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Book of abstracts

EXISTENCE AND ASYMPTOTIC STABILITY OF TRAVELING WAVE SOLUTIONS FOR NONLOCAL VISCOUS CONSERVATION LAWS

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We study a Korteweg-de Vries-Burgers equation with a non-local diffusion term, which arises in the analysis of a shallow water flow in a channel. The resulting non-local operator is a fractional Caputo derivative of order between 1 and 2. We show the well-posedness of the Cauchy problem for essentially bounded initial data. Moreover, we prove the existence and asymptotic stability of traveling wave solutions. In absence of the dispersive term, we show that only traveling wave solutions with monotone profiles exist. In contrast, traveling waves of the non-local KdV-Burgers equation are not in general monotone, as is the case for the corresponding classical KdV-Burgers equation. This requires a more complicated existence proof compared to our previous work. The traveling wave problem for the classical KdV-Burgers equation is usually analyzed via a phase-plane analysis, which is not applicable here due to the presence of the non-local diffusion operator. In addition, we discuss the monotonicity of the waves in terms of a control parameter and prove their dynamic stability in case they are monotone.

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LARGE-TIME BEHAVIOR IN HYPOCOERCIVE BGK-MODELS

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BGK equations are kinetic transport equations with a relaxation operator that drives the phase space distribution towards the spatially local equilibrium, a Gaussian with the same macroscopic parameters. Due to the absence of dissipation w.r.t. the spatial direction, convergence to the global equilibrium is only possible thanks to the transport term that mixes various positions. Hence, such models are hypocoercive.

We shall prove exponential convergence towards the equilibrium with explicit rates for several linear, space periodic BGK-models in dimension 1 and 2. Their BGK-operators differ by the number of conserved macroscopic quantities (like mass, momentum, energy), and hence their hypocoercivity index. Our discussion includes also discrete velocity models, and the local exponential stability of a nonlinear BGK-model.

The proof is based, first, on a Fourier decomposition in space and Hermite function decomposition in velocity. Then, the crucial step is to construct a problem adapted Lyapunov functional, by introducing equivalent norms for each mode.

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(IN)STABILITY OF SHEAR FLOWS AT HIGH REYNOLDS NUMBERS AND LANDAU DAMPING IN KINETIC THEORY

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The plane, periodic Couette flow has served as a canonical problem in the field of hydrodynamic stability since the late 19th century. Obtaining a precise understanding the stability and instability of this flow in the nonlinear equations has however remained elusive in both 2D and 3D. In this talk I will discuss some of the recent progress we have made in this direction in both 2D (joint with Nader Masmoudi, Vlad Vicol, and Fei Wang) and 3D (joint with collaborators Pierre Germain and Nader Masmoudi). The dynamics of solutions have been determined in a variety of settings, and are governed by several important effects: namely inviscid damping and mixing-enhanced dissipation. Comparisons and contrasts will be made with recent positive and negative results on Landau damping in weakly collisional or collisionless plasmas.

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AN EXTENDED ENTROPY POWER INEQUALITY

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The entropy power for a random vector X in $\mathbb{R}^n$ with density f is defined by

$$N(X) = \exp\left\{\frac{2}{n}\,h(X)\right\} = \exp\left\{-\frac{2}{n}\int f\log f\right\},$$

where $h(X)$ is the (Shannon differential) entropy. One of the most remarkable properties of this functional is the entropy power inequality

$$N(X+Y) \geq N(X) + N(Y),$$

which holds true as long as the random vectors X and Y independent (with equality for Gaussian distributions only). We will discuss an extension of this classical result to the more general Rényi entropies

$$N_\alpha(X) = \exp\left\{\frac{2}{n}\,h_\alpha(X)\right\} = \left(\int f^\alpha\right)^{-\frac{2}{n}\frac{1}{\alpha-1}} \qquad (\alpha > 1).$$

However, the extension cannot be of the same form, since it turns out to be false in general, even when X is normal and Y is nearly normal. Therefore, some modifications have to be done. As a natural variant, we consider proper powers of the Renyi entropy power, in analogy with the approach by Savaré and Toscani (2014) to the concavity of the entropy power for solutions of the nonlinear heat (porous medium) equation. Namely, it will be shown that

$$N_\alpha^r(X+Y) \geq N_\alpha^r(X) + N_\alpha^r(Y)$$

whenever $r \geq \frac{\alpha+1}{2}$.

