Boundary value problems for the Dirac equation in fractal domains

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Jump problem
Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain with boundary $\Gamma$ and let be $g$ a continuous complex valued function defined on $\Gamma$. 

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Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain with boundary $\Gamma$ and let be $g$ a continuous complex valued function defined on $\Gamma$.

To find a function $\Phi$, holomorphic in $\mathbb{R}^2 \setminus \Gamma$, with continuous boundary values up to $\Gamma$ satisfying

$$\Phi^+(t) - \Phi^-(t) = g(t), \ t \in \Gamma, \quad (1)$$
Riemann-Hilbert problem

**Riemann Condition**

\[ \Phi^+(t) - G(t)\Phi^-(t) = g(t), \quad t \in \Gamma, \quad (2) \]

**Hilbert Condition**

\[ \Re[F(t)\Phi^+(t)] = f(t), \quad t \in \Gamma. \quad (3) \]
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(3)


Cauchy transform: the main tool

\[ C_{\Gamma}g(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(\tau)d\tau}{\tau - z}, \quad z \notin \Gamma \]  (4)
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Plemelj-Sokhotski Formulae

\[ C_{\Gamma}^{+} g(t) - C_{\Gamma}^{-} g(t) = g(t) \]

\[ C_{\Gamma}^{+} g(t) + C_{\Gamma}^{-} g(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{g(\tau) d\tau}{\tau - t} \]
Holomorphic functions in \( \mathbb{R}^m \)

**Cauchy Riemann Operator**

\[
\partial_z = \frac{1}{2}(\partial_x + i\partial_y), \quad z = x + iy.
\]

\( f : \Omega \to \mathbb{C} \) is **holomorphic in** \( \Omega \subset \mathbb{R}^2 \iff \partial_z f = 0 \) in \( \Omega \)
Holomorphic functions in $\mathbb{R}^m$

**Cauchy Riemann Operator**

$$\partial_z = \frac{1}{2}(\partial_x + i\partial_y), \ z = x + iy.$$  

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**Dirac Operator in $\mathbb{R}^m$**

$$\partial = \sum_{i=1}^{m} e_i \partial_{x_i}$$

$f : \Omega \to \mathbb{R}_{0,m}$ is holomorphic in $\Omega \subset \mathbb{R}^m \iff \partial f = 0$ in $\Omega$

Holomorphic $\iff$ monogenic
Special Instances of Dirac equation $\partial f = 0$

**Solenoidal and irrotational vector fields**

$\vec{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \partial \vec{f} = 0 \iff \begin{cases} \text{div} \vec{f} = 0 \\ \text{curl} \vec{f} = 0 \end{cases}$
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**Harmonic Vector Fields in $\mathbb{R}^m$**

$$\vec{f} : \mathbb{R}^m \mapsto \mathbb{R}^m, \partial \vec{f} = 0 \iff \begin{cases} \text{div} \vec{f} = 0 \\ \partial \wedge \vec{f} = 0 \end{cases}$$

($\wedge$ denotes the usual outer product)
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**Time-harmonic electromagnetic fields**

\[ \begin{align*} \partial_{-\alpha} \phi &= \text{div} \vec{j} + \alpha \vec{j} \\
\partial_{\alpha} \psi &= -\text{div} \vec{j} + \alpha \vec{j} \end{align*} \iff \begin{cases} \text{curl} \vec{H} = -i\omega\varepsilon \vec{E} + \vec{j} \\ \text{curl} \vec{E} = i\omega\mu \vec{H} \\ \text{div} \vec{E} = \frac{\rho}{\varepsilon} \\ \text{div} \vec{H} = 0 \end{cases} \]
Higher Order Dirac Equations

\[ \partial^2 f = 0(\text{Bimonogenic}) \iff \Delta f = 0(\text{Harmonic}) \]
\[ \partial^4 f = 0(\text{Tetramonogenic}) \iff \Delta \Delta f = 0(\text{Biharmonic}) \]
\[ \partial^k f = 0(\text{Polymonogenic}) \iff \Delta^k f = 0(\text{Polyharmonic}) \]
The jump problem for $\partial$ in $\mathbb{R}^m$

Let $\Omega^+ := \Omega$, and $\Omega^- := \mathbb{R}^m \setminus (\Omega \cup \Gamma)$ denote the complementary connected domains separated by $\Gamma$ on $\mathbb{R}^m$. 

