Some mathematical models in neuroscience

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Mathematical models to describe the membrane potential

• **Synapse**: Specialized junctions through which neurons signal to each other and to non-neuronal cells such as those in muscles or glands

• **Crucial** to the biological computations that underlie perception and thought

• A **huge number** in human brain:
  ◦ $10^{16}$ in young children
  ◦ $10^{15} - 5 \times 10^{15}$ for adults

• It happens at the level of the **membranes of the cells**
Mathematical models to describe the membrane potential

How to model membrane potential?

• Membrane potential is the electric potential difference across a cell’s plasma membrane
• Membrane is polarized due to different charges inside and outside of the cell
• Influences in the membrane potential:
  ◦ Permeability of the different ions
  ◦ Different concentrations of the different ions
  ◦ Ion pumps

Membrane potential can be modeled as an electric circuit
Mathematical models to describe the membrane potential

Different scales

- **Microscopic** (Particle Description): Is a very huge number
- **Mesoscopic** (Kinetic Description): $f(t, x, \nu)$ Particle density at time $t$ in variables $(x, \nu)$ (in kinetic theory: position $x$ with velocity $\nu$)
- **Macroscopic** (Hydrodynamic Description): System for the macroscopic quantities: momentum of $f$ (in kinetic theory: density, momentum and temperature)
Outline

• Background

• Mesoscopic models

  ◆ Nonlinear Noisy Leaky Integrate and Fire (NNLIF)
    ○ Blow-up
    ○ Steady states
    ○ Delay
    ○ Refractory state
    ○ Spontaneous activity
    ○ Excitatory-inhibitory populations

• Conclusion
How to model membrane potential?

Membrane potential can be modeled as an electric circuit.
How to model membrane potential?

Neuronal model of L. Lapicque (1907)

\[ V = RI \]

- **V**: Potential difference (Volts)
- **R**: Electrical Resistance (Ohms)
- **I**: Electric Current (Amperes)
How to model membrane potential?

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- \( V \) Potential difference (Volts)
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Rewritten as:

\[ CV = I \quad C = \frac{1}{R} \]

- \( C \) Electrical Conductance (Siemens)
How to model membrane potential?

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Therefore, the evolution in time:

\[ C \frac{dV}{dt} = \frac{dI}{dt} \]
How to model membrane potential?

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Therefore, the evolution in time:

\[ C \frac{dV}{dt} = \frac{dI}{dt} \]

equivalently

\[ C_m \frac{dV}{dt} = I_{\text{apply}} \]

\[ C_m \]

- \( C_m \) Capacitance (Faraday)

\[ I_{\text{apply}} \]

- \( I_{\text{apply}} \) Applied Current (Amperes)
How to model membrane potential?

\[ C_m \frac{dV}{dt} = I_{\text{apply}} \]

Who is \( I_{\text{apply}} \)?
Microscopic level: different models

• Integrate-and-Fire model
  ◦ Simple case: 1 neuron
  ◦ A set of neurons

• Conductance-Integrate-and-fire model

Other models: Hodgkin-Huxley, FitzHugh-Nagumo, Morris-Lecar
...
Recall the equation for the membrane potential

\[ C_m \frac{dV}{dt} = I_{\text{apply}} \]

considering \( I_{\text{apply}} = -I_L + I \)

\[ I_L = g_L(V - V_L) \]

- \( g_L \) leak conductance
- \( V_L \) resting potential
Noisy Leaky Integrate-and-fire model

\[ C_m \frac{dV}{dt} = -g_L(V - V_L) + I \]

Calling \[ \tau_m = C_m/g_L \approx 2ms \]
Noisy Leaky Integrate-and-fire model

\[ \tau_m \frac{dV}{dt} = -(V - V_L) + \frac{I}{g_L} \]
Noisy Leaky Integrate-and-fire model

\[ \tau_m \frac{dV}{dt} = -(V - V_L) + \frac{I}{g_L} \]

- There are firing times \( t_{\text{spike}} \): \( V(t_{\text{spike}}) = V_{\text{threshold}} \)
- Immediately after \( t_{\text{spike}} \) the potential is reset: \( V(t_{\text{spike}}^+) = V_{\text{reset}} \)
- \( V_L < V_{\text{reset}} < V_{\text{threshold}} \)
- Typically \( V_L \approx -70\,\text{mV} \), \( V_{\text{reset}} \approx -60\,\text{mV} \) and \( V_{\text{threshold}} \approx -50\,\text{mV} \)
**Noisy Leaky Integrate-and-fire model**

\[
\tau_m \frac{dV}{dt} = -(V - V_L) + \frac{I}{g_L}
\]

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How is \( I(t) \) modelled?

