Ae2 Mathematics: 1st and 2nd order PDEs

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These notes are not identical word-for-word with my lectures which will be given on a WB. Some of these notes may contain more examples than the corresponding lecture while in other cases the lecture may contain more detailed working. I will not be handing out copies of these notes – you are therefore advised to attend lectures and take your own.

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¹Do not confuse me with Dr J. Gibbons who is also in the Mathematics Dept.
1st order PDEs & the method of characteristics

1.1 The derivation of the auxiliary equations

Consider the semi-linear 1st order partial differential equation \(^2\) (PDE)

\[
P(x, y)u_x + Q(x, y)u_y = R(x, y, u)
\]  

(1.1)

where \(P\) and \(Q\) are continuous functions and \(R\) is not necessarily linear\(^3\) in \(u\).

Consider solutions represented as a family of surfaces (which one depends on our boundary conditions). Below is a picture of one of these surfaces which we’ll call

\[
F(x, y, u) = 0 \quad u = u(x, y)
\]  

(1.2)

in \((x, y, u)\)-space.

\[
F(x, y, u) = 0
\]

Because \(F = 0\) in (1.2), it must be true that \(dF = 0\) and so the chain rule gives

\[
0 = dF = F_x dx + F_y dy + F_u du
\]  

(1.3)

\[
du = u_x dx + u_y dy
\]  

(1.4)

Combining these two gives

\[
0 = F_x dx + F_y dy + (u_x dx + u_y dy) F_u .
\]  

(1.5)

Re-arranging terms we have

\[
u_x[F_u dx] + u_y[F_u dy] = -[F_x dx + F_y dy] .
\]  

(1.6)

Now compare this with our PDE in (1.1): a comparison of coefficients gives

\[
F_u dx = P; \quad F_u dy = Q; \quad -[F_x dx + F_y dy] = R .
\]  

(1.7)

Now, because \(-[F_x dx + F_y dy] = F_u du\) we can represent (1.7) as a series of ratios which are called the auxiliary equations

\(^2\)The subscript notation \(u_x = \partial u/\partial x\) and \(u_{xy} = \partial^2 u/\partial x \partial y\) etc is used throughout.

\(^3\)This PDE is also said to be quasi-linear if \(P\) and \(Q\) are dependent on \(u\).
\[
\begin{align*}
\frac{dx}{P} &= \frac{dy}{Q} = \frac{du}{R} \quad R \neq 0 \quad (1.8) \\
\frac{dx}{P} &= \frac{dy}{Q} = 0 \quad \text{and} \quad du = 0 \quad (R = 0) \quad (1.9)
\end{align*}
\]

1. The first pair in the auxiliary equations can be re-written as a differential equation in \(x, y\) without reference to \(u\)

\[
\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}.
\]

(1.10)

In principle, this can be solved to give

\[
\lambda(x, y) = c_1
\]

(1.11)

where \(c_1\) is a constant of integration. These curves or lines are called the characteristics or characteristic curves of the PDE\(^4\). They form a family of curves because of the arbitrariness of the constant \(c_1\).

2. If \(R = 0\) we have \(du = 0\) as in the second line of (1.8), in which case \(u = \text{const} = c_2\) on characteristics.

If \(R \neq 0\) as in the first line of (1.8) then one of the other pair of differential equations must be solved to get \(u = g(x, y, c_2)\) on characteristics \(\lambda(x, y) = c_1\), where \(c_2\) is another constant of integration.

3. The two arbitrary constants \(c_1\) and \(c_2\) can be thought of as being related by an arbitrary function \(c_2 = f(c_1)\).

1.2 Seven examples

**Example 1:** Consider the simple PDE

\[
u_x + u_y = 0.
\]

(1.12)

**Solution:** Obviously \(P = 1\), \(Q = 1\) and \(R = 0\). Therefore the auxiliary equations (1.8) are

\[
\frac{dx}{1} = \frac{dy}{1} \quad \text{and} \quad du = 0.
\]

(1.13)

Clearly the characteristics are the family of curves \(y = x + c_1\) on which \(u = \text{const} = c_2\). The arbitrary constants \(c_1\) and \(c_2\) are related by \(c_2 = f(c_1)\) in which case \(u = f(x - y)\) for an

\(^4\)They are also sometimes referred to as Riemann invariants,
arbitrary differentiable function $f$: this is the general solution. It can easily be checked that this is indeed a solution of (1.13) by writing $X = x - y$ and $u = f(X)$. Then

$$
\begin{align*}
    u_x &= X_x f'(X) \\
    u_y &= X_y f'(X)
\end{align*}
$$

(1.14)

However, $X_x = 1$ and $X_y = -1$ and so $u_x + u_y = 0$.

For Example 1, the characteristics are the family of straight lines $y = x + c_1$.

Example 2: Consider the simple PDE

$$
x u_x - y u_y = 0.
$$

(1.15)

subject to the boundary conditions $u = x^4$ on the line $y = x$.

Solution: Obviously $P = x$, $Q = -y$ and $R = 0$. Therefore the auxiliary equations (1.8) are

$$
\frac{dx}{x} = -\frac{dy}{y} \quad \text{and} \quad du = 0.
$$

(1.16)

Clearly the characteristics come from

$$
\int \frac{dx}{x} + \int \frac{dy}{y} = \text{const}
$$

(1.17)

from which we discover that $\ln(xy) = \text{const}$. Thus the characteristics are the family of hyperbolae $xy = c_1$. On these characteristics $u = \text{const} = c_2$ in which case

$$
u = f(xy)
$$

(1.18)

for an arbitrary differentiable function $f$: this is the general solution. It can easily be checked that this is indeed a solution of (1.15) by writing $X = xy$ and $u = f(X)$. Then $u_x = X_x f'(X)$ and $u_y = X_y f'(X)$ with $X_x = y$ and $X_y = x$ and so $x u_x - y u_y = 0$.

Application of the BCs $u = x^4$ on the line $y = x$ now determines $f$ because on $y = x$

$$
x^4 = f(x^2)
$$

(1.19)

and so $f(t) = t^2$: however, $f(t)$ is only defined$^5$ for $t \geq 0$. Thus our solution is

$$
u(x, y) = x^2 y^2 \quad \text{for } xy \geq 0,
$$

(1.20)

which means that it is only valid in the 1st and 3rd quadrants of the characteristic plane.

$^5$The variable $t$ is simply the argument of the function $f(t)$: the fact that it is called $t$ has no meaning – we could designate it by any symbol we wish.
For Example 2, the characteristics are the family of hyperbolae $xy = c_1 \geq 0$.

Example 3: Consider the PDE
\[ xu_x + yu_y = u . \]  
subject to the boundary conditions $u = y^2$ on the line $x = 1$.

