PERFECT COMMUTING GRAPHS

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Abstract. We classify the finite quasisimple groups whose commuting graph is perfect and we give a general structure theorem for finite groups whose commuting graph is perfect.

1. Introduction

Let $\Gamma$ be a simple, undirected, finite graph with vertex set $V$. If $U \subseteq V$ then the induced subgraph of $\Gamma$ on $U$ is the graph $\Delta$ with vertex set $U$, and with two vertices connected in $\Delta$ if and only if they are connected in $\Gamma$. The chromatic number $\chi(\Gamma)$ is the smallest integer $k$ such that there exists a partition of $V$ into $k$ parts, each with the property that it contains no two adjacent vertices. The clique number $\text{Cl}(\Gamma)$ is the size of the largest complete subgraph of $\Gamma$. Clearly $\text{Cl}(\Gamma) \leq \chi(\Gamma)$ for any graph $\Gamma$. The graph $\Gamma$ is perfect if $\text{Cl}(\Delta) = \chi(\Delta)$ for every induced subgraph $\Delta$ of $\Gamma$.

Let $G$ be a finite group. The commuting graph $\Gamma(G)$ is the graph $\Gamma(G)$ whose vertices are the elements of $G \setminus Z(G)$, with vertices joined by an edge whenever they commute. Some authors prefer not to exclude the central elements of $G$, and nothing in this paper depends significantly on which definition is used.

We are interested in classifying those finite groups $G$ for which the commuting graph $\Gamma(G)$ is perfect. In this paper we offer, in Theorem 1, a complete classification in the case of quasisimple groups. We use this result to derive detailed structural information about a general finite group with this property, in Theorem 2. The notation for quasisimple groups used in the statements of these theorems is explained in §2.

Theorem 1. Let $G$ be a finite quasisimple group and let $\Gamma(G)$ be the commuting graph of $G$. Then $\Gamma(G)$ is perfect if and only if $G$ is isomorphic to one of the groups in the following list:

$\text{SL}_2(q)$ with $q \geq 4$;
$\text{L}_3(2)$;
$\text{L}_3(4)$, $2.\text{L}_3(4)$, $3.\text{L}_3(4)$, $(2 \times 2).\text{L}_3(4)$, $\text{L}_3(4)$, $(6 \times 2).\text{L}_3(4)$, $(4 \times 4).\text{L}_3(4)$, $(12 \times 4).\text{L}_3(4)$;

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Recall that a group is quasisimple if it is a central extension of a simple group, and equal to its derived subgroup. A component of a finite group $G$ is a quasisimple subnormal subgroup of $G$. If $G$ is a group such that $\Gamma(G)$ is perfect, then every component of $G$ has a perfect commuting graph, and is therefore isomorphic to one of the groups listed in Theorem 1. In fact we can say more.

**Theorem 2.** Let $G$ be a finite group such that $\Gamma(G)$ is perfect. Then $G$ has at most two components, and one of the following statements holds.

1. $G$ has a single component, which is isomorphic to one of the groups in the following list:
   - $\text{SL}_2(q)$, $6.A_6$, $(4 \times 4).\text{L}_3(4)$, $(12 \times 4).\text{L}_3(4)$, $\text{Sz}(2^{2a+1})$, $2.\text{Sz}(8)$, $(2 \times 2).\text{Sz}(8)$.

2. $G$ has a single component $N$, and the centralizer $\text{Cent}_G(N)$ is abelian.

3. $G$ has two components $N_1, N_2$, each of which is isomorphic to one of the groups listed in (i), and such that $\text{Cent}_G(N_1N_2)$ is abelian.

We say that a group $G$ is an AC-group if the centralizer of every non-central element of $G$ is abelian. An AC-group necessarily has a perfect commuting graph. Information about these groups emerges naturally in the course of the proof of Theorem 1, which we summarize in the following result.

**Corollary 3.**

1. The finite quasisimple AC-groups are $6.A_6$ and $\text{SL}_2(q)$ for $q \geq 4$.

2. If a finite AC-group $G$ has a component $N$, then $N$ is the unique component of $G$, and the subgroup $\text{NZ}(G)$ has index at most 2 in $G$.

An equivalent formulation of Corollary 3(i) is that a finite quasisimple group $G$ is an AC-group if and only if $G$ has a central subgroup $N$ such that $G/N \cong \text{SL}_2(q)$ for some $q \geq 4$. The case $G \cong 6.A_6$ conforms to this statement, since this group is isomorphic to $3.\text{SL}_2(9)$.

1.1. **Relation to the literature.** There has been a great deal of recent interest in commuting graphs, in the most part relating to the question of the diameter of $\Gamma(G)$, where $G$ is a finite group. The construction by Hegarty and Zhelezov [14] of a 2-group whose commuting graph has diameter 10, has led Giudici and Parker [12] to a construction of a family of 2-groups with commuting graphs of unbounded diameter. This has answered (negatively) an influential conjecture of Iranmanesh and Jafarzadeh [16], that there was a universal upper bound for the diameter of a connected commuting graph. Morgan and Parker [22] have shown, using the...
classification of finite simple groups, that if \(Z(G)\) is trivial then the diameter of any connected component of \(\Gamma(G)\) is at most 10.

Our interest in proving Theorems 1 and 2 stems in part from earlier work with Azad [4] in which we study a generalization of the commuting graph of a group \(G\), namely the \(c\)-nilpotency graph, \(\Gamma_c(G)\). This is the graph whose vertices are elements of \(G\) with two vertices \(g, h\) being connected if and only if the group \(\langle g, h \rangle\) is nilpotent of rank at most \(c\). Ignoring central elements of \(G\), the commuting graph of \(G\) is the same as the 1-nilpotency graph.

In [4] we calculate the clique-cover number and independence number for the graphs \(\Gamma_c(G)\) for various simple groups \(G\) and observe, in particular, that for these groups the two numbers coincide (allowing the possibility that the graphs are perfect). This observation is part of the motivation for the current paper and, moreover, suggests an obvious direction for further research: the question of which quasisimple groups \(G\) have perfect \(c\)-nilpotency graph, for \(c > 1\).

The question of which groups have perfect commuting graph has recently been asked in a blog entry by Cameron [7]. He gives an example of a finite 2-group whose commuting graph is non-perfect. A further motivation for this paper has been to provide at least a partial answer to his question.

The definition of an AC-group is a generalization of the better known notion of a CA-group, a group for which the centralizer of any non-identity element is abelian. Groups with this property arose naturally in early results towards the classification of finite simple groups, and so they have significant historical importance. It has been shown that every finite CA-group is a Frobenius group, an abelian group, or \(\text{SL}_2(2^a)\) for some \(a\) [25], [6], [23].

To a limited degree, Corollary 3 extends this classical work on CA-groups. Of course our work, unlike the work we have cited on CA-groups, depends on the classification of finite simple groups. A proof of Corollary 3 independent of the classification would be of considerable interest and significance. There has been recent interest in AC-groups, and Corollary 3 is related to results from [1] and [3] in particular.

1.2. Structure and methods. The paper is structured as follows. In §2 we state basic definitions and background results as well as proving some straightforward general lemmas. In §3 we work through the different families of simple groups given by the classification, and we establish which finite simple groups have perfect commuting graph, thereby establishing Theorem 1. In §4 we prove Theorem 2 and discuss briefly how it might be strengthened; we also establish Corollary 3.

For the most part the presentation of our arguments does not depend on computer calculation. We acknowledge, however, that the computational algebra packages GAP [11] and Magma [5] have been indispensable to us in arriving at our results, and also that we have allowed ourselves to state many facts about the
structure of particular groups, without proof or reference, when such statements are easily verified computationally.

By ‘group’ we shall always mean ‘finite group’. By ‘simple group’ we shall always mean ‘non-abelian finite simple group’.

1.3. Acknowledgments. The second author was a visitor at the University of Bristol while this work was undertaken and would like to thank the members of the Department of Mathematics for their hospitality.

2. Background

In this section we gather together relevant background material, as well as proving some basic lemmas.

2.1. Graphs. In this paper all graphs are finite, simple and undirected. Let $\Gamma = (V, E)$ be such a graph (with vertex set $V$ and edge set $E$).

(1) The chromatic number of $\Gamma$, $\chi(\Gamma)$, is the smallest number of colours required to colour every vertex of $\Gamma$ so that neighbouring vertices have different colours.

(2) The order of $\Gamma$ is $|V|$.

(3) The clique number of $\Gamma$, $\text{Cl}(\Gamma)$, is the order of the largest complete subgraph of $\Gamma$.

(4) An induced subgraph of $\Gamma$ is a graph $\Lambda = (V', E')$ such that $V' \subseteq V$ and there is an edge between two vertices $v$ and $v'$ in $\Lambda$ if and only if there is an edge between $v$ and $v'$ in $\Gamma$.

(5) We say that $\Gamma$ is perfect if $\chi(\Lambda) = \text{Cl}(\Lambda)$ for every induced subgraph of $\Lambda$.

We have already noted in the introduction that every graph $\Gamma$ satisfies $\chi(\Gamma) \geq \text{Cl}(\Gamma)$.

To state the two most important theorems concerning perfect graphs, we require some terminology. The complement of $\Gamma$ is the graph $\Gamma^c$ with vertex set $V(\Gamma)$, in which an edge connects two vertices if and only if they are not connected by an edge in $\Gamma$. A cycle is a finite connected graph $\Gamma$ such that every vertex has valency 2. A $k$-cycle is a cycle of order $k$.

