Hyper-Kähler Geometry, Magnetic Monopoles, and Nahm's Equations

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Abstract

In this report we investigate the correspondence between SU(2) magnetic monopoles and solutions to the Nahm equations on $\mathbb{R}^3$. Further, we investigate a variation of this result for singular Dirac monopoles. To this end, the report starts with a review of the necessary spin geometry and hyper-Kähler geometry required, as well as the definition of magnetic monopoles and the Nahm equations. The final chapter consists of a presentation of Nakajima's proof of the monopole correspondence, and an investigation of how these techniques can be applied to the case of singular Dirac monopoles.

Contents

Introduction

It is a well-known observation in classical electromagnetism that all magnetic fields appear to be generated by magnetic dipoles. This is in contrast with electric fields, which may be generated by particles of a single parity. In the landmark paper [Dir31] in 1931, Paul Dirac proposed a resolution to the mystery of why electric charge is quantized by proposing the existence of magnetic *monopoles*. Through Dirac's work, it would be possible to prove that electric charge must be quantized provided the existence of even a single magnetic monopole in the universe. Despite this and other strong theoretical evidence for the existence of such elementary particles, none have yet been observed.

Mathematically, magnetic monopoles may be modelled as solutions of the *Bogomolny equations* on $\mathbb{R}^3$,

$$F_A = \star d_A \Phi.$$

Similarly to the theory of instantons on $\mathbb{R}^4$ modelling other elementary particles, solutions to the Bogomolny equations are gauge-theoretic data on $\mathbb{R}^3$, and are hence of considerable mathematical and geometric interest independent of their physical origins.

In particular, moduli spaces of solutions to equations such as the Bogomolny equations are known to, in many cases, provide powerful invariants of the underlying spaces, and give new tools for solving many problems in geometry and topology. This is exemplified by the famous Donaldson theory and Seiberg-Witten theory in four-manifold topology.

As such, there is much interest in finding new ways of constructing solutions to these gauge-theoretic equations. The first major breakthrough in this area came with the novel ADHM construction developed by Atiyah, Drinfeld, Hitchin, and Manin in [ADHM78], in order to find solutions of the anti-self-dual Yang-Mills equations on $\mathbb{R}^4$ using essentially algebraic data. In [Nah83] Nahm developed, in analogy with the ADHM construction, a method for constructing solutions to the Bogomolny equations from gauge-theoretic data over an interval. In [Hit83] Hitchin showed that this technique can in fact be used to find *all* magnetic monopoles, and considerable work was done by many others around this time to extend these results to other situations, in particular with varying choices of gauge group.

The work of Nahm and Hitchin involved passing through intermediate algebraic-geometric data, spectral curves in twistor spaces. In [Nak93] Nakajima gave a purely differential-geometric proof of the correspondence of Hitchin, phrased in language that is readily generalisable to the theory of *Nahm transforms*.

In this report, we will review the correspondence between magnetic monopoles and solutions of the Nahm equations through the eyes of Nakajima's differential geometric techniques, and investigate a variation of this result for *singular* magnetic monopoles of the simplest type – so called *Dirac* monopoles.

In Chapter 1 we recall the necessary preliminaries for these constructions, including elementary spin geometry and the basics of hyper-Kähler geometry.

In Chapter 2 we describe in more detail the definition of a magnetic monopole and the construction of the hyper-Kähler moduli space of solutions to the Bogomolny equations.

In Chapter 3 the Nahm equations and their relation to the anti-self-duality equations on $\mathbb{R}^4$ are described.

Finally, in Chapter 4 we present (part of) the proof of the correspondence of magnetic monopoles with the Nahm equations as shown by Nakajima, as well as an investigation of the correspondence for singular Dirac monopoles. This involves the theory of b-geometry and scattering calculus, and a mild amalgamation of these concepts applicable in this case.

The authors wish to thank Professor Michael Singer for providing guidance in learning the material presented in this report, and for contributing most of the ideas in determining the correspondence between singular magnetic monopoles and appropriate Nahm data.

Chapter 1

Preliminaries

In this chapter we will give some preliminary results which will be necessary for the development of the project.

1.1 Spin Geometry

Here we will lay out the essential definitions and some useful results about Spin geometry and Dirac operators. Many of these concepts can be studied in greater generality, but we will often restrict ourselves to the case we are interested in. We will go over many of the results without proofs, specially at the beginning. A more detailed and rigorous exposition can be found Lawson and Michelsohn's Spin Geometry [LM89], whose notation we follow closely, and in Friedrich's Dirac Operators in Riemannian Geometry [Fri00].

1.1.1 Clifford algebras and the Spin group

Definition 1.1.1. Let (V, q) be a vector space with a quadratic form. Its *Clifford algebra* is defined as

$$\mathrm{Cl}(V, q) := \mathcal{T}(V)/\mathcal{I}_q(V),$$

where

$$\mathcal{T}(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$$

is the tensor algebra of V and $\mathcal{I}_q(V)$ is the ideal of $\mathcal{T}(V)$ generated by elements of the form $v \otimes v + q(v)$ for $v \in V$.

The product of two elements $\varphi, \psi \in \mathrm{Cl}(V, q)$ will be denoted by $\varphi \cdot \psi$, or simply by $\varphi\psi$.

In other words, the Clifford algebra of a vector space is the algebra generated by V under the relations $v^2 = -q(v)1$ for $v \in V$. Equivalently, if $q(v,w) = \frac{1}{2}(q(v+w)-q(v)-q(w))$ is the polarisation of the quadratic form (if the underlying field has characteristic different from 2), then we have the relations

$$v \cdot w + w \cdot v = -2q(v,w).$$

It can be shown that $\mathrm{Cl}(V,q)$ is isomorphic to $\bigwedge^\bullet V$ as a vector space (it will be isomorphic as an algebra precisely when $q \equiv 0$). In particular, V will be naturally embedded as a vector space.

From now on, for simplicity, we will take our vector space to be finite dimensional and real, and we will take the quadratic form to be positive definite. This means that (V,q) will be isomorphic to $\mathbb{R}^n$ with the standard metric, for some n. Whenever we take (V,q) to be explicitly $\mathbb{R}^n$ with the standard metric, we will write $\mathrm{Cl}(n)$ for the Clifford algebra. It will be useful to consider as well the complexification of the Clifford algebra, which will be denoted by $\mathbb{C}\mathrm{l}(V,q) := \mathrm{Cl}(V,q) \otimes \mathbb{C}$.

Firstly, let us consider

$$\mathrm{Cl}^\times(V,q) = \{\varphi \in \mathrm{Cl}(V,q) : \exists \varphi^{-1} \text{ such that } \varphi\varphi^{-1} = \varphi^{-1}\varphi = 1\},$$

which has a group structure. This group is, in fact, a Lie group of dimension 2^n, and its Lie algebra can be identified with $\mathrm{Cl}(V,q)$, with the Lie bracket being the commutator $[x,y] = xy - yx$. Furthermore, analogously to matrix Lie groups, the adjoint representation on the Lie algebra is given by conjugation.

We can now define the Spin group as a subgroup of $\mathrm{Cl}^\times(V,q)$.

Definition 1.1.2. Given (V,q), we define the *Spin group* as

$$\mathrm{Spin}(V,q) := \{v_1 v_2 \cdots v_{2k} \in \mathrm{Cl}(V,q) : v_i \in V \text{ and } q(v_i) = 1\}.$$

That is, it is the subgroup of $\mathrm{Cl}^\times(V,q)$ formed by products of an even amount of unit vectors. It forms a group, since multiplying two elements clearly gives another element of the group, and the element $v_{2k} \cdots v_2 v_1$ is the inverse of $v_1 v_2 \cdots v_{2k}$. Again, if (V,q) is taken to be explicitly $\mathbb{R}^n$ with the standard metric, then we denote the Spin group by $\mathrm{Spin}(n)$.

The Spin group acts on $\mathrm{Cl}(V,q)$ through the adjoint representation, and can be shown that this action actually preserves $V \subset \mathrm{Cl}(V,q)$. This gives a group homomorphism $\xi\colon \mathrm{Spin}(V,q) \to \mathrm{SO}(V,q)$. It turns out that this is actually a covering.

Theorem 1.1.3. *Given (V,q), the map $\xi\colon \mathrm{Spin}(V,q) \to \mathrm{SO}(V,q)$ is a 2-to-1 covering. Furthermore, if the dimension of V is greater than 1, then this covering is not trivial, and if it is greater than 2, this is the universal cover. In particular, the Lie algebras of both groups are isomorphic (and from now on will be identified).*

Note that, if we have a representation of $\mathrm{SO}(V,q)$, we can pull it back to a representation of $\mathrm{Spin}(V,q)$. However, if we have a representation of $\mathrm{Cl}(V,q)$, then we can restrict it to a representation of $\mathrm{Spin}(V,q)$ which will not factor through $\mathrm{SO}(V,q)$.

There are, in particular, some preferred representations.

Let us first see what the Clifford algebras look like. It turns out that both $\mathrm{Cl}(n)$ and $\mathbb{C}\mathrm{l}(n)$ can be easily identified with matrix algebras over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We are particularly interested in the complexified Clifford algebras.

Proposition 1.1.4. *The complexified Clifford algebras can be written as*

$$\begin{aligned} \mathbb{C}\mathrm{l}(n) &= \mathbb{C}(2^k) && \text{if } n = 2k, \\ \mathbb{C}\mathrm{l}(n) &= \mathbb{C}(2^k) \oplus \mathbb{C}(2^k) && \text{if } n = 2k+1, \end{aligned}$$

where $\mathbb{C}(2^k)$ is the algebra of $2^k \times 2^k$ matrices over the complex numbers.

In view of this result, we can see that there are some obvious representations: for $n = 2k$, the representation on $\mathbb{C}^{2^k}$, and for $n = 2k+1$ the two representations on $\mathbb{C}^{2^k}$ given by acting with each of the two components. It turns out that these are, in fact, the only irreducible (real or complex) representations up to isomorphism. Since Clifford algebra representations split as sums of irreducible representations, these are going to be the most important representations.

Now, if we restrict our representation to $\mathrm{Spin}(n) \subset \mathbb{C}\mathrm{l}(n)$, we obtain representations of the Spin group. In the even case, the representation splits into two inequivalent irreducible representations, and in the odd case, the resulting representation is irreducible and independent of the choice of representations of the Clifford algebra. We denote these representations by $\Delta_{2k}^{\mathbb{C}} = \Delta_{2k}^{\mathbb{C}+} \oplus \Delta_{2k}^{\mathbb{C}-}$ and $\Delta_{2k+1}^{\mathbb{C}}$.

We can establish some further properties of these representations. In particular, we would be interested in knowing how vectors in the original n-dimensional vector space act in these representations.

In the case $n = 2k$, the space of the representation is $\mathbb{C}^{2^k}$. As mentioned before, it can be split as $\mathbb{C}^{2^k} = \mathbb{C}^{2^{k-1}} \oplus \mathbb{C}^{2^{k-1}}$. We can then find a basis of $\mathbb{R}^{2k}$ such that the first $2k-1$ vectors act as $\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$, where the σ_i are the actions of a basis of $\mathbb{R}^{k-1}$ on the space $\mathbb{C}^{2^{k-1}}$ given by the irreducible Clifford action of the case $2k-1$, and where the last vector acts like $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In particular, the action of the elements of $\mathbb{R}^{2k}$ interchanges the spaces of the representations $\Delta_{2k}^{\mathbb{C}+}$ and $\Delta_{2k}^{\mathbb{C}-}$.

In the case $n = 2k+1$, the space of the representation is $\mathbb{C}^{2^k}$, like in the case of dimension $2k$. Then, we can choose a basis of $\mathbb{R}^{2k+1}$ such that the first $2k$ vectors act as a basis of $\mathbb{R}^{2k}$ act in the corresponding representation, and where the last vector acts as $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, following the splitting of the previous case.

1.1.2 Spin and Spinor Bundles

In a similar way to how we developed a theory of Clifford algebras and Spin groups over a vector space with a metric, we can try to develop it for a Riemannian vector bundle E over a manifold M (for example, the tangent bundle of a Riemannian manifold).

If we have such a vector bundle we have can construct its Clifford bundle analogously of how we did before.

Definition 1.1.5. Given a vector bundle E with a metric g over a manifold M, we define the *Clifford bundle* of E as

$$\mathrm{Cl}(E) = \mathcal{T}(E)/\mathcal{I}_g(E),$$

where $\mathcal{T}(E)$ is the tensor bundle of E and $\mathcal{I}_g(E)$ is the bundle of ideals which generated at each fibre by elements of the form $v \otimes v + g(v)$.

Note that this bundle is isomorphic, as a vector bundle, to $\bigwedge^\bullet(E)$.

We can describe this bundle in an alternative way using the theory of associated bundles. Suppose that E is orientable, and let P be its associated principal $\mathrm{SO}(n)$-bundle. Then, since any orthogonal automorphism of $\mathbb{R}^n$ extends to an automorphism of $\mathrm{Cl}(n)$, we can form the associated bundle through the action of $\mathrm{SO}(n)$ on $\mathrm{Cl}(n)$. This will be precisely $\mathrm{Cl}(E)$.

Now we want to construct the analogous to the Spin group, which should be a principal $\mathrm{Spin}(n)$-bundle. Note, however, that simply restricting the Clifford bundle to its Spin group will not always give a principal $\mathrm{Spin}(n)$ structure, since the Clifford bundle is not itself a principal bundle.

The proper notion of the Spin bundle of E will be, rather, a lifting of the $\mathrm{SO}(n)$ structure.

Definition 1.1.6. Let E be an orientable Riemannian vector bundle, and let P be its associated principal $\mathrm{SO}(n)$-bundle. We say Q is a *spin structure* of E if it is a principal $\mathrm{Spin}(n)$-bundle and has a 2-to-1 covering of P which is equivariant through the covering of $\mathrm{SO}(n)$ by $\mathrm{Spin}(n)$.

Such a structure does not always exist, and when it does, it is not necessarily unique. We have the following characterisation.

Theorem 1.1.7. *An orientable Riemannian vector bundle E has a Spin structure if and only if its second Stiefel-Whitney class is zero. Furthermore, in this case, the possible Spin structures are in one to one correspondence with elements of $H^1(M; \mathbb{Z}_2)$.*

Hence, for example, there exists a unique spin structure on the 2-sphere or on $\mathbb{R}^3$ with any finite amount of points removed (these spaces will come up in later chapters).

These spin structures can be used to construct bundles.

Definition 1.1.8. Let $\rho : \mathrm{Cl}(n) \to \mathrm{End}(W)$ be a representation of a Clifford algebra on a vector space W. If we have a $\mathrm{Spin}(n)$ structure $Q \to M$, the associated *real spinor bundle* is defined as $S(E) = Q \times_\rho W$ over M, where ρ is taken to be the restriction to $\mathrm{Spin}(n) \subset \mathrm{Cl}(n)$.

If $\rho : \mathbb{C}\mathrm{l}(n) \to \mathrm{End}(W)$ is a complex representation of the complex Clifford algebra, then the associated bundle is a *complex spinor bundle*.

For example, one may construct a complex spinor bundle using an irreducible representation of the Clifford algebra, as described above. This means that the spinor bundle will be associated through the representation $\Delta_n^{\mathbb{C}}$. If n is even, since this representation splits, it will give a splitting of the vector bundle into the direct sum of two bundles, sometimes called the *positive* and *negative* spinor bundles.

One of the main features of these spinor bundles is that the representation of the Clifford algebra on the vector space gives a fibre-wise representation of the Clifford bundle on the spinor bundle. This fact is easy to see by defining the Clifford bundle as an associated bundle of the Spin bundle (through the adjoint action).

Proposition 1.1.9. *In the conditions above, there exists an action of each fibre of the Clifford bundle* $\mathrm{Cl}(E)$ *on each fibre of the spinor bundle* $S(E)$*. This also holds for complex spinor bundles.*

In particular, each element of the original bundle E gives an automorphism of the corresponding fibre of $S(E)$*, which we call* Clifford multiplication.

Furthermore, if our original bundle E had a connection (compatible with the metric), we can use it to construct a connection on the spinor bundle.

In order to do this, consider the connection 1-form $\omega \in \Omega(P; \mathfrak{so}(n))$. Since Q covers P, we can pull back the 1-form to get a 1-form of the principal $\mathrm{Spin}(n)$-bundle Q, with values in $\mathfrak{spin}(n) \cong \mathfrak{so}(n)$, which will satisfy the appropriate conditions to define a connection on Q. Since $S(E)$ is an associated bundle to Q, we get a connection on $S(E)$. In our case, we will assume that the elements of $\mathrm{Spin}(n)$ (and indeed any unit vector of $\mathrm{Cl}(n)$) act by isometries on W (this can always be achieved by an appropriate choice of metric on W).

We also assume that ρ is faithful, we can embed Q in the principal $\mathrm{SO}(m)$-bundle of of the spinor bundle, using the homomorphism $\mathrm{Spin}(n) \to \mathrm{SO}(m)$ (choosing a basis for W).

Note that this provides another way to construct the spin connection: this gives a map $\mathfrak{spin}(n) \to \mathfrak{so}(m)$, which allows us to extend the 1-form to a $\mathfrak{so}(W)$-valued

1-form on the principal $SO(m)$-bundle of $S(E)$, which will, once again, satisfy the appropriate conditions to be a connection 1-form.

This connection (which will be compatible with the metric of $S(E)$) can be expressed locally in terms of the connection 1-form of the original connection. In order to do that, first pick a local orthonormal frame $e_1, \ldots, e_n$ of E. This gives a local section of P, which lifts (in two different ways) to a section of Q. Through the embedding of Q in the principal $\mathrm{SO}(m)$-bundle of $S(E)$, we obtain a section of the latter, which provides an orthonormal frame $\sigma_1, \ldots, \sigma_m$ of $S(E)$ (note that this frame will depend on the choice of lifting only by a sign).

Now, let $\omega = \omega_i^{\ j}$ be the local connection 1-form (so that $\nabla^E e_i = \omega_i^{\ j} e_j$). With respect to the local frame $\sigma_1, \ldots, \sigma_m$, the connection 1-form will be

$$\omega^S = \frac{1}{4} \sum_{i,j=1}^{n} \omega_i^{\ j} e_i e_j, \qquad \text{(Eq. 1.1)}$$

where $e_i e_j$ represents Clifford multiplication by the corresponding vectors.

1.1.3 Dirac Operators

One of the main motivations for spin geometry and spinor bundles is that we can define on them a certain kind of operators which have very interesting properties.

Definition 1.1.10. Let S be a bundle over a Riemannian manifold M with a (fibrewise) action of the Clifford bundle $\mathrm{Cl}(TM)$. Suppose that S has a metric such that the action by unit tangent vectors is an isometry, and suppose that it has a connection ∇^S compatible with the metric. The *Dirac operator* on S is the operator $D : \Gamma(S) \to \Gamma(S)$ defined locally by

$$D(s) = \sum_{i=1}^{n} e_i \cdot \nabla^S_{e_i} s,$$

where $e_1, \ldots, e_n$ is a local orthonormal frame of vector fields and $s \in \Gamma(S)$.

An important case is that in which the bundle is the spinor bundle constructed from a spin structure of the tangent bundle through the irreducible action described above, and the connection on it is the connection induced by the Levi-Civita connection on the tangent bundle. From now on, we will use $\$$ to refer to this bundle (or $\$_M$ if the manifold M needs to be specified) and we will use $\not{D}$ to refer to its Dirac operator. Therefore, $\$$ is a complex vector bundle of (complex) rank 2^n, where n is the (real) dimension of the manifold. Furthermore, if n is even, we will be able to write it in terms of the positive and negative spinor bundles $\$ = \$^+ \oplus \$^-$. In this case, $\not{D}$ can be split into two operators, $\not{D}^+ : \Gamma(\$^+) \to \Gamma(\$^-)$

and $\not{D}^- : \Gamma(\not{S}^-) \to \Gamma(\not{S}^+)$, since the connection is compatible with the splitting and Clifford multiplication interchanges the two subbundles. Note that we will often refer to these bundles as *the* spinor bundles, although the are not necessarily unique (see Theorem 1.1.7).

Another case we will want to consider is the following: suppose that we have a bundle $E \to M$ over our manifold, with a metric and a compatible connection A. Then, we can consider the bundle $\not{S} \otimes E$, which will have the product connection. We will refer to the resulting Dirac operator as the *coupled Dirac operator* and we will denote it by $\not{D}_A$.

We will now go through some properties of these Dirac operators which will be useful in following chapters. Some of the more general results can be consulted in the mentioned references, and we will only show the computations for the less common ones.

Firstly, note that it follows from the definition and from the properties of connections that, using the notation of Definition 1.1.10, if f is a smooth function, we have the product rule

$$D(fs) = \operatorname{grad}(f) \cdot s + fD(s). \qquad \text{(Eq. 1.2)}$$

The Dirac operator is very closely related to the Laplacian. This relationship is specified in the following theorem.