Solution: Clearly $P = x$, $Q = y$ and $R = u$. Therefore the auxiliary equations (1.8) are
\[ \frac{dx}{x} = \frac{dy}{y} = \frac{du}{u} . \]  
Clearly the characteristics come from
\[ \int \frac{dy}{y} - \int \frac{dx}{x} = \text{const} \]  
from which we discover that $\ln \left( \frac{y}{x} \right) = \text{const}$. Thus the characteristics are the family of lines $y = xc_1$: these are a fan of straight lines all passing through the origin. Now integrate one of the other pair (either will do): $\ln \left( \frac{u}{x} \right) = \text{const}$ which means that $u = xc_2$. Therefore, on characteristics
\[ u = xf \left( \frac{y}{x} \right) \]  
for an arbitrary differentiable function $f$: this is the general solution. Now applying the BCs: $u = y^2$ on $x = 1$ we obtain $f(y) = y^2$. Therefore, with these BCs, the solution is
\[ u = x \left( \frac{y}{x} \right)^2 = y^2 / x . \]  

Example 4: Consider the PDE
\[ yu_x + xu_y = x^2 + y^2 , \]  
subject to the boundary conditions
\[ u = \begin{cases} 1 + x^2 & \text{on } y = 0 \\ 1 + y^2 & \text{on } x = 0 \end{cases} \]  

Solution: $P = y$, $Q = x$ and $R = x^2 + y^2$. Therefore the auxiliary equations (1.8) are
\[ \frac{dx}{y} = \frac{dy}{x} = \frac{du}{x^2 + y^2} . \]
Characteristics come from the integral
\[ \int (x\,dx - y\,dy) = \text{const} \] (1.29)
which gives \( x^2 - y^2 = c_1 \) and
\[
\begin{align*}
 du & = y^{-1}(x^2 + y^2)\,dx \\
 & = y\,dx + x^2y^{-1}\,dx \\
 & = y\,dx + x\,dy \quad \text{on characteristics} \\
 & = d(xy)
\end{align*}
\] (1.30)
which integrates to
\[ u = xy + c_2 . \] (1.31)
Therefore, as the general solution, we have
\[ u = xy + f(x^2 - y^2) . \] (1.32)
Applying the BCs:
\[
\begin{align*}
1 + x^2 & = f(x^2) \quad \Rightarrow \quad f(t) = 1 + t , \quad t \geq 0 , \\
1 + y^2 & = f(-y^2) \quad \Rightarrow \quad f(t) = 1 - t , \quad t \leq 0 .
\end{align*}
\] (1.33)
Thus we end up with
\[ f(t) = 1 + |t| \] (1.34)
so
\[ u = xy + 1 + |x^2 - y^2| . \] (1.35)

**Example 5: (Exam 2001)** Show that the PDE
\[ yu_x - 3x^2yu_y = 3x^2u , \] (1.36)
has a general solution of the form
\[ yu(x, y) = f(x^3 + y) \] (1.37)
where \( f \) is an arbitrary function.

(i) If you are given that
\[ u(0, y) = y^{-1}\tanh y \] (1.38)
on the line \( x = 0 \), show that
\[ yu(x, y) = \tanh (x^3 + y) . \] (1.39)

(ii) If you given that \( u(x, 1) = x^6 \) on \( y = 1 \) show that
\[ yu(x, y) = (x^3 + y - 1)^2 . \] (1.40)
Solution: \( P = y, \ Q = -3x^2y \) and \( R = 3x^2u \). Thus the auxiliary equations are

\[
\frac{dx}{y} = -\frac{dy}{3x^2y} = \frac{du}{3x^2u},
\]

(1.41)

which gives characteristics as solutions of \( \frac{dy}{dx} = -3x^2 \). These are the family of curves \( y + x^3 = c_1 \). Then we also have

\[
\frac{du}{u} = -\frac{dy}{y},
\]

(1.42)

from which we discover that \( \ln uy = \text{const} \) or \( uy = c_2 \). Therefore the general solution is

\[
yu(x, y) = f(y + x^3).
\]

(1.43)

Then, on \( x = 0 \),

\[
f(y) = \frac{\tanh y}{y}
\]

(1.44)

in which case \( f(y) = \tanh y \) and so

\[
yu(x, y) = \tanh(y + x^3).
\]

(1.45)

However, for the other BC \( u(x, 1) = x^6 \), we have \( x^6 = f(1 + x^3) \) from which we find \( f(t) = (t - 1)^2 \) where \( t = 1 + x^3 \). With these BCs, the solution is

\[
yu(x, y) = (y + x^3 - 1)^2.
\]

(1.46)

Example 6: (Exam 2002) Show that the PDE

\[
yu_x + xu_y = 4xy^3,
\]

(1.47)

has a general solution of the form

\[
u(x, y) = y^4 + f(y^2 - x^2)
\]

(1.48)

where \( f \) is an arbitrary function. If you are given that \( u(0, y) = 0 \) and \( u(x, 0) = -x^4 \), show that the solution is

\[
u(x, y) = 2x^2y^3 - x^4.
\]

(1.49)

Solution: The auxiliary equations are

\[
\frac{dx}{y} = \frac{dy}{x} = \frac{du}{4xy^3}.
\]

(1.50)

Characteristics come from the integration of \( xdx = ydy \) thereby giving and so \( y^2 - x^2 = c_1 \). We also have \( du = 4y^2dy \) resulting in \( u = y^4 + c_2 \), thereby giving the general solution

\[
u(x, y) = y^4 + f(y^2 - x^2).
\]

(1.51)
Applying the two boundary conditions gives
\[ 0 = y^4 + f(y^2) \quad \Rightarrow \quad f(t) = -t^2 \quad t \geq 0 \]
\[-x^4 = f(-x^2) \quad \Rightarrow \quad f(t) = -t^2 \quad t \leq 0 \]  
\tag{1.52} \]
Thus we have
\[ u(x, y) = y^4 - (y^2 - x^2)^2 = 2x^2y^2 - x^4. \]  
\tag{1.53} \]
One can check directly that this is indeed a solution.

**Example 7: (Exam 2003)** Show that the PDE
\[ y^2u_x + x^2u_y = 2xy^2, \]  
\tag{1.54} \]
has a general solution of the form
\[ u(x, y) = x^2 + f(y^3 - x^3) \]  
\tag{1.55} \]
where \( f \) is an arbitrary function.

(i) If \( u(0, y) = -y^6 \) and \( u(x, 0) = x^2 - x^6 \) show that
\[ u(x, y) = x^2 - x^6 + 2x^3y^3 - y^6 \]  
\tag{1.56} \]
and
(ii) If \( u(0, y) = \exp(y^3) \) and \( u(x, 0) = x^2 + \exp(-x^3) \) show that
\[ u(x, y) = x^2 + \exp(y^3 - x^3). \]  
\tag{1.57} \]