**Theorem** (Weak Perfect Graph Theorem [21]). A graph $\Gamma$ is perfect if and only if the complement of $\Gamma$ is perfect.

**Theorem** (Strong Perfect Graph Theorem [8]). A graph $\Gamma$ is perfect if and only if it has no induced subgraph isomorphic either to a cycle of odd order at least 5, or to the complement of such a cycle.

We shall say that a subgraph $\Delta$ of $\Gamma$ is forbidden if $\Delta$ is an induced subgraph of $\Gamma$, and either $\Delta$ or $\Delta^c$ is isomorphic to a cycle of odd order at least 5. Figure 1 shows the three smallest forbidden subgraphs. (We note that the complement of a 5-cycle is another 5-cycle.)
The term *Berge graph* has also been used to mean a graph with no forbidden subgraphs (and an alternative statement of the Strong Perfect Graph Theorem is that the class of Berge graphs and the class of perfect graphs are the same). In fact the arguments presented in §3 directly characterize those quasisimple groups $G$ for which $\Gamma(G)$ is a Berge graph. It is the Strong Perfect Graph Theorem which allows us to express these results in terms of perfect graphs.

2.2. **Commuting graphs.** Let $G$ be a finite group. We defined the commuting graph $\Gamma(G)$ in the introduction. It is worth justifying here the assertion that the presence or absence as vertices of the central elements of $G$ has no affect on whether $\Gamma(G)$ is perfect. Let $\Gamma'(G)$ be the graph with vertex set $G$ and an edge $\{g, h\}$ if and only if $gh = hg$. We note that $\Gamma(G)$ is the induced subgraph of $\Gamma'(G)$ on the vertices $G \setminus Z(G)$.

**Lemma 4.** $\Gamma(G)$ is perfect if and only if $\Gamma'(G)$ is perfect.

**Proof.** Suppose that $\Gamma'(G)$ is perfect and $\Lambda$ is an induced subgraph of $\Gamma(G)$. Then $\Lambda$ is an induced subgraph of $\Gamma'(G)$ and so $\chi(\Lambda) = \text{Cl}(\Lambda)$ and $\Gamma(G)$.

Conversely suppose that $\Gamma(G)$ is perfect and $\Lambda'$ is an induced subgraph of $\Gamma'(G)$. Then $V(\Lambda') = V(\Lambda) \cup V_Z$ where $\Lambda$ is an induced subgraph of $\Gamma(G)$ and $V_Z$ is a set of central elements. Now

$$\chi(\Lambda') = \chi(\Lambda) + |V_Z| = \text{Cl}(\Lambda) + |V_Z| = \text{Cl}(\Lambda')$$

and we conclude that $\Gamma'(G)$ is perfect. \qed

It will be convenient to extend our notation in the following way: if $\Omega \subseteq G$, then we write $\Gamma(\Omega)$ for the induced subgraph of $\Gamma(G)$ whose vertices are elements of $\Omega$.

2.3. **The classification of finite simple groups.** Our results are all dependent on the classification of finite simple groups. The principal sources for information on these groups and their covering groups is [9] and [17], which we have used very extensively, without necessarily mentioning it explicitly in every instance. We have used the notation of [9] for finite simple and quasisimple groups, except in a few cases where we believe another usage is less likely to cause confusion.
Alternating groups \( A_n \) \( n \geq 5 \)

Classical groups

- Linear \( L_n(q) \) \( n \geq 2; \) not \( L_2(2) \) or \( L_2(3) \),
- Unitary \( U_n(q) \) \( n \geq 3; \) not \( U_3(2) \),
- Symplectic \( \text{PSp}_{2m}(q) \) \( m \geq 2; \) not \( \text{PSp}_4(2) \),
- Orthogonal \( \text{PO}_{2m+1}(q) \) \( m \geq 3; q \) odd,
\( \text{PO}_{2m}^+(q) \) \( m \geq 4, \)
\( \text{PO}_{2m}^-(q) \) \( m \geq 4, \)

Exceptional groups

- Chevalley \( G_2(q), F_4(q), E_6(q); \) not \( G_2(2) \),
\( E_7(q), E_8(q), \)
- Steinberg \( ^3D_4(q), ^2E_6(q), \)
- Suzuki \( Sz(2^{2a+1}) \) \( a \geq 1, \)
- Ree \( ^2F_4(2^{2a+1})', ^2G_2(3^{2a+1}) \) \( a \geq 0, a \geq 1 \)

Sporadic groups

- \( M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, J_3, J_4, \) \( \text{Co}_3, \text{Co}_2, \text{Co}_1, \text{Fi}_{22}, \)
\( \text{Fi}_{23}, \text{Fi}_{24}, \text{HS}, \text{McL}, \text{He}, \text{Ru}, \text{Suz}, \text{O'N}, \text{HN}, \text{Ly}, \text{Th}, B, M. \)

\begin{table}
\begin{center}
\begin{tabular}{ll}
Alternating groups & \( A_n \) \\
Classical groups & \\
Linear & \( L_n(q) \) \( n \geq 2; \) not \( L_2(2) \) or \( L_2(3) \),
Unitary & \( U_n(q) \) \( n \geq 3; \) not \( U_3(2) \),
Symplectic & \( \text{PSp}_{2m}(q) \) \( m \geq 2; \) not \( \text{PSp}_4(2) \),
Orthogonal & \( \text{PO}_{2m+1}(q) \) \( m \geq 3; q \) odd,
& \( \text{PO}_{2m}^+(q) \) \( m \geq 4, \)
& \( \text{PO}_{2m}^-(q) \) \( m \geq 4, \)
Exceptional groups & \\
Chevalley & \( G_2(q), F_4(q), E_6(q); \) not \( G_2(2) \),
& \( E_7(q), E_8(q), \)
Steinberg & \( ^3D_4(q), ^2E_6(q), \)
Suzuki & \( Sz(2^{2a+1}) \) \( a \geq 1, \)
Ree & \( ^2F_4(2^{2a+1})', ^2G_2(3^{2a+1}) \) \( a \geq 0, a \geq 1 \)
Sporadic groups & \\
& \( M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, J_3, J_4, \) \( \text{Co}_3, \text{Co}_2, \text{Co}_1, \text{Fi}_{22}, \)
& \( \text{Fi}_{23}, \text{Fi}_{24}, \text{HS}, \text{McL}, \text{He}, \text{Ru}, \text{Suz}, \text{O'N}, \text{HN}, \text{Ly}, \text{Th}, B, M. \)
\end{tabular}
\end{center}
\end{table}

The non-abelian finite simple groups are listed in Table 1. In this table, and throughout the paper, \( q \) is a power of a prime \( p \). The parameters in this list have been restricted in order to reduce the number of occurrences of isomorphic groups under different names. The following isomorphisms remain:

\[ A_5 \cong L_2(4) \cong L_2(5), \ L_2(7) \cong L_3(2), \ A_6 \cong L_2(9), \ A_8 \cong L_4(2), \ U_4(2) \cong \text{PSp}_4(3). \]

2.4. Quasisimple groups. Although the notation used in this paper is standard, we recall here some key definitions. A group is perfect if it coincides with its commutator subgroup. (There is no connection between the usages of the word ‘perfect’ as it applies to graphs and to groups.) A quasisimple group is a perfect group \( G \) such that \( G/Z(G) \) is simple.

All of the finite quasisimple groups are known, as a corollary to the Classification of Finite Simple Groups. In most cases a quasisimple group \( G \) has cyclic centre, and for a positive integer \( n \) and a simple group \( S \), we write \( n.S \) for a group \( G \) such that \( Z(G) \) is cyclic of order \( n \) and \( G/Z(G) \cong S \). This notation extends in a natural
way to groups with non-cyclic centres; for instance we write $(2 \times 2).S$ for a group with centre $C_2 \times C_2$ and $G/Z(G) \cong S$.

In principle the notation just described does not specify a group up to isomorphism – for instance, there are many isomorphism classes of groups of type $n.S$ – however in all instances of the notation in this paper, the isomorphism class is in fact unique. This notation for finite quasisimple groups is consistent with, for instance, [9] and [17].

A component of a group $G$ is a quasisimple subgroup $N$ which is subnormal, i.e. there exists a finite chain of subgroups of the form $N = N_0 < N_1 < N_2 < \cdots < N_k = G$, such that $N_i$ is normal in $N_{i+1}$ for all $i$. The significance of the set of components of $G$ has been demonstrated by the seminal work of Bender, in which the notion of the Generalized Fitting Group $F^*(G)$ is defined (see, for instance, [2]).

2.5. Basic lemmas.

**Lemma 5.** If $H \leq G$ and $\Gamma(H)$ is not perfect, then $\Gamma(G)$ is not perfect.

*Proof.* Any induced subgraph of $\Gamma(H)$ is an induced subgraph of $\Gamma(G)$. The result follows immediately. 

**Lemma 6.** Suppose that $g$ is an element in $G$ for which $\text{Cent}_G(g)$ is abelian. Then $g$ does not lie on a forbidden subgraph of $\Gamma(G)$.

*Proof.* The supposition implies that if any two elements $h, k$ are neighbours of $g$ in $\Gamma(G)$, then there is an edge between $h$ and $k$, and the result follows. 

**Lemma 7.** Let $G$ be a group and let $g, h \in G$ be vertices of a forbidden subgraph of $\Gamma(G)$. Then $gh^{-1} \notin Z(G)$.

*Proof.* Suppose that $gh^{-1} \in Z(G)$. Then $\text{Cent}_G(g) = \text{Cent}_G(h)$, and so $g$ and $h$ have the same neighbours in $\Gamma(G)$. But for any two distinct vertices $u, v$ of a forbidden subgraph, there is a third vertex $w$ of the subgraph which is connected to $u$ but not to $v$. 