Theorem 1.1.11 (Weitzenböck formula)**.** *In the above conditions,*

$$D^2 = (\nabla^S)^* \nabla^S + \mathcal{R},$$

where $(\nabla^S)^* \nabla^S$ *is the connection Laplacian,*

$$\mathcal{R} = \frac{1}{2} \sum_{i,j=1}^{n} e_i \cdot e_j \cdot R^S(e_i, e_j),$$

and R^S *is the curvature of the connection* ∇^S*.*

Another important property is that these Dirac operators will be formally self-adjoint with respect to the L^2 metric (using the metric on the bundle and the Riemannian structure).

In future chapters we will be interested in the following case.

Proposition 1.1.12. *Let* (N, h) *be a* $2n$*-dimensional Riemannian manifold and* $\not{S}_N \to N$ *its spinor bundle. Consider the manifold* $M = N \times \mathbb{R}$ *with the metric* $g = e^{2\varphi(t)}h + dt^2$*, where* t *is the coordinate on the* $\mathbb{R}$ *factor and* $\varphi \colon \mathbb{R} \to \mathbb{R}$ *is a smooth function. Then,*

$$\not{D}_M s = e^{-\varphi} \not{D}_N s + \frac{\partial}{\partial t} \cdot \left(\frac{\partial s}{\partial t} + n\varphi' s \right), \qquad \text{(Eq. 1.3)}$$

for $s \in \Gamma(\$_M)$, where $\$_M$ is identified with the pullback of $\$_N$ through the projection in order to apply $\not{D}_N$, and where $\frac{\partial}{\partial t}\cdot$ indicates Clifford multiplication by the vector field in the direction of the coordinate t.

Proof. The identification between the pullback of $\$_N$ and $\$_M$ can be seen through the construction of the spinor bundles, since we can construct the spin bundle of M by taking the pullback of a spin bundle on N and taking the associated bundle using the action of $\mathrm{Spin}(2n)$ on $\mathrm{Spin}(2n+1)$ given by left multiplication (since it is a subgroup). This will give a spin bundle on M, and the corresponding spinor bundle will be isomorphic (as a vector bundle) to the pullback of $\$_N$, since the irreducible representations of $\mathrm{Cl}(2n)$ and $\mathrm{Cl}(2n+1)$ are on the same vector space. Furthermore, vectors orthogonal to $\frac{\partial}{\partial t}$ will act in the same way.

Now, let $f_1, \ldots, f_{2n}$ be local orthonormal vector fields on N. Then, the Levi-Civita connection with respect to h will be give by local 1-forms ${\theta_i}^j$, with $1 \leq i, j \leq 2n$, such that $\nabla^N f_i = {\theta_i}^j f_j$. We denote ${\theta_{ki}}^j = {\theta_i}^j(f_k)$.

The vector fields $f_1, \ldots, f_{2n}$ can also be though at vector fields on M in the direction of N, and by setting $e_i = e^{-\varphi} f_i$, for $1 \leq i \leq 2n$ and letting e_0 be the vector field in the direction of t, we get a local basis of orthonormal vectors on M. Then, the Levi-Civita connection with respect to g will have the form $\nabla^M e_i = {\omega_i}^j e_j$ for some local 1-forms ${\omega_i}^j$, with $0 \leq i, j \leq 2n$. Analogously to before, we write ${\omega_{ki}}^j = {\omega_i}^j(e_k)$.

Some elementary and not too enlightening computations will tell us that

$$
{\omega_{ki}}^j = \begin{cases} e^{-\varphi}{\theta_{ki}}^j & \text{if } i, j, k \neq 0 \\ \varphi' \delta_{kj} & \text{if } i = 0 \text{ and } j, k \neq 0 \\ -\varphi' \delta_{ki} & \text{if } j = 0 \text{ and } i, k \neq 0 \\ 0 & \text{otherwise.} \end{cases}
$$

Note, in particular, that ${\omega_{0i}}^j = {\omega_{k0}}^0 = 0$.

We note, also, that Clifford multiplication in by f_i in $\$_N$ is the same as Clifford multiplication by e_i in $\$_M$ (under the identification).

If $\sigma_1, \ldots, \sigma_m$ is the local orthonormal basis of $\$_M$ (and $\$_N$) corresponding to $e_1, \ldots, e_{2n}$ (and $f_1, \ldots, f_{2n}$), we can compute the Dirac operator using (Eq. 1.1).

We have

$$
\begin{aligned}
\not{D}_M(\sigma_\ell) =& \sum_{k=0}^{2n} e_k \cdot \nabla_{e_k}^{\$_M} \sigma_\ell \\
=& \sum_{k=0}^{2n} e_k \cdot \omega^S(e_k)\sigma_\ell \\
=& \frac{1}{4}\sum_{k=0}^{2n} e_k \cdot \left(\sum_{i,j=0}^{2n} \omega_{ki}{}^j e_i \cdot e_j \cdot \sigma_\ell \right) \\
=& \frac{1}{4}\sum_{k=1}^{2n} e_k \cdot \left(\sum_{i,j=1}^{2n} \omega_{ki}{}^j e_i \cdot e_j \cdot \sigma_\ell \right) \\
&+ \frac{1}{4}\sum_{k=1}^{2n} e_k \cdot \left(\sum_{j=1}^{2n} \omega_{k0}{}^j e_0 \cdot e_j \cdot \sigma_\ell \right) + \frac{1}{4}\sum_{k=1}^{2n} e_k \cdot \left(\sum_{i=1}^{2n} \omega_{ki}{}^0 e_i \cdot e_0 \cdot \sigma_\ell \right) \\
=& \frac{1}{4} e^{-\varphi} \sum_{k=1}^{2n} f_k \cdot \left(\sum_{i,j=1}^{2n} \theta_{ki}{}^j f_i \cdot f_j \cdot \sigma_\ell \right) \\
&+ \frac{1}{4}\varphi' \left(\sum_{k=1}^{2n} e_k \cdot e_0 \cdot e_k \cdot \sigma_\ell - \sum_{k=1}^{2n} e_k \cdot e_k \cdot e_0 \cdot \sigma_\ell \right) \\
=& e^{-\varphi} \not{D}_N(\sigma_\ell) + n\varphi' \sigma_\ell,
\end{aligned}
$$

which is exactly (Eq. 1.3) for elements in the basis. Therefore, it only remains to check that the operator $\not{D}_M$ defined in (Eq. 1.3) satisfies the appropriate product rule (Eq. 1.2). For that, let $s \in \Gamma(\$_M)$ and $f \in C^\infty(\mathbb{R})$. Then,

$$
\begin{aligned}
\not{D}_M(fs) =& e^{-\varphi}\not{D}_N(fs) + \frac{\partial}{\partial t} \cdot \left(\frac{\partial fs}{\partial t} + n\varphi' fs \right) \\
=& e^{-\varphi} \operatorname{grad}_{(N,h)}(f) \cdot s + e^{-\varphi} f \not{D}_N(s) + \frac{\partial f}{\partial t} e_0 \cdot s + f e_0 \cdot s + n\varphi' fs \\
=& \left(e^{-\varphi} \operatorname{grad}_{(N,h)}(f) + \frac{\partial f}{\partial t} e_0 \right) \cdot s + f(\not{D}_N(s) + n\varphi' s) \\
=& \operatorname{grad}_g(f) \cdot s + f \not{D}_M(s),
\end{aligned}
$$

as we wanted (note that, when using the product rule for $\not{D}_N$, we would get Clifford multiplication of the gradient of f with respect to h (which is tangent to the N component and might depend on t), but through the identification with $\$_M$, this is the same as Clifford multiplication by part of the gradient with respect to g which is tangent to the N component multiplied by $e^{-\varphi}$). $\square$

We can write this in a slightly nicer way by considering the relationship between the Clifford algebra representations for $2n$ and $2n+1$.

Corollary 1.1.13. *In the conditions above, we have*

$$\not{D}_M = \begin{pmatrix} i\left(\frac{\partial}{\partial t} + n\varphi'\right) & e^{-\varphi}\not{D}_N^- \\ e^{-\varphi}\not{D}_N^+ & -i\left(\frac{\partial}{\partial t} + n\varphi'\right) \end{pmatrix}.$$

Proof. This follows from Proposition 1.1.12 and the discussion above about the irreducible representations of Clifford algebras. □

We conclude this section by recalling the special case of the spin Dirac operator on a spin Kähler manifold. See [Fri00, §3.4] for more details. It was observed by Hitchin in [Hit74] that a Kähler manifold is spin if and only if the canonical bundle admits a square root (and in fact there is a one-to-one correspondence between spin structures and holomorphic square roots). In this setting we have the following theorem.

Theorem 1.1.14. *Let M be a Kähler manifold of dimension* $\dim_{\mathbb{C}} M = n$ *admitting a holomorphic square root of K_M, and let $L \to M$ any such holomorphic square root. Then with respect to the corresponding spin structure, there is an isomorphism*

$$\not{S} \cong (\textstyle\bigwedge^{0,0} \oplus \bigwedge^{0,1} \oplus \cdots \oplus \bigwedge^{0,n}) \otimes L,$$

with the splitting $\not{S} = \not{S}^+ \oplus \not{S}^-$ corresponding to

$$\not{S}^+ \cong \textstyle\bigwedge^{0,even} \otimes L, \quad \not{S}^- \cong \bigwedge^{0,odd} \otimes L.$$

Furthermore the spin Dirac operator on $\not{S}$ is

$$\not{D} = \sqrt{2}(\overline{\partial} + \overline{\partial}^*).$$

Notice that the factor of $\sqrt{2}$ agrees with the Kähler identity that $\overline{\partial}^2 = 2\Delta_{\overline{\partial}} = \Delta_d$.

This result may be extended to the case of coupled Dirac operators. In particular, if E is a Hermitian holomorphic vector bundle over M with Chern connection ∇ and $S := \not{S} \otimes E$ is a Dirac bundle with coupled spin connection ∇^S, then Theorem 1.1.14 becomes the following.

Theorem 1.1.15. *If D is the coupled Dirac operator on $S = \not{S} \otimes E$ defined by the coupled spin connection ∇^S coming from the Chern connection on $E \to M$, then with respect to the splitting*

$$\begin{aligned} S &\cong (\textstyle\bigwedge^{0,0} \oplus \bigwedge^{0,1} \oplus \cdots \oplus \bigwedge^{0,n}) \otimes L \otimes E \\ &\cong (\textstyle\bigwedge^{0,0}(E) \oplus \bigwedge^{0,1}(E) \oplus \cdots \oplus \bigwedge^{0,n}(E)) \otimes L, \end{aligned}$$

we have the corresponding splitting $S = S^+ \oplus S^-$ with

$$S^+ \cong \bigwedge^{0,even}(E) \otimes L, \quad S^- \cong \bigwedge^{0,odd}(E) \otimes L,$$

and

$$D = \sqrt{2}(\overline{\partial}_{EL} + \overline{\partial}_{EL}^*)$$

where $\overline{\partial}_{EL} = \nabla^{0,1}$ is the Dolbeault operator on $E \otimes L$.

1.2 Kähler and Hyper-Kähler Manifolds

Recall the definition of a Kähler manifold.

Definition 1.2.1. A Kähler manifold is a triple (M, g, I) where (M, g) is a Riemannian manifold, and I is an integrable almost-complex structure on M such that I is orthogonal with respect to g, and if one defines

$$\omega(X, Y) = g(IX, Y)$$

then the 2-form ω is closed.

The condition that $d\omega = 0$ above is called the *Kähler condition.* Kähler manifolds are Riemannian complex manifolds such that the two structures are compatible in the right sense. Namely, there is a list of equivalent conditions to the Kähler condition, which justify this statement.

Proposition 1.2.2. *Let M be a complex manifold with Riemannian metric g, associated integrable almost-complex structure I, and Levi-Civita connection ∇. Then the following are equivalent:*

1. *$h = g + i\omega$ is a Kähler metric,*
2. *$d\omega = 0$,*
3. *$\nabla I = 0$,*
4. *the Chern connection of the Hermitian metric $h := g + i\omega$ on TM agrees with the Levi-Civita connection ∇,*
5. *in terms of local holomorphic coordinates, we have*
$$\frac{\partial g_{j\bar{k}}}{\partial z^\ell} = \frac{\partial g_{\ell\bar{k}}}{\partial z^j},$$
6. *for each point $p \in M$ there is a smooth real function F in a neighbourhood of p such that $\omega = i\partial\overline{\partial}F$, and F is called the* Kähler potential, *and*

7. *for each point $p \in M$ there are holomorphic coordinates centred at p such that $g(z) = \mathbf{1} + O(|z|^2)$.*

Kähler manifolds have many nice properties that general complex manifolds do not have, such as a Hodge decomposition of their complex de Rham cohomology into Dolbeault components. See [WGP80], [GH78], or [Bal06] for more details.

Notice that by the above proposition, for a Kähler manifold (M, g, I) we have

$$\nabla I = 0, \quad \nabla g = 0$$

where ∇ is the Levi-Civita connection. Combining these two identities tells us that the holonomy of the Levi-Civita connection on a Kähler manifold will preserve both the complex and Riemannian structures. Indeed we have

$$\mathrm{Hol}(g) \subseteq \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{O}(2n) = \mathrm{U}(n)$$

where n is the complex dimension of M. The converse is also true.

Theorem 1.2.3. *A manifold (M, g) is Kähler if and only if* $\mathrm{Hol}(g) \subseteq \mathrm{U}(n)$*.*

Kähler manifolds appear as one part of Berger's classification of Riemannian holonomy groups. The classification includes several other subgroups of $\mathrm{GL}(2n, \mathbb{R})$. For example, if we have $\mathrm{Hol}(g) \subseteq \mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$ then M is *Calabi-Yau* (in fact one usually requires compactness in the definition of Calabi-Yau manifolds). If further we have

$$\mathrm{Hol}(g) \subseteq \mathrm{Sp}(m) = \mathrm{SU}(n) \cap \mathrm{Sp}(n, \mathbb{C}),$$

where $n = 2m$, then M is *hyper-Kähler*. The above expression for $\mathrm{Sp}(m)$ corresponds to the Levi-Civita connection preserving a holomorphic symplectic form on M, but using the equality

$$\mathrm{Sp}(m) = \mathrm{SU}(n) \cap \mathrm{Sp}(n, \mathbb{C}) = \mathrm{O}(2n) \cap \mathrm{GL}(m, \mathbb{H})$$

we see that a hyper-Kähler structure implies that the Levi-Civita connection preserves a representation of the quaternions on the tangent bundle of M.

This gives an alternative characterisation of hyper-Kähler manifolds, which we take as the definition.

Definition 1.2.4 (Hyper-Kähler Manifold)**.** A *hyper-Kähler manifold* is a quintuple (M, g, I, J, K) where (M, g) is a Riemannian manifold, and I, J, and K are three integrable almost-complex structures on M, each orthogonal for g, such that

$$I^2 = J^2 = K^2 = IJK = -\mathbf{1}.$$

Notice that the above definitions require that the dimension of M be a multiple of four. The first example of a hyper-Kähler manifold is therefore $\mathbb{H}$ itself. Indeed under the identification $\mathbb{H} = \mathbb{C}^2 = \mathbb{R}^4$ we have the standard Riemannian metric

$$g = dx_1^2 + dy_1^2 + dx_2^2 + dy_2^2$$

with three compatible almost-complex structures coming from writing a vector as $x_1 + iy_1 + jx_2 + ky_2$, and multiplying by i, j, and k respectively. From the hyper-Kähler structure on $\mathbb{H}$ we can obtain a larger class of flat hyper-Kähler manifolds simply by taking the tensor product with a fixed vector space.

In particular, let G be a Lie group admitting a bi-invariant metric on its Lie algebra $\mathfrak{g}$, say B. Let $M = \mathfrak{g} \otimes \mathbb{H}$. Then M admits a hyper-Kähler structure by combining the three Kähler forms on $\mathbb{H}$ with the bi-invariant metric on $\mathfrak{g}$. Such a flat example of a hyper-Kähler manifold comes with an action of G by the adjoint action in the first factor, and bi-invariance implies that this action preserves the hyper-Kähler structure. We will see in the next section how this allows one to construct many examples of interesting hyper-Kähler manifolds as quotients of flat hyper-Kähler manifolds.

1.3 Hyper-Kähler Quotients

At the end of the previous section we saw examples of flat hyper-Kähler manifolds. One way to obtain more interesting examples is by taking quotients. Given a symplectic manifold (M, ω), there is a well-developed theory of symplectic reduction. Namely, if a group G acts on (M, ω) preserving the symplectic form in such a way that a moment map $\mu\colon M \to \mathfrak{g}^*$ exists, where μ is equivariant (using the co-adjoint action on $\mathfrak{g}^*$) and satisfies

$$d\langle \mu, X\rangle = -\iota_{X^\#}\omega,$$

for every $X \in \mathfrak{g}$, then

$$M_{red} := \mu^{-1}(0)/G$$

inherits a symplectic structure, with a form ω_{red} satisfying $\pi^*\omega_{red} = \iota^*\omega$ for $\iota\colon \mu^{-1}(0) \hookrightarrow M$ and $\pi\colon \mu^{-1}(0) \to M_{red}$. This is the *Marsden-Weinstein* quotient of M by G.

When M has a compatible complex structure, that is, when M is Kähler, and when G acts holomorphically, the quotient M_{red} will also inherit a Kähler structure. There are many interesting examples of Kähler manifolds constructed in this way. For example the action of S^1 on $\mathbb{C}^{n+1}$ admits a moment map $\mu(z) = \frac{1}{2}|z|^2 - \frac{1}{2}$, and the Kähler reduction $\mu^{-1}(0)/S^1 = S^{2n+1}/S^1$ is $\mathbb{CP}^n$ with the Fubini-Study Kähler form.

When M is hyper-Kähler, under suitable assumptions we can extend this theory. Namely, suppose a group G acts on M preserving the hyper-Kähler structure, in the sense that G acts holomorphically with respect to I, J, and K, and admits a moment map with respect to each of the three Kähler forms ω_I, ω_J, and ω_K. Then one can build a hyper-Kähler moment map $\mu\colon M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ defined by

$$\mu = (\mu_I, \mu_J, \mu_K).$$

Assuming 0 is a regular value and G acts freely on $\mu^{-1}(0)$, there is a hyper-Kähler reduction

$$M /\!\!/\!\!/ G := \mu^{-1}(0)/G$$

of M by G, with hyper-Kähler structure inherited as in the case of regular symplectic reduction. In particular we have an inclusion $\iota\colon \mu^{-1}(0) \hookrightarrow M$ and a projection $\pi\colon \mu^{-1}(0) \to M/\!\!/\!\!/G$, and again $\iota^*\omega_I = \pi^*\omega_{I,red}$ and similarly for J and K.

Most interesting examples of hyper-Kähler manifolds are obtained by some kind of hyper-Kähler reduction, often-times infinite-dimensional. We will see two fundamental examples (and two of the first examples) of this infinite-dimensional reduction in Chapters 2 and 3, but first we will study a more fundamental finite-dimensional example.

Recall that $M = \mathfrak{g} \otimes \mathbb{H}$ admits a flat hyper-Kähler structure when $\mathfrak{g}$ is the Lie algebra of a compact real Lie group, and so admits a bi-invariant metric. The space M is acted upon by G through the adjoint action preserving the quaternionic structure, and so this action admits a moment map as follows.

Let $X \in \mathfrak{g}$. Then the fundamental vector field $X^\#$ is defined by

$$X^\#\big|_Y := \frac{d}{dt}(\mathrm{ad}(\exp(tX))Y)_{t=0}$$

where $Y = Y_0 + iY_1 + jY_2 + kY_3 \in \mathfrak{g} \otimes \mathbb{H}$. This is just the derivative of ad, which is well-known to be given by the Lie bracket itself. Namely

$$X^\#\big|_Y = [X, Y_0] + i[X, Y_1] + j[X, Y_2] + k[X, Y_3].$$

If B denotes the bi-invariant metric on $\mathfrak{g}$, then the form ω_I is given by

$$\omega_I(Z, W) = \langle iZ, W\rangle = -B(Z_1, W_0) + B(Z_0, W_1) - B(Z_3, W_2) + B(Z_2, W_3)$$

and so if $Z \in T_Y(\mathfrak{g} \otimes \mathbb{H}) \cong \mathfrak{g} \otimes \mathbb{H}$ then

$$\iota_{X^\#}\omega_I\big|_Y(Z) = -B([X, Y_1], Z_0) + B([X, Y_0], Z_1) - B([X, Y_3], Z_2) + B([X, Y_2], Z_3).$$

Theorem 1.3.1. *The action of G on $\mathfrak{g} \otimes \mathbb{H}$ has a moment map $\mu = (\mu_I, \mu_J, \mu_K)$ given by*

$$\begin{aligned}
\mu_I(T_0, T_1, T_2, T_3) &= [T_0, T_1] + [T_2, T_3], \\
\mu_J(T_0, T_1, T_2, T_3) &= [T_0, T_2] + [T_3, T_1], \\
\mu_K(T_0, T_1, T_2, T_3) &= [T_0, T_3] + [T_1, T_2].
\end{aligned}$$

Proof. We will just check μ_I, the other cases being completely analogous. Observe we have

$$\begin{aligned}
\mu_I(Y + sZ) &= [Y_0 + sZ_0, Y_1 + sZ_1] + [Y_2 + sZ_2, Y_3 + sZ_3], \\
\left.\frac{d}{ds}\right|_{s=0} \mu_I(Y + sZ) &= [Y_0, Z_1] + [Z_0, Y_1] + [Z_2, Y_3] + [Y_2, Z_3].
\end{aligned}$$

Taking an inner product with X and using bi-invariance of B, we obtain precisely $\iota_{X^\#}\omega_I|_Y (Z)$ as computed above, showing that

$$d\langle \mu_I, X\rangle = \iota_{X^\#}\omega_I$$

for all $X \in \mathfrak{g}$, as desired. □

Most examples of hyper-Kähler constructions known are some variation of these moment map equations, such as coadjoint orbits, Higgs moduli spaces, monopole moduli spaces, and Nahm's equations. For an exposition of such examples, see the article [Hit92] of Hitchin. We will point out the most immediate and important example of this phenomenon now, and mention some others in passing later.