**Solution:** The auxiliary equations are
\[ \frac{dx}{y^2} = \frac{dy}{x^2} = \frac{du}{2xy^2}. \]  
\tag{1.58} \]
Characteristics come from the integration of \( x^2dx = y^2dy \) thereby giving the family of curves \( y^3 - x^3 = c_1 \). We also have \( du = 2xdx \) giving \( u = x^2 + c_2 \). Thus the general solution is
\[ u(x, y) = x^2 + f(y^3 - x^3). \]  
\tag{1.59} \]
Applying the two boundary conditions gives:
(i) For \( u(0, y) = -y^6 \) and \( u(x, 0) = x^2 - x^6 \)
\[ -y^6 = f(y^3) \quad \Rightarrow \quad f(t) = -t^2 \]
\[ x^2 - x^6 = x^2 + f(-x^3) \quad \Rightarrow \quad f(t) = -t^2 \]  
\tag{1.60} \]
Therefore
\[ u(x, y) = x^2 - (y^3 - x^3)^2 = x^2 - x^6 + 2x^3y^3 - y^6. \]  
\tag{1.61} \]
(ii) For \( u(0, y) = \exp(y^3) \) and \( u(x, 0) = x^2 + \exp(-x^3) \)

\[
\exp(y^3) = f(y^3) \quad \Rightarrow \quad f(t) = \exp t
\]
\[
x^2 + \exp(-x^3) = x^2 + f(-x^3) \quad \Rightarrow \quad f(t) = \exp t
\]

Therefore, with \( f(t) = \exp t \) the solution with these BCs is

\[
u(x, y) = x^2 + \exp \left( y^3 - x^3 \right)
\] (1.63)

2 Characteristics and 2nd order PDEs

2.1 Derivation of two sets of characteristics

Consider the class of 2nd order PDEs

\[
Ru_{xx} + 2Su_{xy} + Tu_{yy} = f
\] (2.1)

where \( u_{xx}, u_{yy} \) & \( u_{xy} \) are 2nd derivatives & \( R, S, T \) and \( f \) are functions of \( x, y, u, u_x \) & \( u_y \).

For motivational purposes let us return to the class of 1st order semi-linear equations

\[
P u_x + Q u_y = R.
\] (2.2)

Together with \( u_x dx + u_y dy = du \), these can be written as

\[
\begin{pmatrix}
P \\
dx
\end{pmatrix}
\begin{pmatrix}
Q \\
dy
\end{pmatrix}
\begin{pmatrix}
u_x \\
u_y
\end{pmatrix}
= \begin{pmatrix}
R \\
du
\end{pmatrix}.
\] (2.3)

However, from the auxiliary equations for (2.2)

\[
\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R}
\] (2.4)

which, can be re-expressed as

\[
\det \begin{pmatrix}
P \\
dx
\end{pmatrix}
\begin{pmatrix}
Q \\
dy
\end{pmatrix}
= 0,
\] (2.5)

the \( 2 \times 2 \) matrix on the LHS in (2.3) has zero determinant. This means that solutions for \( u_x \) and \( u_y \) are not unique: characteristics are a family of curves, so \( u_x \) and \( u_y \) may differ on each curve within the family.

Keeping this property in mind for the 2nd order class in (2.1) we use the chain rule to find \( dF \) for a function

\[
dF = F_x dx + F_y dy
\] (2.6)

and then take \( F = u_x \) and \( F = u_y \) in turn.

\[
d(u_x) = u_{xx} dx + u_{xy} dy
\] (2.7)
\[
d(u_y) = u_{xy} dx + u_{yy} dy.
\] (2.8)
Together with (2.1) we now have a $3 \times 3$ system:

\[
\begin{pmatrix}
R & 2S & T \\
\frac{dx}{dx} & \frac{dy}{dy} & 0 \\
0 & \frac{dx}{dx} & \frac{dy}{dy}
\end{pmatrix}
\begin{pmatrix}
u_{xx} \\
u_{xy} \\
u_{yy}
\end{pmatrix} =
\begin{pmatrix}
f \\
d(u_x) \\
d(u_y)
\end{pmatrix}. \tag{2.9}
\]

Zero determinant of the $3 \times 3$ on the LHS side of (2.9) gives

\[
R(\frac{dy}{dx})^2 - 2S \frac{dx}{dx} + T(\frac{dx}{dx})^2 = 0 \tag{2.10}
\]

which leads to the following formal classification:

**Classification:**

\[
R \left( \frac{dy}{dx} \right)^2 - 2S \left( \frac{dy}{dx} \right) + T = 0, \tag{2.11}
\]

which has two roots

\[
\frac{dy}{dx} = \frac{S \pm \sqrt{S^2 - RT}}{R}. \tag{2.12}
\]

In principle, this provides us with two ODEs to solve: call these solutions $\xi(x,y) = c_1$ and $\eta(x,y) = c_2$; **these are our two sets of characteristic curves.**

1. When $S^2 > RT$ the two roots are real: the PDE is classed as **HYPERBOLIC**;
2. When $S^2 < RT$ the roots form a complex conjugate pair: the PDE is classed as **ELLIPTIC**;
3. When $S^2 = RT$ the double root is real: the PDE is classed as **PARABOLIC**.

A transformation of the PDE from derivatives in $x, y$ into one in $\xi, \eta$ produces the **canonical form** of the PDE:

1. In the hyperbolic case we use $\xi(x,y)$ and $\eta(x,y)$ as the new co-ordinates in place of $x, y$: these arise from integration of the two real solutions of (2.12).
2. The new co-ordinates $\xi(x,y)$ and $\eta(x,y)$ arise from the real and imaginary parts of the complex conjugate pair of solutions of (2.12).
3. In the parabolic there is only one real (double) root $\xi(x,y)$ of (2.12): the other $\eta(x,y)$ may be chosen at will, usually for convenience; for instance, if $\xi = x + y$ then it might be convenient to choose $\eta = x + y$ for simplicity.

\[\text{Note the negative sign on the central term } -2S \frac{dx}{dx} \text{ in contrast to the positive sign in the PDE (2.1).}\]
2.2 Six Examples

Example 1: The standard form of the wave equation is \( u_{xx} - c^{-2}u_{tt} = 0 \) but under the transformation \( y = ct \) we obtain \( u_{xx} - u_{yy} = 0 \).

Solution: \( R = 1 \), \( S = 0 \) and \( T = -1 \). Thus \( R^2 - ST = 1 \) and we have a hyperbolic PDE. (2.11) is

\[
\left( \frac{dy}{dx} \right)^2 - 1 = 0 \tag{2.13}
\]

which has two real roots \( dy/dx = \pm 1 \). Thus our two sets of characteristics are

\[
\xi = x + y = c_1 \quad \eta = x - y = c_2 . \tag{2.14}
\]

Clearly, therefore, the characteristics are two families of straight lines, the first of gradient +1 and the second -1.

For both Examples 1 & 2, the characteristics are the 2 families of straight lines \( x - y = c_2 \) and \( x + y = c_1 \).

Now transform into the new co-ordinates \( \xi = x + y \), \( \eta = x - y \). The chain rule gives

\[
\frac{\partial}{\partial x} = \xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} ; \quad \frac{\partial}{\partial y} = \xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} \tag{2.15}
\]

into which the definitions of \( \xi \), \( \eta \) allow us to write \( \xi_x = 1 \), \( \eta_x = 0 \), \( \xi_y = 1 \) and \( \eta_y = -1 \).