Based on a joint work with Arnaud Marsliglietti (Caltech).

NONLINEAR AND NONLOCAL DEGENERATE DIFFUSIONSON BOUNDED DOMAINS

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We investigate quantitative properties of nonnegative solutions $u(t,x) \geq 0$ to the nonlinear fractional diffusion equation, $\partial_t u + \mathcal{L}F(u) = 0$ posed in a bounded domain, $x \in \Omega \subset \mathbb{R}^N$, with appropriate homogeneous Dirichlet boundary conditions. As $\mathcal{L}$ we can use a quite general class of linear operators that includes the three most common versions of the fractional Laplacian $(-\Delta)^s$, $0 < s < 1$, in a bounded domain with zero Dirichlet boundary conditions; many other examples are included. The nonlinearity F is assumed to be increasing and is allowed to be degenerate, the prototype being $F(u) = |u|^{m-1}u$, with $m > 1$.
We will shortly present some recent results about existence, uniqueness and a priori estimates for a quite large class of very weak solutions, that we call weak dual solutions.
We will devote special attention to the regularity theory: decay and positivity, boundary behavior, Harnack inequalities, interior and boundary regularity, and asymptotic behavior. All this is done in a quantitative way, based on sharp a priori estimates. Although our focus is on the fractional models, our techniques cover also the local case s = 1 and provide new results even in this setting.
A surprising instance of this problem is the possible presence of nonmatching powers for the boundary behavior: for instance, when $\mathcal{L} = (-\Delta)^s$ is a spectral power of the Dirichlet Laplacian inside a smooth domain, we can prove that, whenever $2s \geq 1 - 1/m$, solutions behave as $\mathrm{dist}^{1/m}$ near the boundary; on the other hand, when $2s < 1 - 1/m$, different solutions may exhibit different boundary behaviors even for large times. This unexpected phenomenon is a completely new feature of the nonlocal nonlinear structure of this model, and it is not present in the elliptic case. The above results are contained on a series of recent papers [1, 2, 3, 5, 4] in collaboration with A. Figalli, Y. Sire, X. Ros-Oton and J. L. Vázquez.

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NONLOCAL PARTICLE SYSTEMS WITH WASSERSTEIN INTERACTION

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Nonlocal particle systems with Wasserstein interaction arise in electrical engineering (quantization and image processing), economics (optimal location problems), and materials science (crystallization and energy-driven pattern formation). The interaction energy of these systems can be written either in terms of atomic measures and Wasserstein distances or in terms of generalised Voronoi diagrams. In this talk I will present some numerical and analytical results about optimal particle configurations. This is joint work with Mark Peletier, Steven Roper, and Florian Theil.

N-PARTICLE GROUND STATES OF THE INTERACTION ENERGY APPROXIMATE CONTINUOUS GROUND STATES

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We study lowest-energy distributions (ground states) of N classical particles interacting through a pair potential and show that in the limit of N large these ground states converge to lowest-energy distributions of the continuum interaction energy functional. The shape of these N-particle minimisers is of interest both in statistical mechanics and in more recent collective behaviour models. In particular we show the following: if the potential is H-stable (in the sense of statistical mechanics), these N-particle ground states spread without limit as N grows; while if the potential is not H-stable, the N-particle ground states stay within a ball of a fixed radius, independently of N. The singularity of the potential is one of the main difficulties in showing the convergence of discrete minimisers to continuum minimisers.
This is a joint work with Francesco Patacchini.

NON-LOCAL ISOPERIMETRIC PROBLEMS AND PHASE SEGREGATION

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Droplets of one phase of a substance in another – gas in a liquid for example – tend to be round when an isotropic surface tension enforces a near minimal bounding surface area. In many physically realistic models, the interactions responsible for the surface tension are non-local, and this gives rise to a class of non-local isoperimetric problems. This talk presents recent work with Maggi on such problems, and recent work applying the inequalities to phase segregation problems that has been done with Carvalho, Esposito and Marra. Equilibrium and non-equilibrium problems will be discussed.