Remark 1.3.2. If we replace T_i by ∇_i where ∇ is a connection arising from a principal G-bundle over $\mathbb{R}^4$ then the resulting moment map equations are

$$\begin{aligned}
F_{01} &= F_{32}, \\
F_{02} &= F_{13}, \\
F_{03} &= F_{21},
\end{aligned}$$

which are just the anti-self dual Yang-Mills equations viewed as a single equation $\mu = 0$. See Section 3.1 for more discussion of this example.

Chapter 2

Magnetic Monopoles

2.1 Yang-Mills-Higgs Equations

Let M be a 3-dimensional Riemannian manifold, G a semisimple compact Lie group, $P \to M$ a principal G-bundle, and let $\mathscr{A} \subset \Omega^1(P;\mathfrak{g})$ denote the space of connections on P.

Definition 2.1.1. The *Yang-Mills-Higgs functional* is

$$\mathrm{YMH} : \mathcal{A} \times \Gamma(\mathrm{ad}(P)) \to \mathbb{R}, \quad \mathrm{YMH}(A, \Phi) := \int_M |F_A|^2 + |d_A\Phi|^2,$$

where the inner product on $\mathfrak{g}$ is given by minus the Killing form.

Proposition 2.1.2. *The Euler-Lagrange equations for* YMH *read:*

$$\begin{aligned} d_A \star F_A &= -[\Phi, d_A\Phi], \\ d_A \star d_A\Phi &= 0 \end{aligned}$$

When (A, Φ) solves the equations above, Φ is referred to as the *Higgs field*.

Remark 2.1.3. Note that, if M be compact, there cannot be non trivial solutions, since, integrating by parts,

$$\int_M |d_A\Phi|^2 = \int_M (d_A \star d_A\Phi, \Phi) = 0.$$

Thus the Higgs field is covariantly constant and the connection must simply satisfy the usual Yang-Mills equations. Hence, one should consider M non compact.

Henceforth, we shall work with $M = \mathbb{R}^3$ and $M = \mathbb{R}^3 \setminus \mathcal{P}$, where $\mathcal{P}$ is some finite set of points.

We shall focus on solutions subject to the further conditions:

$$\mathrm{YMH}(A, \Phi) < \infty, \quad |\Phi(x)| \to 1 \text{ as } r := |x| \to \infty \tag{Eq. 2.1}$$

The second of these is referred to as the Prasad-Sommerfield limit. It is shown in [JT80] that these conditions imply

$$\begin{aligned} |F_A| &= O(r^{-2}), \\ |d_A\Phi| &= O(r^{-2}), \\ \left|\frac{\partial \Phi}{\partial \Omega}\right| &= O(r^{-2}), \text{and} \\ |\Phi(x)| &= 1 + \frac{m}{r} + O(r^{-2}), \end{aligned}$$

where $m \in \mathbb{R}$ is some constant.

2.2 SU(2) Monopoles

Firstly, we shall look at the simplest example of $M = \mathbb{R}^3$ and $G = SU(2)$.

To start off, we make some remarks about the asymptotic behaviour of the Higgs field. Given a solution (A, Φ) to the Yang-Mills-Higgs equations satisfying the conditions above, fix some small $\varepsilon > 0$ and let $R > 0$ be large enough so that, for any $|x| \geq R$, $|\Phi(x)|$ has two distinct eigenvalues, say λ_1, λ_2 which satisfy $|i - \lambda_1| < \varepsilon$ and $|i + \lambda_2| < \varepsilon$. The values of λ_1, λ_2 may of course depend on x, but the point is that one may speak of the eigenvalue *near* $\pm i$. The reason this is possible is because any element of $\mathfrak{su}(2)$ with unit length squares to -1, so its eigenvalues are $\pm i$, and it is also tracefree, so that it have distinct eigenvalues.

Granted this, define a line bundle $L_\Phi \to S_R$, where the fibre over x consists of the eigenspace of $\Phi(x)$ with eigenvalue near i. Note that, even though the bundle $P \to \mathbb{R}^3$ is trivial, the line bundle L_Φ need not be trivial. Let $k \in \mathbb{Z} \cong H^2(S_R)$ denote the degree of L_Φ. It can be shown that k is independent of R.

Definition 2.2.1. The integer k is called *magnetic charge* of the solution (A, Φ).

Remark 2.2.2. The charge k can be computed simply as the degree of the map $|\Phi|^{-1}\Phi : S_R \to S^2$, where $R > 0$ is large and $S^2 \subset \mathfrak{su}(2)$ is the unit sphere.

For large R, let B_R be the ball of radius R and consider:

$$\int_{B_R} |F_A|^2 + |d_A\Phi|^2 = \int_{B_R} |F_A - \star d_A\Phi|^2 + 2(\star d_A\Phi, F_A)$$

Where $(\cdot,\cdot)$ denotes the combination of the inner product on $\mathfrak{su}(2)$ with the inner product of forms. Now:

$$d(\Phi, F_A) = (d_A\Phi, F_A) - (\Phi, d_A F_A) = (d_A\Phi, F_A) = \star(\star d_A\Phi, F_A)$$

Hence, by Stokes:

$$\int_{B_R} (\star d_A\Phi, F_A) = \int_{S_R} (\Phi, F_A)$$

Meanwhile, one can use Chern-Weil theory to show that:

$$\lim_{R\to\infty} \int_{S_R} (\Phi, F_A) = 4\pi k$$

Hence, if we add the assumption that $k \geq 0$:

$$\mathrm{YMH}(A,\Phi) = \int_{\mathbb{R}^3} |F_A - \star d_A\Phi|^2 + 8\pi k \geq 8\pi k$$

Thus, the functional is minimized whenever $F_A = \star d_A\Phi$.

Definition 2.2.3. Let M be a 3-manifold, $P \to M$ be a principal G-bundle, A a connection on P and $\Phi \in \Gamma(\mathrm{ad}P)$. The equation $F_A = \star d_A\Phi$ is called the *Bogomolny equation*.

Definition 2.2.4. In the case $M = \mathbb{R}^3$ and $G = SU(2)$, a solution of the Bogomolny equation subject to $k \geq 0$, where k is the magnetic charge, and the Prasad-Sommerfield boundary conditions above is referred to as an $SU(2)$ *magnetic monopole* of charge k.

Remark 2.2.5. The constant m in the asymptotic behaviour of Φ turns out to only depend on the charge k. Indeed, $m = -\frac{k}{2}$.

Example 2.2.6. It is possible to write down explicit solutions which have a relatively simple form. For example, the pair (A,Φ) with

$$A = \sum_{i,j,\ell} \left(\frac{|x|}{\sinh|x|} - 1\right) \varepsilon_{ij\ell} \frac{x_\ell}{|x|} \frac{e_i}{\sqrt{2}} \otimes dx_j$$
$$\Phi = \sum_i (|x|\coth|x| - 1) \frac{x_i}{|x|^2} \frac{e_i}{\sqrt{2}}$$

is a monopole of charge 1, where e_i denote the Pauli matrices. Note it has an $SO(3)$ symmetry. This solution is called the Prasad-Sommerfield monopole; see [TM79] for a derivation. It can be shown that this is the *only* monopole of charge 1 up to translation and gauge equivalence.

Remark 2.2.7. It is worth noting that a solution (A, Φ) to the Bogomolny equation defines via:

$$\Phi dx_0 + A_1 dx_1 + A_2 dx_2 + A_3 dx_3$$

an ASD connection on the trivial $SU(2)$-bundle over $\mathbb{R}^4$; conversely, any ASD connection on the trivial bundle which be constant on some direction defines a monopole. This "dimensional reduction" is another way to think of monopoles and it is how Taubes proves the necessary analytical results in [Tau83]. See Section 3.2 for more discussion of this phenomenon.

2.3 U(1) Monopoles

Consider, now, the case of $G = U(1)$. Notice that, if $M = \mathbb{R}^3$, then, since $U(1)$ is abelian, the Yang-Mills-Higgs equations imply $d_A \star d_A \Phi = d \star d\Phi = 0$. Together with the Prasad-Sommerfield limit, this means that Φ is a bounded harmonic function, and hence it is constant. Therefore, solutions on $\mathbb{R}^3$ are uninteresting, and consequently we turn our attention to $M = \mathbb{R}^3 \setminus \mathcal{P}$, for $\mathcal{P} \subset \mathbb{R}^3$ a finite set of points. Notice that, in this case, the bundle $P \to M$ need not be trivial.

There is a notion of magnetic charge here as well, and the definition is analogous: select some large ball $B_R \subset \mathbb{R}^3$ so as to have $\mathcal{P} \subset B_R$. Consider the restriction $P|_{M \setminus B_R}$, and define the *magnetic charge* $k \in \mathbb{Z} \cong H^2(\mathbb{R}^3 \setminus B_R)$ to be the degree of this line bundle. The Bogomolny equation has the same relevance here as it does in the $SU(2)$ case, providing a minimum of the Yang-Mills-Higgs action; the proof is similar. At this point it is worth noting that the concept of magnetic charge can be generalised to the case of a general Lie group G at the expense of much complexity. As, in this text, we concentrate only on $SU(2)$ and $U(1)$ monopoles, it was decided to leave out the general definition.

Definition 2.3.1. Solutions to the Bogomolny equation satisfying the conditions (Eq. 2.1) on $M = \mathbb{R}^3 \setminus \mathcal{P}$ are referred to as *singular* $U(1)$ *monopoles*.

Remark 2.3.2. Since $U(1)$ is abelian, Φ is simply i times a real valued function on the base manifold; furthermore, the Bogomolny equation reduces to $F_A = \star d\Phi$. Moreover, upstairs in the principal bundle P, one may write $F_A = dA$, and hence one has $dA = \star d\Phi$.

Remark 2.3.3. Due to the Bianchi identity, $d \star d\Phi = 0$; that is to say that Φ is harmonic. Whence, it follows that Φ must have the form:

$$\Phi(x) = i\left(1 + \sum_{p\in\mathcal{P}} \frac{k_p}{2|x-p|}\right) \qquad \text{(Eq. 2.2)}$$

where $k_p \in \mathbb{R}$ are constants. One can easily see, by use of Chern-Weil theory, that the k_p must be integers.

2.3.1 Dirac Monopole

The primordial magnetic monopole is the one discovered, albeit not quite as we shall describe, by Dirac. We now describe it in detail. To set the stage, fix an integer $k \in \mathbb{Z}$ and consider the Higgs field:

$$\Phi : \mathbb{R}^3 \setminus 0 \to i\mathbb{R}, \quad \Phi(x) = i\left(1 - \frac{k}{2|x|}\right)$$

we seek a principal $U(1)$-bundle $P \to \mathbb{R}^3 \setminus 0$ equipped with connection A satisfying the Bogomolny equation $dA = \star d\Phi$.

A few basic remarks are in line before we determine what P and A are going to be. Identify $\mathbb{R}^3 \setminus 0$ with $S^2 \times (0, \infty)$ and let $\pi : \mathbb{R}^3 \to S^2$ be the evident projection; this is a deformation retraction; hence, the complex line bundles over $\mathbb{R}^3$ are precisely $\pi^*\mathcal{O}(\ell)$, for $\ell \in \mathbb{Z}$. In what follows, I shall use the usual chart with domain $\{[z : 1] \in \mathbb{CP}^1 = S^2 \mid z \in \mathbb{C}\}$; the same statements can be checked analogously on $\{[1 : z] \in S^2 \mid z \in \mathbb{C}\}$. Consider, firstly, the tautological bundle $\mathcal{O}(-1) \to S^2$; this is a holomorphic line bundle; moreover, it has a preferred metric h coming from the embedding $\mathcal{O}(-1) \hookrightarrow S^2 \times \mathbb{C}^2$; explicitly:

$$h \in \Gamma(\mathcal{O}(-1) \otimes \mathcal{O}(1)) = \Gamma(S^2 \times \mathbb{C}), \quad h([1 : z]) = 1 + |z|^2$$

therefore, let C denote the associated Chern connection; the familiar formula for line bundles gives:

$$\tilde{C}|_{[1:z]} = (\partial \log h)|_{[1:z]} = \frac{\bar{z} dz}{1 + |z|^2}$$

where $\tilde{C}$ denotes the connection matrix; thus, the curvature is:

$$F_C = \bar{\partial}\partial \log h = \frac{d\bar{z} \wedge dz}{(1 + |z|^2)^2}$$

Letting $\alpha \in \Omega^2(S^2)$ denote the usual Euclidean area form, if one care to check, $F_C = -\frac{i}{2}\alpha$. Now, let $\hat{P} \to S^2$ be the principal $U(1)$-bundle to which $\mathcal{O}(-1)$ is associated via the standard representation of $U(1)$ on $\mathbb{C}$; that is $g \cdot z = gz$; this is simply the Hopf fibration. One shows that $\mathcal{O}(\ell)$ is associated to $\hat{P}$ as well for any $\ell \in \mathbb{Z}$; in this case, one uses the representation $g \cdot z = g^{-\ell} z$. The connection C above can be thought of as a connection form on $\hat{P}$; thus, it defines a connection on $\mathcal{O}(\ell)$; call it C_ℓ; one easily checks that $F_{C_\ell} = -\ell F_C$. At last, one pulls back everything to $\mathbb{R}^3 \setminus 0$ via π, thereby obtaining line bundles $\pi^*\mathcal{O}(\ell)$ equipped with connections $\pi^* C_\ell$.

Now, we return to the Dirac monopole. Define $P \to \mathbb{R}^3 \setminus 0$ to be principal $U(1)$-bundle to which $\pi^*\mathcal{O}(k)$ is associated *via the standard representation* $g \cdot z = gz$;

and let A be the connection $\pi^* C_k$; we must verify the Bogomolny equation. One directly verifies that

$$\star d\Phi = \frac{ik}{2|x|^2}(\star d|x|) = \frac{ik}{2|x|^2}|x|^2\pi^*\alpha = -k\frac{i}{2}\pi^*\alpha = -kF_C = F_A$$

Definition 2.3.4. The pair (A, Φ) thus constructed is called the *Dirac monopole of charge k.*

2.3.2 General Case

In the general case, that is over $\mathbb{R}^3 \setminus \mathcal{P}$ for $\mathcal{P} \subset \mathbb{R}^3$ finite, the Higgs field is as in (Eq. 2.2). Notice that, due to $U(1)$ being abelian, the Bogmolny equations are linear in the sense that, locally, it can be written as $d\tilde{A} = \star d\Phi$ where $\tilde{A}$ denotes the connection matrix. This allows one to construct the bundle P and the connection A by using the Dirac monopole as follows. Let S_p be the homology class generated by a sphere of small radius centered at $p \in \mathcal{P}$; hence, $H^2(\mathbb{R}^3 \backslash \mathcal{P}; \mathbb{Z}) = \langle S_p \mid p \in \mathcal{P}\rangle_{\mathbb{Z}}$; define the principal $U(1)$-bundle P to be the unique one such that the complex line bundle associated via the standard representation $g \cdot z = gz$ has first Chern class $\sum_{p\in\mathcal{P}} k_p S_p$. Now, let (A_p, Φ_p) be the Dirac monopole centered at $p \in \mathcal{P}$; fix some atlas of $\mathfrak{U}$ of $\mathbb{R}^3 \setminus \mathcal{P}$ over which P and each of the principal bundles of the Dirac monopoles tthat rivialise and set:

$$\tilde{A}_U = \sum_{p\in\mathcal{P}} \tilde{A}_{p,U}$$

where $U \in \mathfrak{U}$, $\tilde{A}_U$ denotes the connection matrix of A over U and $\tilde{A}_{p,U}$ denotes the connection matrix of A_p over U. By construction, A is a connection and satisfies the Bogomolny equation.

Definition 2.3.5. The integer k_p is called the *charge of the monopole at p.*

Remark 2.3.6. One can show that the charge k of the monopole as defined above satisfies $k = \sum_{p\in\mathcal{P}} k_p$.

2.4 Moduli Space of SU(2) Monopoles

We now make a small detour to demonstrate one way in which monopoles relate to hyper-Kähler geometry; namely, the moduli space of $SU(2)$ monopoles on $\mathbb{R}^3$.

First, the following remark is necessary. Let (A', Φ') denote the Dirac monopole of charge k; from it, one can construct an $SU(2)$ monopole over $\mathbb{R}^3 \setminus 0$ by using the trivial principal $SU(2)$-bundle $P \to \mathbb{R}^3 \setminus 0$ and setting:

$$\Phi = \begin{pmatrix} \Phi' & 0 \\ 0 & \bar{\Phi}' \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} \tilde{A}' & 0 \\ 0 & \tilde{\bar{A}}' \end{pmatrix}$$

where the tilde denotes the connection matrix with respect to the obvious trivialisations. When there be no risk of confusion, we refer to (A, Φ) also as the Dirac monopole. The point to be made here is that, *any* $SU(2)$-monopole over $\mathbb{R}^3$ behaves *asymptotically* like the Dirac monopole of equal charge. To be more precise: an $SU(2)$-monopole over $\mathbb{R}^3$, perhaps after a gauge transformation, differs from the Dirac monopole of equal charge, on the complement of a large ball, by some $(a, \phi) \in \Omega^1(\mathbb{R}^3; \mathfrak{su}(2)) \times \Omega^0(\mathbb{R}^3; \mathfrak{su}(2))$ such that $|a| = |\phi| = O(r^{-2})$; see [AH88, Ch. 4] for details.

To begin, let N_k denote the moduli space of $SU(2)$ monopoles of charge $k \geq 0$; that is the quotient of the space of all $SU(2)$ monopoles of charge k over $\mathbb{R}^3$ by the action of the gauge group $\mathcal{G}$. It is not obvious even that N_k is a manifold, but this does turn out to be the case and its dimension is $4k-1$. Consider, for example the case of charge 1; as remarked above, the Prasad-Sommerfield monopole is the unique solution up to translation; thus, the moduli space N_1 is isomorphic to $\mathbb{R}^3$. Due to its dimension, already, N_k cannot be hyper-Kähler; hence, if one desires a hyper-Kähler space, N_k must be enlarged. Moreover, the analytical difficulties encountered favour the definition of this larger space. We shall provide two definitions of this space, denoted M_k; the first is simpler, the second is more elegant.

Firstly, note that, for any $SU(2)$ monopole (A, Φ), one can fix the gauge so that

$$\lim_{t\to\infty} \Phi(0,0,t) = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

a Higgs field that satisfies this condition is said to be *framed*. Define M_k to be the quotient of the space of $SU(2)$ monopoles (A, Φ) with Φ framed by the action of the gauge transformations $g : \mathbb{R}^3 \to SU(2)$ satisfying

$$\lim_{t\to\infty} g(0,0,t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The topology on the spaces of monopoles and gauge transformations is that of local uniform convergence in all derivatives. Taubes proves in [Tau83], by considering the Hessian of the functional YMH, that M_k is a manifold of dimension $4k$; and this manifold, verily, turns out to be hyperKähler. Now, notice that subgroup $U(1) \subset SU(2)$ of diagonal matrices acts on the space of framed Higgs fields (as constant gauge transformations); this is because the framing is defined so as to make the value of Φ tend to a diagonal matrix in the limit above; hence, one can check that $U(1)$ acts on M_k; this action is free (notice that these gauge transformations were not allowed when we took the quotient to obtain M_k). If one care to check, quotienting M_k by this $U(1)$ action recovers N_k; thus, $M_k \to N_k$ is a circle bundle. The manifold M_k turns out to be familiar; Donaldson proved:

Theorem 2.4.1. *There is a natural diffeomorphism between M_k and the space of rational functions $\mathbb{CP}^1 \to \mathbb{CP}^1$ of degree k.*

Now we shall give the more sophisticated and more useful (at least in what follows), definition of M_k. Start by defining V_k to be the vector space of pairs $(a, \phi) \in \Omega^1(\mathrm{ad}(P)) \times \Omega^0(\mathrm{ad}(P))$ such that $|a| = |\phi| = O(|x|^{-2})$. Secondly, define C_k to be the space of pairs $(A, \Phi) \in \mathscr{A} \times \Omega^0(\mathrm{ad}(P))$ where Φ has charge k and (A, Φ) differs, in the complement of some large ball, from the Dirac monopole (embedded into $SU(2)$ as outlined above) of charge k by some element of V_k. As remarked, any $SU(2)$ monopole is gauge equivalent to some element of C_k. Clearly, C_k is an affine space under the action of the vector space V_k. Notice that the elements of C_k are *not* required to be monopoles; indeed, let $\tilde{M}_k \subset C_k$ be the subset of elements which do satisfy the Bogomolny equation. Thirdly, define $\mathcal{G}' \subset \mathcal{G} \equiv \Gamma(\mathrm{Ad}(P))$ to be the subset of the full gauge group which has as its Lie algebra the elements $X \in \Omega^0(\mathrm{ad}(P))$ satisfying $|X| = O(|x|^{-1})$. At last, define $M_k := \tilde{M}_k/\mathcal{G}'$. Notice that the subgroup $U(1) \hookrightarrow SU(2)$ of diagonal elements acts on C_k via the *constant* gauge transformations; one does well to notice that the rest of $SU(2)$ does not act, in this way, on C_k; this action descends to the quotient M_k essentially because, if $g \in \mathcal{G}'$ and $u \in \mathcal{G}$ be a constant gauge transformation, then $ugu^{-1} \in \mathcal{G}'$; one can prove that quotienting M_k by this $U(1)$ action recovers N_k; thus, the circle bundle structure $M_k \to N_k$ is evident in this definition as well.