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} ; \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \tag{2.16}
\]

Thus we have \( u_x = u_\xi + u_\eta \) and \( u_y = u_\xi - u_\eta \). Moreover,

\[
u_{xx} = \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) (u_\xi + u_\eta) ; \quad u_{yy} = \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right) (u_\xi - u_\eta) \tag{2.17}
\]

Thus we have

\[
u_{xx} = u_{\xi \xi} + 2u_{\xi \eta} + u_{\eta \eta} ; \quad u_{yy} = u_{\xi \xi} - 2u_{\xi \eta} + u_{\eta \eta} \tag{2.18}
\]

and so our PDE transforms to

\[
0 = u_{xx} - u_{yy} = 4u_{\xi \eta} . \tag{2.19}
\]
The canonical form is $u_{\xi \eta} = 0$. This can be integrated wrt $\xi$ directly to give

$$u_{\eta} = F(\eta),$$

(2.20)

where $F$ is an arbitrary function of $\eta$, and then again wrt $\eta$

$$u(\xi, \eta) = \int F(\eta) \, d\eta + g(\xi)$$

$$= f(\eta) + g(\xi).$$

(2.21)

Both $f$ and $g$ are arbitrary functions. Thus we have the general solution

$$u(x, y) = f(x - y) + g(x + y).$$

(2.22)

**Example 2:** Consider the PDE $u_{xx} + 2u_{xy} + u_{yy} = 0$: in this case $R = 1$, $S = 1$ and $T = 1$ so $R^2 - ST = 0$. Thus the PDE is parabolic: (2.11) is

$$\left( \frac{dy}{dx} - 1 \right)^2 = 0$$

(2.23)

which has a double real root $dy/dx = 1$. Thus one characteristic curve is

$$\eta = x - y$$

(2.24)

and we have a free choice with the other: for convenience we choose this as $\xi = x + y$, which makes $(\xi, \eta)$ the same as Example 1. Then we have

$$u_{xx} = u_{\xi \xi} + 2u_{\xi \eta} + u_{\eta \eta}$$

$$u_{yy} = u_{\xi \xi} - 2u_{\xi \eta} + u_{\eta \eta}$$

$$u_{xy} = u_{\xi \xi} - u_{\eta \eta}$$

(2.25)

and so our PDE transforms to

$$0 = u_{xx} + 2u_{xy} + u_{yy} = 4u_{\xi \xi}.$$

(2.26)

Integration wrt $\xi$ gives

$$u_{\xi} = f(\eta)$$

(2.27)

for arbitrary $f$, and again

$$u(\xi, \eta) = \xi f(\eta) + g(\eta)$$

(2.28)

for arbitrary $g$. In terms of $x$, $y$ this becomes

$$u(x, y) = (x + y)f(x - y) + g(x - y).$$

(2.29)

One can check by direct differentiation – provided $f$, $g$ have continuous second derivatives – that (2.29) is a solution.
Example 3: Consider the PDE $u_{xx} + x^2u_{yy} = 0$: in this case $R = 1$, $S = 0$ and $T = x^2$ so $R^2 - ST = -x^2 < 0$. Thus the PDE is elliptic. (2.11) is

$$
\left( \frac{dy}{dx} \right)^2 + x^2 = 0.
$$

(2.30)

Is there a natural canonical form? The formal solution of (2.30) is the complex function

$$
y \pm \frac{1}{2}ix^2 = c_{1,2}.
$$

(2.31)

We could choose $\xi$ and $\eta$ as the real and imaginary parts respectively (or v-v). Take $\xi = \frac{1}{2}x^2$ and $\eta = y$, then

$$
\frac{\partial}{\partial x} = x \frac{\partial}{\partial \xi} \quad \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \eta}.
$$

(2.32)

Thus $u_x = xu_\xi$ and $u_y = u_\eta$. Differentiating again is tricky because we have mixed old/new derivatives on the RHS of $u_x = xu_\xi$. To find $u_{xx}$ we use the product rule, differentiating wrt $x$ first and then using the chain rule

$$
u_{xx} = u_\xi + x \frac{\partial}{\partial x} u_\xi = u_\xi + x^2 u_{\xi\xi},
$$

$$u_{yy} = u_{\eta\eta}.
$$

(2.33)

Thus the PDE is

$$0 = u_{xx} + x^2 u_{yy} = x^2 (u_{\xi\xi} + u_{\eta\eta}) + u_\xi
$$

(2.34)

and so the canonical form is

$$u_{\xi\xi} + u_{\eta\eta} + \frac{1}{2\xi} u_\xi = 0.
$$

(2.35)

Example 4 (exam 05): Consider the PDE $8u_{xx} - 6u_{xy} + u_{yy} + 4 = 0$. Show that this is hyperbolic and that the characteristics are $\xi = x + 2y$ and $\eta = x + 4y$. Hence show the canonical form is $u_{\xi\eta} = 1$. If $u = \cosh x \& u_y = 2\sinh x$ on $y = 0$, show that the solution is

$$u = \xi \eta - \frac{1}{2}(\xi^2 + \eta^2) + \cosh \xi.
$$

(2.36)

Solution: In this case $R = 8$, $S = -3$ and $T = 1$ so $S^2 - RT = 1$. Thus the PDE is hyperbolic. (2.11) is

$$8 \left( \frac{dy}{dx} \right)^2 + 6 \frac{dy}{dx} + 1 = 0,
$$

(2.37)

which factorizes to

$$
\left( 4 \frac{dy}{dx} + 1 \right) \left( 2 \frac{dy}{dx} + 1 \right) = 0,
$$

(2.38)

so $\xi = x + 2y$ and $\eta = x + 4y$ as required. Now we transform to canonical variables

$$u_x = u_\xi + u_\eta \quad \quad u_y = 2u_\xi + 4u_\eta
$$

(2.39)
and
\[
\begin{align*}
    u_{xx} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \\
    u_{xy} &= 2u_{\xi\xi} + 6u_{\xi\eta} + 4u_{\eta\eta}, \\
    u_{yy} &= 4u_{\xi\xi} + 16u_{\xi\eta} + 16u_{\eta\eta}.
\end{align*}
\]  
\text{(2.40)}

Therefore
\[
0 = 8u_{xx} - 6u_{xy} + u_{yy} + 4
= u_{\xi\xi}(8 - 12 + 4) + u_{\xi\eta}(16 - 36 + 16) + u_{\eta\eta}(8 - 24 + 16) + 4
= 4 - 4u_{\xi\eta}.
\]  
\text{(2.41)}

Thus we have the \textbf{canonical form} \(u_{\xi\eta} = 1\) which integrates to
\[
u = \xi\eta + F(\eta) + G(\xi).
\]  
\text{(2.42)}

Applying the BCs: on \(y = 0\) we have \(\xi = \eta = x\): with \(u = \cosh x\)
\[
cosh x = F(x) + G(x) + x^2.
\]  
\text{(2.43)}

and with \(u_y = 2\sinh x\)
\[
2\sinh x = \left\{ 2\left( \frac{\partial}{\partial \xi} + 2 \frac{\partial}{\partial \eta} \right) \left[ F(\eta) + G(\xi) + \xi\eta \right] \right\}_{y=0}
= 2 \left\{ G'(x) + 3x + 2F'(x) \right\}
\]  
\text{(2.44)}