The next lemma helps us to pass between simple groups and their quasisimple covers.

**Lemma 8.** Let $G$ be a group, let $Z$ be a central subgroup of $G$, let $K = G/Z$, and let $\varphi : G \rightarrow K$ be the natural projection. Let $\Omega \subseteq K$ and suppose that for each $\omega \in \Omega$, the elements of $\varphi^{-1}(\omega)$ are pairwise non-conjugate. Then $\Gamma(\Omega)$ is perfect if and only if $\Gamma(\varphi^{-1}(\Omega))$ is perfect.

*Proof.* For each $\omega \in \Omega$, let $\omega'$ be a pre-image in $G$ of $\omega$, and let $\Omega'$ be the set $\{\omega' \mid \omega \in \Omega\}$. It is clear that if two elements $h, h' \in \Omega'$ commute, then $\varphi(h)$ and $\varphi(h')$ commute. On the other hand if $\varphi(h)$ and $\varphi(h')$ commute, then $[h', h] \in Z$, 

and so $hh'h^{-1} = zh'$ for some $z \in Z(G)$. But now the condition on pairwise non-conjugacy implies that $z = 1$, and so $h$ and $h'$ commute. Hence $h$ and $h'$ commute if and only if $\varphi(h)$ and $\varphi(h')$ commute, and so we conclude that $\Gamma'(\Omega') \equiv \Gamma(\Omega)$. Now $\Gamma'(\Omega')$ is an induced subgraph of $\Gamma(\varphi^{-1}(\Omega))$, from which it follows that if $\Gamma(\varphi^{-1}(\Omega))$ is perfect then so is $\Gamma(\Omega)$.

For the converse suppose that $\Gamma(\varphi^{-1}(\Omega))$ contains a forbidden subgraph $\Delta$. Lemma 7 tells us that the vertices of $\Delta$ have distinct images under $\varphi$. Now $\Delta$ can be extended to a set $\Omega'$ as described above, and we have seen that $\Gamma'(\Omega') \equiv \Gamma(\Omega)$. It follows that $\Gamma(\Omega)$ contains a forbidden subgraph, as required. □

Note that if $\omega$ is an element of $G$ with order coprime to $|Z(G)|$, then all elements of $\phi^{-1}(\omega)$ are pairwise non-conjugate.

3. Commuting graphs of quasisimple groups

In this section we study the commuting graphs of quasisimple groups. We go through the various families given by the classification of finite simple groups, establishing which groups have Berge graphs as their commuting graphs. These are precisely the perfect commuting graphs, by the Strong Perfect Graph Theorem.

A technique we use frequently to show that a group $G$ has a non-perfect commuting graph, is to exhibit a subgroup for which this is already known, and then invoke Lemma 5. Both the Strong Perfect Graph Theorem and Lemma 5 will therefore be in constant use in this section. We shall usually suppress explicit references to them, in order to avoid tedious repetitions.

In the cases where no subgroup of $G$ is known to have a non-perfect commuting graph, it is necessary to determine whether $\Gamma(G)$ contains a forbidden subgraph directly. We use a variety of techniques for exhibiting odd length cycles. In most cases these have length 5, but we have made no particular effort to describe the shortest cycle possible. Indeed, for the infinite families $2.A_n$ and $2.G_2(q)$ our arguments yield 7-cycles in the commuting graphs, although it is known that 5-cycles exist in almost all of the groups in these families. In fact we know of only three finite quasisimple groups $G$ such that $\Gamma(G)$ is non-perfect, but contains no induced subgraph isomorphic to a 5-cycle; these are

$2.A_7$, $L_2(13)$, $L_2(17)$,

each of which can be shown to have a 7-cycle as a forbidden subgraph. We believe that there exist no further examples, but we have not attempted to prove this.

3.1. Alternating groups.

Lemma 9. Let $G = S_5$, the symmetric group on 5 letters. Then $\Gamma(G)$ is not perfect.
Proof. The induced subgraph on the vertices (1 5), (2 1), (2 3), (3 4), (4 5) is a 5-cycle. □

Lemma 10. Let $G = A_n$, the alternating group on $n$ letters. Then $\Gamma(G)$ is perfect if and only if $n \leq 6$.

Proof. The centralizer of every non-identity element of $A_5$ is abelian, and so $\Gamma(A_5)$ is perfect by Lemma 6. On the other hand, if $n \geq 7$ then $A_n$ has a subgroup isomorphic to $S_5$, and so Lemma 9 tells us that $\Gamma(A_n)$ is not perfect.

It remains to deal with the case $n = 6$. The only non-trivial elements of $A_5$ with non-abelian centralizers have order 2, and so if $\Lambda$ is a forbidden subgraph of $\Gamma(G)$, then every vertex of $\Lambda$ is an involution. Let $g \in G$ be a vertex of $\Lambda$. Observe that $\text{Cent}_G(g) \cong D_8$, the dihedral group of order 8. Since there are only five involutions in $D_8$ we conclude that $\Lambda$ is either a cycle of odd order, or else the complement of a 7-cycle. (Recall that the complement of a 5-cycle is another 5-cycle.)

Suppose that $\Lambda$ is the complement of a 7-cycle. We may assume that one of its vertices is $(1 2)(3 4)$. The neighbours of this vertex in $\Lambda$ can only be the other four involutions in its centralizer, namely

$$(1 3)(2 4), \quad (1 2)(5 6), \quad (1 4)(2 3), \quad (3 4)(5 6).$$

Similarly the neighbours of $(1 4)(2 3)$ must be

$$(1 2)(3 4), \quad (1 3)(2 4), \quad (1 4)(5 6), \quad (2 3)(5 6).$$

But we have now listed seven involutions (not including repetitions), and it is easily checked that the induced subgraph on these vertices is not the complement of a 7-cycle; this is a contradiction.

We have still to show that $\Lambda$ cannot be a cycle of odd order at least 5. The group $G$ has two conjugacy classes of subgroups isomorphic to $C_2 \times C_2$. Exactly one subgroup from each class is contained in $\text{Cent}_G(g)$; let these subgroups be $A$ and $B$. If $h$ and $h'$ are the neighbours of $g$ in $\Lambda$, then since $h$ and $h'$ do not commute, we see that the subgroups $\langle g, h \rangle$ and $\langle g, h' \rangle$ are distinct; so one of them is $A$ and the other $B$. It follows that if we colour each edge of $\Lambda$ according to whether the vertices it connects generate a conjugate of $A$ or a conjugate of $B$, then we have a 2-colouring of the edges of $\Lambda$. But this is a contradiction, since a cycle of odd length is not 2-colourable. □

Corollary 11. Let $G = S_n$, the symmetric group on $n$ letters. Then $\Gamma(G)$ is perfect if and only if $n \leq 4$.

Proof. Lemma 9 implies that $\Gamma(G)$ is not perfect if $n \geq 5$. On the other hand $A_6$ has a subgroup isomorphic to $S_4$ and so Proposition 12 implies that $\Gamma(S_4)$ is perfect; hence the same is true of $\Gamma(S_n)$ with $n < 4$. □
Now we generalize Lemma 10 to deal with quasisimple covers of alternating groups.

**Proposition 12.** Let $G$ be a quasisimple group such that $G/Z(G) \cong A_n$ for some $n \geq 5$. Then $\Gamma(G)$ is perfect if and only if $G$ is equal to one of the groups in the following list:


**Proof.** If $G$ is simple then it is either $A(5)$ or $A(6)$ by Lemma 10, and so we may suppose that $G$ is not simple. If $n \geq 7$ and $|Z(G)| = 2$, then the induced subgraph in $A_n$ on the vertices

$$(123), (456), (127), (345), (167), (234), (567)$$

is a 7-cycle. Since all of these elements have order 3, while $Z(G)$ has order 2, each element lifts to elements of order 3 and 6 in $G$. Now the lifts commute exactly when their projective images commute, and so the induced subgraph of $\Gamma(G)$ on the lifts of order 3 is a 7-cycle.

Since the Schur multiplier of $A(n)$ has order 2 for $n \geq 8$, we may now suppose that $n \leq 7$. We deal with the remaining groups one by one.

The groups $2.A_5$, $2.A_6$, and $6.A_6$ are AC-groups (being isomorphic to $\text{SL}_2(5)$, $\text{SL}_2(9)$ and $3.\text{SL}_2(9)$ respectively), and so have perfect commuting graphs by Lemma 6.

In $3.A_6$, the only non-central elements with non-abelian centralizers are elements whose square is central (i.e. they are lifts of involutions in $A_6$). No forbidden subgraph can contain two vertices in the same coset of the centre, as these would have the same set of neighbours. So we may restrict our attention to the commuting graph on the involutions of $3.A_6$. But this commuting graph is isomorphic to the commuting graph on the involutions of $A_6$ (see Lemma 8), which we have already seen to be perfect.

To exclude $3.A_7$ we recall that $\Gamma(A_7)$ contains a 5-cycle whose vertices are involutions. Each of these involutions lifts to a unique involution in $3.A_7$ and the induced subgraph of $\Gamma(3.A_7)$ on these involutions is, again, a 5-cycle.