Now, we shall define the Riemannian metric on M_k; to do this, we must consider the *linearised* Bogomolny equations. Let (A, Φ) be an $SU(2)$ monopole; define $T_{(A,\Phi)}M_k \subset V_k$ to be the elements satisfying:

$$\begin{aligned} \star d_A a - d_A \phi + [\Phi, a] &= 0 \\ \star d_A \star a + [\Phi, \phi] &= 0 \end{aligned} \qquad \text{(Eq. 2.3)}$$

The first of these equations is the linearised Bogomolny equation; the second represents the condition of (a, ϕ) being "orthogonal to the gauge directions". As it turns out, $T_{(A,\Phi)}M_k$ is the tangent space to M_k at (A, Φ) and so $\dim T_{(A,\Phi)}M_k = 4k$. The Riemannian metric on M_k is given by the L^2-inner product on each $T_{(A,\Phi)}M_k$; [AH88] show that this metric is complete. Given an $SU(2)$ monopole (A, Φ), note that Φ itself is an infinitesimal gauge transformation (not square integrable though, as it does not vanish at ∞); it gives rise to the element $(d_A\Phi, 0) \in T_{(A,\Phi)}M_k$; the orthogonal complement, in $T_{(A,\Phi)}M_k$, to the span of $(d_A\Phi, 0)$, is the tangent space to $p(A, \Phi) \in N_k$ where $p : M_k \to N_k$ denotes the quotient. If it be of interest, [Tau83] covers the analytical details.

To define the almost complex structures on M_k, think of a $(a, \phi) \in V_k$ as a section of $\mathrm{ad}(P) \otimes \mathbb{H}$ by writing:

$$\phi + a_1 I + a_2 J + a_3 K$$

This turns $\mathrm{ad}(P) \otimes \mathbb{H}$ into an $\mathbb{H}$-module bundle; moreover, if one care to check, equations (Eq. 2.3) are invariant under this $\mathbb{H}$-action; hence, $T_{(A,\Phi)} M_k$ also inherits this $\mathbb{H}$-module structure and thus we obtain the almost complex structures.

The fact that M_k is hyperKähler with respect to these almost complex structures I, J, K can be seen by exposing the quotient that defines M_k as a hyperKähler quotient. Verily, the Bogomolny equation suggests the moment map should be:

$$\mu : C_k \to \mathrm{Lie}\,(\mathcal{G}') \otimes \mathbb{R}^3, \quad (A, \Phi) \mapsto F_A - \star d_A \Phi$$

we shall now verify that this is indeed a moment map. Consider the first component μ_1. Equivariance is immediate. What remains is to check that, for any $(A, \Phi) \in C_k$, $(a, \phi) \in V_k$, $X \in \mathrm{Lie}\,(\mathcal{G}')$:

$$-\iota_{X^\sharp}\omega_1 \equiv -\left\langle I \cdot (a, \phi), \left.\frac{d}{dt}\right|_{t=0} \exp(tX) \cdot (A, \Phi) \right\rangle =^? \left.\frac{d}{dt}\right|_{t=0} \langle \mu_1(A + ta, \Phi + t\phi), X \rangle \tag{Eq. 2.4}$$

Where $\langle \cdot, \cdot \rangle$ denote the combination of the inner product on $\mathbb{H}$, the invariant inner product on $\mathfrak{su}(2)$, and the L^2-inner product. Expand the terms on the left hand side:

$$I \cdot (a, \phi) = \phi I - a_1 + a_2 K - a_3 J$$

And:

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \exp(tX) \cdot (A, \Phi) =& [X, \Phi] + [X, A] - dX \\ =& [X, \Phi] + [X, A_1]I + [X, A_2]J + [X, A_3]K \\ & - (\partial_{x_1} X)I - (\partial_{x_2} X)J - (\partial_{x_3} X)K\end{aligned}$$

Hence:

$$\begin{aligned}\iota_X^\sharp \omega_1 =& -\langle a_1, [X, \Phi] \rangle + \langle \phi, [X, A_1] \rangle - \langle a_3, [X, A_2] \rangle + \langle a_2, [X, A_3] \rangle \\ & - \langle \phi, \partial_{x_1} X \rangle + \langle a_3, \partial_{x_2} X \rangle - \langle a_2, \partial_{x_3} X \rangle\end{aligned}$$

Now, the derivative on the right hand side of (Eq. 2.4) becomes simply the linearised Bogomolny equation; thus, the right hand side of (Eq. 2.4) becomes:

$$\langle \partial_{x_2} a_3 - \partial_{x_3} a_2 + [A_2, a_3] - [A_3, a_2] - \partial_{x_1} \phi - [A_1, \psi] + [\Phi, a_1], X \rangle$$

A few applications of integration by parts to pass the derivatives to the side of X and the fact that the inner product on $\mathfrak{su}(2)$ satisfies $\langle [A, B], C\rangle = \langle A, [B, C]\rangle$ immediately gives the desired equality. The same argument can be applied to μ_2 and μ_3.

Remark 2.4.2. One should note the similarity between the derivation of the moment map above and Theorem 1.3.1.

Chapter 3

Nahm's Equations

3.1 Instantons

As seen in Chapter 2, one often obtains interesting gauge-theoretic objects by considering functionals defined in terms of the curvature of a connection. The case of magnetic monopoles corresponds to the Yang-Mills-Higgs functional. In some sense this may be viewed as a special case of the Yang-Mills function in four dimensions (cf. Remark 2.2.7).

Suppose initially that we have any orientable manifold M with a principal G-bundle P over it, where G is some compact real Lie group admitting a bi-invariant inner product. Then a connection A on P has a curvature form $F_A \in \Omega^2(M, \operatorname{ad} P)$. Fixing a Riemannian metric g on M, we obtain a norm $|F_A| \in C^\infty(M)$ and we define the *Yang-Mills functional* by

$$\mathrm{YM}(A) := \int_M |F_A|^2 \, d\mathrm{vol}_g.$$

Definition 3.1.1. An *instanton* on P is a critical point of the Yang-Mills functional

The Euler-Lagrange equations for the Yang-Mills functional are given by

$$d_A^* F_A = 0 \qquad \text{(Eq. 3.1)}$$

where d_A is the induced exterior covariant derivative on $\Omega^2(M, \operatorname{ad} P)$, and so instantons are by definition solutions to (Eq. 3.1).

Initially the Yang-Mills equations are a second order system of equations in the connection A. However, if we consider just the case of $\dim M = 4$, it is possible to find solutions to (Eq. 3.1) by solving a related first-order system. In particular, on a four-manifold M we know $\star : \Omega^2(M) \to \Omega^2(M)$ and $\star\star = 1$. Thus the Hodge star has eigenvalues ± 1 and there is a splitting $\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M)$ of two-forms into self-dual and anti-self-dual parts. Furthermore we have the expression $d_A^* = \pm \star d_A \star$ so the Yang-Mills equations are equivalent to

$$d_A \star F_A = 0.$$

It is well known that any connection on P satisfies the *Bianchi identity*

$$d_A F_A = 0,$$

so if we suppose $F_A \in \Omega^2_\pm(M)$ then A would automatically satisfy (Eq. 3.1).

Definition 3.1.2. The *anti-self-duality (ASD) equations* for a connection A on a principal bundle $P \to M$ over a four-manifold M are

$$\star F_A = -F_A.$$

We could just as well have considered the *self*-duality equations, but for our purposes we will be interested in the anti-self-dual variant. As was mentioned in Remark 1.3.2, if in local coordinates we write $F_A = F_{ij} dx^i \wedge dx^j$, where

$$F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} + [A_i, A_j],$$

then the ASD equations can be given by the triple of first-order relations

$$\begin{aligned} F_{01} &= F_{32}, \\ F_{02} &= F_{13}, \\ F_{03} &= F_{21} \end{aligned} \qquad \text{(Eq. 3.2)}$$

on the connection A.

3.2 Dimensional Reduction and Nahm's Equations

The ASD equations are of profound importance in four-dimensional geometry and topology, due primarily to the pioneering work of Donaldson since the early 1980s. However, through the process of dimensional reduction these equations also have important implications in lower dimensions.

Consider first a principal G-bundle $P \to \mathbb{R}^4$ (which we may take to be trivial) with global connection form given by $A = A_0 dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3$ for $A_i \in \Gamma(\mathbb{R}^4, \operatorname{ad} P)$. To dimensionally reduce, take a subgroup Λ of translations of

$\mathbb{R}^4$ and require the data A_i to be invariant under Λ. Then the connection A will correspond to a connection on the quotient $\mathbb{R}^4/\Lambda$, and one can consider structures on this quotient satisfying the dimensionally reduced ASD equations.

For example, if we suppose Λ consists of translations in the x^0 direction then the connection A corresponds to a pair $(\tilde{A}, \Phi)$ where $\tilde{A} = A_1 dx^1 + A_2 dx^2 + A_3 dx^3$ is a connection form on a G-bundle over $\mathbb{R}^3$, and $\Phi = A_0$ is an endomorphism $\Phi \in \Gamma(\mathbb{R}^3, \operatorname{ad} P)$. Recall from Remark 2.2.7 that in this case it may be easily checked that the Bogomolny equations discussed in Chapter 2 are precisely the dimensionally reduced ASD equations for this data $(\tilde{A}, \Phi)$.

If we suppose Λ consists of translations in both the x^0 and x^1 directions then again we obtain a Higgs field Φ with two components, which we hence view as complex-valued. In this case it is more apt to consider the self-dual equations, and these are in fact conformally invariant, and so make sense on a compact Riemann surface Σ_g. In this setting the SD equations are *Hitchin's equations* first investigated by Nigel Hitchin, and the solutions give rise to *Higgs bundles.* These equations have important applications in algebraic geometry and integrable systems.

Finally suppose Λ consists of translations in coordinates x^1, x^2, and x^3. Then the connection A corresponds to data which we denote (T_0, T_1, T_2, T_3) where $T_0 = A_0$ is the connection form on a G-bundle over $\mathbb{R}$ and the $T_i, i = 1, 2, 3$, are endomorphisms $T_i = A_i \in \Gamma(\mathbb{R}, \operatorname{ad} P)$. In this case the ASD equations become *Nahm's equations*, which we now describe more explicitly.

Firstly we will consider the case of our Nahm data (T_0, T_1, T_2, T_3) being defined over any interval I, rather than all of $\mathbb{R}$. If this interval has coordinate t, then identify t with the coordinate x^0 on $\mathbb{R}^4$. Then this data corresponds to a connection $A = T_0 dx^0 + T_1 dx^1 + T_2 dx^2 + T_3 dx^3$ with curvature

$$F_{ij} = \begin{cases} 0 & i, j = 0, \\ [T_i, T_j] & i, j > 0, \\ \frac{dT_j}{dx^0} + [T_0, T_j] & i = 0, j > 0, \\ -(\frac{dT_i}{dx_0} + [T_0, T_i]) & i > 0, j = 0. \end{cases}$$

In this case the ASD equations (Eq. 3.2) become (switching from x^0 back to t),

$$\begin{aligned} \frac{dT_1}{dt} + [T_0, T_1] + [T_2, T_3] = 0, \\ \frac{dT_2}{dt} + [T_0, T_2] + [T_3, T_1] = 0, \\ \frac{dT_3}{dt} + [T_0, T_3] + [T_1, T_2] = 0, \end{aligned} \qquad \text{(Eq. 3.3)}$$

the so-called *Nahm equations.*

Remark 3.2.1. Recall that $\nabla = \frac{d}{dt} + T_0$ is a connection on over the interval, and as operators $[\nabla, T_i] = \frac{dT_i}{dt} + [T_0, T_i]$ and one will often see the Nahm equations stated in this slightly more invariant form (which does not depend on an explicit trivilisation of the G-bundle over I).

Remark 3.2.2. The triple of equations (Eq. 3.3) can be reduced to a pair of equations as follows. If we let $\alpha := T_0 + iT_1$ and $\beta := T_2 + iT_3$ then $\mu(T_0, T_1, T_2, T_3) = 0$ is equivalent to the system

$$\frac{d\beta}{dt} + [\alpha, \beta] = 0,$$

$$\frac{d(\alpha + \alpha^*)}{dt} + [\alpha, \alpha^*] + [\beta, \beta^*] = 0,$$

where $\alpha^* = T_0^* - iT_1^* = -T_0 + iT_1$ because we are assuming G to be compact semi-simple so $\mathfrak{g}$ consists of matrices which are skew-adjoint.

As was pointed out in Remark 1.3.2, there is an obvious similarity between ASD equations and the equations for hyper-Kähler reduction of a flat manifold $M = \mathfrak{g} \otimes \mathbb{H}$. Indeed this can be made precise for the Nahm equations. If we let

$$\mathscr{C}(I) := \{(T_0, T_1, T_2, T_3) \mid T_i \in C^\infty(I, \mathfrak{g})\}$$

and identify (T_0, T_1, T_2, T_3) with $T_0 + iT_1 + jT_2 + kT_3$ then $\mathscr{C}(I)$ becomes a quaternionic vector space (of infinite dimension). Indeed this space can be thought of as

$$\mathscr{C}(I) = C^\infty(I, \mathfrak{g}) \otimes \mathbb{H}$$

and the gauge group $\mathscr{G} = C^\infty(I, G)$ acts on the Nahm data (T_0, T_1, T_2, T_3) as

$$g \cdot (T_0, T_1, T_2, T_3) := (gT_0g^{-1} - \frac{dg}{dt}g^{-1}, gT_1g^{-1}, gT_2g^{-1}, gT_3g^{-1}),$$

where the action on the first factor comes from the fact that T_0 is a connection form. If the Nahm data is framed appropriately (for example if one fixes the values of the T_i at the ends of the interval I and requires the gauge transformations g to approach the identity at these same endpoints), then we have

Theorem 3.2.3. *The Nahm equations* (Eq. 3.3) *are the components of a hyper-Kähler moment map for the action of* $\mathscr{G}$ *on* $\mathscr{C}(I)$.

The proof of this result is essentially exactly the same as Theorem 1.3.1, except one must take care of certain intergration by parts that are allowed due to the framing of the Nahm data described above.

As a consequence, the moduli space of solutions to Nahm's equations has a natural hyper-Kähler structure, and indeed variations on Nahm's equations can be used to endow other spaces of interest with hyper-Kähler metrics. For example one may view both coadjoint orbits in complex Lie groups as well as moduli spaces of rational maps as hyper-Kähler manifolds through Nahm's equations. For more details see [Hit92].

Remark 3.2.4. If one describes the Nahm equations as a complex and real equation as in Remark 3.2.2 then the complex equation gives a *holomorphic* moment map for the space $\mathscr{C}(I)$ viewed with the i complex structure, and the real equation gives a further symplectic moment map. This lines up with the observation for finite-dimensional hyper-Kähler quotients that they may be viewed as a holomorphic symplectic quotient followed by a real symplectic quotient on a "semistable locus." For more details on this perspective see [Hit92].

Chapter 4

Correspondence between Monopoles and Nahm's Equations

In 1983 Nahm gave in [Nah83] a construction of magnetic monopoles using a novel adaption of the ADHM construction of instantons for four-manifolds. Nahm demonstrated that magnetic monopoles could be obtained as solutions to a certain system of ordinary differential equations, the Nahm equations, defined on a vector bundle over an interval.

In the same year Hitchin proved in [Hit83] a correspondence between certain solutions of Nahm's equations for a rank k Hermitian bundle over an interval and SU(2) magnetic monopoles on $\mathbb{R}^3$ of magnetic charge k satisfying certain asymptotic properties.

Nakajima in [Nak93] has given a proof of the correspondence of Hitchin using purely differential-geometric techniques. Furthermore, Nakajima verified that this correspondence induces a hyper-Kähler isometry of the corresponding moduli spaces of solutions. In this chapter we will present the correspondence as proven by Nakajima, including how to pass between Nahm data and monopoles, and discuss variations on this result for singular Dirac monopoles.

4.1 $\mathrm{SU}(2)$ Monopoles and Nahm's Equations

Recall that the dimensional reduction of the ASD equations from $\mathbb{R}^4$ to $\mathbb{R}^3$ give the Bogomolny equations

$$F_A = \star d_A \Phi$$

describing magnetic monopoles. This dimensional reduction was obtained by taking a group Λ of translations in the x^0 direction in $\mathbb{R}^4$ and requiring the ASD equations be invariant under Λ.

Similarly, to obtain the Nahm equations one dimensionally reduces to $\mathbb{R}$ by defining Λ' to be translations in the x^1, x^2, and x^3 directions.

Hence the correspondence discussed in the introduction to this chapter may be viewed as a duality or correspondence between solutions of the (anti-)self duality equations invariant in one coordinate or in three coordinates.

Originally this correspondence was proved by Hitchin by passing through a third object, a spectral curve in twistor space $T\mathbb{CP}^1$. The curve corresponding to the given Nahm data is defined by

$$\det(\lambda \mathbf{1} + A(\zeta, t)) = 0$$

where $A(\zeta) = A_0 + \zeta A_1 + \zeta^2 A_2$, and $A_0 = T_1 + iT_2, A_1 = -2iT_3, A_2 = T_1 - iT_2$, and we choose a gauge where $T_0 = 0$. In this set up the Nahm equations are equivalent to the single equation

$$\frac{dA}{dt} + [A, B] = 0$$

for $B(\zeta) = \frac{1}{2}A_1 + \zeta A_2$.

Donaldson in [Don84] built further upon this to prove that solutions to the monopole equations (of charge k) may also be identified with moduli spaces of rational maps of degree k from $\mathbb{CP}^1 \to \mathbb{CP}^1$.

The correspondence between solutions to the (anti-)self duality equations which are invariant in one or three directions generalises to solutions to the ASD equations invariant under other dual groups of translations. A *Nahm transform* allows one to compare such solutions invariant under dual groups of transformations of a four-manifold (the simplest examples are on $\mathbb{R}^4$, but in principle the construction works for a more general class of four-manifolds, as discussed in the survey [Jar04]).

If Λ denotes the translation group on $\mathbb{R}^4$ which the connection is invariant under, then the case $\Lambda = \mathbb{R}$ and $\Lambda' = \mathbb{R}^3$ corresponds to monopoles and Nahms equations, as mentioned above. The duality for $\Lambda = 0$ is closely related to the ADHM construction (see [DK90]). For $\Lambda = \mathbb{Z}^4$ one obtains a correspondence between instantons on dual four-dimensional tori. The case $\Lambda = \mathbb{Z}$ gives rise to calorons, which correspond to Nahm-type equations on a circle, and the case $\Lambda = \mathbb{Z}^2$ gives a correspondence between periodic instantons and tame solutions of Hitchin's equations on a 2-torus. Finally, the case $\Lambda = \mathbb{R} \times \mathbb{Z}$ gives rise to periodic monopoles considered by Cherkis and Kapustin. The dual in this case are solutions to Hitchin's equations on a cylinder. Again, further discussion can be found in [Jar04].

The correspondence result we will discuss in this chapter is the following.