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The jump problem for $\partial$ in $\mathbb{R}^m$

Let $\Omega^+ := \Omega$, and $\Omega^- := \mathbb{R}^m \setminus (\Omega \cup \Gamma)$ denote the complementary connected domains separated by $\Gamma$ on $\mathbb{R}^m$.

The jump problem associated to $\partial$ is the problem of reconstructing a $\mathbb{R}_0,m$-valued function $\Psi$ satisfying in $\mathbb{R}^m \setminus \Gamma$ the Dirac equation $\partial \Psi = 0$, vanishing at infinity and having a prescribed jump $g$ across $\Gamma$, i.e.,

$$\psi^+(x) - \psi^-(x) = g(x), \quad x \in \Gamma, \quad \Psi(\infty) = 0,$$

(5)

where $\psi^\pm(x) = \lim_{\Omega^\pm \ni y \to x} \Psi(y)$.

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Boundary value problems in fractal domains
Clifford Cauchy transform: the main tool

\[ C_\Gamma g(x) := \int_{\Gamma} E_0(y - x) \kappa(y) g(y) \, dy, \quad x \notin \Gamma, \]

where

\[ E_0(x) = -\frac{1}{\sigma_m} \frac{x}{|x|^m}. \]

\( \kappa(y) \) denotes the exterior normal vector at \( y \in \Gamma \) and \( \sigma_m \) the surface area of the unit sphere in \( \mathbb{R}^m \).
(5) is solvable $\iff C_{\Gamma}g(x)$ has continuous extension

- Clifford-Cauchy transform for compact Liapunov surfaces.
- Plemelj-Sokhotski Formula

$$\left( C_{\Gamma}g \right)^+(x) - \left( C_{\Gamma}g \right)^-(x) = g(x) \quad x \in \Gamma$$

Non-smooth setting

Continuous extension of the Clifford-Cauchy transform \( C_\Gamma g(x) \) on a rectifiable and Ahlfors-David regular surface \( \Gamma \) in \( \mathbb{R}^m \) is optimally answered under assumption that

\[
\int_{\Gamma \setminus \{|y-x| \leq \epsilon\}} E_0(y-x) \kappa(y)(g(y) - g(x))dy, \quad x \in \Gamma
\]

converges uniformly on \( \Gamma \) as \( \epsilon \to 0 \).
Non-smooth setting

- Continuous extension of the Clifford-Cauchy transform $C_{\Gamma}g(x)$ on a rectifiable and Ahlfors-David regular surface $\Gamma$ in $\mathbb{R}^m$ is optimally answered under assumption that

$$\int_{\Gamma \setminus \{|y-x| \leq \epsilon\}} E_0(y-x)\kappa(y)(g(y) - g(x))dy, \ x \in \Gamma$$

converges uniformly on $\Gamma$ as $\epsilon \to 0$.

Non-rectifiable setting

Let $\mathcal{H}^{m-1}(\Gamma) < \infty$. If $g$ satisfies a Hölder condition on $\Gamma$ with exponent $\nu$, $0 < \nu \leq 1$, then under condition

$$\nu > \frac{m-1}{m},$$

(6)

jump problem (5) permits a solution given by $C_\Gamma g$. 

R. Abreu Blaya; J. Bory Reyes and T. Moreno García.

Cauchy Transform on non-rectifiable surfaces in Clifford Analysis.


