell^{-11/18}V_\alpha^{35/36}\,\mathrm{Re}^{17/12}\,. \tag{3.12}$$

We can also estimate

$$\begin{aligned}(3.13)\qquad \langle \|\boldsymbol{u}\|_\infty^2\rangle &\le \langle H_1\rangle^{1/2}\langle H_2\rangle^{1/2}\\ &\le c\,\nu^2 L^{-2}(1+\alpha^2\ell^{-2})^{1/2} V_\ell^{1/3} V_\alpha^{4/3}\ \mathrm{Re}^{5/2},\end{aligned}$$

These all give estimates for $\langle \kappa_{n,r}^2\rangle$. By choosing $r = 1$ one can achieve an improvement for the bound on $\langle \kappa_{n,0}^2\rangle$ by writing

$$(3.14)\qquad \langle \kappa_{n,0}^2\rangle = \langle \kappa_{n,1}^{2(n-1)/n}\kappa_{1,0}^{2/n}\rangle \le \langle \kappa_{n,1}^2\rangle^{(n-1)/n}\langle \kappa_{1,0}^2\rangle^{1/n}$$

and using estimates for $\langle \kappa_{1,0}^2\rangle$. This is the origin of the n-dependence in the exponents in the Table. An explicit example is the calculation for LANS-α given in [21].

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