ENTROPY PRODUCTION INEQUALITIES FOR THE KAC WALK

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We investigate new functional inequalities for the well-known Kac's Walk, and largely resolve the 'Almost' Cercignani Conjecture on the sphere. A new notion of chaoticity plays an essential role. The results we obtain validate Kac's suggestion that functional inequalities for the Kac walk could be used to quantify the rate of approach to equilibrium for the Kac-Boltzmann equation. This is joint work with E. Carlen and A. Einav.

A BLOB METHOD FOR (DEGENERATE) DIFFUSION

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For a range of physical and biological processes, from biological swarming to dynamics of granular media, the evolution of a large number of interacting agents can be described in terms of the competing effects of drift, diffusion, and nonlocal interaction. The resulting partial differential equations are gradient flows with respect to the Wasserstein metric, and this gradient flow structure provides a natural framework for numerical particle methods. However, developing deterministic particle methods for problems involving diffusion poses unique challenges, particularly when nonlocal interaction terms are also present. In this talk, I will present new work on a blob method for diffusion and degenerate diffusion, inspired by blob methods from classical fluid mechanics. This is joint work with Francesco Patacchini and José Antonio Carrillo.

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INTERACTION ENERGY MINIMIZERS AND PARTIAL REGULARITY FOR HIGHER ORDER OBSTACLE PROBLEMS

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Motivated by swarming models, we analyse local minimizers to the Interaction Energy. Using classical results of potential theory, we show a surprising relationship between local minimizers of the Interaction Energy and solutions to Obstacle Problems. Exploiting this relationship we translate regularity results from one setting onto the other.

RECENT ADVANCES IN THE STUDY OF THE BECKER-DÖRING EQUATIONS

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The Becker-Döring equations are a fundamental set of equations that describe the kinetics of first order phase transition, and are applicable to a wide variety of phenomena such as crystallisation, vapour condensation and aggregation of lipids.
While the equations can be traced back to a model from 1935, a rigorous systematic study of them, and the long time behaviour of their solutions, only appeared around the 80's in works by Ball, Carr and Penrose.
In our talk we will focus on the *subcritical case*, in which it has been shown the existence, and convergence to, a state of equilibrium. We will use the so-called entropy method to give *quantitative* rates of convergence to equilibrium, and see how this relates to the question of a *uniform in time* bounds of the moments of the solution - another new result we shall discuss.
This talk is based on a couple of papers that are a joint work with José Cañizo and Bertrand Lods.

NONLINEAR CROSS-DIFFUSION AND NONLOCAL INTERACTION SYSTEMS

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I will discuss a joint work with M. Di Francesco and S. Fagioli [3]. We are dealing with systems of PDEs with nonlinear cross-diffusion and nonlocal interactions under a uniform "coercivity" assumption on the diffusion part. This kind of systems has several applications in the social sciences, finance, biology and real world problems. In the context of Wasserstein spaces [1], we provide existence of weak measure solutions by means of a semi-implicit version of the JKO scheme [4, 5] and the flow interchange technique [6].

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SORTING PHENOMENA IN A MATHEMATICAL MODEL FOR TWO MUTUALLY ATTRACTING/REPELLING SPECIES

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Macroscopic models for systems involving diffusion, short-range repulsion, and long-range attraction have been studied extensively in the last decades. In the paper [1] we extend the analysis to a system for two species interacting with each other according to different inner- and intra-species attractions. Under suitable conditions on this self- and crosswise attraction an interesting effect can be observed, namely phase separation into neighboring regions, each of which contains only one of the species. We prove that the intersection of the support of the stationary solutions of the continuum model for the two species has zero Lebesgue measure, while the support of the sum of the two densities is simply connected. Preliminary results indicate the existence of phase separation, i.e. spatial sorting of the different species. A detailed analysis in one spatial dimension follows. The existence and shape of segregated stationary solutions is shown via the Krein-Rutman theorem. Moreover, for small repulsion/nonlinear diffusion, also uniqueness of these stationary states is proved.