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Boundary value problems in fractal domains
Non-rectifiable setting

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$$\nu > \frac{m - 1}{m},$$

jump problem (5) permits a solution given by $C_{\Gamma}g$.

Domains with fractal boundary

Alternative way of defining the Clifford-Cauchy transform, where a central role is played by the Teodorescu transform involving fractal dimensions, is described.

\[(C^*_f)(x) := \tilde{g}(x)\chi_\Omega(x) + \int_{\Omega} E_0(y-x)\partial \tilde{g}(y)dy\]

- $\tilde{g} \rightarrow$ a Whitney extension of $g$
- $\chi_\Omega(x) \rightarrow$ characteristic function of $\Omega$. 
If $f$ satisfies a Hölder condition on $\Gamma$ with exponent $\nu$, $0 < \nu \leq 1$, then a solution of (1) can be obtained by using $C^*_\Gamma g$ under assumption

$$\nu > \frac{M(\Gamma)}{m}.$$  

(7)
Fractal dimension setting

- If $f$ satisfies a Hölder condition on $\Gamma$ with exponent $\nu$, $0 < \nu \leq 1$, then a solution of (1) can be obtained by using $C_{(\Gamma)}^{*}g$ under assumption

$$\nu > \frac{M(\Gamma)}{m}.$$  \hspace{1cm} (7)

The condition (7) cannot be improved on the whole class of surfaces with fixed upper Minkowski fractal dimension.
Fractal dimension setting (cont.)

The condition (7) cannot be improved on the whole class of surfaces with fixed upper Minkowski fractal dimension.

**Theorem**

For any \( m \in [m - 1, m) \) and \( 0 < \nu \leq \frac{m}{m} \) there exists a surface \( \Gamma_* \subset \mathbb{R}^m \) such that \( M(\Gamma) = m \) and a \( \nu \)-Hölder continuous function \( g_* \) in \( \Gamma \), such that the jump problem (5) has no solution.
The condition (7) cannot be improved on the whole class of surfaces with fixed upper Minkowski fractal dimension.

**Theorem**

*For any* $m \in [m - 1, m)$ *and* $0 < \nu \leq \frac{m}{m}$ *there exists a surface* $\Gamma_* \subset \mathbb{R}^m$ *such that* $M(\Gamma) = m$ *and a* $\nu$-Hölder continuous function* $g_*$ *in* $\Gamma$, *such that the jump problem (5) has no solution.*

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R. Abreu Blaya; J. Bory Reyes and T. Moreno García.  
$d$-summable setting

- If $g$ satisfies the $\nu$-Hölder condition, $0 < \nu \leq 1$
$d$-summable setting

- If $g$ satisfies the $\nu$-Hölder condition, $0 < \nu \leq 1$
- $\Gamma$ is $d$-summable

$$\int_{0}^{1} N_{\Gamma}(\tau)\tau^{d-1}d\tau \text{ converges}$$

where $N_{\Gamma}(\tau)$ denotes the number of $\tau$-balls needed to cover $\Gamma$. 
If $g$ satisfies the $\nu$-Hölder condition, $0 < \nu \leq 1$

- $\Gamma$ is $d$-summable

$$\int_{0}^{1} N_{\Gamma}(\tau) \tau^{d-1} d\tau$$

converges

where $N_{\Gamma}(\tau)$ denotes the number of $\tau$-balls needed to cover $\Gamma$.

- Then, under condition

$$\nu > \frac{d}{m} \quad (8)$$

the jump problem (1) permits a solution given by $C_{\Gamma}^* g$. 

$d$-summable setting
The condition (8) cannot be weakened on the whole class of \( d \)-summable surfaces.
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Time-Harmonic Maxwell equation in fractal domains

Let $\rho$ and $\vec{j}$ belong to $L^p(\Omega)$ $(p > 3)$. Let $\vec{e}$ and $\vec{h}$ be complex vector valued functions in $C^{0,\nu}(\Gamma)$, $\nu > \frac{d}{3}$.