Finally suppose that $G = 6.A_7$. There are two conjugacy classes of non-abelian subgroups which arise as centralizers in $G$. Let $T$ be the set of elements with non-abelian centralizers and $\Gamma(T)$ the induced subgraph of $\Gamma(G)$ with vertices in $T$. It is a straightforward computation that if $g, h \in G$ are conjugate elements of $T$ which commute, then their centralizers are equal. Let $\Gamma(T)/\sim$ be the quotient of $\Gamma(G)$ obtained by identifying vertices with the same centralizer. Then this graph is bipartite, and hence perfect. It follows easily that $\Gamma(G)$ is perfect. \qed

3.2. **Linear groups of dimension 2.**

**Lemma 13.** Suppose that $G \leq \text{GL}_2(q)$. Then $\Gamma(G)$ is perfect.
Proof. If \( g \) is a non-trivial element of \( G \), then \( \text{Cent}_G(g) \) is abelian. Now the result follows from Lemma 6. \( \square \)

Lemma 13 implies, in particular, that the quasisimple groups \( \text{SL}_2(q) \) have perfect commuting graphs. The next result deals with most of the remaining 2-dimensional quasisimple groups.

**Lemma 14.** If \( G = \text{L}_2(q) \) with \( q \) odd and \( q > 9 \), then \( \Gamma(G) \) is not perfect.

**Proof.** Define \( \epsilon \) by

\[
\epsilon = \begin{cases} 
1 & \text{if } q \equiv 1 \mod 4, \\
-1 & \text{if } q \equiv 3 \mod 4.
\end{cases}
\]

The group \( G \) has a single class \( T \) of involutions, of size \( q(q + \epsilon)/2 \). This is the only conjugacy class of \( G \) whose elements have non-abelian centralizers. The centralizer of each involution is a dihedral group of order \( q - \epsilon \).

The graph \( \Gamma(T) \) is a regular graph of degree \( (q - \epsilon)/2 \). Let \( t \in T \), and let \( \Omega_d \) be the set of vertices in \( \Gamma(T) \) at distance \( d \) from \( t \). Then \( |\Omega_1| = (q - \epsilon)/2 \). Each vertex \( s \) in \( \Omega_1 \) is connected to exactly one other vertex in \( \Omega_1 \), and so \( s \) has \( (q - \epsilon - 4)/2 \) neighbours in \( \Omega_2 \).

We claim that \( \Gamma(T) \) contains no subgraph (induced or otherwise) isomorphic to a 4-cycle. To prove this claim, let us suppose that \( T' = \{t_1, t_2, t_3, t_4\} \) is a subset of \( T \) such that \( \Gamma(T') \) contains a 4-cycle, with vertices in the order listed. Recall that the intersection of the centralizers of distinct involutions in \( \text{L}_2(q) \) is abelian, being a subgroup of a Klein 4-group. Since each of \( t_2 \) and \( t_4 \) centralizes both \( t_1 \) and \( t_3 \), we see that \( t_2 \) and \( t_4 \) commute. So \( \Gamma(T') \) is not a 4-cycle, and the claim is proved.

An immediate corollary to the claim is that if \( \Gamma(T) \) contains a 5-cycle as a subgraph, then that subgraph is an induced subgraph. The claim implies, moreover, that no two vertices in \( \Omega_1 \) have a common neighbour in \( \Omega_2 \), and so we have

\[
|\Omega_2| = \frac{1}{4}(q - \epsilon)(q - \epsilon - 4).
\]

Each vertex \( r \) in \( \Omega_2 \) has a neighbour \( s \in \Omega_2 \) such that \( r \) and \( s \) have a common neighbour in \( \Omega_1 \). Suppose that \( r \) has another neighbour \( u \in \Omega_2 \). If \( r' \) and \( u' \) are the neighbours of \( r \) and \( u \), respectively, in \( \Omega_1 \), then it is clear that the induced subgraph on the vertices \( t, r', r, u, u' \) is a 5-cycle.

We may suppose, then, that any two neighbours in \( \Omega_2 \) have a common neighbour in \( \Omega_1 \), and hence no common neighbour in \( \Omega_3 \). Let \( x \in \Omega_3 \), and suppose that \( r \) and \( s \) are neighbours of \( x \) in \( \Omega_2 \). Then \( r \) and \( s \) are not adjacent, and have no common neighbour in \( \Omega_1 \). Let \( r' \) and \( s' \) be the neighbours of \( r \) and \( s \), respectively, in \( \Omega_1 \). If \( r' \) and \( s' \) are adjacent, then the induced subgraph on \( \{r', r, x, s, s'\} \) is a 5-cycle, and so again here, \( \Gamma(G) \) is not perfect.

We may therefore assume that for any vertex \( x \) in \( \Omega_3 \), and for any pair \( u, v \) of neighbours in \( \Omega_1 \), there exists at most one \( r \in \Omega_2 \) such that \( r \) is joined to \( x \) and...
to either of \( u \) or \( v \). It follows that the number of neighbours for \( x \) in \( \Omega_2 \) cannot be greater than \(|\Omega_1|/2 = (q - \epsilon)/4\). Now each element of \( \Omega_2 \) has \((q - \epsilon - 4)/2\) neighbours in \( \Omega_3 \), and so we have

\[
|\Omega_3| \geq \frac{2(q - \epsilon - 4)}{q - \epsilon} - |\Omega_2| = \frac{1}{2} (q - \epsilon - 4)^2.
\]

Now clearly \(|T| \geq 1 + |\Omega_1| + |\Omega_2| + |\Omega_3|\); but asymptotically the left-hand side of this inequality is \(q^2/2\), whereas the right-hand side is \(3q^2/4\). We may therefore bound \( q \) above; specifically, we obtain the inequality

\[
q^2 - (18 + 8\epsilon)q + (39 + 18\epsilon) \leq 0.
\]

When \( \epsilon = -1 \) this implies that \( q \leq 7 \), and for \( \epsilon = 1 \) that \( q \leq 17 \).

It remains only to check the cases \( q = 13 \) and \( q = 17 \), which require separate treatment since \( \Gamma(L_2(q)) \) does not have a 5-cycle as an induced subgraph in these cases. A straightforward computation shows that in each group there exists an involution \( t \), and an element \( g \) of order \((q + 1)/2\), such that \([t, t^g] = 1\). Now it is not hard to show that the induced subgraph on the conjugates \( t, t^9, t^{16}, \ldots \) is a cycle of order \((q + 1)/2\).

We see that if \( q \) is even or at most 9, then \( \Gamma(L_2(q)) \) is perfect. For even \( q \) this follows immediately from Lemma 13. For \( L_2(5) \cong A_5 \) and \( L_2(9) \cong A_6 \), see Proposition 12, and for \( L_2(7) \cong SL_3(2) \) see Lemma 17 below. (The commuting graph of the non-quasisimple group \( L_2(3) \) is easily seen to be perfect.)

The exceptional covers of \( L_2(9) \) have also been dealt with in Proposition 12 above. We thus have a complete classification of those quasisimple groups \( G \) such that \( G/Z(G) \cong L_2(q) \) and such that \( \Gamma(G) \) is perfect.

**Remark 15.** It follows from Lemmas 13 and 14 that the commuting graph of \( PGL_2(q) \) is not perfect when \( q \) is odd and \( q > 9 \). We remark that \( \Gamma(PGL_2(q)) \) is not perfect when \( q \in \{5, 7, 9\} \) either; we omit the proof of this fact, which is straightforward to establish computationally.

### 3.3. Classical groups of dimension 3.

**Lemma 16.** Let \( G \) be isomorphic to \( SL_3(q) \) or \( L_3(q) \) with \( q \neq 2, 4 \). Then the commuting graph of \( G \) is not perfect.

**Proof.** Let \( \alpha \) and \( \beta \) be distinct non-zero elements of \( \mathbb{F}_q \). Then the five matrices

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}
, \quad
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
, \quad
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
, \quad
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
, \quad
\begin{pmatrix}
\alpha & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & \beta
\end{pmatrix}
\]
constitute a 5-cycle subgraph of the commuting graph of GL_3(q). For the last of these matrices to lie in SL_3(q) we require that \( \alpha \beta^2 = 1 \); this equation is soluble (by distinct elements \( \alpha, \beta \)) unless \( q \) is 2 or 4.

It remains only to observe that the images of these matrices in L_3(q) induce a 5-cycle in \( \Gamma(L_3(q)) \).

\[ \square \]

Lemma 17. Let \( G \) be isomorphic to one of SL_3(2), SL_3(4) or L_3(4). Then \( \Gamma(G) \) is perfect.

\[ \text{Proof.} \] It suffices to prove the lemma for SL_3(4) and L_3(4), since SL_3(2) is contained in each as a subgroup.

The only non-central elements of SL_3(4) whose centralizers are non-abelian are the transvections. These have order 2 or 6, and fall in three conjugacy classes. These classes merge into one class of involutions in L_3(4), and this class contains all of the non-trivial elements of L_3(4) with non-abelian centralizers. From these facts, it easily follows that \( \Gamma(SL_3(4)) \) and \( \Gamma(L_3(4)) \) are perfect if and only if the induced subgraph \( \Gamma(T) \), common to both, on the set \( T \) of involutory transvections, is perfect.

Let \( V \) be the natural module for SL_3(4). For a transvection \( t \), we write \( H(t) \) for the hyperplane of \( V \) fixed by \( t \). We write \( L(t) \) for the image of \( t - I \) (equivalently, the unique 1-dimensional \( t \)-invariant subspace \( \langle v \rangle \) of \( V \) such that \( t \) acts trivially on \( V/\langle v \rangle \)). It is easy to show that elements \( t \) and \( u \) of \( T \) commute if and only if either \( H(t) = H(u) \) or \( L(t) = L(u) \).