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A SYMMETRIZATION RESULT FOR FRACTIONAL NONLOCAL ORNSTEIN–UHLENBECK EQUATION

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In this talk the fractional nonlocal Ornstein–Uhlenbeck equation is analyzed. More precisely for $0 < s < 1$, we consider the following Dirichlet problem

$$\begin{cases} (-\Delta + x \cdot \nabla)^s u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a possibly unbounded open subset of $\mathbb{R}^n$, $n \geq 2$. The aim is to derive a concentration comparison estimate for solutions using Gaussian symmetrization techniques. As a consequence, L^p and $L^p(\log L)^\alpha$ regularity estimates in terms of the datum f are obtained by comparing u with half-space solutions. The appropriate functional settings for this nonlocal equation and its corresponding extension problem are also presented.

This talk is based on a joint work with P. R. Stinga and B. Volzone.

ON THE SHAPE OF ALMOST-FLAT d-CONES

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A developable-cone (also "d-cone" in short) is the shape one obtains when pushing an elastic sheet at its center into a hollow cylinder by an amount $\epsilon > 0$. Starting from a nonlinear model depending on the thickness $h > 0$ of the sheet, we can first prove a rigorous Γ-convergence result as $h \to 0$, and then study the exact shape of minimizers for ϵ small enough. In particular, we can rigorously justify previous results in the physics literature. This is a joint work with Connor Mooney.

LIPSCHITZ METRICS FOR NONLINEAR PDES

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Solutions of both the Hunter–Saxton and the Camassa–Holm equation might enjoy wave breaking in finite time. This means that solutions in general do not exist globally, but only locally in time since their spatial derivative might become unbounded from below pointwise in finite time, while the solution itself remains bounded. In addition, energy concentrates on sets of measure zero when wave breaking occurs. The prolongation of solutions beyond wave breaking is non-unique. We show how the stability of conservative solutions, i.e., solutions where the energy is not manipulated at breaking time, can be analyzed by constructing a Lipschitz metric, which is based on the use of the Wasserstein metric, in the case of the Hunter–Saxton and the Camassa–Holm equation.

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QUANTITATIVE IMPROVEMENT OF THE FRACTIONAL SOBOLEV INEQUALITY VIA DUALITY

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his work is concerned with an improvement of the critical Sobolev's inequality with a remainder term that involves a dual inequality, namely Hardy-Littlewood-Sobolev. In particular, we're interested in bounds on the optimal constant appearing in the improved inequality. We can provide some by studying a linearized problem on the one hand, and by considering a non linear fractional diffusion flow on the other hand. As a by product, we derive an inequality of Moser-Trudinger type.
This is a joint work with Van Hoang Nguyen.

OPTIMAL CONTROL PROBLEM FOR INTERACTION EQUATIONS: MEAN-FIELD LIMIT AND GAMMA CONVERGENCE.

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In this talk I will describe a mean-field limit approximation of an optimal control problem for a family of interaction equations. By means of a suitable Gamma-convergence I will show that the the mean-field limit for the equation commutes with the optimization. Techniques of optimal transportation of measures are used.

COMPACTNESS OF CRITICAL POINTS FOR ELLIPTIC ISOPERIMETRIC PROBLEMS

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We prove that unions of Wulff shapes are the only accumulation points of sequences of sets whose anisotropic mean curvature converge to a constant. As applications we can characterize local minimizers of energies consisting of an elliptic surface tension plus a potential energy, and we can prove a weak version of Alexandrov's theorem for non-smooth and non-elliptic surface tension energies. This is a joint work with Matias Delgadino, Cornelia Mihaila and Robin Neumayer.

ATOMISTIC POTENTIALS AND THE CAUCHY-BORN RULE FOR CARBON NANOTUBES

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Carbon nanotubes are modeled as point configurations and investigated by minimizing Tersoff interaction potentials. Optimal configurations are identified with local minima and their fine geometry is fully characterized in terms of lower-dimensional problems. Under moderate tension, such local minimizers are proved to be periodic.
This is a joint work with Manuel Friedrich, Paolo Piovano, and Ulisse Stefanelli.