If there exists a pair of vector fields $\vec{E}$ and $\vec{H}$, both in $C^{0,\nu}(\Omega \cup \Gamma)$, satisfying in $\Omega$ the time-harmonic Maxwell equations and on $\Gamma$ the boundary conditions

$$\vec{E}|_{\Gamma} = \vec{e}, \quad \vec{H}|_{\Gamma} = \vec{h},$$

(9)

Then

$$\mathcal{T}_{-\alpha} \partial_{-\alpha}(-i\omega\varepsilon\vec{e} + \alpha\vec{h})|_{\Gamma} = \mathcal{T}_{-\alpha}(\text{div}\vec{j} + \alpha\vec{j})|_{\Gamma},$$

(10)

$$\text{Sc}(\mathcal{T}_{-\alpha} \partial_{-\alpha}(-i\omega\varepsilon\vec{e} + \alpha\vec{h})) = \text{Sc}(\mathcal{T}_{-\alpha}(\text{div}\vec{j} + \alpha\vec{j})) \text{ in } \Omega$$

(11)

and

$$\mathcal{T}_\alpha \partial_\alpha (i\omega \varepsilon \tilde{e} + \alpha \tilde{h})|_\Gamma = \mathcal{T}_\alpha (-\text{div}\tilde{j} + \alpha \tilde{j})|_\Gamma$$ (12)

$$\text{Sc}(\mathcal{T}_\alpha \partial_\alpha (i\omega \varepsilon \tilde{e} + \alpha \tilde{h})) = \text{Sc}(\mathcal{T}_\alpha (-\text{div}\tilde{j} + \alpha \tilde{j})) \text{ in } \Omega$$ (13)

where $\alpha = \omega \sqrt{\varepsilon \mu}$. 
Conversely, if (10)-(13) are satisfied, then the vector fields

\[ \vec{E} = \vec{\tilde{e}} - \frac{1}{2i\omega\varepsilon} \left\{ i\omega\varepsilon (T_\alpha \partial_\alpha + T_{-\alpha} \partial_{-\alpha}) (\vec{\tilde{e}}) + \alpha (T_{-\alpha} \partial_{-\alpha} - T_\alpha \partial_\alpha) \right\} \]

\[ \vec{H} = \vec{\tilde{h}} - \frac{1}{2\alpha} \left\{ i\omega\varepsilon (T_{-\alpha} \partial_\alpha - T_\alpha \partial_{-\alpha}) (\vec{\tilde{e}}) - \alpha (T_{-\alpha} \partial_{-\alpha} + T_\alpha \partial_\alpha) \right\} \]

satisfy Maxwell equations together with the boundary conditions (9)
Time-Harmonic Maxwell equation in fractal domains.

Cont.

Higher order Lipschitz data

Let $k$ be a non-negative integer and $0 < \alpha \leq 1$. We shall say that a function $g$, defined in $\Gamma$, belongs to the higher order Lipschitz class $\text{Lip}(\Gamma, k + \nu)$ if there exist bounded functions $g^{(j)}$, $0 < |j| \leq k$, defined on $\Gamma$, with $g^{(0)} = g$, and so that

$$R_j(x, y) = g^{(j)}(x) - \sum_{|l| \leq k-|j|} \frac{g^{(j+l)}(y)}{l!}(x - y)^l, \quad x, y \in \Gamma \quad (14)$$

satisfies

$$|R_j(x, y)| = O(|x - y|^{k+\nu-|j|}), \quad x, y \in \Gamma, |j| \leq k. \quad (15)$$
Such a function $g$ can be extended to a $C^{k,\nu}$ function $g$ on all of $\mathbb{R}^m$ with

$$\frac{\partial^{(j)} g}{\partial x^j} := \frac{\partial |j| g}{\partial x_1^{j_1} \partial x_2^{j_2} \ldots \partial x_m^{j_m}} = g^{(j)} \text{ in } \Gamma,$$

for all $|j| := j_1 + j_2 + \cdots + j_m \leq k$. 
Let be \( g \in \text{Lip}(\Gamma, k - 1 + \nu) \). We are interested in the following boundary value problem:

Find a polymonogenic function \( \Phi \), i.e., \( \partial^k \Phi = 0 \) in \( \mathbb{R}^m \setminus \Gamma \) satisfying the boundary conditions

\[
(\partial^i \Phi)^+(x) - (\partial^i \Phi)^-(x) = \partial^i g(x), \quad x \in \Gamma \text{ for } 0 \leq i \leq k - 1
\]

\[
(\partial^i \Phi)^-(\infty) = 0, \quad \text{for } 0 \leq i \leq k - 1.
\]

(16)
Solution. Higher order Teodorescu operator

**Theorem**

If \( g \in \text{Lip}(\Gamma, k - 1 + \nu) \), with \( \nu > \frac{d}{m} \), then the jump problem (16) has a solution given by

\[
\Phi(x) = \begin{cases} 
\tilde{g}(x) - T_k[\partial^k \tilde{g}](x), & x \in \Omega_+ \\
- T_k[\partial^k \tilde{g}](x), & x \in \Omega_-
\end{cases}
\]  

(17)

where

\[
T_k f(x) := \frac{(-1)^k}{\sigma_m} \int_{\Omega} \frac{(\xi - x)(\xi - x + \xi - x)^{k-1}}{2^{k-1}(k - 1)!|\xi - x|^m} f(\xi) d\xi
\]

is a higher order Teodorescu operator.
Uniqueness

Theorem

Let be \( g \in \text{Lip}(\Gamma, k-1+\nu) \), with \( \nu > \frac{d}{m} \) and let

\[
\dim_H(\Gamma) - (m-1) < \beta < \frac{m\nu - d}{m - d}.
\]

Then the function given by (17) is the unique solution of the jump problem (16) which belongs to the class

\[
\text{Lip}_\beta^k := \{ \phi \in \text{Lip}(\overline{\Omega_+}, k+\beta) \cap \text{Lip}(\overline{\Omega_-}, k+\beta), \partial^i \phi(\infty) = 0, 0 < i \leq k-1 \}
\]

Problems for the sandwich equation $\partial \Phi \bar{\partial} = 0$

**Inframonogenic Functions**

\[
\partial \Phi \bar{\partial} = 0 \quad \implies \quad \begin{cases} 
\Phi \text{ is biharmonic} \\
\Phi \text{ two-sided 3-monogenic}
\end{cases}
\]
Problems for the sandwich equation $\partial \Phi \bar{\partial} = 0$

**Inframonogenic Functions**

$\partial \Phi \bar{\partial} = 0 \implies \begin{cases} 
\Phi \text{ is biharmonic} \\
\Phi \text{ two-sided } 3\text{-monogenic} 
\end{cases}$

**Inframonogenic Vector Fields**

$\partial \vec{F} \bar{\partial} = 0 \iff \begin{cases} 
\partial \cdot (\partial \cdot \vec{F}) = 0 \\
\partial \wedge (\partial \cdot \vec{F}) - \partial \cdot (\partial \wedge \vec{F}) = 0 \\
\partial \wedge (\partial \wedge \vec{F}) = 0 
\end{cases}$

($\cdot$ denotes the usual inner product (up to a minus sign)).
Inframonogenic Teodorescu operator

\[ T\phi(x) = \frac{1}{2} \left[ \int_{\Omega} E_0(y - x)\phi(y)(y - x)dy + \sum_{i=1}^{n} e_i \int_{\Omega} E_1(y - x)\phi(y)dye_i \right], \]

where

\[ E_1(x) = \frac{1}{(m - 2)\sigma_m|x|^{m-2}}, \quad x \neq 0 \]

is the fundamental solution of the Laplace operator \( \triangle \) in \( \mathbb{R}^m \).
THANK YOU