Let \( S \) be a subset of \( T \) of size at least 5. Suppose that \( S \) contains distinct vertices \( t \) and \( u \) such that \( H(t) = H(u) \) and \( L(t) = L(u) \). Then for each \( v \in S \) distinct from \( t \) and \( u \), we see that \( v \) is adjacent to \( t \) if and only if it is adjacent to \( u \). It follows that the induced subgraph of \( \Gamma(SL_3(4)) \) on \( S \) cannot be a cycle or its complement. We shall therefore suppose that \( S \) contains no such elements \( t \) and \( u \). Now we may colour each edge \((t, u)\) of the induced subgraph of \( \Gamma(T) \) on \( S \) with colours \( H \) and \( L \), depending on whether \( H(t) = H(u) \) or \( L(t) = L(u) \).

Suppose that the induced subgraph on \( S \) is a cycle \((t_1, \ldots, t_k)\). For \( t_i \in S \), we see that the colour of \((t_i, t_{i-1})\) is not the colour of \((t_i, t_{i+1})\), or else \( t_{i-1} \) and \( t_{i+1} \) would commute. It follows immediately that the cycle is 2-colourable, and hence that it has even length.

Suppose on the other hand that the induced subgraph on \( S \) is the complement of a \( k \)-cycle. We may assume that \( k > 5 \), since the complement of a 5-cycle is another 5-cycle. Let \( t_1, t_2 \) be two connected vertices in \( S \) so either \( H(t_1) = H(t_2) \) or else \( L(t_1) = L(t_2) \). Now \( \{t_1, t_2\} \) is a subset of \( k - 5 \) distinct triangles in \( \Gamma(S) \) and so at least \( k - 3 \) vertices in \( \Gamma(S) \) have the same colour. But this implies that \( S \) contains a complete subgraph on \( k - 3 \) vertices, which is impossible. \( \square \)
By reference to [17, Table 5.1.D] we see that the results given above attend to almost all quasisimple covers of the groups $L_3(q)$. When $q = 2$ the group $L_3(2)$ has an exceptional cover isomorphic to $SL_2(7)$; however this has been dealt with in the previous section and so can be excluded here. The remaining exceptions occur when $q = 4$, and the next lemma deals with this situation.

**Lemma 18.** Suppose that $G$ is a quasisimple group such that $G/Z(G) \cong L_3(4)$. Then $\Gamma(G)$ is perfect if and only if $G$ is one of the groups in the following list.

$$L_3(4), \ 2SL_3(4), \ 3L_3(4), \ (2 \times 2)L_3(4), \ 6L_3(4),$$

$$(6 \times 2)L_3(4), \ (4 \times 4)L_3(4), \ (12 \times 4)L_3(4).$$

**Proof.** By reference to [17, Table 5.1.D], we observe that the Schur multiplier of $L_3(4)$ is $C_{12} \times C_4$. The extension $3L_3(4)$ is isomorphic to $SL_3(4)$. The elements of $L_3(4)$ have orders from the set $\{1, 2, 3, 4, 5, 7\}$; there is a unique conjugacy class $T$ of involutions.

Let $G$ be a quasisimple extension of $L_3(4)$. If $g$ is an element of $G$ whose image in $L_3(4)$ has order 3,4,5 or 7, then the centralizer of $g$ in $G$ is abelian. For each involution $t \in L_3(4)$, let $t_G$ be an element of $G$ which projects onto $t$, and let $T_G = \{t_G \mid t \in T\}$. Then Lemma 7 implies that $\Gamma(G)$ is perfect if and only if its induced subgraph $\Gamma(T_G)$ on $T_G$ is perfect.

Since the graphs $\Gamma(T_G)$, for the various quasisimple extensions $G$, all have vertex sets in natural bijection to one another, we can represent them all using a single ornamented graph. Let $M = (12 \times 4)L_3(4)$ be the full covering group. Let $\Gamma(T)$ be the commuting graph on $T$. We endow $\Gamma(T)$ with an edge-labelling, where the label of the edge $(s,t)$ is determined by the commutator $[s_M, t_M]$, an element of $Z(M)$.

In fact only four labels are needed, since if $s$ and $t$ are commuting elements of $L_3(4)$ then $[s_M, t_M]$ has order at most 2 [11], and hence lies in the unique subgroup $V$ of $Z(M)$ isomorphic to $C_2 \times C_2$.

Each quasisimple group $G$ is a central quotient of $M$, and it is clear that the commuting graph $\Gamma(T_G)$ is determined by the image $V_G$ of $V$ in this quotient. If $V_G$ is trivial, then $\Gamma(T_G)$ is isomorphic to $\Gamma(T)$, which by Lemma 17 is perfect. If $V_G \cong C_2 \times C_2$ then $T_G \cong T_M$, which consists of 105 connected components, each isomorphic to the triangle graph $K_3$; so clearly $\Gamma(T_G)$ is perfect in this case also.

For the remaining cases, recall that the elements of $T$ correspond to transvections in $SL_3(4)$, and that each transvection has associated with it a hyperplane $H(t)$ of fixed points, and a line $L(t)$, which is the image of $I - t$. Transvections $s$ and $t$ commute if and only if $H(s) = H(t)$ or $L(s) = L(t)$. For any hyperplane $H$ in $F_4^3$ there are 15 transvections $t$ such that $H(t) = H$, and for each line $L$ there are 15 such that $L(t) = L$. Furthermore, for each pair $(H, L)$ such that $L < H$, there are three transvections $t$ such that $H(t) = H$ and $L(t) = L$ (which yield the triangles...
Perfect Commuting Graphs

in $\Gamma(T_M)$ described above). There are 21 lines and 21 hyperplanes in $F_4^1$, and so $\Gamma(T)$ may be expressed as a union of 42 copies of the complete graph $K_{15}$.

The vertices of each of these copies of $K_{15}$ generate an elementary abelian group $A$ of order 16. The group $A$ is naturally endowed with the structure of a 2-dimensional vector space over the field $F_4$, scalar multiplication being given by the rule $(\lambda, t) \mapsto I + \lambda(t - I)$ for $\lambda \in F_4$.

We are now in a position to deal with groups $G$ such that $V_G \cong C_2$. Let $v \in V$ be the non-identity element of the kernel of the map $V \longrightarrow V_G$. We associate the group $V$ with the additive group $F_4$, with $I$ as 0 and with $v$ as 1. It is not hard to show that the map $(s, t) \mapsto [s_M, t_M]$ defines a non-degenerate alternating form on $A$. Let $s$ and $t$ be a hyperbolic pair with respect to this form; so $(s, t) = v = 1$. Let $\alpha$ be a primitive element of $F_4$, and consider the induced subgraph of $\Gamma(T)$ on the vertices

$s, \alpha s, \alpha^{-1}s + \alpha^{-1}t, at, t$.

We see that in the order listed above, edges between consecutive vertices receive labels 0 or 1, whereas other edges receive labels $\alpha$ or $\alpha^{-1}$. It follows that these vertices induce a 5-cycle in $\Gamma(T_G)$, and so $\Gamma(G)$ is not perfect.

Thus the commuting graph of $G$ is perfect if and only if $G \cong M/Z_0$ where $Z_0 \leq Z(M)$ and either $V \leq Z_0$ or $V \cap Z_0 = \{1\}$. It is an easy matter to ascertain which groups $Z_0 \leq Z(M)$ satisfy this condition and one obtains quotients as listed. Note that for some of these quotients, $2L_3(4)$ for instance, there is more than one choice for the subgroup $Z_0$; in such cases we can appeal to [13, Theorem 6.3.2] to see that they are all isomorphic.

Lemma 19. Let $G$ be a quotient of $SU_3(q)$ by a central subgroup, where $q > 2$. Then $\Gamma(G)$ is not perfect.

Proof. For details about the dilatation and transvection mappings used in this argument, we refer the reader to [10, Chapter 2].

Suppose first that $G = SU_3(q)$. Let $V$ be the natural module for $G$, and let $F$ be the underlying Hermitian form on $V$. For any 1-dimensional subspace $L$ of $V$, there is a non-central element $X$ of $G$ such that $L$ is $X$-invariant, and such that $X$ acts as a scalar on $L^\perp$.

If $L$ is non-singular with respect to $F$, then $X$ is a scalar multiple of a dilatation in $GU_3(q)$, with axis $L$ and centre $L^\perp$. The transformation $X$ may be chosen to have order 2 if $q$ is odd, or order $q + 1$ if $q$ is even. The centralizer of $X$ in $G$ is equal to the stabilizer of $L$.

On the other hand if $L$ is singular with respect to $F$ then $X$ is a scalar multiple of a transvection, again with axis $L$ and centre $L^\perp$. In this case $X$ may be chosen to have order $p$, where $p$ is the characteristic. In this case the centralizer of $X$ in $G$ is a proper subgroup of the stabilizer of $L$, of index $q^2 - 1$. 


Let $\Omega$ be a set of transformations $X$ of the types described above, one for each line in $V$. We write $L(X)$ for the axis of $X$. Suppose that $X, Y \in \Omega$ be distinct elements which commute. Then $Y$ stabilizes $L(X)$, and since $L(X) \neq L(Y)$, it is easy to see that $L(X) \in L(Y)^\perp$. Conversely, suppose that $L(X) \in L(Y)^\perp$; then $L(X)$ and $L(Y)$ cannot both be singular, since $V$ has no totally singular 2-dimensional subspace. We may suppose without loss of generality that $L(X)$ is non-singular; now since $Y$ acts as a scalar on $L(Y)^\perp$ we see that $Y$ is in the stabilizer of $L(X)$, which is equal to the centralizer of $X$.

