AE2 Mathematics
Solutions to Example Sheet 2: Fourier Series

1) \( f(x) = |\sin x| \) on \((-\pi, \pi)\) with \( L = \pi \): \( f(x) \) is an even function so \( b_n = 0 \). On \([0, \pi]\) we have \( |\sin x| = \sin x \).

\[ a_0 = \frac{2}{\pi} \int_{0}^{\pi} |\sin x| \, dx = \frac{2}{\pi} \int_{0}^{\pi} \sin x \, dx = \frac{4}{\pi} \]

where \( \cos n\pi = (-1)^n \).

\[ a_n = \frac{2}{\pi} \int_{0}^{\pi} \sin x \cos nx \, dx \]

We also know that \( 2 \sin x \cos nx = \sin[(n+1)x] - \sin[(n-1)x] \) so for \( n \geq 2 \) (note: \( a_1 = 0 \))

\[ a_n = \frac{1}{\pi} \int_{0}^{\pi} \sin[(n+1)x] \, dx - \frac{1}{\pi} \int_{0}^{\pi} \sin[(n-1)x] \, dx \]

\[ = -\frac{1}{\pi} \left[ \cos[(n+1)x] \right]_0^\pi + \frac{1}{\pi} \left[ \cos[(n-1)x] \right]_0^\pi \]

\[ = -\frac{1}{\pi} \left[ (-1)^{n+1} - 1 \right] \left[ \frac{1}{n+1} - \frac{1}{n-1} \right] = \frac{2}{\pi} \left[ (-1)^{n+1} - 1 \right] \frac{1}{n^2 - 1} \]

Therefore

\[ a_n = \begin{cases} 
0 & n = 2m + 1 \text{ (odd)} \\
\frac{4}{\pi(4m^2-1)} & n = 2m \text{ (even)} 
\end{cases} \]

and so

\[ |\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\cos 2mx}{4m^2-1} \]

2) \( f(x) = \begin{cases} 
x(1-x) & 0 \leq x \leq 1 \\
0 & -1 \leq x \leq 0 
\end{cases} \)

\( L = 1 \) & the function is neither odd nor even.

\[ a_0 = \int_{-1}^{1} f(x) \, dx = \int_{0}^{1} x(1-x) \, dx = 1/6 \]

\[ b_n = \int_{0}^{1} x(1-x) \sin n\pi x \, dx \]

By evaluating these integrals, one finds

\[ a_n = \begin{cases} 
0 & n = 2m + 1 \text{ (odd)} \\
-\frac{1}{2m^2\pi} & n = 2m \text{ (even)} 
\end{cases} \]

\[ b_n = \begin{cases} 
\frac{4}{(2m+1)^2\pi} & n = 2m + 1 \text{ (odd)} \\
0 & n = 2m \text{ (even)} 
\end{cases} \]
thus giving the answer. The odd extension of \( f(x) = x(1 - x) \), originally defined on \( 0 \leq x \leq 1 \), on the range \(-1 \leq x \leq 1\), has \( b_n = 2 \int_0^1 x(1 - x) \sin(n\pi x) \, dx \). Thus the answer is an odd sine-series with coefficient twice that above, namely \( \frac{8}{(2m+1)^2\pi^2} \).

3) \( x \sin x \) is an even function over \((-\pi, \pi)\) so \( b_n = 0 \) and \( a_n = \frac{2}{\pi} \int_0^\pi x \sin nx \, dx \). Using the fact that \( 2 \sin x \cos nx = \sin[(n + 1)x] - \sin[(n - 1)x] \), we have (except for \( n = 1 \))
\[
a_n = \frac{1}{\pi} \int_0^\pi x \sin[(n + 1)x] - \sin[(n - 1)x] \, dx = \frac{2(-1)^{n+1}}{n^2 - 1} \quad \text{by parts}
\]
Thus \( a_0 = 2 \) and \( a_1 \) is
\[
\therefore \quad a_1 = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x \, dx = \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx = -\frac{1}{2} \quad \text{(by parts)}
\]
\[
x \sin x = 1 - \frac{1}{2} \cos x + 2 \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2 - 1} \cos nx
\]

4)

The figure shows that \( f(x) \) is even about \( x = 0 \); thus \( L = \pi \) and \( b_n = 0 \) and
\[
a_0 = \frac{2}{\pi} \left\{ \int_0^{\pi/2} x \, dx + \int_{\pi/2}^{\pi} (x - \pi) \, dx \right\} = 0
\]
\[
a_n = \frac{2}{\pi} \left\{ \int_0^{\pi/2} x \cos nx \, dx + \int_{\pi/2}^{\pi} (x - \pi) \cos nx \, dx \right\}
\]
\[
= \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx - 2 \int_{\pi/2}^{\pi} \cos nx \, dx
\]
By parts
\[
\int_0^{\pi} \cos nx \, dx = \frac{1}{n^2} \left[ nx \sin nx + \cos nx \right]_0^{\pi} = \frac{(-1)^n - 1}{n^2} = -\frac{2}{(2m + 1)^2}
\]
when \( n = 2m + 1 \) and zero when \( n \) is even. Moreover,
\[
\int_{\pi/2}^{\pi} \cos nx \, dx = \frac{1}{n} \left[ \sin nx \right]_{\pi/2}^{\pi} = -\frac{\sin \frac{1}{2} n \pi}{n} = -\frac{(-1)^m}{2m + 1}
\]
thus giving the answer as advertised.