Theorem 4.1.1 (Hitchin [Hit83], Nakajima [Nak93])**.** *There is a correspondence between*

1. $\mathrm{SU}(2)$ *monopoles* (A, Φ) *on a rank 2 Hermitian vector bundle over* $\mathbb{R}^3$ *such that as* $r := |x| \to \infty$ *we have*

   (a)
   $$|d_A \Phi| = O(r^{-2}),$$

   (b)
   $$\frac{d|\Phi|}{d\Omega} = O(r^{-2}), \text{ and}$$

   (c)
   $$\Phi = \begin{pmatrix} i(1 - \frac{k}{2r}) & 0 \\ 0 & -i(1 - \frac{k}{2r}) \end{pmatrix} + O(r^{-2}),$$

   and,

2. *skew-Hermitian solutions* (T_0, T_1, T_2, T_3) *to the Nahm equations on a rank* k *Hermitian vector bundle over the open interval* $I = (-1, 1)$ *such that*

   (a) *Each* T_i *has at most simple poles at* $t = \pm 1$ *but is otherwise analytic on a neighourhood of* I *in* $\mathbb{C}$*, and*

   (b) *at each pole the residues of the triple* (T_1, T_2, T_3) *define an irreducible representation of* $\mathfrak{su}(2)$.

Remark 4.1.2. We require that the T_i define an irreducible representation (say at $t = 1$) in the sense that if

$$T_i(t) = \frac{a_i}{t - 1} + b_i(t)$$

where b_i is analytic on a neighbourhood of $1 \in \mathbb{C}$ then

$$x_1 e_1 + x_2 e_2 + x_3 e_3 \mapsto -2(x_1 a_1 + x_2 a_2 + x_3 a_3)$$

is a k-dimensional irreducible representation of $\mathfrak{su}(2)$, where

$$e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Note that, to prove this condition, we will only have to prove that this gives an irreducible representation up to a multiplicative constant, since the specific constant of -2 will be forced from the Nahm equations and the commutator relations of the matrices.

Remark 4.1.3. Condition (c) of the magnetic monopole is to be understood in the following way. Regard $\mathbb{R}^3 \setminus 0$ as $S^2 \times (0, \infty)$ by letting $S^2 \times \{r\}$ correspond to the sphere of radius r centred at 0. Let $U = \{[z : 1] \mid z \in \mathbb{C}\} \subset \mathbb{CP}^1 = S^2$, $\phi : U \to \mathbb{C}$ be the usual stereographic projection chart. We obtain a chart on $\mathbb{R}^3 \setminus 0$ by setting $V := U \times (0, \infty)$, $\psi : V \to \mathbb{C} \times \mathbb{R}$, $\psi(x, r) = (\phi(x), r)$. This chart misses out a ray emanating from the origin corresponding to the pole $[1 : 0]$ of S^2. The expression of Φ as a matrix in (c) is to be understood as being with respect to this chart. We found this worth remarking for the following reason. If Φ could be written as in (c) in a *global* gauge, then it would have charge zero as it would define a null homotopic map from the sphere at infinity to the unit sphere of $\mathfrak{su}(2)$ and the magnetic charge is determined by the homotopy type of this map. In order to write Φ as in (c), it is crucial to allow gauge transformations which be singular along the aforementioned ray; the matrix in (c) may seem to extend over this ray, but such is not the case due to Φ transforming under the adjoint representation $\Phi \mapsto g\Phi g^{-1}$, which causes the singularities along the ray to cancel out.

4.1.1 Monopoles to Solutions of Nahm's Equations

Given a monopole (A, Φ), we wish to construct a solution (T_0, T_1, T_2, T_3) of Nahm's equations satisfying the conditions of Theorem 4.1.1.

The strategy is as follows. Using analytical results of Callias, a monopole gives rise to a vector bundle over the interval $(-1, 1)$ defined as the cokernel of a Dirac operator. This vector bundle lies inside a trivial vector bundle of infinite rank, but with a natural inner product. The Nahm data is defined by multiplication by coordinate functions, and orthogonal projection back onto the cokernel. It is essentially formal calculation that these operators satisfy the Nahm equations. The key step is showing that the residues of the Nahm data define an irreducible representation of $\mathfrak{su}(2)$. To do this, one finds an approximate trivialisation of the vector bundle near ± 1 using $H^0(\mathbb{CP}^1, \mathcal{O}(k-1))$, noting that the latter space admits an irreducible representation of $\mathfrak{su}(2)$ of dimension k. Estimates show that the approximation gives a trivialisation as one approaches the ends of the interval, and therefore the residues still define an irreducible representation.

Let $\not{D}_A$ denote the twisted Dirac operator on $\mathbb{R}^3$ obtained from the SU(2) monopole (A, Φ), and define

$$\not{D}_{A,t} := \not{D}_A + (\Phi - it) \colon \Gamma(\mathbb{R}^3, \not{S} \otimes E) \to \Gamma(\mathbb{R}^3, \not{S} \otimes E)$$

where $\not{S}$ is the spinor bundle on $\mathbb{R}^3$ and E is the vector bundle on which A and Φ are defined. Since Φ is skew-Hermitian and $\not{D}_A$ is self-adjoint, we note

$$\not{D}_{A,t}^* = \not{D}_A - (\Phi - it),$$

and hence, combining the Weitzenböck formula with the Bogomolny equations satisfied by (A, Φ), we obtain the following.

Lemma 4.1.4.

$$\not{D}^*_{A,t}\not{D}_{A,t} = \mathbf{1}_S \otimes (\nabla^*_A \nabla_A - (\Phi - it)^2). \qquad \text{(Eq. 4.1)}$$

Proof. Recall the Weitzenböck formula Theorem 1.1.11. In our case we obtain

$$\begin{aligned}
\not{D}^*_{A,t}\not{D}_{A,t} &= \not{D}^2_A + \not{D}_A \circ (\Phi - it) - (\Phi - it) \circ \not{D}_A - (\Phi - it)^2 \\
&= \nabla^*_A \nabla_A + \sum_{i<j} c(e_i)c(e_j)F_A(e_i, e_j) + \not{D}_A(\Phi - it) - (\Phi - it)^2 \\
&= \nabla^*_A \nabla_A + \sum_{i<j} c(e_i)c(e_j)F_A(e_i, e_j) + \sum_{k=1}^{3} c(e_k)(d_A)_{e_k}(\Phi) - (\Phi - it)^2 \\
&= \nabla^*_A \nabla_A + \sum_{i<j} c(e_i)c(e_j)F_A(e_i, e_j) + \sum_{i<j} -c(e_i)c(e_j)F_A(e_i, e_j) - (\Phi - it)^2 \\
&= \nabla^*_A \nabla_A - (\Phi - it)^2.
\end{aligned}$$

Here we have used $\star F_A = d_A(\Phi)$ and the simple computation that if $\star e_i = e_j \wedge e_k$ on $\mathbb{R}^3$ then $c(e_i) = -c(e_j)c(e_k)$ as can easily be verified from the Pauli matrices. □

In [Cal78] Callias proves that the index of $\not{D}_{A,t}$ is $-k$ whenever $t \in (-1, 1)$, where k is the charge the magnetic monopole. Furthermore, the formula above implies $\not{D}_{A,t}$ is a positive operator, and hence has no kernel. Thus, if we define

$$V_t := \ker_{L^2} \not{D}^*_{A,t},$$

then $\dim V_t = k$ for all $t \in (-1, 1)$. This defines a vector bundle of rank k on $(-1, 1)$, which sits as a subbundle of the trivial, infinite rank bundle $\underline{L^2(\mathbb{R}^3, S \otimes E)}$ with fibre $L^2(\mathbb{R}^3, S \otimes E)$.

Define the Nahm data corresponding to (A, Φ) as follows. Let π denote the orthogonal projection onto V_t. Then

$$\nabla_t \psi := \pi\left(\frac{\partial \psi}{\partial t}\right), \qquad T_\alpha(\psi) := \pi(ix_\alpha \psi); \quad \alpha = 1, 2, 3.$$

The matrices T_0, T_1, T_2, T_3 are then defined by taking a trivialisation of V_t over $(-1, 1)$. In particular T_0 is the connection matrix of ∇_t. We remark that this definition makes sense because ψ decays exponentially as $r \to \infty$; this is due to it being a solution to the PDE $\not{D}_{A,t}\not{D}^*_{A,t}\psi = 0$ which is a Schrödinger-type PDE with potential going to zero at infinity since $|\Phi| \to 1$; solutions to such equations always decay exponentially; the reader is referred to [Agm82] for details. As a consequence, $ix_\alpha\psi$ is still square integrable.

Integration by parts shows that T_0 is skew-Hermitian, and this property is also immediate for the T_α. First we will verify that this data satisfies Nahm's equations. That is, that

$$\nabla_t T_1 + [T_2, T_3] = 0,$$

and similarly for the cyclic permutations of (123).

Recall from Remark 3.2.2 that the Nahm equations may be rephrased as a complex and real equation. Namely, if we fix a trivialisation of V so that $\nabla = \frac{d}{dt} + T_0$, and write

$$\alpha := \frac{1}{2}(T_0 + iT_1), \quad \beta := \frac{1}{2}(T_2 + iT_3),$$

then the Nahm equations become equivalent to the pair

$$\begin{aligned} \frac{d\beta}{dt} + 2[\alpha, \beta] = 0 \\ \frac{d(\alpha + \alpha^*)}{dt} + 2([\alpha, \alpha^*] + [\beta, \beta^*]) = 0 \end{aligned}$$

where here $\alpha^* = -T_0 + iT_1$ and similarly for β^*. For our purposes, we need only solve the complex equation, since by relabelling T_1, T_2, and T_3 this implies all three of Nahm's equations.

Theorem 4.1.5. *The data (T_0, T_1, T_2, T_3) defined from a monopole (A, Φ) satisfies the Nahm equations.*

Proof. Given (A, Φ), define an $\mathbb{R}$-invariant instanton on $\mathbb{R}^4$ (invariant in the x^0 direction) by

$$B = \Phi dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3$$

where A_i is the ith component of A on $\mathbb{R}^3$. Then corresponding to B is a twisted Dirac operator $\not{D}_B$ on $\mathbb{R}^4$, and we can further twist this operator by the flat connection $-itdx^0$ to obtain $\not{D}_{B,t}$. Now the spinor bundle on $\mathbb{R}^4$ splits as a sum $\not{S} = \not{S}^+ \oplus \not{S}^-$, and the spinor bundle on $\mathbb{R}^3$ may be identified with either summand. Recall that the Clifford multiplication on $\not{S}$ can be obtained from that on $\mathbb{R}^3$ by taking

$$\tilde{e}_0 = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \tilde{e}_\alpha = \begin{pmatrix} 0 & e_\alpha \\ e_\alpha & 0 \end{pmatrix},$$

where the e_α are the Pauli matrices defining the Clifford multiplication on $\mathbb{R}^3$. Thus we may write

$$\not{D}_{B,t} = \begin{pmatrix} 0 & \not{D}^-_{B,t} \\ \not{D}^+_{B,t} & 0 \end{pmatrix}$$

and notice that (in our sign convention),

$$\not{D}^-_{B,t} = \not{D}^*_{A,t} - \frac{\partial}{\partial x^0}, \quad \not{D}^+_{B,t} = \not{D}_{A,t} + \frac{\partial}{\partial x^0}.$$

There is a natural identification

$$\not{S}^+ \cong \bigwedge^{0,0} \oplus \bigwedge^{0,2}, \quad \not{S}^- \cong \bigwedge^{0,1}$$

on $\mathbb{R}^4 \cong \mathbb{C}^2$, and with respect to this decomposition the Dirac operator becomes the Dolbeault operator on $\mathbb{R}^4$ (see Theorem 1.1.15). Namely we have

$$\begin{aligned}
\not{D}^-_{B,t} &= \sqrt{2}\begin{pmatrix} \overline{\partial}^*_{B,t} \\ \overline{\partial}_{B,t} \end{pmatrix} : \Omega^{0,1}(E) \to \Omega^{0,0}(E) \oplus \Omega^{0,2}(E), \\
\not{D}^+_{B,t} &= \sqrt{2}(\overline{\partial}_{B,t}, \overline{\partial}^*_{B,t})\colon \Omega^{0,0}(E) \oplus \Omega^{0,2}(E) \to \Omega^{0,1}(E).
\end{aligned}$$

Note the $\sqrt{2}$ to agree with the Kähler identity $\Delta_d = 2\Delta_{\overline{\partial}}$ on $\mathbb{R}^4$. Applying these operators to sections which are independent of the x^0 coordinate defines a corresponding decomposition of $\not{D}^*_{A,t}$ and $\not{D}_{A,t}$, whose components we denote, analogously, by $\overline{\partial}_{A,t}$ and $\overline{\partial}^*_{A,t}$.

If we write $\Delta_{A,t}$ for the $\nabla^*_A \nabla_A - (\Phi - it)^2$, it follows from (Eq. 4.1) that

$$\begin{pmatrix} \overline{\partial}^*_{A,t} \\ \overline{\partial}_{A,t} \end{pmatrix} (\overline{\partial}_{A,t}, \overline{\partial}^*_{A,t}) = \begin{pmatrix} \overline{\partial}^*_{A,t}\overline{\partial}_{A,t} & \overline{\partial}^*_{A,t}\overline{\partial}^*_{A,t} \\ \overline{\partial}_{A,t}\overline{\partial}_{A,t} & \overline{\partial}_{A,t}\overline{\partial}^*_{A,t} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \Delta_{A,t} & 0 \\ 0 & \Delta_{A,t} \end{pmatrix}. \qquad \text{(Eq. 4.2)}$$

Recall that since $\Delta_{A,t}$ had no kernel, we have a Green's operator $G := \Delta^{-1}_{A,t}$ on L^2.

Lemma 4.1.6. *The orthogonal projection onto V_t is given by*

$$\pi = \mathbf{1} - \not{D}_{A,t}(\mathbf{1}_S \otimes G)\not{D}^*_{A,t}.$$

Proof. From the expression $\not{D}^*_{A,t}\not{D}_{A,t} = \mathbf{1}_S \otimes \Delta_{A,t}$ one can verify that $\pi^* = \pi, \pi^2 = \pi$, and that $\pi(\psi) \in V_t$ for any $\psi \in L^2(\mathbb{R}^3, S \otimes E)$. Clearly $\pi(\psi) = \psi$ when $\not{D}^*_{A,t}\psi = 0$, so π is the orthogonal projection onto V_t. ■

Finally, note that on $\mathbb{R}^4 = \mathbb{C}^2$ we have the commutators

$$[\overline{\partial}_B, x^0 + ix^1] = [\overline{\partial}_B, x^2 + ix^3] = 0 \qquad \text{(Eq. 4.3)}$$

thinking of $x^0 + ix^1$ and $x^2 + ix^3$ as complex coordinates. If we assume our data is invariant in the x^0 direction and twist by $itdx^0$, then we have the following equalities on $\mathbb{R}^3$.

Lemma 4.1.7. *On $\mathbb{R}^3$, we have commutator relations*

$$[\overline{\partial}_{A,t}, -i\frac{d}{dt} + ix^1] = 0, \tag{Eq. 4.4}$$

$$[\overline{\partial}_{A,t}, x^2 + ix^3] = 0, \tag{Eq. 4.5}$$

where x^j is understood to mean multiplication by the coordinate function x^j. Note that x^0 invariance allows us to write $\overline{\partial}_{A,t}$ on the left (i.e. the commutators vanish on $\mathbb{R}^3$).

Proof. Write $\zeta_1 = x_0 + ix_1$, $\zeta_2 = x_2 + ix_3$. Recall:

$$d_{B,t} = d_B - itdx_0 = d_B - \frac{it}{2}(d\zeta_1 + d\bar{\zeta}_1)$$

Hence:

$$\bar{\partial}_{B,t} = \bar{\partial}_B - \frac{it}{2}d\bar{\zeta}_1$$

Now, $\bar{\partial}_{A,t}$ is defined simply as $\bar{\partial}_{B,t}$ restricted to acting on sections independent of x_0; therefore, what has to be proven is that, if $s \in \Omega^{0,k}(E)$ is independent of x_0, then:

$$\left[\bar{\partial}_{B,t}, -i\frac{d}{dt} + ix_1\right] s = 0, \quad \left[\bar{\partial}_{B,t}, x_2 + ix_3\right] s = 0$$

The second of these is a triviality. Consider the first:

$$\begin{aligned}\left[\bar{\partial}_B - \frac{it}{2}d\bar{\zeta}_1, -i\frac{d}{dt} + ix_1\right] = &\left[\bar{\partial}_B, -i\frac{d}{dt}\right] + \left[\bar{\partial}_B, ix_1\right] \\ &+ \left[-\frac{it}{2}d\bar{\zeta}_1, -i\frac{d}{dt}\right] + \left[-\frac{it}{2}d\bar{\zeta}_1, +ix_1\right]\end{aligned}$$

The first term vanishes as differentiation by t commutes with $\bar{\partial}_B$; the last also obviously vanishes. Now, $ix_1 = 1/2(\zeta_1 - \bar{\zeta}_1)$; hence:

$$\begin{aligned}&= \left[\bar{\partial}_B, \frac{1}{2}(\zeta_1 - \bar{\zeta}_1)\right] + \left[-\frac{it}{2}d\bar{\zeta}_1, -i\frac{d}{dt}\right] \\ &= \frac{1}{2}\bar{\partial}(\zeta_1 - \bar{\zeta}_1) + \frac{1}{2}d\bar{\zeta}_1 \\ &= 0\end{aligned}$$

∎

Lemma 4.1.7 allows us to treat $-i\frac{d}{dt} + ix^1$ like a complex variable on $\mathbb{R}^3$. To that end, define

$$z^1 := -i\frac{d}{dt} + ix^1, \qquad z^2 := x^2 + ix^3$$

on $\mathbb{R}^3$, so (Eq. 4.4) and (Eq. 4.5) become $[\overline{\partial}_{A,t}, z^1] = 0$ and $[\overline{\partial}_{A,t}, z^2] = 0$, respectively. We will use this notation to simplify the rest of the proof.

To complete the proof we combine these previous identities as follows. Let $\psi \in \Gamma((-1,1), V)$ so that $\not{D}^*_{A,t}\psi = 0$. Then by the definition of the operators T_α, α, β, and Lemma 4.1.6, we have formulae

$$iz^1\psi - \left(\frac{d}{dt} + 2\alpha\right)\psi = (\mathbf{1} - \pi)(iz^1\psi) = \not{D}_{A,t}(\mathbf{1}_S \otimes G)\not{D}^*_{A,t}(iz^1\psi), \qquad \text{(Eq. 4.6)}$$

$$iz^2\psi - 2\beta\psi = (\mathbf{1} - \pi)(iz^1\psi) = \not{D}_{A,t}(\mathbf{1}_S \otimes G)\not{D}^*_{A,t}(iz^2\psi). \qquad \text{(Eq. 4.7)}$$

Now if we multiply iz^1 by (Eq. 4.7) and subtract iz^2 multiplied by (Eq. 4.6) we obtain

$$iz^2\left(\frac{d}{dt} + 2\alpha\right)\psi - 2iz^1\beta\psi = 2\overline{\partial}_{A,t}(iz^1 G\overline{\partial}^*_{A,t}(iz^2\psi) - iz^2 G\overline{\partial}^*_{A,t}(iz^1\psi))$$

where on the right we have used the expression

$$\not{D}_{A,t}(\mathbf{1}_S \otimes G)\not{D}^*_{A,t} = 2(\overline{\partial}_{A,t}G\overline{\partial}^*_{A,t} + \overline{\partial}^*_{A,t}G\overline{\partial}_{A,t}),$$

the commutators in Lemma 4.1.7 and the fact that $\not{D}^*_{A,t}\psi = 0$.

Applying the operator π to the right-hand side and using the expressions from (Eq. 4.2) we see that this vanishes. Applying π to the left-hand side and using the definition of the T_α, α, and β, we arrive at

$$\frac{d\beta}{dt} + 2[\alpha, \beta] = 0$$

as desired. □

What remains is to show that the residue of the Nahm data we have defined gives irreducible representations of $\mathfrak{su}(2)$ as one approaches $t \to \pm 1$. We will follow [Nak93] and study just the case $t \to -1$, the $t \to 1$ case being essentially the same argument. The asymptotic behaviour as $t \to -1$ is dictated by the properties of the monopole as $r \to \infty$ in $\mathbb{R}^3$, so it is necessary to study the structure of our Dirac operator $\not{D}_{A,t}$ in this limit.

Recall that a monopole (A, Φ) comes with a magnetic charge $k \in \mathbb{Z}_{\geq 0}$. The asymptotic assumptions on Φ and the Bogomolny equations $F_A = \star d_A\Phi$ imply that "on the sphere at infinity," the monopole (A, Φ) gives a decomposition of the rank 2 hermitian bundle into a direct sum of line bundles with connections. These two line bundles are the $+i$ and $-i$ eigenspaces of Φ at infinity, and the connection matrix splits as

$$A = \begin{pmatrix} A_0^* & 0 \\ 0 & A_0 \end{pmatrix}.$$

for a connection form A_0 on a line bundle over S^2. The magnetic charge is simply the degree of this line bundle, which may be computed by Chern-Weil theory applied to A_0.