Let $\Delta_F$ be the graph whose vertices are 1-dimensional subspaces of $V$, with edges connecting lines which are perpendicular with respect to $F$. Then we have shown that $\Delta_F$ is isomorphic to the subgraph of $\Gamma(G)$ induced on the vertices $\Omega$. Let $(v_1, v_2, v_3)$ be a basis for $V$; we may take $F$ to be the hermitian form given by

$$F(v_i, v_j) = \begin{cases} 1 & \text{if } (i, j) = (1, 1), (2, 3) \text{ or } (3, 2), \\ 0 & \text{otherwise.} \end{cases}$$

Now it is clear that the set of lines containing the points $v_1, v_2, v_1 + v_2, v_1 - v_3, v_3$ induces a 5-cycle in $\Delta_F$ and we are done.

Now suppose that $G = SU_3(q)/A$, where $A$ is a central subgroup of $SU_3(q)$. Let $X$ and $Y$ be in $\Omega$. It is straightforward to check that the images of $X$ and $Y$ in $U_3(q)$ commute if and only if $X$ and $Y$ commute. It follows immediately that $\Gamma(G)$ is not perfect.

3.4. Classical groups of dimension at least 4. We start with a general result for all classical groups of large enough dimension over almost all fields.

Lemma 20. Let $q$ be a prime power with $q \neq 2, 4$. Let $G$ be a quasisimple classical group with $G/Z(G)$ isomorphic to $L_n(q)$ or $U_n(q)$ with $n \geq 4$, or to $PSp_{2m}(q)$, $PO_{2m}^\pm(q)$ or $PO_{2m+1}(q)$ with $m \geq 3$. Then $\Gamma(G)$ is not perfect.

Proof. If $G/Z(G) \not\cong U_n(q)$, then it is a standard result that $G$ contains a parabolic subgroup $P$ for which a Levi complement $L$ contains a normal subgroup isomorphic to either $L_3(q)$ or $SL_3(q)$. Next suppose that $G/Z(G) \cong U_n(q)$. The group $SU_n(q)$ contains a subgroup $H_0$ that stabilizes a non-degenerate subspace of dimension 3; now let $H_1$ be the lift of $H_0$ in the universal version of $U_n(q)$ (in all cases except $(n, q) = (4, 3)$, this universal version is just $SU_n(q)$ itself and so $H_1 = H_0$) and let $H$ be the projective image of $H_1$ in $G$. Then $H$ contains a normal subgroup isomorphic to either $U_3(q)$ or $SU_3(q)$ (see, for instance, [17, §§4.1 and 4.2]).

The result now follows from Lemmas 16 and 19.

We now work through the families of classical groups one by one; the force of Lemma 20 is that we have only to deal with the case that $q$ is 2 or 4, and with groups $G$ such that $G/Z(G) \cong PSp_4(q)$. 


Proposition 21. Let $G$ be a quasisimple group with $G/Z(G)$ isomorphic to $\text{PSp}_{2m}(q)$, with $m \geq 2$. Then $\Gamma(G)$ is not perfect.

Proof. By Lemma 20, it will be sufficient to deal with the case that $q$ is even, and with the case $m = 2$.

Suppose that $q$ is even. Since $\text{Sp}_4(2) \cong S_6$, we know from Corollary 11 that the commuting graph of $\text{Sp}_4(2)$ is not perfect. Since $\text{Sp}_{2m}(q)$ has $\text{Sp}_4(q)$ as a subgroup for $m \geq 2$, it follows that $\Gamma(\text{Sp}_{2m}(q))$ is not perfect. Referring to [17, Table 5.1.D], we see that the only quasisimple group left to consider is the double cover of $\text{Sp}_6(2)$. But reference to [9] shows that this group contains $U_3(3)$ as a subgroup, and hence it has a non-perfect commuting graph by Lemma 19.

Suppose next that $q$ is odd and that $m = 2$. Referring to [9] we see that $\text{PSp}_4(3)$ contains a subgroup isomorphic to $S_6$. If $q > 3$, then [17, Proposition 4.3.10] tells us that $G$ contains a field extension subgroup isomorphic to $L_2(q^2).2$: the commuting graph of this subgroup is not perfect by Lemma 14. It follows that, in either case, $G$ has a non-perfect commuting graph.

It remains to deal with the groups $\text{Sp}_4(q)$ for odd $q$. Our argument is similar to that for the unitary groups $U_3(q)$ in Lemma 19 above, and we again refer the reader to [10, Chapter 2] for facts about transvections. Let $G = \text{Sp}_4(q)$, and let $V$ be the natural module for $G$, with $F$ the underlying alternating form on $V$. For any non-zero $v \in V$, the transvection map $T_v : x \mapsto x + F(x, v)v$ lies in $G$. The maps $T_v$ and $T_w$ commute if and only if $F(v, w) = 0$. Let $\{e_1, f_1, e_2, f_2\}$ be a hyperbolic basis for $V$; so $F(e_1, f_1) = F(e_2, f_2) = 1$, and $\langle e_2, f_2 \rangle = \langle e_1, f_1 \rangle$. Define

$$v = e_1, \quad w = e_2, \quad x = f_1, \quad y = f_1 + f_2, \quad z = e_1 - e_2 + f_2.$$  

Then the induced subgraph of $\Gamma(G)$ on the vertices $T_v, T_w, T_x, T_y, T_z$ is a 5-cycle, and so $\Gamma(G)$ is not perfect. \qed

Proposition 22. Let $G$ be a quasisimple group with $G/Z(G)$ isomorphic to $\text{PO}_{2m+1}(q)$ with $m \geq 3$ or to $\text{PO}_{2m}^\perp(q)$ with $m \geq 4$. Then $\Gamma(G)$ is not perfect.

Proof. By Lemma 20, it is sufficient to deal with the case that $q$ is even. Furthermore, we may suppose that $G/Z(G) \cong \text{PO}_{2m}^\perp(q)$, because of the isomorphism $\text{PO}(2m+1, 2^k) \cong \text{PSp}(2m, 2^k)$. Now [17, Proposition 4.1.7] implies that $G$ contains a subgroup isomorphic to a quasisimple cover of $\text{Sp}_{n-2}(q)$, and the result follows from Proposition 21. \qed

Proposition 23. Let $n \geq 4$ and let $G$ be a quasisimple group with $G/Z(G)$ isomorphic to $L_n(q)$ or $U_n(q)$. Then $\Gamma(G)$ is not perfect.

Proof. We suppose that $q$ is even, since otherwise Lemma 20 gives the result.

Suppose that $n$ is even. Then [17, Propositions 4.5.6 and 4.8.3] imply that $\text{Sp}_n(q)$ is a subgroup of both $L_n(q)$ and $U_n(q)$. If $(n, q) \neq (4, 2)$, then $\text{Sp}_n(q)$ is simple and so some quasisimple cover of $\text{Sp}_n(q)$ is a subgroup of $G$. The result now follows from
Proposition 21. If \( G/Z(G) \cong L_4(2) \), then the result follows from Proposition 12 since \( L_4(2) \cong A_8 \). If \( G/Z(G) \cong U_4(2) \), then the result follows from Proposition 21, since \( U_4(2) \cong \text{PSp}_4(3) \).

Suppose, on the other hand, that \( n \) is odd. If \( G = L_n(q) \), then \( G \) contains a subgroup \( H \) such that \( H/Z(H) \cong L_{n-1}(q) \). Similarly, if \( G = U_n(q) \), then \( G \) contains a subgroup \( H \) such that \( H/Z(H) \cong U_{n-1}(q) \). Since \( n - 1 \) is even, the result in each case follows from above.

\[ \square \]

3.5. Ree and Suzuki groups.

Proposition 24. If \( G = Sz(q) \) with \( q > 2 \), then \( \Gamma(G) \) is perfect.

The result is also true for \( q = 2 \), but we omit it from the statement since \( Sz(2) \) is not simple.

Proof. We refer to [24] and observe that the only non-trivial elements in \( G \) which have non-abelian centralizer are the involutions (and there is a single conjugacy class of these). Let \( \Lambda \) be a forbidden subgraph of \( \Gamma(G) \) and observe that all of its vertices correspond to involutions in \( G \). Let \( g \) be one such. Then the set of involutions which commute with \( g \) lie in an elementary abelian subgroup of \( \text{Cent}_G(g) \) and hence any two neighbours of \( g \) in \( \Lambda \) must themselves be neighbours, a contradiction.

\[ \square \]

Proposition 25. If \( G = 2F_4(q)' \) with \( q \geq 2 \), then \( \Gamma(G) \) is not perfect.

Proof. We consult [9] to see that \( 2F_4(2)' \) contains \( S_6 \) and hence, by Lemma 9, \( 2F_4(2)' \) is not perfect. Since \( G \) contains \( 2F_4(2)' \) as a subgroup, the result follows.

\[ \square \]

Proposition 26. If \( G = 2G_2(q) \), then \( \Gamma(G) \) is not perfect.