- \( I(t) \): Interactions of the neuron with the network (synapse)
- Each neuron spikes \( \sim \) spike trains (stochastic process)
**Statistical Description**

- **Microscopic (Particle Description):** Is a very huge number
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Some mesoscopic models: PDE models

The unknown, $p$, is the probability of finding a neuron at time $t$

- If spike trains follow Poisson processes $\leadsto$ Nonlinear Fokker-Planck equations
  - $p(v, t)$ is the probability of finding a neuron at time $t$ with voltage $v$
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- If spike trains follow point processes $\sim$ age-structured partial differential equations
  - (Pakdaman-Perthame-Salort)
  - $p(s, t)$ is the probability of finding a neuron at time $t$ with ‘state’ $s$. ‘state’ is the time elapsed since last spike
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Chevallier, preprint
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Recently:
Dumont, Henry and Tarniceriu give some relations between both mesoscopic models

Nonlinear Noisy Leaky Integrate and Fire (NNLIF)
(Nonlinear Fokker-Planck equations)
Nonlinear Fokker-Planck equation

\[
\frac{\partial p}{\partial t}(v, t) + \frac{\partial}{\partial v} [h(v, N(t))p(v, t)] - a(N(t)) \frac{\partial^2 p}{\partial v^2}(v, t) = \delta(v - V_R)N(t)
\]
Nonlinear Fokker-Planck equation

\[
\frac{\partial p}{\partial t}(v, t) + \frac{\partial}{\partial v} \left[ h(v, N(t))p(v, t) \right] - a(N(t))\frac{\partial^2 p}{\partial v^2}(v, t) = \delta(v - V_R)N(t)
\]

- \( v \in (-\infty, V_F] \)
- \( h(v, N(t)) = -v + bN \)
- \( a(N) = a_0 + a_1N \quad a_0 > 0, \ a_1 \geq 0 \)
- \( V_{\text{reset}} = V_R \)
Nonlinear Fokker-Planck equation

\[
\frac{\partial p}{\partial t}(v, t) + \frac{\partial}{\partial v} [h(v, N(t))p(v, t)] - a(N(t)) \frac{\partial^2 p}{\partial v^2}(v, t) = \delta(v - V_R)N(t)
\]

- \(v \in (-\infty, V_F]\)
- \(h(v, N(t)) = -v + bN\) excitatory/inhibitory network
- \(a(N) = a_0 + a_1N\) \(a_0 > 0, a_1 \geq 0\)
- \(V_{reset} = V_R\)
Nonlinear Fokker-Planck equation

\[ \frac{\partial p}{\partial t}(v, t) + \frac{\partial}{\partial v} \left[ h(v, N(t))p(v, t) \right] - a(N(t)) \frac{\partial^2 p}{\partial v^2}(v, t) = \delta(v - V_R)N(t) \]

- \( v \in (-\infty, V_F] \)
- \( h(v, N(t)) = -v + b N \) excitatory/inhibitory network
- \( a(N) = a_0 + a_1 N \) \( a_0 > 0, a_1 \geq 0 \)
- \( V_{\text{reset}} = V_R \) Boundary conditions
- \( p(V_F, t) = 0 \)
- \( p(-\infty, t) = 0 \)
Nonlinear Fokker-Planck equation

\[
\frac{\partial p}{\partial t}(v, t) + \frac{\partial}{\partial v} \left[ h(v, N(t))p(v, t) \right] - a(N(t)) \frac{\partial^2 p}{\partial v^2}(v, t) = \delta(v - V_R)N(t)
\]

\begin{itemize}
  \item \( v \in (-\infty, V_F] \)
  \item \( h(v, N(t)) = -v + b N \) excitatory/inhibitory network
  \item \( a(N) = a_0 + a_1 N \quad a_0 > 0, a_1 \geq 0 \)
  \item \( V_{\text{reset}} = V_R \) \underline{Boundary conditions}
  \item \( p(V_F, t) = 0 \)
  \item \( p(-\infty, t) = 0 \)
\end{itemize}