More precisely, asymptotically, A can be written as

$$A = \begin{pmatrix} A_0^* & 0 \\ 0 & A_0 \end{pmatrix} + O(r^{-2}),$$

where A_0 is a homogenous connection on a line bundle of degree k over S^2, extended radially to $\mathbb{R}^3 \setminus \{0\}$. Let (r, θ) denote a radial coordinate system on $\mathbb{R}^3 \setminus \{0\}$. Recall that $\mathbb{R}^3 \setminus \{0\}$ is isometric to $S^2 \times (0, \infty)$ with the metric $g = dr^2 + r^2 d\theta^2$ where $d\theta^2$ is the standard metric on S^2. Therefore we are in a similar setting to Proposition 1.1.12: the spinor bundle on $\mathbb{R}^3$ can be identified with the pullback of the spinor bundle of S^2 through the projection. Furthermore, the spinor bundle on S^2 splits as a direct sum $S = S^+ \oplus S^-$, and hence so does the spinor bundle on $S^2 \times (0, \infty) = \mathbb{R}^3 \setminus \{0\}$. We can write the Dirac operator over the spinor bundle using this splitting.

Lemma 4.1.8. *If $\not{D}$ is the Dirac operator on $\mathbb{R}^3 \setminus \{0\}$, then*

$$\not{D}\psi = \begin{pmatrix} i\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) & \frac{1}{r}\not{D}^- \\ \frac{1}{r}\not{D}^+ & -i\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) \end{pmatrix} \psi$$

where $\not{D}^\pm$ denote the Dirac operators on S^2.

Proof. This is obtained simply by applying Corollary 1.1.13, setting $n = 1$ and $e^\varphi = r$. □

We can twist the Dirac operators $\not{D}^\pm$ by A_0 or A_0^*, and consequently in our gauge we get

$$\not{D}_{A,t}^* = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} + O(r^{-2}) \qquad \text{(Eq. 4.8)}$$

where

$$B_1 = \begin{pmatrix} i\left(\frac{\partial}{\partial r} + t - 1 + \frac{k+2}{2r}\right) & \frac{1}{r}\not{D}_{A_0^*}^- \\ \frac{1}{r}\not{D}_{A_0^*}^+ & -i\left(\frac{\partial}{\partial r} - t + 1 - \frac{k-2}{2r}\right) \end{pmatrix},$$
$$B_2 = \begin{pmatrix} i\left(\frac{\partial}{\partial r} + t + 1 - \frac{k-2}{2r}\right) & \frac{1}{r}\not{D}_{A_0}^- \\ \frac{1}{r}\not{D}_{A_0}^+ & -i\left(\frac{\partial}{\partial r} - t - 1 + \frac{k+2}{2r}\right) \end{pmatrix}.$$

Recall that for the Kähler manifold $\mathbb{CP}^1$, we have isomorphisms

$$S^+ = \bigwedge\nolimits^{0,0} \otimes H^* \cong \mathcal{O}(-1), \quad S^- = \bigwedge\nolimits^{0,1} \otimes H^* \cong \mathcal{O}(1),$$

where $H^* = \mathcal{O}(-1)$ is the square root of the canonical bundle $K = \mathcal{O}(-2)$ (see Theorem 1.1.14). Furthermore we had a decomposition $E \cong \mathcal{O}(-k) \oplus \mathcal{O}(k)$ (for sufficiently big spheres). Combining these we obtain

$$S \otimes E \cong \mathcal{O}(-k-1) \oplus \mathcal{O}(-k+1) \oplus \mathcal{O}(k-1) \oplus \mathcal{O}(k+1).$$

Now let $R > 0$ be fixed and big enough, and let χ be a smooth cut off function which vanishes on $[0, R]$ and is identically 1 on $[R+1, \infty)$. Then, given some $f \in H^0(\mathbb{CP}^1, \mathcal{O}(k-1))$, we may specify a section of $S \otimes E$ by

$$\psi := (0, 0, \chi(r)e^{-(t+1)r} r^{\frac{k-2}{2}} f(\theta), 0), \qquad \text{(Eq. 4.9)}$$

where f depends only on the angular variable θ.

Since the Dirac operator on $S^2 \cong \mathbb{CP}^1$ is the Dolbeault operator, and furthermore since A_0 was the homogenous connection on $\mathcal{O}(k) \to \mathbb{CP}^1$, we have $\not{D}^+_{A_0} = \overline{\partial}_{\mathcal{O}(k-1)}$, so that this operator kills f. This implies that the first term in (Eq. 4.8) applied to ψ disappears, so

$$|\not{D}^*_{A,t}\psi| \leq Ce^{-(t+1)r}(r+1)^{\frac{k-2}{2}-2}, \qquad \text{(Eq. 4.10)}$$

for some constant C which does not depend on t. Thus ψ is approximately a solution to $\not{D}^*_{A,t}\psi = 0$. We will search for an actual solution by looking at a variation of the kind

$$\psi - \not{D}_{A,t}\varphi,$$

where φ is a solution to

$$\not{D}^*_{A,t}\not{D}_{A,t}\varphi = \not{D}^*_{A,t}\psi.$$

This elliptic equation for φ has a unique solution, which is the minimum of the functional

$$S(\varphi) = \|\nabla_A\varphi\|^2_{L^2} + \|(\Phi - it)\varphi\|^2_{L^2} - 2\langle\varphi, \not{D}^*_{A,t}\psi\rangle_{L^2}.$$

See [JT80, Prop. IV.4.1] for more details. This functional has a unique minimum which satisfies

$$S(\varphi) \leq S(0) = 0$$

giving the inequality

$$\|\not{D}_{A,t}\varphi\|^2_{L^2} = \|\nabla_A\varphi\|^2_{L^2} + \|(\Phi - it)\varphi\|^2_{L^2} \leq 2\langle\varphi, \not{D}^*_{A,t}\psi\rangle_{L^2}.$$

Note that we cannot conclude a similar equality here using integration by parts since we don't know the asymptotic behaviour φ.

For $R \gg 0$ and t near -1, the eigenvalue of $\Phi - it$ providing the bundle $\mathcal{O}(k)$ approaches $i(1+t)$, so we have an estimate

$$(1+t)|\varphi| \leq 2|(\Phi - it)\varphi|$$

on the complement of a ball $\mathbb{R}^3 \backslash B_{\frac{R}{1+t}}$. Thus over this complement we obtain

$$(1+t)^2 \int_{\mathbb{R}^3 \backslash B_{\frac{R}{1+t}}} |\varphi|^2 dx \leq 4\|(\Phi - it)\varphi\|_{L^2}^2.$$

In what follows, C_i are used to denote constants independent of t. Inside the ball $B_{\frac{R}{1+t}}$, we have the estimate

$$(1+t)^2 \int_{B_{\frac{R}{1+t}}} |\varphi|^2 dx \leq C_1 \|\varphi\|_{L^6(B_{R/(t+1)})}^2$$

from Hölder's inequality applied to $f = |\varphi|^2$ and $g = (t+1)^2$ with $p = 3$. The Gagliardo-Nirenberg-Sobolev inequality on $B_{R/(t+1)}$ gives

$$\|\varphi\|_{L^6(B_{R/(t+1)})}^2 \leq C_2 \|d|\varphi|\|_{L^2(B_{R/(t+1)})}^2,$$

and Kato's inequality gives

$$\|d|\varphi|\|_{L^2(B_{R/(t+1)})}^2 \leq C_3 \|\nabla_A \varphi\|_{L^2(B_{R/(t+1)})}^2 \leq C_3 \|\nabla_A \varphi\|_{L^2(\mathbb{R}^3)}^2.$$

Putting these together, we obtain

$$\begin{aligned} \|\not{D}_{A,t}\varphi\|_{L^2}^2 &\leq 2\langle \varphi, \not{D}_{A,t}^* \psi \rangle_{L^2} \\ &\leq 2\|\varphi\|_{L^2} \|\not{D}_{A,t}^* \psi\|_{L^2} \\ &\leq \frac{C_4}{1+t} \|\not{D}_{A,t}\varphi\|_{L^2} \|\not{D}_{A,t}^* \psi\|_{L^2}, \end{aligned}$$

and in particular

$$\|\not{D}_{A,t}\varphi\|_{L^2} \leq \frac{C_5}{1+t} \|\not{D}_{A,t}^* \psi\|_{L^2}. \qquad \text{(Eq. 4.11)}$$

Meanwhile, we have:

$$\begin{aligned} \|\psi\|_{L^2}^2 &\geq C_6 \int_{R+1}^{\infty} e^{-2(t+1)r} r^{k-2} dr \\ &= \frac{C_6}{(2(t+1))^{k-2}} \int_{2(R+1)(t+1)}^{\infty} e^{-r} r^{k-2} dr \end{aligned}$$

Whereas, from (Eq. 4.10), we have:

$$\begin{aligned} \|\not{D}_{A,t}^* \psi\|_{L^2}^2 &\leq C_7 \int_{R+1}^{\infty} e^{-2(t+1)r} r^{k-6} dr \\ &= \frac{C_7}{(2(t+1))^{k-6}} \int_{2(R+1)(t+1)}^{\infty} e^{-r} r^{k-6} dr \end{aligned}$$

Note that, as $t \to -1$:

$$\int_{2(R+1)(t+1)}^{\infty} e^{-r} r^{k-\ell} dr \longrightarrow \Gamma(k-\ell+1)$$

Therefore, it follows that:

$$\|\psi\|_{L^2} \geq \frac{C_8}{(t+1)^{(k-2)/2}}, \quad \|\not{D}^*_{A,t}\psi\|^2_{L^2} \leq \frac{C_9}{(t+1)^{(k-6)/2}},$$

Whence one infers that:

$$\|\not{D}^*_{A,t}\psi\|_{L^2} \leq C_{10}(1+t)^2\|\psi\|_{L^2}.$$

Combining this with (Eq. 4.11), we obtain:

$$\|\not{D}_{A,t}\varphi\|_{L^2} \leq C_{11}(t+1)\|\psi\|_{L^2}$$

Whereby, we see that the correction term $\not{D}_{A,t}\varphi$ is small relative to ψ when t is near -1. In particular, if we choose an orthonormal basis of $H^0(\mathbb{CP}^1, \mathcal{O}(k-1))$, then the correction terms are small enough that these vectors remain a basis when we pass to $V \to (-1, 1)$, and, therefore, give a trivialisation near $t = -1$. Furthermore, since, in the limit $t \to -1$, the correction term gets arbitrarily small, we should expect that the trivialisation of V act exactly like $H^0(\mathbb{CP}^1, \mathcal{O}(k-1))$. This is promising as the latter space gives an irreducible representation of $\mathfrak{su}(2)$. To make this last statement precise, we prove the following lemma.

Lemma 4.1.9. *Define endomorphisms a_α of $H^0(\mathbb{CP}^1, \mathcal{O}(k-1))$ by*

$$\langle a_\alpha f_1, f_2\rangle_{L^2} = \int_{\mathbb{CP}^1} \langle i x_\alpha f_1, f_2\rangle$$

for $\alpha = 1, 2, 3$ and $f_1, f_2 \in H^0(\mathbb{CP}^1, \mathcal{O}(k-1))$. Then a_α gives the residue of the simple pole of T_α at $t = -1$, up to a multiplicative constant.

Proof. Recall that the operator T_α is defined by $T_\alpha\psi = \pi(ix_\alpha\psi)$ where π is the orthogonal projection onto V_t. In the following we will ignore correction terms $\not{D}_{A,t}\varphi$, since in the limit as $t \to -1$ these terms become insignificant.

In order to find the residue, we would be interested in looking at the expression $(t+1)\pi(ix_\alpha\psi_1)$ as $t \to -1$. Now, suppose that ψ_1 is constructed near $t = -1$ from some $f_1 \in H^0(\mathbb{CP}^1, \mathcal{O}(k-1))$ by the expression (Eq. 4.9). If we let ψ_1' be constructed by the same expression from the function $a_\alpha f_1 \in H^0(\mathbb{CP}^1, \mathcal{O}(k-1))$, then we want to check that these expressions tend to the same limit as $t \to -1$. In order to avoid looking at the projection map, we can simply look at their product with elements in V_t. In other words, if ψ_2 is constructed from $f_2 \in H^0(\mathbb{CP}^1, \mathcal{O}(k-1))$, we want to prove that

$$\lim_{t\to -1}\langle \psi_1', \psi_2\rangle_{L^2} = K \cdot \lim_{t\to -1}\langle (t+1)ix_\alpha\psi_1, \psi_2\rangle_{L^2},$$

for some constant K.

The first product will be

$$\begin{aligned}
\langle \psi_1', \psi_2\rangle_{L^2} &= \int_{\mathbb{R}^3} \chi(r)^2 e^{-2(1+t)r} r^{k-2}\langle f_1', f_2\rangle_{S\otimes E} dvol \\
&= \int_0^\infty \chi(r)^2 e^{-2(1+t)r} r^{k-2}\langle f_1', f_2\rangle_{L^2} r^2 dr \\
&= \langle a_\alpha f_1, f_2\rangle_{L^2}\int_0^\infty \chi(r)^2 e^{-2(1+t)r} r^k dr.
\end{aligned}$$

The second one will be

$$\begin{aligned}
\langle (t+1)ix_\alpha\psi_1, \psi_2\rangle_{L^2} &= \int_{\mathbb{R}^3} \chi(r)^2 e^{-2(1+t)r} r^{k-2}\langle (t+1)ix_\alpha f_1, f_2\rangle_{S\otimes E} dvol \\
&= \int_{\mathbb{R}^3} \chi(r)^2 e^{-2(1+t)r} r^{k-2} r\langle (t+1)i\frac{x_\alpha}{r} f_1, f_2\rangle_{S\otimes E} dvol \\
&= \int_0^\infty \chi(r)^2 e^{-2(1+t)r} r^{k-1} r\langle (t+1)ix_\alpha f_1, f_2\rangle_{L^2} r^2 dr \\
&= \langle ix_\alpha f_1, f_2\rangle_{L^2}\int_0^\infty \chi(r)^2 (t+1) e^{-2(1+t)r} r^{k+1} dr.
\end{aligned}$$

We know that $\langle ix_\alpha f_1, f_2\rangle_{L^2} = \langle a_\alpha f_1, f_2\rangle_{L^2}$ by the definition of a_α, and a computation will show that the integrals of both expressions differ by a positive constant in the limit (they both tend to infinity in the limit, but their quotient tends to a constant, which is the same as saying that limits differ by a multiplicative constant after normalising appropriately). In the limit, the cut off function will become irrelevant, since the integrals of r^{k-1} and r^k diverge, so the exponential $e^{-(t+1)r}$ will allow an arbitrarily big proportion of the integrals to be away far from 0, for t close enough to -1. Then, result follows from the fact that

$$\int_0^\infty e^{-2(1+t)r} r^k dr = \frac{k!}{2^{k+1}(t+1)^{k+1}}$$

and

$$\int_0^\infty (t+1)e^{-2(1+t)r} r^{k+1} dr = \frac{(k-1)!}{2^k(t+1)^{k+1}},$$

which differ by the multiplicative constant $\frac{k}{2}$ (which, of course, doesn't depend on t).

$\square$

Furthermore, we have the following result.

Theorem 4.1.10. *The map*

$$x_1e_1 + x_2e_2 + x_3e_3 \mapsto x_1a_1 + x_2a_2 + x_3a_3$$

gives an irreducible representation of $\mathfrak{su}(2)$ *of dimension* k *(modulo a multiplicative constant).*

Proof. Recall the representation theory of $\mathfrak{su}(2)$. There is a unique irreducible (complex) representation V_k of $\mathbb{C}$-dimension $k+1$ for each $k \in \mathbb{N}$. This representation is naturally identified with the space of order k homogeneous complex polynomials in two variables. Consider the line bundle $\mathcal{O}(k) \to \mathbb{CP}^1$. The fibre of $\mathcal{O}(k)$ above a point $p \in \mathbb{CP}^1$ is the space of degree k maps $p \to \mathbb{C}$ where we view $p \subset \mathbb{C}^2$ as a line. If z_1, z_2 denote the coordinates of $\mathbb{C}^2$, then, the holomorphic sections of $\mathcal{O}(k)$ are given by restricting the degree k homogeneous polynomials in the variables z_1, z_2; hence, it is clear that $\mathrm{H}^0\big(\mathbb{CP}^1; \mathcal{O}(k)\big)$ is naturally isomorphic to V_k and we have an action of $\mathfrak{su}(2)$ on it. More generally, given any section $s \in \Gamma(\mathcal{O}(k))$ (not necessarily holomorphic), we view it as a map $\mathbb{C}^2 \to \mathbb{C}$ satisfying $s(tz) = t^k s(z)$ for all $z \in \mathbb{C}^2$; in which case, the action of $X \in \mathfrak{su}(2)$ is defined by:

$$(X \cdot s)(z) = -s(Xz)$$

Thus, we $\mathrm{L}^2\big(\mathbb{CP}^1; \mathcal{O}(k)\big)$ also inherits an $\mathfrak{su}(2)$ action. Naturally, we have the holomorphic sections as a subrepresentation:

$$V_k \cong \mathrm{H}^0\big(\mathbb{CP}^1; \mathcal{O}(k)\big) \hookrightarrow \mathrm{L}^2\big(\mathbb{CP}^1; \mathcal{O}(k)\big)$$

Now, consider the coordinate functions x_1, x_2, x_3 of $\mathbb{R}^3$. We have $\mathbb{CP}^1 \hookrightarrow \mathbb{R}^3$; therefore, x_j can be thought of as a smooth function on $\mathbb{CP}^1$. There also is a natural action of $\mathfrak{su}(2)$ on $\mathrm{span}_{\mathbb{C}}\{x_1, x_2, x_3\} \subset \Omega^0(\mathbb{CP}^1)$ by identifying it with $\mathfrak{su}(2)$ itself via

$$\sum_j \alpha_j x_j \mapsto \sum_j \alpha_j e_j$$

and acting via the adjoint representation. This representation is V_2. Define the map

$$m : \mathrm{L}^2\big(\mathbb{CP}^1; \mathcal{O}(k)\big) \otimes V_2 \to \mathrm{L}^2\big(\mathbb{CP}^1; \mathcal{O}(k)\big),$$

which simply multiplies by the coordinate functions; that is $m(s, x_j) = x_j s$. One can check that m is $\mathfrak{su}(2)$-equivariant. Let

$$\pi : \mathrm{L}^2\big(\mathbb{CP}^1; \mathcal{O}(k)\big) \to V_k$$

be the L^2 orthogonal projection onto the holomorphic sections; this is also $\mathfrak{su}(2)$-equivariant. In summary, we have the $\mathfrak{su}(2)$-equivariant composite:

$$\phi := V_k \otimes V_2 \hookrightarrow \mathrm{L}^2\big(\mathbb{CP}^1; \mathcal{O}(k)\big) \otimes V_2 \xrightarrow{m} \mathrm{L}^2\big(\mathbb{CP}^1; \mathcal{O}(k)\big) \xrightarrow{\pi} V_k$$

Now, by decomposing into irreducibles, one finds that $V_k \otimes V_2 \cong V_{k+2} \oplus V_k \oplus V_{k-2}$; thus, equivariance of ϕ implies that it must be a scalar multiple of the projection $\mathrm{pr}_2 : V_{k+2} \oplus V_k \oplus V_{k-2} \to V_k$ onto the V_k summand; that is $\phi = \alpha \mathrm{pr}_2$ for some $\alpha \in \mathbb{C}$. We claim that $\alpha \neq 0$. To show this, it suffices to find an element outside its kernel; consider the image of $z_1^{k-1} z_2 \otimes x_1 \in V_k \otimes V_2$ under ϕ; note that:

$$\langle x_1 z_1^{k-1} z_2, z_1^{k-1} z_2 \rangle_{\mathrm{L}^2} = \frac{i}{2} \int_{\mathbb{C}} \frac{2z \operatorname{Re} z}{\big(1+|z|^2\big)^{1+k}} dz \wedge d\bar{z} \neq 0$$

So the projection of $x_1 z_1^{k-1} z_2$ onto $z_1^{k-1} z_2$ is nonzero; which implies that $\phi(z_1^{k-1} z_2 \otimes x_1) \neq 0$ as claimed. To finish off, notice that the projection $V_k \otimes V_2 \to V_k$ is familiar; observe the map:

$$V_k \otimes V_2 = V_k \otimes \mathfrak{su}(2) \to V_k, \quad v \otimes X = X \cdot v$$

Where X acts on V_k as it is an $\mathfrak{su}(2)$ representation to begin with. This also is equivariant and non zero; therefore, it is a multiple of the projection as well; therefore, ϕ is a (non-zero) scalar multiple of the unique $k+1$ dimensional irreducible representation. $\square$

Combining these results, we can conclude that, indeed, the residues of the Nahm data give an irreducible representation.

4.1.2 Solutions of Nahm's Equations to Monopoles

We now shall present the opposite direction: given Nahm data $T_0, T_1, T_2, T_3 : (-1,1) \to \mathfrak{su}(k)$ satisfying the conditions in Theorem 4.1.1, we construct an $SU(2)$ monopole of charge k.