Proof. We first deal with the case \( q = 3 \), when \( G \) is not quasisimple, but isomorphic to the automorphism group of \( SL_2(8) \). Let \( F \) be the automorphism of \( SL_2(8) \) induced by the field automorphism \( x \mapsto x^2 \). Let \( \alpha \) be an element of \( F_8 \) such that \( \alpha^3 + \alpha = 1 \). We define the following elements of \( SL_2(8) \):

\[
J = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} \alpha & 0 \\ \alpha^6 & \alpha^6 \end{pmatrix}, \quad Y = \begin{pmatrix} \alpha^4 & \alpha \\ 0 & \alpha^3 \end{pmatrix}.
\]

Now it is a straightforward computation to verify that the induced subgraph of \( \Gamma(G) \) on the vertices

\[ F^X, J^X, J, F, K, K^X, F^Y \]

is isomorphic to a 7-cycle, and so \( \Gamma(G) \) is not perfect. (The point of this construction is that the matrices \( X \) and \( Y \) lie in opposite Borel subgroups in \( SL_2(8) \), and that the commutator \([XY^{-1}, F]\) is fixed by \( F \). It is perhaps worth noting that there is no induced subgraph of \( \Gamma(G) \) isomorphic to a 5-cycle in this case.)

We now observe that if \( q > 3 \) then \( G \) contains a subgroup isomorphic to \( 2G_2(3) \), and so the result follows.

\[ \square \]
There are two non-simple quasisimple groups whose quotients are Ree or Suzuki groups and we deal with these in the final result of this section.

**Lemma 27.** If \( G = 2.Sz(8) \) or \( (2 \times 2).Sz(8) \), then \( \Gamma(G) \) is perfect.

**Proof.** Using Magma [5] we establish that \( G \) has precisely one non-central conjugacy class \( C \) of involutions. What is more \( C \) is the only non-central conjugacy class whose members have non-abelian centralizers. Thus, by Lemma 6, it is enough to show that \( \Gamma(C) \) is perfect.

Let \( g \in C \) and suppose that \( \Lambda \) is a forbidden subgraph of \( \Gamma(C) \). The set of involutions which commute with \( g \) lie in an elementary abelian subgroup of \( \text{Cent}_G(g) \) and hence any two neighbours of \( g \) in \( \Lambda \) must themselves be neighbours, a contradiction. \( \square \)

### 3.6. The remaining exceptional groups.

**Proposition 28.** If \( G \) is a quasisimple group with \( G/Z(G) \) isomorphic to \( G_2(q) \) or to \( ^3D_4(q) \), then \( \Gamma(G) \) is not perfect.

**Proof.** Suppose first that \( G \) is simple. Referring to [19] we see that \( G_2(q) < ^3D_4(q) \) for all \( q \). Furthermore [9] and [18] imply that \( U_3(3) = G_2(2)^{\prime} < G_2(q) \) for all \( q \), and the result follows from Lemmas 5 and 19.

If \( G \) is not simple, then \( G = 2.G_2(4) \) or \( 3.G_2(3) \). In both cases \( G \) contains a subgroup isomorphic to \( U_3(3) \) and the result follows as before. \( \square \)

**Proposition 29.** Let \( G \) be a quasisimple group with \( G/Z(G) \) isomorphic to one of \( F_4(q) \), \( ^2E_6(q) \), \( E_6(q) \), \( E_7(q) \) or \( E_8(q) \). Then \( \Gamma(G) \) is not perfect.

**Proof.** Referring to [20] we see that
\[
^3D_4(q) < F_4(q) < E_6(q), \ ^2E_6(q).
\]

Furthermore, the *universal* version of \( E_6(q) \) is a subgroup of the *adjoint* version of \( E_7(q) \), and likewise the universal version of \( E_7(q) \) is a subgroup of the adjoint version of \( E_8(q) \). Since the Schur multiplier of \( ^3D_4(q) \) is trivial we conclude that all quasisimple covers of the (simple) adjoint versions of \( F_4(q), ^2E_6(q), E_6(q), E_7(q) \) and \( E_8(q) \) contain a subgroup isomorphic to \( ^3D_4(q) \), and the result follows from Proposition 28. \( \square \)

### 3.7. Sporadic groups.

**Proposition 30.** If \( G \) is a quasisimple group with \( G/Z(G) \) isomorphic to a sporadic simple group, then \( \Gamma(G) \) is not perfect.

**Proof.** Our strategy here is to find, for each sporadic simple group, a subgroup which has already been shown to have non-perfect commuting graph. The result will then follow from Lemma 5. Our essential reference is [9], which provides lists
of maximal subgroups of these groups. For reasons of transparency, we use only subgroup inclusions which are immediately visible from the structural information these lists provide (though the subgroups need not themselves be maximal).

We deal first with the simple groups. We have the following subgroup inclusions:

\[
\begin{align*}
S_5 & < M_{11}, M_{12}, \text{Th}, B, M, \\
A_7 & < M_{22}, M_{23}, M_{24}, \text{HS}, \text{McL}, \text{Co}_1, \text{Fi}_{23}, \text{Fi}'_{24}, O'N, \\
A_8 & < \text{Ru}, \\
M_{12} & < \text{Suz}, \text{Fi}_{22}, \text{HN}, \\
M_{23} & < \text{Co}_3, \text{Co}_2, \\
M_{24} & < J_4, \\
L_2(11) & < J_1, \\
L_2(19) & < J_3, \\
U_3(3) & < J_2, \\
\text{Sp}_4(4) & < \text{He}, \\
G_2(5) & < \text{Ly}.
\end{align*}
\]

We deal now with the case that \(G\) is non-simple. The following subgroup inclusions cover most possibilities:

\[
\begin{align*}
M_{11} & < 2.M_{12}, 2.\text{HS}, 3.\text{McL}, 3.O'N, \\
U_3(3) & < 2.J_2, \\
L_2(19) & < 3.J_3, \\
2^{3}.F_4(2)' & < 2.\text{Ru}, 2.\text{Fi}_{22}, 3.\text{Fi}_{22}, 6.\text{Fi}_{22}, 2.B, \\
\text{Co}_2 & < 2.\text{Co}_1, \\
\text{Fi}_{23} & < 3.\text{Fi}'_{24}.
\end{align*}
\]

In addition all quasisimple covers of \(\text{Suz}\) contain a quasisimple cover of \(G_2(4)\).

We are left with the possibility that \(G\) that \(G/Z(G) \cong M_{22}\). Note that the simple group \(M_{22}\) contains a subgroup isomorphic to \(A_7\) and all quasisimple covers of \(A_7\) have non-perfect commuting graph, except \(6.A_7\). Thus, for \(\Gamma(G)\) to be perfect, the subgroup \(A_7\) in \(M_{22}\) must lift to a subgroup \(6.A_7\) in \(G\). This implies immediately that \(|Z(G)| = 6\) or 12. Now we consult [9] to see that elements of order 4 in \(M_{22}\) do not lift to elements of order 24 when \(|Z(G)| = 6\). Since elements of order 4 in \(A_7\) lift to elements of order 24 in \(6.A_7\) we conclude that \(6.M_{22}\) does not contain \(6.A_7\). Thus we must have \(G = 12.M_{22}\).

Now we refer to [15, Table 1], to see that a maximal subgroup of \(M_{22}\) which is isomorphic to \(L_3(4)\) lifts in \(12.M_{22}\) to a cover whose centre is cyclic and has order
divisible by 4. All such covers of $L_3(4)$ have non-perfect commuting graph and the result follows by Lemma 18.

4. Components in finite groups

In this section we prove Theorem 2 and Corollary 3. The next result is required for the proof of Theorem 2, and also illustrates a diagrammatic method we have found helpful.

**Proposition 31.** (1) Let $K$, $L$ and $M$ be finite non-abelian groups. Then $\Gamma(K \times L \times M)$ is not perfect.

(2) Let $K, L, M$ be three distinct finite non-abelian subgroups of a group $G$, each of which centralizes the other two. Then $\Gamma(G)$ is not perfect.

**Proof.** (1) Define $(k, k'), (\ell, \ell')$ and $(m, m')$ to be pairs of non-commuting elements from $K$, $L$ and $M$ respectively. Now the five elements

$(1) (1, l, m), (k', 1, 1), (1, l', 1), (k, 1, m'), (k, l, 1),$

induce a 5-cycle in $\Gamma(K \times L \times M)$ and we are done.

(2) If $K, L$ and $M$ are subgroups of $G$ which centralize one another, then there is a natural homomorphism $K \times L \times M \rightarrow G$ given by $(x, y, z) \mapsto xyz$. It is easy to check that the images under this map of the five elements constructed in part (i), induce a 5-cycle in $\Gamma(G)$.

Before we proceed, let us take a moment to understand more clearly why the elements listed at (1) induce a 5-cycle. To do this we refer to Figure 2 in which we draw the commuting graphs of the three projections of the listed tuples. Note that we maintain the same orientation for each graph, so that the vertex corresponding to the entry from the first tuple is at the ‘east’ of the graph, and entries from the following tuples are written anticlockwise around the graph. Now it is clear that the commuting graph of the three 5-tuples listed at (1) has edges between two vertices precisely when all three projections have edges between the corresponding vertices. This observation immediately implies that the tuples listed at (1) form a 5-cycle, as required. In the arguments below we shall use the same convention for representing projections.

Our next result is in similar vein and to state it we need some terminology: We say that $\Gamma(G)$ contains a 4-chain if there is an induced subgraph of $\Gamma(G)$ isomorphic to a path graph on four vertices.

**Proposition 32.** Let $K$ and $L$ be subgroups of a group $G$ such that $\Gamma(K)$ contains a 4-chain, $L$ is non-abelian, and $K$ and $L$ centralize one another. Then $\Gamma(G)$ is not perfect.
Proof. Let $k_1, k_2, k_3, k_4$ be the vertices of a 4 chain in $K$, and let $\ell, \ell'$ be non-commuting elements in $L$. Then the projection graphs in Figure 3 illustrate that the five elements

$k_1, k_2 \ell, k_3 \ell, k_4, \ell'$

in $KL$ induce a 5-cycle in $\Gamma(G)$.