Since \( \int_{-\infty}^{V_F} p(v, t) \, dv = \int_{-\infty}^{V_F} p(0, v) \, dv = 1 \) for all \( t \geq 0 \)

\[
N(t) := -a(N(t)) \frac{\partial p}{\partial v}(V_F, t) \geq 0 \quad V_F = V_{\text{threshold}}
\]
Blow up
Blow up

Theorem (C-Carrillo-Perthame) Assuming:

- $h(v, N) + v \geq bN$, $0 < a_m \leq a(N)$ ($-\infty < v \leq V_F$ and $N \geq 0$)
- $b > 0$ average-excitatory network

If the initial data is concentrated enough around $v = V_F$ there are no global-in-time weak solutions
**Blow up**

Theorem (C-Carrillo-Perthame) **Assuming:**

- \( h(v, N) + v \geq bN, \ 0 < a_m \leq a(N) \) \((-\infty < v \leq V_F \text{ and } N \geq 0)\)
- \( b > 0 \) average-excitatory network

If the initial data is concentrated enough around \( v = V_F \) there are no global-in-time weak solutions

The solutions blow up when ... 

- the initial condition is concentrated enough around \( V_F \), for \( b \) fixed
- \( b \), the connectivity parameter, is large enough, for \( p^0 \) fixed

\[
\int_{-\infty}^{V_F} e^{\mu v} p^0(v) \, dv \geq \frac{e^{\mu V_F} - e^{\mu V_R}}{b \mu} =: \lambda
\]

Choosing \( \mu > \max\left(\frac{V_F}{a_m}, \frac{1}{b}\right) \)
**Blow up**

Carrillo-González-Gualdani-Schonbek* analyze the global existence of classical solutions

- For inhibitory networks: Global classical solutions
- For excitatory networks: Local well-posedness of classical solutions
- Blow up criterium:

\[ T^* = \sup \{ t > 0 : N(t) < \infty \} \]

\( T^* \): maximal existence time
- For inhibitory networks: \( T^* = \infty \)
- For excitatory networks: there exist classical solutions which blow up at finite time \( T^* \) and \( N \) diverges

**Blow up**

Blow-up

Distribution functions $p(v, t)$ for $a \equiv 1$ and $b = 0.5$ at different times

Case: a unique steady state [video]
Delay avoids blow-up

\[\frac{\partial p}{\partial t}(v, t) + \frac{\partial}{\partial v} \left[ h(v, N(t))p(v, t) \right] - a(N(t)) \frac{\partial^2 p}{\partial v^2}(v, t) = \delta(v - V_R)N(t)\]

- \(v \in (-\infty, V_F]\)
- \(h(v, N(t)) = -v + bN\)
- \(a(N) = a_0 + a_1N \quad a_0 > 0, a_1 \geq 0\)

Work in progress in collaboration with Ricarda Schneider
Delay avoids blow-up

\[
\frac{\partial p}{\partial t}(v, t) + \frac{\partial}{\partial v} \left[ h(v, N(t-D))p(v, t) \right] - a(N(t-D)) \frac{\partial^2 p}{\partial v^2}(v, t) = \delta(v - V_R)N(t)
\]

- \( v \in (-\infty, V_F] \)
- \( h(v, N(t-D)) = -v + bN(t-D) \)
- \( a(N(t-D)) = a_0 + a_1 N(t-D) \quad a_0 > 0, a_1 \geq 0 \)