THE UNIFICATION OF EARLY-TIME INFLATION WITH LATE-TIME ACCELERATION IN MODIFIED GRAVITY

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We review the unified description of the universe evolution from the inflation till dark energy in frames of modified gravity. The special role of F(R) gravity will be discussed. We show that in modified gravity the field equations are higher-derivative ones. Even in simple situations like Friedmann universe it is hard or even impossible to solve these equations explicitly. Due to lack of rigorous mathematical results, the method of reconstruction is considered. It represents kind of inverse problem, where instead of explicit solution of equations we search for realistic function F(R) which provides the correct approach to the universe evolution. Some consequences are described.

CONSTRUCTION OF GRADIENT FLOWS IN ABSTRACT METRIC SPACES VIA BDF2

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In this talk I will discuss the construction of λ-contractive gradient flows in abstract metric spaces by means of a semi discretization of second order in time. In the smooth setting, our scheme is simply a variational formulation of the BDF2 method; in the metric setting, it can be considered as the natural second order analogue of the Minimizing Movement or JKO scheme. In difference to the JKO method, our scheme does not necessarily decrease the energy of the discrete solution in each time step, but we can still prove a suitable *almost diminishing* property.

It is well-know that in smooth situations, the BDF2 method converges to order τ^2. We prove that our variational scheme converges at least to order $\tau^{1/2}$ in the general non-smooth setting, provided a certain convexity hypothesis is satisfied. Specifically, that hypothesis is equivalent to λ-uniform convexity in the flat case, and is implied by λ-convexity along generalized geodesics in the L^2-Wasserstein case. The key element in the convergence proof is the derivation of step-size independent bounds on potential and kinetic energy of the discrete solution. The combination of the local bounds and a discrete Gronwall type argument into a global error estimate is significantly harder than for the Minimizing Movement scheme.

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REGULARITY THEORY FOR SINGULAR NONLOCAL DIFFUSION EQUATIONS

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We prove continuity for bounded weak solutions of a nonlinear nonlocal parabolic problem associated to a Dirichlet form with a rough kernel. The equation is allowed to be singular at the level zero, and solutions may change sign. Under additional assumptions on the oscillation of the nonlinearity at the origin, the solution is proved to be moreover Hölder continuous.

The results are new even when the Dirichlet form is the one corresponding to the fractional Laplacian.
Joint work with Arturo de Pablo and Ana Rodríguez.

DETERMINISTIC PARTICLE APPROXIMATION FOR NONLOCAL TRANSPORT EQUATIONS WITH NONLINEAR MOBILITY

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The aim of this work is to describe the one dimensional rearrangement of the components of a biological population under the action of a nonlocal attractive potential and the restriction that no overcrowding can occur. This setting can be expressed by a scalar nonlinear conservation law of the form $\partial_t \rho = -\partial_x(\rho v(\rho)\nabla V * \rho)$ with strictly monotone velocity v and potential V. We prove that a weak solution of the related Cauchy problem, for nonnegative initial condition $\bar{\rho} \in BV(\mathbb{R})$, can be rigorously obtained as the large particle limit of a suitable microscopic follow-the-leader type model, which is interpreted formally as the discrete Lagrangian approximation of the nonlinear scalar conservation law. We finally discuss possible generalizations of our argument to the case of aggregation-diffusion equations.

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MANY PARTICLE APPROXIMATIONS FOR CONSERVATION LAWS

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In this talk we present recent results on the deterministic particle approximation of nonlinear conservation laws. The unique entropy solution to the scalar conservation law

$$\rho_t + \big[\rho\, v(\rho)\big]_x = 0 \tag{CL}$$

with a given initial datum in $\mathbf{L}^\infty$ and with strictly monotone v was rigorously approximated in [1, 3] by the discrete density constructed from the follow-the-leader particle system