Consider the following two spaces: $\mathrm{L}^2\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big)$ consisting of sections of $V \otimes \underline{\mathbb{C}^2}$ where $\underline{\mathbb{C}^2}$ denotes the trivial bundle, and the Sobolev space $\mathcal{H}_0^1\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big)$ of sections of $V \otimes \underline{\mathbb{C}^2}$ which vanish at the endpoints ± 1 of the interval and whose derivatives are in L^2. Fix some $x \in \mathbb{R}^3$ and define an operator:

$$\mathfrak{D}_x : \mathcal{H}_0^1\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big) \to \mathrm{L}^2\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big),$$
$$\mathfrak{D}_x = \mathbf{1}_{\mathbb{C}^2} \otimes \nabla_t + \sum_{i=1}^{3} (e_i \otimes T_i - i x_i e_i \otimes \mathbf{1}_V)$$

Using the complex notation as in Section 4.1.1, this can be written as:

$$\mathfrak{D}_x = \begin{pmatrix} \frac{d}{dt} + 2\alpha & 2\beta^* \\ 2\beta & \frac{d}{dt} - 2\alpha^* \end{pmatrix} - \begin{pmatrix} -x_1 & -ix_2 - x_3 \\ ix_2 - x_3 & x_1 \end{pmatrix}$$

where the matrix notation reflects the $\underline{\mathbb{C}}^2$ factor. Likewise, the format adjoint has the form:

$$\mathfrak{D}_x^* = \begin{pmatrix} -\frac{d}{dt} + 2\alpha^* & 2\beta^* \\ 2\beta & -\frac{d}{dt} - 2\alpha \end{pmatrix} - \begin{pmatrix} -x_1 & -ix_2 - x_3 \\ ix_2 - x_3 & x_1 \end{pmatrix}$$

Now, using Nahm's equations, it immediately follows that:

$$\mathfrak{D}_x^*\mathfrak{D}_x = \mathbf{1}_{\mathbb{C}^2} \otimes \left(\nabla_t^* \nabla_t + \sum_{i=1}^{3} (T_i - ix_i)^*(T_i - ix_i) \right) \qquad \text{(Eq. 4.12)}$$

Whence it follows that $\mathfrak{D}_x$ is a positive operator and $\mathrm{Ker}(\mathfrak{D}_x) = 0$. We now proceed to determine the index of $\mathfrak{D}_x$; to do this, we shall express it in terms of a simpler operator.

Firstly, let a_1, a_2, a_3 denote the residues of T_1, T_2, T_3 respectively at $t = -1$; these are assumed to define an irreducible representation of $\mathfrak{su}(2)$ via:

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \mapsto -2(\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3)$$

this is a representation of dimension k. Recall the representation theory of $\mathfrak{su}(2)$; the representations are classified by their complex dimension; verily, the representation of dimension k, is simply the representation on the space of homogeneous polynomials of degree $k-1$ on two variables; denote this representation by $\rho_{k-1} : \mathfrak{su}(2) \to V_{k-1}$. Hence, the representation defined by the residues of T_i is V_{k-1}. Meanwhile, the residue of $\sum_{i=1}^3 e_i \otimes T_i$ at $t = -1$, i.e. $\sum_{i=1}^3 e_i \otimes a_i$, acts on the tensor product of representations $V_1 \otimes V_{k-1}$. Reducing this as a direct sum of irreducibles, gives $V_1 \otimes V_{k-1} = V_k \oplus V_{k-2}$. Now, using $C(V)$ to denote the Casimir operator of a representation V, it follows that:

$$C(V_{k-1} \otimes V_1) = C(V_{k-1}) \otimes \mathbf{1} + 2\sum_{i=1}^{3} \rho_{k-1}(e_i) \otimes e_i + \mathbf{1} \otimes C(V_1)$$

moreover, the Casimir operator of the representation V_k is given by scalar multiplication by $-k(k+2)$. So $a := \sum_{i=1}^3 e_i \otimes a_i$ may be expressed, on each of the direct summands, in terms of these Casimir operators:

$$\begin{aligned} a|_{V_k \oplus 0} &= \frac{1}{4}(-(k-1)(k+1) - 3 + k(k+2)) = \frac{1}{2}(k-1), \\ a|_{0 \oplus V_{k-2}} &= \frac{1}{4}(-(k-1)(k+1) - 3 + (k-2)k) = -\frac{1}{2}(k+1) \end{aligned}$$

Define a new operator:

$$\widetilde{\mathfrak{D}} : \mathcal{H}^1_0\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big) \to \mathrm{L}^2\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big)$$
$$\widetilde{\mathfrak{D}} = \mathbf{1}_{\mathbb{C}^2} \otimes \frac{d}{dt} + \left(\frac{1}{t-1} + \frac{1}{t+1}\right) a$$

The index of $\widetilde{\mathfrak{D}}$ can easily be determined.

Proposition 4.1.11. *The index of $\widetilde{\mathfrak{D}}$ is -2.*

Proof. On $V_k \oplus 0$, solving $\widetilde{\mathfrak{D}}u = 0$ amounts to solving the ODE:

$$\left(\frac{d}{dt} + \frac{1}{2}\left(\frac{1}{t-1} + \frac{1}{t+1}\right)(k-1)\right) u = 0$$

If one care to check, all solutions are of the form:

$$u_0\big((t+1)(t-1)\big)^{-(k-1)/2}$$

where $u_0 \in V_k \otimes 0$ is a constant. None of these are in $\mathcal{H}^1_0\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big)$ as they do not vanish at the end points. On the other hand, on $0 \oplus V_{k-2}$, one solves the ODE:

$$\left(\frac{d}{dt} - \frac{1}{2}\left(\frac{1}{t-1} + \frac{1}{t+1}\right)(k+1)\right) u = 0$$

finding that all solutions are of the form:

$$u_0\big((t+1)(t-1)\big)^{(k+1)/2}$$

and these do lie in $\mathcal{H}^1_0\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big)$, leading to a space of solutions of dimension $\dim_{\mathbb{C}} V_{k-2} = k-1$; hence, $\dim_{\mathbb{C}} \mathrm{Ker}(\widetilde{\mathfrak{D}}) = k-1$. Meanwhile, the adjoint operator is:

$$\widetilde{\mathfrak{D}}^* : \mathrm{L}^2\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big) \to \mathcal{H}^1_0\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big)$$
$$\widetilde{\mathfrak{D}}^* = -\mathbf{1}_{\mathbb{C}^2} \otimes \frac{d}{dt} + \left(\frac{1}{t-1} + \frac{1}{t+1}\right) a$$

Applying the same reasoning as for $\widetilde{\mathfrak{D}}$ now yields the same results but in reverse; that is one gets a space of dimension $k+1$ of solutions of the form:

$$u_0\big((t+1)(t-1)\big)^{+(k-1)/2}$$

and a space of dimension $k-1$ of solutions of the form:

$$u_0\big((t+1)(t-1)\big)^{-(k+1)/2}$$

If one care to check, only the latter lie in $\mathrm{L}^2\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big)$. All in all, the index is $(k-1) - (k+1) = -2$. $\square$

Using this, we can now deduce the index of the original operator.

Proposition 4.1.12. *The index of $\mathfrak{D}_x$ is the same as the index of $\widetilde{\mathfrak{D}}$.*

Proof. By Schur's lemma, there exists some $g \in U(k)$ such that:

$$\operatorname*{Res}_{t=-1} T_i = \operatorname*{Res}_{t=1} gT_i g^{-1}$$

Furthermore, one can find some skew-adjoint X such that $g = e^{2X}$; hence, one can write:

$$\mathfrak{D}_x = e^{-(t+1)X}\widetilde{\mathfrak{D}}e^{(t+1)X} + K$$

where K is some section of the endomorphism bundle. Note that, due to the conditions imposed on the T_i, K must be analytic on some neighbourhood of $[-1,1]$; this is key, for it implies that it is C^∞ and its all its derivatives are bounded on $[-1,1]$; whence it follows that it defines a *compact* operator $\mathcal{H}_0^1\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big) \to \mathrm{L}^2\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big)$. Recall that the Fredholm index is not affected by adding a compact operator; hence, the index of $\mathfrak{D}_x$ coincides with that of $e^{-(t+1)X}\widetilde{\mathfrak{D}}e^{(t+1)X}$. Finally, notice that $e^{\pm(t+1)X}$ is invertible; so, verily, the index of $\mathfrak{D}_x$ is the same as that of $\widetilde{\mathfrak{D}}_x$. $\square$

One concludes that $\dim_{\mathbb{C}} \operatorname{Ker}(\mathfrak{D}_x^*) = 2$ and proceeds in a reverse fashion to Section 4.1.1. Define a vector bundle $E \to \mathbb{R}^3$ of rank 2 whose fibre over x is $\operatorname{Ker}(\mathfrak{D}_x^*)$; this is a subbundle of the trivial bundle over $\mathbb{R}^3$ with fibre $\mathrm{L}^2\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big)$; the L^2 inner product on $\mathrm{L}^2\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big)$ defines a hermitian structure on E; a connexion and Higgs bundle are also inherited by setting:

$$d_A := \pi \circ d, \quad \Phi = \pi \circ (u \mapsto itu)$$

Where d denotes the distinguished product connexion on $\mathbb{R}^3 \times \mathrm{L}^2\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big)$ and $\pi : \mathrm{L}^2\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big) \to \operatorname{Ker}(\mathfrak{D}_x^*)$ is the L^2 orthogonal projection.

Proposition 4.1.13. *The pair (A, Φ) satisfies the $SU(2)$ Bogomolny equation.*

Proof. Let $\Delta_x : W_0^{1,2}(I;V) \to W_0^{1,2}(I;V)^*$ be $\Delta_x := \nabla_t^*\nabla_t + \sum_{i=1}^3 (T_i - ix_i)^*(T_i - ix_i)$. Hence, similarly to 4.1.6, one can show that:

$$\pi = \mathbf{1} - \mathfrak{D}_x(\mathbf{1}_{\mathbb{C}^2} \otimes {\Delta_x}^{-1})\mathfrak{D}_x^*$$

Now, write:

$$\sigma_x = \begin{pmatrix} \frac{d}{dt} + 2\alpha + x_1 \\ 2\beta - ix_2 + x_3 \end{pmatrix}, \quad \tau_x = \begin{pmatrix} 2\beta - ix_2 + x_3 & -\frac{d}{dt} - 2\alpha - x_1 \end{pmatrix}$$

So as to be able to decompose:

$$\mathfrak{D}_x = \begin{pmatrix} \sigma_x & \tau_x^* \end{pmatrix}, \quad \mathfrak{D}_x^* = \begin{pmatrix} \sigma_x^* \\ \tau_x \end{pmatrix}$$

Whence it follows that:

$$\mathfrak{D}_x(\mathbf{1}_{\mathbb{C}^2} \otimes F_x)\mathfrak{D}_x^* = \sigma_x F_x \sigma_x^* + \tau_x^* F_x \tau_x$$

Further, due to equation (Eq. 4.12), one has:

$$\mathfrak{D}_x^*\mathfrak{D}_x = \begin{pmatrix} \sigma_x^*\sigma_x & \sigma_x^*\tau_x \\ \tau_x\sigma_x^* & \tau_x\tau_x^* \end{pmatrix} = \begin{pmatrix} \Delta_x & 0 \\ 0 & \Delta_x \end{pmatrix} \tag{Eq. 4.13}$$

It turns out, as in Section 4.1.1, that it is easier to work over $\mathbb{R}^4$ in order to be able to use a complex structure; hence, write $B = \Phi dx_0 + A_1 dx_1 + A_2 dx_2 + A_3 dx_3$; thus, what must be shown is that the connexion B is ASD. For that end, let $L \to \mathbb{R}^4$ denote the trivial vector bundle with fibre the Hilbert space $\mathrm{L}^2\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big)$; then, define a Dirac operator on this bundle:

$$D\!\!\!\!/'^{\pm} : \Gamma(S^{\pm} \otimes L) \to \Gamma(S^{\mp} \otimes L)$$

$$D\!\!\!\!/'^{\pm} = D\!\!\!\!/^{\pm} \pm it\tilde{e}_0$$

Where $D\!\!\!\!/$ is the usual Dirac operator with respect to the product connexion on L and t is to be understood as sending a section $f \in \mathrm{L}^2\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big)$ to $t \mapsto tf(t)$. In line with Section 4.1.1, write:

$$D\!\!\!\!/'^{+} = \sqrt{2}\begin{pmatrix} \bar{\partial} & \bar{\partial}^* \end{pmatrix}, \quad D\!\!\!\!/'^{-} = \sqrt{2}\begin{pmatrix} \bar{\partial}^* \\ \bar{\partial} \end{pmatrix}$$

One can see easily that $\bar{\partial}_B = \pi \circ \bar{\partial}$. Now, the operators σ_x and τ_x define operators σ and τ on the relevant vector bundles over $\mathbb{R}^4$ by letting x vary. Then, using juxtaposition to denote composition, observe:

$$\begin{aligned}
\bar{\partial}_B^2 =& \pi\bar{\partial}\pi\bar{\partial} \\
=&(1 - \sigma\Delta^{-1}\sigma^* - \tau^*\Delta^{-1}\tau)\bar{\partial}(1 - \sigma\Delta^{-1}\sigma^* - \tau^*\Delta^{-1}\tau)\bar{\partial} \\
=& -\bar{\partial}\sigma\Delta^{-1}\sigma^*\bar{\partial} - \bar{\partial}\tau^*\Delta^{-1}\tau\bar{\partial} \\
&+ \sigma\Delta^{-1}\sigma^*\bar{\partial}\sigma\Delta^{-1}\sigma^*\bar{\partial} + \sigma\Delta^{-1}\sigma^*\bar{\partial}\tau^*\Delta^{-1}\tau\bar{\partial} \\
&+ \tau^*\Delta^{-1}\tau\bar{\partial}\sigma\Delta^{-1}\sigma^*\bar{\partial} + \tau^*\Delta^{-1}\tau\bar{\partial}\tau^*\Delta^{-1}\tau\bar{\partial}
\end{aligned}$$

By a direct calculation, one can verify that $[\bar{\partial}, \sigma] = 0$ and $[\bar{\partial}, \tau] = 0$. Using these facts, together with $\bar{\partial}^2 = 0$ and the relations deduced from equation (Eq. 4.13), one can simplify:

$$
\begin{aligned}
=& - \bar{\partial}\sigma\Delta^{-1}\sigma^*\bar{\partial} - \bar{\partial}\tau^*\Delta^{-1}\tau\bar{\partial} \\
&+ \sigma\Delta^{-1}\sigma^*\sigma\bar{\partial}\Delta^{-1}\sigma^*\bar{\partial} + \sigma\Delta^{-1}\sigma^*\bar{\partial}\tau^*\Delta^{-1}\tau\bar{\partial} \\
&+ \tau^*\Delta^{-1}\tau\sigma\bar{\partial}\Delta^{-1}\sigma^*\bar{\partial} + \tau^*\Delta^{-1}\bar{\partial}\tau\tau^*\Delta^{-1}\tau\bar{\partial} \\
=& - \bar{\partial}\tau^*\Delta^{-1}\tau\bar{\partial} + \sigma\Delta^{-1}\sigma^*\bar{\partial}\tau^*\Delta^{-1}\tau\bar{\partial} \\
=& - (1 - \sigma^*\Delta^{-1}\sigma)\bar{\partial}\tau^*\Delta^{-1}\tau\bar{\partial} \\
=&(1 - \sigma^*\Delta^{-1}\sigma)\bar{\partial}(1 - \tau^*\Delta^{-1}\tau)\bar{\partial}
\end{aligned}
$$

Now, note that the expression $1 - \tau^*\Delta^{-1}\tau$ is, in fact, the orthogonal projection onto the kernel of τ; since $[\bar{\partial}, \tau] = 0$, $\bar{\partial}$ also commutes with this projection and one concludes:

$$\bar{\partial}_B^2 = 0$$

Which is to say that ∂_B defines a holomorphic structure on E no matter the complex structure chosen for $\mathbb{R}^4$. By varying the complex structure on $\mathbb{R}^4$, it follows that B is ASD. $\square$

At this point, what remains to be done of the construction is to confirm that the asymptotic behaviour of (A, Φ) is as it must be in order to be a monopole of charge k; that is, we must verify that $\mathrm{YMH}(A, \Phi) < \infty$, $|\Phi(x)| \to 1$ and that the eigenbundles of Φ away from some large ball have Chern classes $\pm k$.

For that, we must define yet another operator. Fix $x \in \mathbb{R}^3$ and set:

$$
\begin{aligned}
&\mathfrak{D}'_x : \mathcal{H}^1_0\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big) \to \mathrm{L}^2\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big) \\
&\mathfrak{D}'_x = e^{-(t+1)X}\widetilde{\mathfrak{D}}e^{(t+1)X} + i\sum_{i=1}^{3} x_i e_i \otimes \mathbf{1}_{\mathbb{C}^k}
\end{aligned}
$$

Note that $\mathfrak{D}_x = \mathfrak{D}'_x + K$ for some K smooth in a neighbourhood of $[-1, 1]$; whence it follows that the index of $\mathfrak{D}'_x$ coincides with that of $\mathfrak{D}_x$ (c.f. proposition Proposition 4.1.12). Integrating by parts, one can show, for $f \in \mathcal{H}^1_0\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big)$ (recall that f vanishes at ± 1), that:

$$
\begin{aligned}
\langle \mathfrak{D}_x^*\mathfrak{D}_x f, f\rangle_{L^2} &= \left\| \frac{d}{dt} f \right\|^2_{L^2} + \left\langle \left(|x|^2 + \sum_i T_i^* T_i \right) f, f \right\rangle_{L^2} \\
&\geq |x|^2 \, \|f\|^2_{L^2}
\end{aligned}
$$

Therefore, for sufficiently large $|x|$, it follows that there is some constant $c \in \mathbb{R}$ such that $\langle {\mathfrak{D}'_x}^*\mathfrak{D}'_x f, f\rangle \geq c^2|x|^2 \, \|f\|^2_{L^2}$. Which means that ${\mathfrak{D}'_x}^*\mathfrak{D}'_x$ is also a positive operator for large $r = |x|$. Together with the remark above about the index, this leads to $\dim_{\mathbb{C}} \mathrm{Ker}({\mathfrak{D}'_x}^*) = 2$.

Use P_x to denote the orthogonal projection onto the kernel of ${\mathfrak{D}'_x}^*$; then, let G_x be the Green's function of $\mathfrak{D}'_x$; so $\mathfrak{D}'_x G_x = \mathbf{1} - P_x$ and $G_x \mathfrak{D}'_x = \mathbf{1}$. Notice that:

$$\|f - P_x f\|^2 = \|\mathfrak{D}'_x G_x f\|^2 = \langle {\mathfrak{D}'_x}^* \mathfrak{D}'_x G_x f, G_x f\rangle \geq c^2 r^2 \|G_x f\|^2$$

So:

$$\|G_x f\| \, cr \leq \|f - P_x f\| \leq \|f\|$$

Which is to say that:

$$\|G_x\| \leq 1/cr, \quad \|G_x^*\| \leq 1/cr \qquad \text{(Eq. 4.14)}$$

Consider a solution f to $\mathfrak{D}_x^* f = 0$. Since $G_x^* \mathfrak{D}'^* = \mathbf{1} - P_x$, one has:

$$f - P_x f = G_x^*(\mathfrak{D}_x^* - K^*) f = -G_x^* K^* f$$

Therefore, one can find some constant $c' \in \mathbb{R}$ such that:

$$\|f - P_x f\| \leq \frac{c'}{r} \|f\| \qquad \text{(Eq. 4.15)}$$

Which means that we can approximate, up to order r^{-1}, solutions to $\mathfrak{D}^* f = 0$ by solutions to $\mathfrak{D}'^* f = 0$.