\( D \geq 0 \) is the transmission delay

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video

Work in progress in collaboration with Ricarda Schneider
Delay avoids blow-up

Distribution functions $p(v, t)$ and firing rate for $a \equiv 1$ and $b = 0.5$ with different delays

Work in progress in collaboration with Ricarda Schneider
Delay avoids blow-up

Firing rate for $a \equiv 1$ and $b = 1.5$ with $D = 0.1$

Work in progress in collaboration with Ricarda Schneider
Steady states
Steady states

A steady state satisfies

\[
\frac{\partial}{\partial v} \left[ (v - V_0(N)) p + a(N) \frac{\partial}{\partial v} p(v) + NH(v - V_R) \right] = 0
\]

in the sense of distributions, with \( H \) being the Heaviside function and \( V_0(N) = bN \).
Steady states

\[ p(v) = \frac{N}{a(N)} e^{-\frac{(v-V_0(N))^2}{2a(N)}} \int_{\max(v,V_R)}^{V_F} e^{-\frac{(w-V_0(N))^2}{2a(N)}} \, dw \]
**Steady states**

\[ p(v) = \frac{N}{a(N)} e^{-\frac{(v-V_0(N))^2}{2a(N)}} \int_{\max(v,V_R)}^{V_F} e^{\frac{(w-V_0(N))^2}{2a(N)}} \, dw \]

\[ \frac{a(N)}{N} = \int_{-\infty}^{V_F} \left[ e^{-\frac{(v-V_0(N))^2}{2a(N)}} \int_{\max(v,V_R)}^{V_F} e^{\frac{(w-V_0(N))^2}{2a(N)}} \, dw \right] \, dv \]
**Steady states**

\[
p(v) = \frac{N}{a(N)} e^{-\frac{(v-V_0(N))^2}{2a(N)}} \int_{\text{max}(v,V_R)}^{V_F} e^{-\frac{(w-V_0(N))^2}{2a(N)}} \, dw
\]

\[
a(N) = \frac{1}{N} \int_{-\infty}^{V_F} \left[ e^{-\frac{(v-V_0(N))^2}{2a(N)}} \int_{\text{max}(v,V_R)}^{V_F} e^{-\frac{(w-V_0(N))^2}{2a(N)}} \, dw \right] \, dv
\]

**Linear case**: unique stationary state \( p_\infty \) given by

\[
p_\infty(v) = \frac{N_\infty}{a_0} e^{-\frac{v^2}{2a_0}} \int_{\text{max}(v,V_R)}^{V_F} e^{\frac{w^2}{2a_0}} \, dw
\]
Question: Number of solutions for

\[ a(N) = \int_{-\infty}^{V_F} \left[ e^{-\frac{(v-V_0(N))^2}{2a(N)}} \int_{\max(v,V_R)}^{V_F} e^{\frac{(w-V_0(N))^2}{2a(N)}} \, dw \right] \, dv \]
Steady states

Change of variables:

\[ z = \frac{v - V_0}{\sqrt{a}}, \quad u = \frac{w - V_0}{\sqrt{a}}, \quad w_F = \frac{V_F - V_0}{\sqrt{a}}, \quad w_R = \frac{V_R - V_0}{\sqrt{a}} \]

\[
\begin{cases} 
  \frac{1}{N} = I(N), \\
  I(N) := \int_{-\infty}^{w_F} \left[ e^{-\frac{z^2}{2}} \int_{\max(z,w_R)}^{w_F} e^{\frac{u^2}{2}} du \right] dz 
\end{cases}
\]

In the case \( h(v, N) = bN - v, \quad a(N) = a_0 \) and \( V_0(N) = bN \), we can rewrite \( I(N) \) as

\[
I(N) = \int_{0}^{\infty} e^{-s^2/2} e^{-\frac{sbN}{\sqrt{a_0}}} e^{\frac{s V_F}{\sqrt{a_0}}} - e^{\frac{s V_R}{\sqrt{a_0}}} ds
\]
Steady states

Number of steady states in terms of connectivity parameter $b$

(C-Carrillo-Perthame):