$$\dot{x}_i(t) = v\left(\frac{m}{x_{i+1}(t) - x_i(t)}\right). \tag{PS}$$

Said result is based on a discrete version of the classical Oleinik one-sided jump condition for $\mathbf{L}^\infty$ initial data and on a $\mathbf{BV}$ contraction estimate for $\mathbf{BV}$ initial data. The former requires some additional conditions on v. The IBVP for (CL) has been considered in [5]. The results in [1] have been extended to the 2×2 system of conservation laws describing the Aw-Rascle-Zhang (ARZ) model for vehicular traffic in [2], where a similar $\mathbf{BV}$ contraction estimate has been proven, based on the interpretation of the ARZ model as a multi-population model. Finally, we present an extension of these techniques to the one dimensional version of the Hughes model for pedestrians

$$\rho_t - \left[\rho\, v(\rho)\, \frac{\phi_x}{|\phi_x|}\right]_x = 0\,, \qquad |\phi_x| = c(\rho)\,, \tag{H}$$

on a bounded interval with Dirichlet boundary conditions. In [4] we prove the rigorous convergence of a suitable adaptation of the particle scheme (PS) to the unique entropy solution to the IBVP for (H).

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THE EQUILIBRIUM MEASURE FOR A NONLOCAL DISLOCATION ENERGY

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In this talk I will present a recent result on the characterisation of the equilibrium measure for a nonlocal and anisotropic energy arising as the Gamma-limit of discrete interacting dislocations, and an extension to more general anisotropies. This is joint work with J.A. Carrillo, J. Mateu, M.G. Mora, L. Rondi and J. Verdera.

PATTERN FORMATION IN NON-LOCAL SYSTEMS WITH CROSS-DIFFUSION

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Multi-agent systems in nature oftentimes exhibit emergent behaviour, *i.e.* the formation of patterns in the absence of a leader or external stimuli such as light or food sources. We present a non-local two species cross-interaction model with cross-diffusion and explore its long-time behaviour. We observe a rich zoology of behaviours exhibiting phenomena such as mixing and/or segregation of both species and the formation of travelling pulses.

A SECOND-ORDER NUMERICAL METHOD FOR THE AGGREGATION EQUATION

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We propose a (formally) second-order numerical method for the aggregation equation

$$\partial_t \rho = \nabla \cdot \big((\nabla W * \rho) \rho \big), \tag{1}$$

where $\rho = \rho(t) \in \mathcal{P}(\mathbb{R}^d)$ is a probability measure representing the density of individuals and W is an interaction potential.

The method is motivated by the equivalence between a Burgers-type equation and the one-dimensional aggregation equation with $W = \pm|x|$, see [1]. Our starting point is a second-order accurate numerical method for solutions to the Burgers-type equation which is "translated" into a scheme for the aggregation equation with $W = \pm|x|$. The resulting scheme is then extended to a large class of potentials and to $\mathbb{R}^d$.

When W is a pointy attractive potential, the numerical approximation converges to the unique gradient flow solution in the 1–Wasserstein metric (d_1). Furthermore, numerical experiments verify the expected second-order convergence rate in d_1 for sufficiently smooth data, and show that the method also works well for potentials that are not necessarily pointy attractive.

This is joint work with José A. Carrillo and Ulrik S. Fjordholm.

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DIFFUSION IN STRONGLY UNDULATING STRUCTURES

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Within many cell organelles and tissues chemical signals diffuse through strongly layered and structured compartments. Via a homogenization approach we want to understand how the respective geometry affects the densitiy distribution of these signals. Different limiting equations are obtained, which indicates that the structure of the domain strongly influences the diffusion process.

This is joint work with *Benedikt Jost*, Münster.

INFORMATION THEORETIC INEQUALITIES FOR STABLE DENSITIES

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We consider the central limit theorem for stable laws in the case of the standardized sum of independent and identically distributed random variables with regular probability density function. By showing decay of different entropy functionals along the sequence we prove convergence with explicit rate in various norms to a LÃ©vy centered density of parameter $\lambda > 1$. This introduces a new information-theoretic approach to the central limit theorem for stable laws, in which the main argument is shown to be the relative fractional Fisher information, recently introduced in [1]. In particular, it is proven that, with respect to the relative fractional Fisher information, the Lévy density satisfies an analogous of the logarithmic Sobolev inequality, which allows to pass from the monotonicity and decay to zero of the relative fractional Fisher information in the standardized sum to the decay to zero in relative entropy with an explicit decay rate [2].