Integrating this gives
\[
G(x) + 2F(x) = \cosh x - \frac{3}{2}x^2 + c
\]  
\text{(2.45)}

Solving for \(F(x)\) and \(G(x)\) between (2.45) and (2.43) gives
\[
F(x) = c - \frac{1}{2}x^2 \quad G(x) = \cosh x - \frac{1}{2}x^2 - c
\]  
\text{(2.46)}

in which case (2.42) becomes
\[
u(\xi, \eta) = \cosh \xi - \frac{1}{2}(\xi^2 + \eta^2) + \xi\eta.
\]  
\text{(2.47)}

Expressing this in \(x, y\)-coordinates it is found that
\[
u(x, y) = \cosh (x + 2y) - 2y^2.
\]  
\text{(2.48)}

\textbf{Example 5 (exam 2003)}: Consider the 2nd order PDE
\[
y^2 \frac{\partial^2 u}{\partial x^2} = x^2 \frac{\partial^2 u}{\partial y^2}.
\]  
\text{(2.49)}
Show firstly that is is hyperbolic in nature. Secondly show that it has characteristics
\[ \xi = y^2 + x^2 = \text{const} , \quad \eta = y^2 - x^2 = \text{const} . \] (2.50)

Thirdly, show that its canonical form in characteristic variables is given by
\[ \frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{1}{2(\xi^2 - \eta^2)} \left( \eta \frac{\partial u}{\partial \xi} - \xi \frac{\partial u}{\partial \eta} \right) . \] (2.51)

**Solution:** (i) \( R = y^2, \ S = 0 \) and \( T = -x^2 \). Thus
\[ y^2 \left( \frac{dy}{dx} \right)^2 = x^2 \] (2.52)
so we have a hyperbolic PDE with two roots: \( \xi = y^2 + x^2 = \text{const} \) and \( \eta = y^2 - x^2 = \text{const} \).

(ii) Using the chain rule we have
\[ u_x = \xi_x u_\xi + \eta_x u_\eta = 2x(u_\xi - u_\eta) \quad u_y = \xi_y u_\xi + \eta_y u_\eta = 2y(u_\xi + u_\eta) \] (2.53)

Using the product rule, and the fact that
\[ \frac{\partial}{\partial x} = 2x \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right) \quad \frac{\partial}{\partial y} = 2y \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \] (2.54)
we have
\[ u_{xx} = 2(u_\xi - u_\eta) + 4x^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) \]
\[ u_{yy} = 2(u_\xi + u_\eta) + 4y^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) \] (2.55)

Substituting this into \( y^2 u_{xx} - x^2 u_{yy} = 0 \), we get the answer, using the fact that \( y^2 = \frac{1}{2}(\xi + \eta) \) and \( x^2 = \frac{1}{2}(\xi - \eta) \) so \( 4x^2y^2 = \xi^2 - \eta^2 \).

**Example 6 (exam 2004):** Consider the 2nd order PDE
\[ y^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 . \] (2.56)

Show firstly that is is hyperbolic in nature. Secondly show that it has characteristics
\[ \xi = \frac{1}{2}y^2 + x = \text{const} \quad \eta = \frac{1}{2}y^2 - x = \text{const} . \] (2.57)

Thirdly, show that its canonical form in characteristic variables is given by
\[ \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{1}{4(\xi + \eta)} \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \right) = 0 . \] (2.58)

**Solution:** (i) \( R = y^2, \ S = 0 \) and \( T = -1 \). Thus \( S^2 - RT = y^2 > 0 \) so we have a hyperbolic PDE with
\[ y^2 \left( \frac{dy}{dx} \right)^2 = 1 . \] (2.59)
Integration gives two roots: \( \xi = \frac{1}{2}y^2 + x = \text{const} \) and \( \eta = \frac{1}{2}y^2 - x = \text{const} \).

(ii) Using the chain rule we have
\[
 u_x = \xi_x u_\xi + \eta_x u_\eta = u_\xi - u_\eta \quad \quad u_y = \xi_y u_\xi + \eta_y u_\eta = y(u_\xi + u_\eta) \quad (2.60)
\]

Using the product rule, and the fact that
\[
 \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \quad \quad \frac{\partial}{\partial y} = y \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \quad (2.61)
\]
we have
\[
 u_{xx} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} \quad \quad u_{yy} = (u_\xi + u_\eta) + y^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) \quad (2.62)
\]

Substituting this into \( y^2u_{xx} - u_{yy} = 0 \), we get the answer, using the fact that \( y^2 = \xi + \eta \) and \( x = \frac{1}{2}(\xi - \eta) \).

3 The wave equation – a hyperbolic PDE

3.1 Physical derivation

In the figure consider a string in motion whose vertical displacement is \( u(x,t) \) at the point \( x \) is taken as a snapshot at time \( t \): it is assumed that (i) the vertical displacement is very small so that the angles \( |\alpha| \) and \( |\beta| \) are small; (ii) stretching of the string is sufficiently negligible that there is no horizontal motion. Thus, resolving horizontally, \( T_1 \cos \alpha = T_2 \cos \beta \approx T \) (the tension). Now consider the small arc-length of string \( \delta s \) between the co-ordinate points \( x \) and \( x + \delta x \). Because the angles are small \( \delta s \simeq \delta x \). If \( \rho \) is the string mass/unit density then the vertical equation of motion for our small element of string of mass \( \rho \delta x \) is
\[
 \rho \delta x \frac{\partial^2 u}{\partial t^2} = T_2 \sin \beta - T_1 \sin \alpha \quad (3.1)
\]
The smallness of \( |\alpha| \) and \( |\beta| \) allow us to write \( \sin \alpha \approx \tan \alpha \) and \( \sin \beta \approx \tan \beta \) to convert (3.1) to
\[
 \rho \delta x \frac{\partial^2 u}{\partial t^2} = T(\tan \beta - \tan \alpha) \quad (3.2)
\]
However
\[
\tan \alpha = \left( \frac{\partial u}{\partial x} \right)_x \quad \tan \beta = \left( \frac{\partial u}{\partial x} \right)_{x+\delta x}
\] (3.3)

Thus (3.2) can be written as
\[
\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \left( \frac{\partial u}{\partial x} \right)_{x+\delta x} - \frac{\partial u}{\partial x} \frac{\delta x}{\delta x}
\] (3.4)

Therefore, in the limit \( \delta x \to 0 \) (3.4) becomes
\[
\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 u}{\partial x^2}
\] (3.5)

\( T\rho^{-1} \) has the dimensions of a squared velocity, denoted as \( c^2 \), which is constant for a chosen string with a fixed tension \( T \). With
\[
c^2 = \frac{T}{\rho}
\] (3.6)

(3.5) becomes the wave equation
\[
\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0.
\] (3.7)