- **Average-inhibitory network** ($b < 0$): there is a unique steady state

- **Average-excitatory network** ($b > 0$):
  - High connectivity: there is no steady state
  - Small connectivity: there is a unique steady state
  - Between small and high connectivity: at least one or at least two steady states
Steady states

Function $I(N)$ is plotted against the function $1/N$

$a \equiv 1, \ V_R = 1, \ V_F = 2$
Steady states

Function $NI(N)$ is plotted against the function $1$

$a \equiv 1$, $V_R = 1$, $V_F = 2$
Are the steady states stable?
Case linear $b=0$

For any smooth convex function $G : \mathbb{R}^+ \longrightarrow \mathbb{R}$

$$-\frac{d}{dt} \int_{-\infty}^{V_F} p_\infty(v) G \left( \frac{p(v,t)}{p_\infty(v)} \right) dv =$$

$$N_\infty \left[ G \left( \frac{N(t)}{N_\infty} \right) - G \left( \frac{p(v,t)}{p_\infty(v)} \right) \right] + a_0 \int_{-\infty}^{V_F} p_\infty(v) G'' \left( \frac{p(v,t)}{p_\infty(v)} \right) \left[ \frac{\partial}{\partial v} \left( \frac{p(v,t)}{p_\infty(v)} \right) \right]^2 dv \geq 0$$
Case linear $b=0$

Theorem [Exponential decay] (C-Carrillo-Perthame)

Fast-decaying solutions verifying $p^0(v) \leq C^0 p_\infty(v)$ for some $C^0 > 0$, satisfy

$$\int_{-\infty}^{V_F} p_\infty(v) \left( \frac{p(v, t) - p_\infty(v)}{p_\infty(v)} \right)^2 dv \leq e^{-2a_0 vt} \int_{-\infty}^{V_F} p_\infty(v) \left( \frac{p^0(v) - p_\infty(v)}{p_\infty(v)} \right)^2 dv.$$
Case non linear \( b \neq 0 \)

- C-Carrillo-Perthame: Numerical analysis
- Carrillo-Perthame-Salort-Smets*: Local asymptotic stability for stationary states for \(|b|\) small

Numerical results
Firing rates $N(t)$ for $a \equiv 1$

Top left: $b = 0.5$  Top right: $b = 3$
Bottom left: $b = 1.5$  Bottom right: $b = -1.5
Extensions of NNLIF

- Refractory state
- Randomness on the discharge potential
- Network consisting of excitatory and inhibitory neurons, as different populations
Extension: NNLIF with refractory

We include a **refractory state**

\[
\begin{aligned}
\frac{\partial p}{\partial t}(v, t) + \frac{\partial}{\partial v} \left[ \left( -v + bN(t) \right) p(v, t) \right] - a(N(t)) \frac{\partial^2 p}{\partial v^2}(v, t) &= \frac{R(t)}{\tau} \delta(v - V_R) \\
\frac{dR(t)}{dt} &= N(t) - \frac{R(t)}{\tau} \\
N(t) &:= -a(N(t)) \frac{\partial p}{\partial v}(V_F, t) \geq 0 \\
p(V_F, t) &= 0, \ p(-\infty, t) = 0, \ p(v, 0) = p^0(v) \geq 0, \ R(0) = R^0 > 0
\end{aligned}
\]

- \( p(v, t) \): density of active neurons at potential \( v \in (-\infty, V_F) \)
- \( R(t) \): density of neurons in the refractory state
- \( \tau \): mean time of the refractory period
- \( N(t) \): flux of firing neurons.

**Conservation of the total number of neurones:**

\[
R(t) + \int_{-\infty}^{V_F} p(v, t) \, dv = R^0 + \int_{-\infty}^{V_F} p^0(v) \, dv = 1
\]
Extension: NNLIF with refractory

Results (C-Perthame)

- Finite time blow-up: weak solution is not global in time:
  - For any $b > 0$: appropriate initial data (concentrated enough near $V_F$) produce blow-up
  - For any initial data: values $b > 0$ large enough produce blow-up
Extension: NNLIF with refractory

Results (C-Perthame)