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CROSS-DIFFUSION HERDING MODEL

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A cross-diffusion system modeling the information herding of individuals is analyzed in a bounded domain with no-flux boundary conditions. The variables are the species' density and an influence function which modifies the information state of the individuals. The cross-diffusion term may stabilize or destabilize the system. Furthermore, it allows for a formal gradient-flow or entropy structure. Exploiting this structure, the global-in-time existence of weak solutions and the exponential decay to the constant steady state is proved in certain parameter regimes. This approach does not extend to all parameters. We investigate local bifurcations from homogeneous steady states analytically to determine whether this defines the validity boundary. This analysis shows that generically there is a gap in the parameter regime between the entropy approach validity and the first local bifurcation. Next, we use numerical continuation methods to track the bifurcating non-homogeneous steady states globally and to determine non-trivial stationary solutions related to herding behaviour. In summary, we find that the main boundaries in the parameter regime are given by the first local bifurcation point, the degeneracy of the diffusion matrix and a certain entropy decay validity condition.

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ON NONLOCAL KELLER-SEGEL TYPE EQUATIONS

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In the first part of the talk we will investigate a Keller-Segel model with quorum sensing and a fractional diffusion operator. This model describes the collective cell movement due to chemical sensing with flux limitation for high cell densities and with anomalous media represented by a nonlinear, degenerate fractional diffusion operator. The purpose here is to introduce and prove the existence of a properly defined entropy solution. In the second part of the talk we will analyze an equation that is gradient flow of a functional related to Hardy-Littlewood-Sobolev inequality in whole Euclidean space $\mathbb{R}^d, d \geq 3$. Under the hypothesis of integrable initial data with finite second moment and energy, we show local-in-time existence for any mass of "free-energy solutions", namely weak solutions with some free energy estimates. We exhibit that the qualitative behavior of solutions is decided by a critical value. The motivation for this part is to generalize Keller-Segel model to higher dimensions.

This is a joint work with K. H. Karlsen(Oslo) and E. A. Carlen(Rutgers).

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LARGE TIME BEHAVIOR IN DEFECTIVE FOKKER-PLANCK EQUATIONS

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The Fokker-Planck equation is an important equation in mathematical analysis with roots in statistical physics and probability. Mostly motivated by recent work of Arnold and Erb, we will discuss optimal long-time behavior for solutions to the hypocoercive Fokker-Planck equation with focus on a defective drift matrix (i.e. lacking eigenvectors). In analogy to defective ODE systems, the true decay rate is not purely exponential, but rather of the form $(1 + t^{2n})e^{-\mu t}$. In order to gain sharp decay estimates, we make use of a new entropic hypercontractivity result, which allows us to take advantage of the geometric nature of the Fokker-Planck operator in the L^2 setting. This is joint work with Anton Arnold and Amit Einav.

EXPONENTIAL RELAXATION OF MEASURE SOLUTIONS TO AGE-STRUCTURED NEURON POPULATION MODELS

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We consider two different age-structured partial differential equation models for structured neuron populations and give some improved results for long time asymptotics. The first model we study is a nonlinear version of the renewal equation, while the second model is a conservative drift-fragmentation equation which adds adaptation and fatigue effects to the neural network model. These problems were introduced in [1] and [2].

We prove that both the problems are well-posed in a measure setting. Both have steady states which may or may not be unique depending on further assumptions. In order to show the exponential convergence to steady states we use a technique from the theory of Markov processes called *Doeblin's method*. This method was used in [3] for demonstrating exponential convergence of solutions of the renewal equation to its equilibrium. It is based on the idea of finding a positive quantitative bound for solutions to the linear problem. This leads us to prove the spectral gap property in the linear setting. Then by exploiting this property we prove that both models converge exponentially to a steady state.

This is a joint work with José A. Cañizo.

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