### 3.2 d’Alembert’s solution of the wave equation

We now wish to solve the wave equation (3.7) subject to initial conditions on the initial shape \( u(x,0) \) and the initial velocity \( \partial u(x,0)/\partial t \)
\[
u(x,0) = h(x) \quad \frac{\partial}{\partial t} u(x,0) = \left. \frac{\partial}{\partial t} u(x,t) \right|_{t=0} = v(x)
\] (3.8)

where \( h(x) \) and \( v(x) \) are given functions. In example 1 in §2.2 we found the general solution of \( u_{xx} - u_{yy} = 0 \) in (2.22). With \( y = ct \) this is
\[
u(x,t) = f(x - ct) + g(x + ct)
\] (3.9)

where, so far, \( f \) and \( g \) are arbitrary functions. Applying (3.8)
\[
f(x) + g(x) = h(x) \quad g'(x) - f'(x) = \frac{1}{c} v(x).
\] (3.10)

Integrating the latter equation from an arbitrary point \( x = a \) to \( x \) and then adding and subtracting, it is found that
\[
f(x) = \frac{1}{2} h(x) - \frac{1}{2c} \int_a^x v(\xi)d\xi - \frac{1}{2c} [g(a) - f(a)]
\]
\[
g(x) = \frac{1}{2} h(x) + \frac{1}{2c} \int_a^x v(\xi)d\xi + \frac{1}{2c} [g(a) - f(a)]
\] (3.11)
Now substitute this into (3.9) with $x \rightarrow x - ct$ in $f(x)$ and $x \rightarrow x + ct$ in $g(x)$ to get

$$u(x, t) = \frac{1}{2} \left\{ h(x - ct) + h(x + ct) \right\} + \frac{1}{2c} \int_{x - ct}^{x + ct} v(\xi) \, d\xi. \quad (3.12)$$

This is the d’Alembert’s solution which is valid on an infinite domain: note that the pair of terms that contain the point $x = a$ cancel leaving no trace.

### 3.3 Waves on a guitar string: Separation of variables

The same initial conditions as above in (3.2) are now used but now with boundary conditions that fix the ends of a finite string down at $x = 0$ and $x = L$.

Now try a solution in the form

$$u(x, t) = X(x)T(t) \quad (3.13)$$

which is substituted into the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0 \quad (3.14)$$

to get

$$\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T}. \quad (3.15)$$

Note that the LHS is a function of $x$ but not $t$ while the RHS is a function of $t$ but not $x$. Thus we can write

$$\frac{X''}{X} = -\lambda^2 \quad \frac{T''}{T} = -\lambda^2 c^2, \quad (3.16)$$

where $-\lambda^2$ is an arbitrary constant$^7$. The ODE for $X$ is $X'' + \lambda^2 X = 0$ which has a solution

$$X(x) = A \cos \lambda x + B \sin \lambda x. \quad (3.17)$$

Applying the BC that $u(x, 0) = 0$ for all values of $t$ means that $X(0) = 0$ from which it is deduced that $A = 0$: likewise from $X(L) = 0$ it is deduced that

$$B \sin \lambda L = 0. \quad (3.18)$$

$^7$The choice of a negative constant is explained lower down.
\[ B = 0 \] is the trivial solution: \( \sin \lambda L = 0 \) gives an infinite number of solutions for \( \lambda \), namely
\[ \lambda_n = \frac{n\pi}{L} \quad n = 0, \pm 1, \pm 2 \ldots \] (3.19)
giving an infinite set of solutions
\[ X_n(x) = B_n \sin \left( \frac{n\pi x}{L} \right). \] (3.20)

Here is the reason for a negative choice of the constant in (3.16): a positive choice of constant \( +\lambda^2 \) would have made \( \sin(\lambda L) \) into \( \sinh(\lambda L) \). This has only one root at \( \lambda = 0 \) which corresponds to the trivial solution.

The time part in (3.16) can now be easily solved
\[ T_n = C_n \sin (\omega_n t) + D_n \cos (\omega_n t). \] (3.21)
where the infinite set of frequencies \( \omega_n \) are defined by \( \omega_n = \frac{n\pi c}{L} \). This means that there is an infinite set of solutions \( u_n = X_n T_n \) which can be summed to form the general solution. In so doing the products of arbitrary constants \( B_n C_n \) etc are re-labelled
\[ u(x, t) = \sum_{n=1}^{\infty} \sin \left( \frac{n\pi x}{L} \right) \left[ c_n \sin (\omega_n t) + d_n \cos (\omega_n t) \right]. \] (3.22)

Now apply the initial conditions from (3.8)
\[ u(x, 0) = h(x); \quad \frac{\partial}{\partial t} u(x, 0) = v(x). \] (3.23)
The first says that
\[ h(x) = \sum_{n=1}^{\infty} d_n \sin \left( \frac{n\pi x}{L} \right) \] (3.24)
This is the half-range Fourier series of \( h(x) \) on \([0, L]\) which was discussed regarding “periodic extension”; this means that the series can be inverted to find \( d_n \)
\[ d_n = \frac{2}{L} \int_0^L h(x) \sin \left( \frac{n\pi x}{L} \right) \, dx. \] (3.25)
Applying the second initial condition gives
\[ v(x) = \sum_{n=1}^{\infty} \tilde{c}_n \sin \left( \frac{n\pi x}{L} \right) \] (3.26)
where \( \tilde{c}_n = c_n n\pi c / L \). We have
\[ \tilde{c}_n = \frac{2}{L} \int_0^L v(x) \sin \left( \frac{n\pi x}{L} \right) \, dx. \] (3.27)

---

\[ \omega_1 \) is the fundamental frequency; \( \omega_2 \) is the 1st harmonic etc. Note that all harmonics are summed in the solution. It is the balance of these that gives a musical instrument its quality.
Question: Is this consistent with d’Alembert’s solution? For simplicity, take \( v = 0 \) so the string is released from rest. The solution in (3.22) is

\[
\begin{align*}
  u(x, t) &= \sum_{n=1}^{\infty} d_n \sin \left( \frac{n\pi x}{L} \right) \cos \left( \frac{n\pi ct}{L} \right). 
\end{align*}
\] (3.28)

Now use a standard trig formula to write this as

\[
\begin{align*}
  u(x, t) &= \sum_{n=1}^{\infty} \frac{1}{2} d_n \left\{ \sin \left( \frac{n\pi x + ct}{L} \right) + \sin \left( \frac{n\pi (x - ct)}{L} \right) \right\} 
\end{align*}
\] (3.29)

which is in the D’Alembert form.

Example Take the string from rest \( (v = 0) \) and \( h(x) \) as a “tent function” of height \( d \) at the mid-point \( x = \frac{1}{2}L \).

\[
\begin{align*}
  h(x) &= \begin{cases} 
    2d & 0 \leq x \leq \frac{1}{2}L \\
    2d(1 - \frac{x}{L}) & \frac{1}{2}L \leq x \leq L.
  \end{cases}
\end{align*}
\]

The Fourier series for this – with no working – contains only odd sine-terms

\[
\begin{align*}
  u(x, t) &= \frac{8d}{\pi^2} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r + 1)^2} \sin \left( \frac{(2r + 1)\pi x}{L} \right) \cos \left( \frac{(2r + 1)\pi ct}{L} \right).
\end{align*}
\] (3.30)

Note that the coefficients of the higher harmonics die off as \( n^{-2} \).