Lemma 33. (1) If $G$ is isomorphic to one of the groups in the following list, then $G$ contains a 4-chain:


(2) If $G$ is a quasisimple group such that $\Gamma(G)$ is perfect and contains a 4-chain, then $G$ is one of the groups listed in (i).

Proof. (1) If $G \cong A_6$, we can take $g_1 = (15)(34), g_2 = (15)(26), g_3 = (12)(56), g_4 = (12)(34)$. If $G \cong 3.A_6$, then we can take pre-images of these four elements. If $G = 6.A_7$ then we can take pre-images in $G$ of

$g_1 = (1234)(56), g_2 = (13)(24), g_3 = (567), g_4 = (12)(34)(567)$.

If $G = SL_3(2)$ then we can take

\[
g_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\]
For the remaining cases we refer back to the proof of Lemma 18. It is clearly sufficient to consider the induced subgraph $\Gamma(T_G)$ introduced there. In fact this graph is the same for any of the extensions of $L_3(4)$ listed here (since the central elementary abelian 2-subgroup $A$ of the full covering group $M$ of $L_3(4)$ is contained in the kernel of the quotient homomorphism $M \longrightarrow G$ in each case.) It is sufficient, therefore, to find a 4-chain in any one of these groups. Since $3.L_3(4)$ is isomorphic to $SL_3(4)$, it contains $SL_3(2)$ as a subgroup, and so the four elements given above for $SL_3(2)$ can be used in this case also.

(2) For this part we must show that every group listed in Theorem 1 but not in Lemma 33 does not contain a 4-chain. This is obviously the case for all of the AC-groups, which are listed in Corollary 3.

If $G$ is $(4 \times 4).L_3(4)$ or $(12 \times 4).L_3(4)$, and if a 4-chain existed in $\Gamma(G)$, then there would be a 4-chain in the graph $\Gamma(T_G)$ constructed in the proof of Lemma 18. But it was seen in that proof that $\Gamma(T_G)$ is a union of pairwise disconnected triangles, and so clearly no 4-chain exists there.

Finally, suppose that $G$ is equal either $Sz(q)$ for some $q = 2^{2n+1}$, or else to $2.Sz(8)$ or $(2 \times 2).Sz(8)$. In any of these cases $G$ has a single class of non-central elements with non-abelian centralizers, consisting of involutions. Furthermore, for each involution $g$ in this class, the involutions commuting with $g$ generate an elementary abelian subgroup of $G$. From these facts it is clear that $\Gamma(G)$ can have no 4-chain.

\[\square\]

Proof of Theorem 2. Proposition 31 tells us that $G$ has at most two components, and Lemma 5 implies that the commuting graphs of the components are perfect, and so each component of $G$ are isomorphic to one of the quasisimple groups listed in Theorem 1.

Suppose first that $G$ has a unique component $N$ and that case (i) of the theorem does not hold. Then $G$ appears in the list of Lemma 33, and so $\Gamma(G)$ contains a 4-chain. It follows from Proposition 32 that no non-abelian subgroup of $G$ can centralize $N$, and so $\text{Cent}_G(N)$ is abelian and (ii) holds.

Next suppose that $G$ has two components $N_1$ and $N_2$. Since $N_1$ and $N_2$ centralize one another, and since they are both non-abelian, it follows from Proposition 32 that neither contains a 4-chain, and that $\Gamma(N_1)$ contains a 4-chain. Therefore each is isomorphic to one of the groups listed in case (i) of the theorem.

Let $C = \text{Cent}_G(N_1N_2)$. Then $N_1$, $N_2$ and $C$ are three subgroups of $G$ which centralize one another, and it follows from Proposition 31 that one of them is abelian. Since $N_1$ and $N_2$ are quasisimple, we see that $C$ is abelian and (iii) holds. \[\square\]
Improvements on Theorem 2 are certainly possible. An obvious first step would be to study almost quasisimple groups with perfect commuting graphs. To see how this might be useful, we offer the following result which pertains to a specific situation.

**Proposition 34.** Let $G$ be a finite group such that $\Gamma(G)$ is perfect, and suppose that $G$ has a component $N$ isomorphic to $A_6$. Let $C = \text{Cent}_G(N)$. Then $C$ is abelian, and the quotient group $G/C$ is isomorphic either to $A_6$ or to the Mathieu group $M_{10}$.

**Proof.** We observe first that $G$ must fall under case (ii) of Theorem 2, and so $N$ is the unique component of $G$, and $C$ is abelian. Since $G/NC$ is isomorphic to a subgroup of $\text{Out}(N)$, we see that $G/C$ is an almost simple group with socle $N$. Reference to [9] tells us that $G/C$ is isomorphic to one of $A_6$, $S_6$, $M_{10}$ or $\text{PGL}_2(9)$, or to the projective semilinear group $\text{PGL}_2(9)$ which contains all of the others as subgroups.

If $G = NC$ then $G/C \cong A_6$. So we suppose that $G \neq NC$. Then there exists $g \in G$ is such that $gNC$ has order 2 in the quotient $G/NC$. Let $H = (N, g)$, and observe that $HC/C$ is an almost simple group of order $2|N|$. Since $H/N$ is cyclic, we have $[h_1, h_2] \in N$ for all $h_1, h_2 \in H$, and now since $N \cap C$ is trivial, it follows that any two conjugate elements of $H$ lie in distinct cosets of $H \cap C$. Therefore the conjugacy action of $H$ on its normal subgroup $H \cap C$ is trivial, and so $H \cap C$ is central in $H$.

Now $H$ is a subgroup of $G$, and so $\Gamma(H)$ is perfect; so Lemma 8 tells us that $\Gamma(H/H \cap C)$ is perfect. But $H/H \cap C \cong HC/C$, and so $HC/C$ cannot be isomorphic to $S_6$ or to $\text{PGL}_2(9)$, since we know that neither of these has a perfect commuting graph (by Corollary 11 and Remark 15). We therefore see that $HC/C \cong M_{10}$.

It is now also clear that $G/C$ cannot be isomorphic to $\text{PGL}_2(9)$, since otherwise there would be a subgroup $H < G$ such that $H/C \cong S_6$. So we have shown that $G/C$ is congruent either to $A_6$ or to $M_{10}$, as claimed. □

**Proof of Corollary 3.** Lemma 6 implies that any finite AC-group $G$ has a perfect commuting graph. If $G$ is quasisimple, then $G$ is one of the groups listed in Theorem 1. It is easy to check that $\text{SL}_2(q)$ and $6.A_6$ are AC-groups.

We observe that a centreless AC-group has abelian Sylow $p$-subgroups for all $p$. Now $\text{Sz}(2^{2a+1})$, $A_6$, $\text{SL}_3(2)$ and $L_3(4)$ have non-abelian Sylow 2-subgroups, and hence they are not AC-groups. It is not hard to check that if $G$ is isomorphic to $3.A_6$, to a quasisimple cover of $L_3(4)$, or to a quasisimple cover of $\text{Sz}(q)$, then the centralizer of a non-central involution is non-abelian. In $6.A_7$ the centralizer of an element of order 4 is non-abelian. This is sufficient to prove (i).

To establish (ii), let $G$ be an arbitrary finite AC-group. Since components are non-abelian, and since any two distinct components centralize one another, it is
clear that $G$ must have a unique component $N$. It is also clear that $N$ must itself be an AC-group, and so $N$ is one of the groups listed in part (i). Let $C = \text{Cent}_G(N)$, and let $Z = Z(G)$. Suppose that $g \in C \setminus Z$. Then $N \leq \text{Cent}_G(g)$, and since $N$ is non-abelian we have a contradiction. Hence $\text{Cent}_G(N) = Z$, and so $G/Z$ is isomorphic to a subgroup of $\text{Aut}(N)$.

Suppose first that $N = \text{SL}_2(q)$, and that $G$ contains an element $g$ whose action on $N/Z(N)$ induces a field automorphism. Then $\text{Cent}_{N/Z(N)}(g) \cong \text{L}_2(q_0)$, where $q = q_0^a$ for some $a > 1$. If $q$ is even then $Z(N)$ is trivial and, since $\text{L}_2(q_0)$ is non-abelian, we immediately obtain a contradiction. If $q$ is odd, then $|Z(N)| = 2$, and hence $\text{Cent}_N(g)$ contains a subgroup isomorphic to a subgroup of $\text{L}_2(q_0)$ of index at most 2. Once again we conclude that $\text{Cent}_N(g)$ is non-abelian, which is a contradiction. So $G/Z(G)$ contains no element acting as a field automorphism, and we conclude that $G/Z(G)$ is isomorphic either to $\text{L}_2(q)$, or to an extension of $\text{L}_2(q)$ of degree 2. So we see that $|G : NZ(G)| \leq 2$ in this case.

Suppose next that $N = 6.A_6$. We refer to [13, Table 6.3.1], which asserts that the action of $\text{Out}(A_6)$ on $Z(N)$ is non-trivial. So if $G/Z \cong \text{Aut}(A_6)$ then $G$ contains an element $g$ which acts non-trivially on $Z(N)$. Thus not all non-trivial elements of $Z(N)$ are central in $G$. But since all non-trivial elements of $Z(N)$ have non-abelian centralizer, this is a contradiction. So $G/Z(G)$ is a proper subgroup of $\text{Aut}(A_6)$, and since $|\text{Out}(A_6)| = 4$ we have $|G : NZ(G)| \leq 2$ in this case too. \hfill $\square$

References


Gerhard Hiss, William J. Husen, and Kay Magaard. Imprimitive irreducible modules for finite quas