- Stationary solutions:
  - Inhibitory case ($b < 0$): there is a unique steady state
  - Excitatory case ($b > 0$): there is an odd number of steady states (considering multiplicity)

\[ V_F = 2, \; V_R = 1, \; b = 1.5, \; a = 1, \; \tau = 0.025 \]
**Extension: NNLIF with refractory**

Long time behaviour for the linear case (C-Perthame):

\[ E(t) \leq E(0)e^{-\nu t} \]

where:

- the energy

\[
E(t) := \int_{-\infty}^{V_F} \frac{(p(v, t) - p_\infty(v))^2}{p_\infty(v)} dv + \frac{(R(t) - R_\infty)^2}{R_\infty}
\]

for the unique steady state \((p_\infty, R_\infty)\)

- \(\nu > 0\) only depends of the model parameters

\((volver)\)
**Extension:** NNLIF with spontaneous activity

- Randomness on the discharge potential
- Avoids the blow-up phenomena and generates bounded oscillations
- Spontaneous activity of the network
- Strong controls on the total activity $N(t)$
**Extension:** NNLIF with spontaneous activity

Without refractory state

\[
\frac{d}{dt} p + \frac{d}{dv} \left[ (-v + bN)p \right] - \frac{d^2}{dv^2} (a(N)p) + \phi_\epsilon(v)p = N(t)\delta(v - V_R), \quad v \in \mathbb{R}
\]

\[
N(t) = \int_{-\infty}^{+\infty} \phi_\epsilon(v)p(v, t) \, dv
\]

- \( \phi_\epsilon(v) \) represents the discharge rate at potential \( v \)
- \( \phi_\epsilon(v) = \frac{1}{\epsilon} (v - V_F)_+ \)
- \( \phi_\epsilon(v) = \frac{1}{\epsilon} \mathbf{1}_{\{v > V_F\}} \)
**Extension:** NNLIF with spontaneous activity

With refractory state

\[
\frac{\partial p}{\partial t} + \frac{\partial}{\partial v} [(-v + bN)p] - a(N) \frac{\partial^2 p}{\partial v^2} + \phi_\varepsilon(v)p = \frac{R(t)}{\tau} \delta(v - V_R), \quad v \in \mathbb{R},
\]

\[
\frac{dR(t)}{dt} = N(t) - \frac{R(t)}{\tau},
\]

\[
N(t) = \int_{-\infty}^{+\infty} \phi_\varepsilon(v)p(v, t) \, dv
\]
Extension: NNLIF with spontaneous activity

\[ V_F = 2, \, a = 1, \, b = 1.5, \, \tau = 0.025, \, \epsilon = 1/14000 \]

\[ V_R = 1, \, V_F = 2, \, a = 1, \, b = 1.5, \, \tau = 0.025 \]
Extension: NNLIF excitatory-inhibitory

\[ \frac{\partial p}{\partial t}(v, t) + \frac{\partial}{\partial v} \left[ h(v, N(t))p(v, t) \right] - a(N(t)) \frac{\partial^2 p}{\partial v^2}(v, t) = \delta(v - V_R)N(t) \]

- \( v \in (-\infty, V_F] \)
- \( h(v, N(t)) = -v + bN \) excitatory/inhibitory network
- \( a(N) = a_0 + a_1 N \quad a_0 > 0, a_1 \geq 0 \)

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Extension: NNLIF excitatory-inhibitory

\[
\frac{\partial \rho_I}{\partial t}(v, t) + \frac{\partial}{\partial v} [h^I(v, N_E, N_I) \rho_I(v, t)] - a_I(N_E, N_I) \frac{\partial^2 \rho_I}{\partial v^2}(v, t) = N_I(t) \delta(v - V_R)
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For \( \alpha = I, E \):

- \( h^\alpha(v, N_E, N_I) = -v + b^\alpha_E N_E - b^\alpha_I N_I + (b^\alpha_E - b^\alpha_E) v_{E, ext} \)
- \( a^\alpha(N_E, N_I) = d^\alpha_E v_{E, ext} + d^\alpha_E N_E - d^\alpha_I N_I \)
- \( N^\alpha(t) = -a^\alpha(N_E(t), N_I(t)) \frac{\partial \rho^\alpha}{\partial v}(V_F, t) \)
- \( b^\alpha_i > 0, i = E, I, \) is the connectivity of the network \( \alpha \),