4 Laplace’s equation – an elliptic PDE

The simplest elliptic PDE is Laplace’s equation in cartesian co-ordinates where \( R = T = 1 \) and \( S = 0 \)

\[
\begin{align*}
  u_{xx} + u_{yy} &= 0 \\
  S^2 - RT &= -1 < 0.
\end{align*}
\] (4.1)

In two-dimensions, the method of separation of variables is useful but needs to be considered in the context of the BCs. Solutions in terms of polar co-ordinates will be our concern of the subsection §4.2 concerning flow around a cylinder. First we look at a simpler problem.

4.1 An infinite strip

Physically Laplace’s equation often occurs in situations where the diffusive flow of heat or some other scalar in a two-dimensional piece of material is governed by the diffusion or heat equation \( u_t = \alpha \nabla^2 u \) where \( \nabla^2 \) is the Laplacian \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \). When the system has reached a steady state – so \( u_t = 0 \) – we are left with the problem of solving Laplace’s equation (4.1). The strip below is an example of how to solve this with a set of given boundary conditions (BCs).
Figure: The region is a strip bounded between $x = 0$ (y-axis) and $x = \pi$ on which $u = 0$ while $u = f(x)$ on $y = 0$.

The infinite strip, as in the figure above, has $u = 0$ on the sides and $u = f(x)$, a given function, on the bottom edge. To remain physical it is also necessary to insist that $u \to 0$ as $y \to \infty$. Inside the strip $u$ satisfies Laplace’s equation (4.1) which we attempt to solve by the method of separation of variables

$$u(x, y) = X(x)Y(y)$$ (4.2)

and thus (4.1) becomes $X''Y + XY'' = 0$. Therefore

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$$ (4.3)

the choice of $\pm$ on the far RHS is dependent on the BCs. Clearly we have the two ODEs

$$X'' + \lambda^2 X = 0 \quad Y'' - \lambda^2 Y = 0$$ (4.4)

whose solution is

$$X = A \cos \lambda x + B \sin \lambda x, \quad Y = Ce^{\lambda y} + De^{-\lambda y}.$$ (4.5)

The BC at $x = 0$ insists that $A = 0$ and at $x = \pi$ that $\sin \lambda \pi = 0$. Thus $\lambda_n = n$ where $n$ is an integer. For $n > 0$ we must also choose $C = 0$ to be sure that there is no exponential growth as $y \to \infty$. We are left with a summed infinite set of solutions

$$u(x, y) = \sum_{n=1}^{\infty} b_n e^{-ny} \sin nx$$ (4.6)

To find the $b_n$ requires the use of the last BC $u = f(x)$ on $y = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$ (4.7)
This is the Fourier sine-series expansion of $f(x)$ on $[0, \pi]$ which inverts to

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx. \quad (4.8)$$

For example, if $f(x) = 1$ – that is, a uniform value – then

$$b_n = \begin{cases} 
0 & n \text{ even} \\
\frac{4}{n\pi} & n \text{ odd}
\end{cases} \quad (4.9)$$

With $n = 2r + 1$, our solution is

$$u(x, y) = \frac{4}{\pi} \sum_{r=1}^{\infty} e^{-(2r+1)y} \left( \frac{\sin(2r+1)x}{2r+1} \right). \quad (4.10)$$

Note that this solution correctly decays exponentially as $y \to \infty$ and is zero at $x = 0$ and $x = \pi$.

### 4.2 Fluid flow around a cylinder

#### 4.2.1 Laplace’s equation in polar co-ordinates

Consider Laplace’s equation in polar co-ordinates (see handout on The Chain Rule)

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} = 0. \quad (4.11)$$

Looking for separable solutions of the form $\Phi(r, \theta) = R(r)H(\theta)$ we find

$$\frac{r^2}{R} \left( R'' + \frac{1}{r} R' \right) = -\frac{H''}{H} = \lambda^2. \quad (4.12)$$

Choosing the separation constant negative anticipates solutions for $H(\theta)$ that need to be periodic. Solving $H'' + \lambda^2 H = 0$ gives

$$H(\theta) = A \cos \lambda \theta + B \sin \lambda \theta. \quad (4.13)$$

When $\lambda \neq 0$ solving $R'' + \frac{1}{r} R' - \frac{\lambda^2}{r^2} R = 0$ gives

$$R(r) = a r^\lambda + b r^{-\lambda}. \quad (4.14)$$

If we require $\Phi(r, \theta)$ to be continuous\(^9\) in $\theta$; that is, $\Phi(r, \theta) = \Phi(r, \theta + 2n\pi)$, then $\lambda = n$ (an integer). The general $2\pi$-periodic solution of (4.11) is

$$\Phi(r, \theta) = \sum_{n=1}^{\infty} \left( a_n r^n + b_n r^{-n} \right) \left( A_n \cos n\theta + B_n \sin n\theta \right). \quad (4.15)$$

\(^9\)The case with $\lambda = 0$ where $H(\theta) = \tilde{A}\theta + \tilde{B}$ and $R(r) = \tilde{a} \ln r + \tilde{b}$ is not $2\pi$-periodic in $\theta$. 
4.2.2 Calculating the flow around the cylinder

Consider an incompressible irrotational 2D fluid with velocity vector \( \mathbf{u} \) flowing past a cylinder of radius \( a \), as in the figure: the centre of the cylinder can be considered to be at \( r = 0 \). At \( r = \pm \infty \) the flow is laminar: that is, \( \mathbf{u} = (0, U) \) where \( U \) is a constant.

\[
U \rightarrow r = a
\]

(i) The divergence-free condition \( \text{div} \, \mathbf{u} = 0 \) means that a stream function \( \psi(x, y) \) exists

\[
\mathbf{u} = (\psi_y, -\psi_x) = \hat{i} \psi_y - \hat{j} \psi_x.
\]

Irrotational flow (\( \text{curl} \, \mathbf{u} = 0 \)) means that

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\partial_x & \partial_y & \partial_z \\
\psi_y & -\psi_x & 0 \\
\end{vmatrix} = 0
\]

Thus we have Laplace’s equation for the stream function

\[
\psi_{xx} + \psi_{yy} = 0 \quad \Rightarrow \quad \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0. \quad (4.16)
\]

(ii) The alternative way, using the potential, starts from \( \text{curl} \, \mathbf{u} = 0 \). This means that a potential function \( \phi \) exists such that \( \mathbf{u} = \nabla \phi = \hat{i} \phi_x + \hat{j} \phi_y \). From \( \text{div} \, \mathbf{u} = 0 \), we have Laplace’s equation \( \nabla^2 \phi = \phi_{xx} + \phi_{yy} = 0 \) which is also \( (4.11) \) in polar co-ordinates.

