We classify groups of order 20 up to isomorphism.

Let $G$ be a group of order 20. Then $|\text{Syl}_5(G)| \equiv 1 \mod 5$, and since $|\text{Syl}_5(G)|$ divides 20, it must be 1. Hence $G$ has a normal subgroup $N$ isomorphic to $C_5$. Now if $H \in \text{Syl}_2(G)$ then $|H| = 4$, and $N \cap H = \{e\}$, so we have $G = NH$. It follows that $G \cong N \rtimes H$ for some homomorphism $\phi : H \to \text{Aut}(N)$. Now $\text{Aut}(N) \cong C_4$, with a generator $\alpha$ being given by $\alpha(g) = g^2$ for $g \in N$.

**Case i.** The first possibility to consider is that $H \cong C_4$. In this case let $h$ be a generator of $H$. There are three subcases:

a) $\text{Im } \phi = \{e\}$. In this case $G \cong N \times H \cong C_5 \times C_4 \cong C_{20}$.

b) $\text{Im } \phi = \langle \alpha^2 \rangle$. Then we must have $\phi_h = \alpha^2$, and since this determines $\phi$ completely, we get exactly one group in this case. The automorphism $\alpha^2$ acts on $N$ by $g \mapsto g^{-1}$, and so we see that $G$ has the presentation $\langle g, h \mid g^5 = h^4 = e, hg = g^3 \rangle$.

c) $\text{Im } \phi = \langle \alpha \rangle$. Then $g$ maps to a generator of $\langle \alpha \rangle$, and so there are two possible homomorphisms, $\phi_1 : h \mapsto \alpha$ and $\phi_2 : h \mapsto \alpha^3$. But if $\beta \in \text{Aut}(H)$ is given by $\beta : h \mapsto h^{-1}$, then $\phi_2 = \phi_1 \circ \beta$, and so these homomorphisms give isomorphic semidirect products, by Proposition 26(a) from lectures. So this case gives just one group, which has the presentation $\langle g, h \mid g^5 = h^4 = e, hg = g^2 \rangle$.

**Case ii.** The second possibility is that $H \cong C_2 \times C_2$. In this case $\text{Im } \phi$ cannot be $\langle \alpha \rangle$, since an element of $\phi(H)$ has order at most 2. There are two further subcases to consider:

d) $\text{Im } \phi = \{e\}$. Then $G \cong N \times H \cong C_5 \times C_2 \times C_2 \cong C_{10} \times C_2$.

e) $\text{Im } \phi = \langle \alpha^2 \rangle$. Then $|\text{Ker } \phi| = 2$, and so we can find elements $h, k \in H$ such that $H = \langle h, k \rangle$, with $\phi(h) = \alpha^2$ and $\phi(k) = e$; these elements determine $\phi$. Note that for any two distinct non-identity element $h', k'$ of $H$, there is an automorphism $\beta$ of $H$ with $\beta(h) = h'$ and $\beta(k) = k'$ (since $\text{Aut}(C_2 \times C_2) \cong S_3$, acting on the three elements of order 2). So by Proposition 26(a), any
two homomorphisms will give isomorphic semidirect products, and so there is just one group arising from this case. Let $g$ be a generator of $N$. Since $\phi_k$ is the identity of $\text{Aut}(N)$, we have $kg = gk$ in $G$. This is an element of order 10; and we observe that since $k = k^{-1}$ we have

$$h(kg) = hkhg = k^{-1}g^{-1} = (kg)^{-1},$$

and so $h$ acts by inversion on the normal cyclic subgroup $\langle kg \rangle$ of order 10. Hence $G \cong D_{20}$.

It remains to check that the five groups we have found are distinct from one another. Clearly a), b), c) are distinct from d), e) since they have non-isomorphic Sylow 2-subgroups. The groups a) and d) are abelian, and the others are not. This rules out any isomorphisms except between b) and c). For this case note that in the group b), the element $gh^2$ has order 10, since $gh^2 = h^2g$. But in c) we observe that the four non-identity elements of $N$ are all conjugate in $G$ (by powers of $h$); hence if $g \in N \setminus \{e\}$ then $|C_G(g)| = 4$, and so $|\text{Cent}_{G}(g)| = 5$. Therefore $G$ has no cyclic subgroup of order 10, and so the groups b) and c) are not isomorphic.

**Question.** Using similar arguments, can you classify the groups of order 12 up to isomorphism?