Extension: NNLIF excitatory-inhibitory

Blow-up

The system can blow-up in finite time in two cases:

- for $\rho^0_E \neq 0$ fixed and $b_E^E > 0$ large enough
- for $b_E^E > 0$ fixed, when $\rho^0_F$ is concentrated enough around $V_F$

\[ V_F = 2, \ V_R = 1, \ b_E^E = 0.5, \ b_I^E = 0.25, \ b_I^I = 0.25, \ b_I^I = 1, \ a_E = a_I = 1, \ v_{E,ext} = 0 \]
Extension: NNLIF excitatory-inhibitory

Steady states

- \( \rho_\alpha = \frac{N_\alpha}{a_\alpha(N_E,N_I)} e^{-\frac{(v-V_0^\alpha(N_E,N_I))^2}{2a_\alpha(N_E,N_I)}} \int_{V_F}^{\max(v,V_R)} e^{\frac{(w-V_0^\alpha(N_E,N_I))^2}{2a_\alpha(N_E,N_I)}} dw \) where \( V_0^\alpha(N_E,N_I) := b_\alpha^E N_E - b_\alpha^I N_I + b_\alpha^E v_{E,ext} \)

- The steady states analysis is more complicated than the case with only one population

- The problem is equivalent to find the solutions of the system

\[
\begin{align*}
\frac{1}{N_E} &= I_E(N_E,N_I), \\
\frac{1}{N_I} &= I_I(N_E,N_I),
\end{align*}
\]

where \( I_\alpha(N_E,N_I) = \int_{-\infty}^{\infty} e^{-\frac{w_\alpha^E}{2}} \int_{\max(z_\alpha,w_\alpha^E)}^{w_\alpha^E} e^{-\frac{w_\alpha^E}{2}} du_\alpha \, dz_\alpha, \quad \alpha = E, I \)


**Extension: NNLIF excitatory-inhibitory**

- For every fixed $N_E$ there is a unique solution, $N_I(N_E)$, such that $N_I(N_E) I_I(N_E, N_I(N_E)) = 1$. Thus, the steady state analysis is reduced to determine the number of solutions to $F(N_E) = 1$, where $F(N_E) = N_E I_E(N_E, N_I(N_E))$

$$F(N_E) \text{ for } b^E_I = 0.5, b^I_I = 0.25, V_F = 2, V_R = 1, v_{E,\text{ext}} = 0, a_E = a_I = 1$$

Left: $b^E_I = 0.75$ and different values for $b^E_E$. Right: $b^E_E = 3$ and different values for $b^E_I$

- For small connectivity parameters the unique steady state is asymptotically stable

(volver)
Summary about NNLIF model

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  - For any $b > 0$: appropriate initial data produce blow-up
  - For any initial data: values $b > 0$ large enough produce blow-up
- No blow-up for $b < 0$ or for $b > 0$ with delay
- The firing rate $N(t)$ blows-up in finite time whenever a singularity in the system occurs (related to synchronization of the network)
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- When random discharge potential is considered spontaneous activity can appear
Works in collaboration with ...

- Nonlinear Noisy Leaky Integrate and Fire (NNLIF)
  - The Journal of Mathematical Neuroscience (2011), 1:7 in collaboration with Carrillo (Imperial College London) and Perthame (Université Pierre et Marie Curie)

- Nonlinear Fokker-Planck Equation including conductance
  - Journal of Computational Physics 230 (2011) 1084-1099 in collaboration with Carrillo and Tao (Center for Bioinformatics, Peking University, Center for Neural Science, NYU)

- From the microscopic to mesoscopic models
  - Mathematical Models and Methods in Applied Sciences in collaboration with Chevallier (Laboratoire J. A. Dieudonne), Doumic (INRIA Paris-Rocquencourt), Reynaud-Bouret (Laboratoire J. A. Dieudonne)