Thus we want to solve \( (4.16) \) under the circumstance where the fluid, of constant horizontal speed \( U \) at infinity, flows past a solid cylinder of radius \( a \) centred at the origin. The fact that no fluid can cross the surface of the cylinder translates into the boundary condition

\[
\left. \frac{\partial \psi}{\partial \theta} \right|_{r=a} = 0. \quad (4.17)
\]

Since the flow at \( r = \pm \infty \) is horizontal we have \( \mathbf{u} = U \hat{i} + 0 \hat{j} \) there, which means that

\[
\psi = U y = U r \sin \theta \quad \text{at} \quad r = \infty. \quad (4.18)
\]

We want to solve Laplace’s equation \( (4.16) \) in the infinite domain around the cylinder of radius \( a \) with prescribed BCs \( (4.17) \) and \( (4.18) \). Separating the \( n = 1 \) term from the rest of the
infinite sum in (4.15) we have
\[ \psi(r, \theta) = \left( a_1 r + b_1 r^{-1} \right) \left( A_1 \cos \theta + B_1 \sin \theta \right) + \sum_{n=2}^{\infty} \left( a_n r^n + b_n r^{-n} \right) \left( A_n \cos n\theta + B_n \sin n\theta \right). \] (4.19)

Applying the BC in (4.18) we find that
\[ a_1 B_1 = U A_1 = 0 \] (4.20)

and all coefficients \( A_n = B_n = 0 \) for \( n \geq 2 \). This leaves us with
\[ \psi = U \left( r + \frac{b_1}{a_1 r} \right) \sin \theta. \] (4.21)

Finally applying the BC (4.17) at \( r = a \) we find \( b_1/a_1 = -a^2 \) giving the stream function as
\[ \psi = U \left( r - \frac{a^2}{r} \right) \sin \theta. \] (4.22)

5 The diffusion equation – a parabolic PDE

Consider a very thin metal bar on the \( x \)-axis on \([0, L]\), as in the figure below, with temperature \( u = 0 \) at both ends. For standard materials, the equation that normally governs heat flow is the diffusion equation\(^{10}\)
\[ u_t = \kappa u_{xx} \] (5.1)
where \( \kappa \) is a material constant (thermal conductivity) which has the dimensions \((\text{length})^2/\text{time}\). In this section we solve two problems: on a finite one-dimensional domain \([0, L]\) and similarity solutions on an infinite domain.

5.1 Separation of variables on a finite domain

<table>
<thead>
<tr>
<th>( x = 0 )</th>
<th>( x = L )</th>
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<tbody>
<tr>
<td>( u = 0 )</td>
<td>( u = 0 )</td>
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<tr>
<td>( u(x, 0) = f(x) ) at ( t = 0 )</td>
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</table>

The BCS are \( u = 0 \) on both \( x = 0 \) and \( x = L \) with\(^{11}\) an initial distribution of temperature \( u(x, 0) = f(x) \). Separation of variables
\[ u(x, t) = X(x)T(t) \] (5.2)

\(^{10}\)In 2 dimensions the equivalent is \( u_t = \kappa \nabla^2 u \) where \( \nabla^2 \) is the Laplacian \( \nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 \).

\(^{11}\)If the end conditions are different, say \( u = 0 \) at \( x = 0 \) and \( u = u_0 \) at \( x = L \), then the following trick is useful: define \( u(x, t) = u_0 x/L + v(x, t) \) with \( v = 0 \) on \( x = 0 \) and \( x = L \) with \( v \) satisfying \( v_t = \kappa v_{xx} \), then the problem reduces to the one solved above with \( u = 0 \) at both ends.
gives

\[
\frac{X''}{X} = \frac{1}{\kappa T'} = -\lambda^2
\]  

(5.3)

for which we write

\[
X'' + \lambda^2 X = 0 \quad \text{with} \quad X(0) = X(L) = 0.
\]  

(5.4)

This we have solved before: (5.4) gives \( X = A \cos \lambda x + B \sin \lambda x \) in which \( A = 0 \) because \( X(0) = 0 \), whereas

\[
\sin \lambda L = 0 \quad \Rightarrow \quad \lambda = \frac{n\pi}{L} \quad \text{with} \quad X_n(x) = B_n \sin \left( \frac{n\pi x}{L} \right).
\]  

(5.5)

The time part \( T' = -\lambda^2 \kappa T \) solves to become

\[
T_n(t) = T_{n,0} \exp \left( -\frac{n^2 \pi^2 \kappa t}{L^2} \right)
\]  

(5.6)

Thus the general solution is a linear sum of all the solutions for each \( n \)

\[
\sum_{n=1}^{\infty} b_n \exp \left( -\frac{n^2 \pi^2 \kappa t}{L^2} \right) \sin \left( \frac{n\pi x}{L} \right),
\]  

(5.7)

where the constants \( B_n T_{n,0} = b_n \). Applying the ICs gives

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right),
\]  

(5.8)

and, as before, this Fourier half-range series can be inverted to give the \( b_n \)

\[
b_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx.
\]  

(5.9)

### 5.2 Similarity solutions on an infinite domain

The diffusion equation in one-dimension is \( u_t = \kappa u_{xx} \) has been solved above on a domain of finite length. What if \( L = \infty \)? Clearly, the method of separation of variables no longer works and we need a different approach. The key lies in \( \kappa \), the diffusion coefficient, which has dimension \( L^2 T^{-1} \). If we are looking for solutions on an infinite domain \( -\infty \leq x \leq \infty \) where there is no natural length scale, then we can use the dimensionless variable

\[
\eta = \frac{x}{\sqrt{\kappa t}}
\]  

(5.10)

and look for solutions in the form

\[
u(x, t) = \nu^0 g(\eta)
\]  

(5.11)
where the number $p$ and the function $g(\eta)$ are to be determined. Substituting (5.11) into $u_t = \kappa u_{xx}$ we find that

$$t^{p-1} \left( pg - \frac{\eta}{2} g' - g'' \right) = 0$$

(5.12)

and so

$$g'' + \frac{\eta}{2} g' = pg.$$  

(5.13)

This is difficult to solve for arbitrary values of $p$ but for special values we can do something.

1. Take $p = 0$ and (5.13) is easily solved to give

$$g'(\eta) = A e^{-\eta^2/4}$$

(5.14)

where $A$ is a constant. Integrating again we have

$$g(\eta) = A \int_{-\infty}^{\eta} e^{-\eta'^2/4} d\eta'.$$

(5.15)

This gives a full solution for $u(x, t)$

$$u(x, t) = A \int_{-\infty}^{x/\sqrt{\kappa t}} e^{-\eta'^2/4} d\eta' = 2A \sqrt{\pi} \operatorname{erf} \left( \frac{x}{2\sqrt{\kappa t}} \right)$$

(5.16)

where the error function $\operatorname{erf}(\xi)$ is defined as $\operatorname{erf}(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\xi} e^{-y^2} dy$. This has the property that $\operatorname{erf}(\infty) = 1$.

2. Now define $G = g e^{\eta^2/4}$ and we observe that (5.13) can be transformed into

$$G'' - \frac{\eta}{2} G' = (p + 1/2)G.$$  

(5.17)

This has the trivial solution $G = b = \text{const}$ provided $p = -1/2$. Hence

$$g(\eta) = b e^{-\eta^2/4}.$$  

(5.18)

This gives a full solution for $u(x, t)$ in the form

$$u(x, t) = b t^{-1/2} e^{-x^2/4\kappa t}.$$  

(5.19)