We now proceed to find a basis of solutions for $\mathfrak{D}'^* f = 0$. Fix a $u \in \mathbb{R}^3$ such that $|u| = 1$. Then, u generates a circle subgroup of $SU(2)$ by exponentiating $\sum_{i=1}^3 u_i e_i \in \mathfrak{su}(2)$; furthermore, this subgroup is a maximal toral subgroup and, with respect to it, one can decompose any representation as a direct sum of weight spaces. We are interested in the representation $V_1 \otimes V_{k-1}$ because of the form of the operator $\widetilde{\mathfrak{D}}$ which contains the term $a = \sum_{i=1}^3 e_i \otimes a_i$ which acts via this representation. Recall that $V_1 \otimes V_{k-1} \cong V_k \oplus V_{k-2}$ as $\mathfrak{su}(2)$ representations; therefore, by knowledge of the representation theory of $\mathfrak{su}(2)$, one knows that the weight space decomposition has weights:

$$-k, -(k-2), \ldots, (k-2), k \quad \text{in } V_k \oplus 0$$

$$-(k-2), -(k-4), \ldots, (k-4), (k-2) \quad \text{in } 0 \oplus V_{k-2}$$

Each weight, in both cases, has multiplicity 1; thus, one sees that the weight spaces of weight $\pm k$ have dimension 1; all others have dimension 2. Now, the operator $\mathfrak{D}'_u$ contains the term $i \sum_{i=1}^3 u_i e_i \otimes \mathbf{1}_{V_{k-1}}$; the action of this term commutes with the action of a; whence it follows that it preserves the weight space decomposition above; furthermore, since the weight spaces of weight $\pm k$ are one dimensional, it must act as a scalar on each of these two weight spaces. The scalars are, in fact, respectively ± 1; this can be seen by noting:

$$\left(i \sum_{i=1}^3 u_i e_i \otimes \mathbf{1}_{V_{k-1}} \right)^2 = \mathbf{1}_{V_1 \otimes V_{k-1}}$$

Pick vectors $v_\pm(u) \neq 0$ in the weight spaces of weight $\pm k$ respectively. For $x \in \mathbb{R}^3 \setminus 0$, write $g_\pm(x) := g(t)e^{-(t+1)X}v_\pm(x/r)$; then:

$$\mathfrak{D}'^*_x g_\pm(x) = e^{(t+1)X}\left(-\frac{d}{dt}g + \frac{k-1}{2}\left(\frac{1}{t-1} + \frac{1}{t+1}\right) \pm rg\right) v_\pm(x/r)$$

If one care to check, the choice

$$g_\pm(t) = \big((t+1)(t-1)\big)^{(k-1)/2} e^{\pm rt}$$

gives $\mathfrak{D}'^*_x g_\pm(x) = 0$; moreover, note $g_\pm(x) \in \mathcal{H}^1_0\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big)$. Since $g_\pm(x)$ are linearly independent and the dimension of the kernel of $\mathfrak{D}'^*_x$ is two, we have obtained a basis for this kernel. By equation (Eq. 4.15), one can approximate, to order r^{-1}, elements of $E_x = \mathrm{Ker}\mathfrak{D}^*_x$ by elements of $\mathrm{Ker}\mathfrak{D}'^*_x$; thus, one can obtain $f_\pm(x) \in \mathcal{H}^1_0\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big)$ such that $\mathfrak{D}^*_x f_\pm(x) = 0$ and:

$$f_\pm(x) = \frac{g_\pm(x)}{\|g_\pm(x)\|_{L^2}} + O(r^{-1})$$

Where the norm is the L^2 norm with respect to t alone. For sufficiently large r, this provides a basis for E_x for all x outside some large ball. What we desire, however, is an actual *trivialisation* of E; this requires the choice of $v_\pm(u)$ to vary *smoothly* in $u \in S^2$. Regarding this, we have the following:

Proposition 4.1.14. *A smooth assignment $u \mapsto v_\pm(u)$ corresponds naturally to a smooth section of the complex line bundle $\mathcal{O}(\pm k)$ over $S^2 = \mathbb{CP}^1$.*

Proof. Consider the circle bundle over S^2 whose fibre over u is the circle generated by exponentiating $\sum_i u_i e_i$; this bundle is the unitary frame bundle of $\mathcal{O}(1)$. Associated to this circle bundle, construct the complex line bundle whose fibre at u is the weight space with weight k in the weight space decomposition above; then, clearly, the association is given by the degree k representation of $U(1)$ on $\mathbb{C}$; therefore, this line bundle is $\mathcal{O}(k)$. By construction, a choice of $v_+(u)$ varying smoothly in u is a section of this complex line bundle. By the same argument, one sees that $v_-(u)$ is a section of $\mathcal{O}(-k)$. $\square$

As a consequence, there is not a smooth choice of v_+ which not vanish at any point of S^2; thus, we must make two choices v^j_+, $j = 1,2$; each vanishing at a single point and the points not coinciding. It is, then, clear that $\mathrm{span}\{g^j_+ \mid j = 1,2\}$ is isomorphic to the line bundle of Chern class k over $\mathbb{R}^3 \setminus 0$; similarly, $\mathrm{span}\{g^j_- \mid j = 1,2\}$ is the line bundle of Chern class $-k$ over $\mathbb{R}^3 \setminus 0$. When we pass over to E, we get a similar result, except that, now, we must work outside some large ball B_R; that is: $\mathrm{span}\{f^j_+ \mid j = 1,2\}$ is isomorphic to the line bundle of Chern class k over $\mathbb{R}^3 \setminus B_R$ and likewise for the f^j_-.

Now, we are in position to address the asymptotic behaviour of Φ. Recall that Φ is defined as $\pi \circ (u \mapsto itu)$; therefore, by the above, one has:

$$\Phi = \begin{pmatrix} \frac{\langle itg_+, g_+ \rangle}{\|g_+\|^2} & \frac{\langle itg_+, g_- \rangle}{\|g_+\|\|g_-\|} \\ \frac{\langle itg_-, g_+ \rangle}{\|g_-\|\|g_+\|} & \frac{\langle itg_-, g_- \rangle}{\|g_-\|^2} \end{pmatrix} + O(r^{-1})$$

Where all norms and inner products are those of $\mathrm{L}^2\big((-1,1); V \otimes \underline{\mathbb{C}^2}\big)$. The trivialisation used for E is the one defined by the sections $\{f_\pm\}$; Now:

$$\langle itg_\pm, g_\mp \rangle = \int_{-1}^{1} it\big((t+1)(t-1)\big)^{k-1} dt = 0$$

Meanwhile, a straightforward computation yields:

$$\frac{\langle itg_\pm, g_\pm \rangle}{\|g_\pm\|^2} = \frac{\int_{-1}^{1} it\big((t+1)(t-1)\big)^{k-1} e^{\pm 2rt} dt}{\int_{-1}^{1} \big((t+1)(t-1)\big)^{k-1} e^{\pm 2rt} dt} \longrightarrow \pm i \quad \text{as } r \to \infty$$

As a result, in this trivialisation:

$$\Phi \to \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{as } r \to \infty$$

So $|\Phi| \to 1$ as $r \to \infty$ as needed. Furthermore, due to proposition Proposition 4.1.14, the eigenbundles of Φ have Chern classes $\pm k$; therefore, the charge of (A, Φ) is k.

All that remains to be checked is the condition of finite energy, i.e. $\mathrm{YMH}(A, \Phi) < \infty$; then, as remarked in chapter Chapter 2, all other asymptotic conditions follow. For this, we must prove that the curvature is square integrable.

Proposition 4.1.15. *The curvature of A is given by the expression:*

$$F_A = \pi dx G G^* d\bar{x}$$

Where π_x is the projection onto the kernel of $\mathfrak{D}_x^*$; G_x is the Green's function of $\mathfrak{D}_x$; that is:

$$\mathfrak{D} G = \mathbf{1} - \pi, \quad G\mathfrak{D} = \mathbf{1}$$

And the notation $dx, d\bar{x}$ signifies:

$$dx = \sum_{i=1}^{3} e_i \otimes dx_i, \quad d\bar{x} = -dx$$

A wedge product is implicit wherever this appears.

Proof. By definition, the curvature is $F_A = \pi d\pi d$. The key facts in this proof are the following:

$$[d, \mathfrak{D}] = dx, \quad [d, \mathfrak{D}^*] = d\bar{x}$$

These are trivially verified. Hence, consider:

$$\begin{aligned}\pi dx GG^* d\bar{x} &= \pi[d, \mathfrak{D}]GG^*[d, \mathfrak{D}^*] \\ &= \pi(d\mathfrak{D} - \mathfrak{D}d)GG^*(d\mathfrak{D}^* - \mathfrak{D}^*d)\end{aligned}$$

The term with $\mathfrak{D}^*$ on the right vanishes as we are acting on elements of the kernel of $\mathfrak{D}^*$; whereas the term with $\pi\mathfrak{D}$ on the left vanishes because π is the orthogonal projection onto the kernel of $\mathfrak{D}^*$ which is the orthogonal complement of the image of $\mathfrak{D}$. What remains is:

$$\begin{aligned}&= -\pi d\mathfrak{D}GG^*\mathfrak{D}^*d \\ &= -\pi d(1-\pi)(1-\pi)d \\ &= \pi d\pi d \\ &= F_A\end{aligned}$$

□

To conclude, notice that the bound obtained for the Green's function of $\mathfrak{D}'$ in (Eq. 4.14) and the ability to approximate the kernel of $\mathfrak{D}$ by the kernel of $\mathfrak{D}'$ as shown in (Eq. 4.15) imply that we also have a bound on the norms of the Green's function of $\mathfrak{D}$:

$$\|G\| \leq 1/cr, \quad \|G^*\| \leq 1/cr$$

So:

$$\|F_A\| \leq \|\pi\| \, \|dx\| \, \|G\| \, \|G^*\| \, \|d\bar{x}\| = \|G\| \, \|G^*\| \leq \frac{1}{c^2r^2}$$

Where these norms are being thought of as the operator norms of bounded linear maps. Since we work in $\mathbb{R}^3$ this places F_A in L^2 and

$$\mathrm{YMH}(A, \Phi) = \int_{\mathbb{R}^3} \|F_A\|^2 + \|d_A\Phi\|^2 < \infty$$

In summary, we have proved:

Theorem 4.1.16. *The pair (A, Φ) where A is the connexion on the hermitian bundle $E = \mathrm{Ker}\mathfrak{D}^*$ defined as $d_A = \pi \circ d$ and Φ is the endomorphism of E defined as $\pi \circ (u \mapsto itu)$ is an $SU(2)$ monopole of charge k.*

The construction presented in this section is the inverse of the construction of Nahm data from a monopole presented in section Section 4.1.1. This is certainly something that must be checked, but, due to time constraints, the proof was left out of this project; we refer the reader to [Nak93] for details.

4.2 Singular Monopoles and Nahm's Equations

The Nahm transform between SU(2) monopoles and solutions of Nahm's equations is a specific case of a more general collection of correspondences between invariant instantons on four-manifolds. One particular example of this type of correspondence should be given by the case of *singular* monopoles on $\mathbb{R}^3$. In this section we will investigate this correspondence.

The correspondence we will investigate is between Dirac monopoles on $\mathbb{R}^3$ as described in Section 2.3 and solutions of Nahm's equations given by the following data:

1. A Hermitian bundle V of rank k over $(0, \infty)$ where $k = k_1 + \cdots + k_n$ is the sum of charges of the Dirac monopoles, and
2. a unitary connection ∇_t on V and three skew-Hermitian endomorphisms T_1, T_2, T_3 satisfying Nahm's equations, such that
    (a) The T_α are analytic for all $s \in (0, \infty)$,
    (b) as $s \to 0$ the residues of the T_α define an irreducible representation of $\mathfrak{su}(2)$, and
    (c) as $s \to \infty$, the operators T_α approach a commuting triple given by

$$i\,\mathrm{diag}(\underbrace{(p_1)_\alpha, \ldots, (p_1)_\alpha}_{k_1\text{times}}, \ldots, \underbrace{(p_n)_\alpha, \ldots, (p_n)_\alpha}_{k_n\text{times}})$$

    where $p_i = ((p_i)_1, (p_i)_2, (p_i)_3)$ gives the coordinates of a singularity of the corresponding monopole with charge k_i.

In the case of SU(2) monopoles the bundle $V \to (-1, 1)$ upon which the Nahm data correponding to the monopole lived was constructed as the kernel of a twisted Dirac operator on $\mathbb{R}^3$. This same approach can be attempted in the case of Dirac monopoles. However, the presence of the singularities requires that one applies the techniques of b-geometry and scattering calculus, as detailed for example in [Mel93].

In particular, let Φ_s denote the Higgs field of a Dirac monopole of mass s, given explicitly by

$$\Phi_s := i\left(s + \sum_{i=1}^{n} \frac{k_i}{2|x - p_i|}\right),$$

where the p_i denote the n singularities of Φ_1 and the k_i are the charges of these singularities. We will often denote the distance function $|x - p_i|$ on $\mathbb{R}^3$ by r_i.

As in the case of SU(2) monopoles, we are interested in the operator

$$\not{D}_{A,s} := \not{D}_A + \Phi_s$$

where A is the connection form corresponding to the Higgs field. Again we have that the formal adjoint of $\not{D}_{A,s}$ is

$$\not{D}^*_{A,s} = \not{D}_A - \Phi_s$$

due to the skew-Hermiticity of Φ_s, and by the Weitzenböck formula we have

$$\not{D}^*_{A,s}\not{D}_{A,s} = \nabla^*_A\nabla_A - \Phi_s^2$$

which is a positive operator.

Remark 4.2.1. A special case of this is where we have a single singularity at the origin. Here, we have a very manageable exact formula for the Dirac operator, namely

$$D^*_{A,s} = \begin{pmatrix} i\left(\frac{\partial}{\partial r} - s - \frac{k-2}{2r}\right) & \frac{1}{r}\not{D}^- \\ \frac{1}{r}\not{D}^+ & -i\left(\frac{\partial}{\partial r} + s + \frac{k+2}{2r}\right) \end{pmatrix}.$$

We can directly try to compute the kernel of this operator.

If we start by looking for functions of the form $g(r)f(\theta)$, where we view $\mathbb{R}^3\setminus\{0\}$ as $S^2\times\mathbb{R}^+$ and g a function on $\mathbb{R}$ and f a section of the bundle $\mathcal{O}(k-1)\oplus\mathcal{O}(k+1)$ over $S^2 = \mathbb{CP}^1$, then we can easily check that we have a k-dimensional space of solutions, given by $(e^{-sr}r^{\frac{k-2}{2}}f, 0)$, where $f \in H^0(\mathbb{CP}, \mathcal{O}(k-1))$.

Using some further argument involving decomposing the spaces of sections over the sphere into eigenspaces of the Laplacian one would attempt to prove that these are indeed all the solutions.

The general case, however, is more complicated. We can, however, write some asymptotic behaviours (cf. (Eq. 4.8)) The behaviour of our operators near infinity can be written as

$$\not{D}_{A,s} = \begin{pmatrix} i\left(\frac{\partial}{\partial r} + s + \frac{k+2}{2r}\right) & \frac{1}{r}\not{D}^- \\ \frac{1}{r}\not{D}^+ & -i\left(\frac{\partial}{\partial r} - s - \frac{k-2}{2r}\right) \end{pmatrix} + O(r^{-2}), \tag{Eq. 4.16}$$

$$\not{D}^*_{A,s} = \begin{pmatrix} i\left(\frac{\partial}{\partial r} - s - \frac{k-2}{2r}\right) & \frac{1}{r}\not{D}^- \\ \frac{1}{r}\not{D}^+ & -i\left(\frac{\partial}{\partial r} + s + \frac{k+2}{2r}\right) \end{pmatrix} + O(r^{-2}), \tag{Eq. 4.17}$$

and the behaviour near the singularity p_i is given by

$$\not{D}_{A,s} = \begin{pmatrix} i\left(\frac{\partial}{\partial r_i} + \frac{k_i+2}{2r_i}\right) & \frac{1}{r_i}\not{D}^- \\ \frac{1}{r_i}\not{D}^+ & -i\left(\frac{\partial}{\partial r_i} - \frac{k_i-2}{2r_i}\right) \end{pmatrix} + O(1) \tag{Eq. 4.18}$$

$$\not{D}^*_{A,s} = \begin{pmatrix} i\left(\frac{\partial}{\partial r_i} - \frac{k_i-2}{2r_i}\right) & \frac{1}{r_i}\not{D}^- \\ \frac{1}{r_i}\not{D}^+ & -i\left(\frac{\partial}{\partial r_i} + \frac{k_i+2}{2r_i}\right) \end{pmatrix} + O(1) \tag{Eq. 4.19}$$

(note that here the s is absorbed into the $O(1)$, since it is a constant).

Now, in order to define the Nahm data, we want to know what the kernels of these operators are. In order to do this, we will need to apply ideas from b- and sc-geometry. In the following sections we will give a concise introduction to these ideas and explain how the apply to our case.

The underlying idea to both is the following. Let M be a smooth manifold with boundary. A smooth function x on M such that $x|_{\partial M} = 0$, $x \neq 0$ away from ∂M, and $dx|_{\partial M} \neq 0$ is called a *boundary-defining function* on M. In a neighbourhood of a boundary point of M, one may use x to define a local coordinate system $(x, y^1, \ldots, y^n)$ where $(y^1, \ldots, y^n)$ are local coordinates on the boundary ∂M.

Consider $\mathbb{R}^3 \backslash \mathcal{P}$ where $\mathcal{P}$ is the finite set of points where Φ_s has singularities. One may perform the real blowup of this space at each point $p_i \in \mathcal{P}$ and at ∞ to obtain a compact manifold $M = \overline{\mathbb{R}^3 \backslash \mathcal{P}}$ with boundary, such that

$$\partial M = \partial_\infty M \sqcup \bigsqcup_{p_i \in \mathcal{P}} \partial_i M,$$

where $\partial_\infty M \cong S^2$ is a sphere at ∞, and $\partial_i M \cong S^2$ is a sphere about each singularity p_i of Φ_s.

Furthermore, in our case, $x_i := \frac{r_i}{r_i+1}$ is a boundary defining function for each $\partial_i M$ and $x_\infty := \frac{1}{r}$ (properly cut off for small r) is a boundary defining function for $\partial_\infty M$. We denote $x = \Pi \frac{r_i}{1+r_i}$, which is a boundary defining function for all the boundaries at the singularities (not to confuse the notation with the one for the coordinates on $\mathbb{R}^3$).

4.2.1 b-Geometry and the case $s = 0$

Given M a manifold with boundary, construct the b-vector fields by

$$\mathrm{Vect}_b(M) := \{X \in \Gamma(M, TM) \mid X \text{ is tangent to } \partial M\}.$$

If M has a boundary-defining function x, then in local coordinates near the boundary a b-vector field is a smooth linear combination of

$$x\frac{\partial}{\partial x}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}.$$

Remark 4.2.2. If one makes a change of variables $t := \log x$ then $\frac{\partial}{\partial t} = x\frac{\partial}{\partial x}$ and the boundary $\partial M = \{x = 0\}$ corresponds to the limit as $t \to -\infty$. Further, the local coordinate system of $(t, y^1, \ldots, y^n)$ identifies a small neighbourhood of ∂M in M with the infinite cylinder $\partial_M \times (-\infty, a)$ for some $a \in \mathbb{R}$. In this sense, b-geometry is the study of differential operators on manifolds with boundary by viewing the boundary as a cylindrical end (and considering vector fields which are asymptotically translation invariant).

These b-vector fields are first-order differential operators on M, and analogously to the regular definition of differential operators, one may define the algebra $\mathrm{Diff}_b^d(M)$ of b-differential operators of order d on M.

Using the differential operator $x\frac{\partial}{\partial x}$ instead of $\frac{\partial}{\partial x}$ near the boundary, one may also analogously define the b-Sobolev spaces $\mathcal{H}_b^k$ of M. These Sobolev spaces rely on the usual metric on the manifold. In a way, a function is in $\mathcal{H}_b^m$ if its first k b-derivatives are in (usual) L^2. Hence, if P is a b-differential operator of order d, then

$$P : \mathcal{H}_b^m \to \mathcal{H}_b^{m-d}$$

for all k.

Locally, a b-differential operator of order m has the form

$$\begin{aligned} P &= p\left(x, y; x\frac{\partial}{\partial x}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right) \\ &= p_0(x, y) + \sum_{i=1}^{d} p_i(x, y, x\frac{\partial}{\partial x}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}), \end{aligned}$$

where p is some polynomial of degree d in the given arguments, and p_i is homogenous of degree i.

Just as in the case of regular differential operators, one may define the b-symbol from the highest order term of P by

$$\sigma_b(P)(x, y, \xi, \eta_1, \dots, \eta_n) := p_d(x, y, \xi, \eta_1, \dots, \eta_n)$$

for formal variables ξ, η_j corresponding to $x\frac{\partial}{\partial x}$ and the $\frac{\partial}{\partial y^j}$. A b-differential operator is b-elliptic when $\sigma_b(P)$ is invertible for $(\xi, \eta_j) \neq 0$.

Unlike in the case of regular differential operators on a smooth manifold M, it is not true that a b-elliptic b-differential operator is Fredholm on b-Sobolev spaces. In the setting of b-geometry there are obstructions to Fredholmness that are detected by the normal operator of P. Define the normal operator $N(P)$ by

$$N(P) := \sum_{i=0}^{d} p_i(0, y; x\frac{\partial}{\partial x}, \frac{\partial}{\partial y^j}).$$

A complex number α is an *indicial root* of $N(P)$ if there exists a non-zero function u on ∂M such that

$$N(P)(x^\alpha u(y)) = 0.$$

In this case one has the following theorem.

Theorem 4.2.3 (See [Mel93, §5.17], or [LMO85]). *If P is a b-elliptic b-differential operator and $\delta \in \mathbb{R}$ is not the real part of any indicial root of $N(P)$, then*

$$P : x^\delta \mathcal{H}_b^m(M,E) \to x^\delta \mathcal{H}_b^{m-d}(M,E)$$

is Fredholm. Here $f \in x^\delta \mathcal{H}_b^m$ if $x^{-\delta} f \in \mathcal{H}_b^m$. Denote the index of P at weight δ by $\operatorname{Ind}(P,\delta)$.

In addition to this result, there is a wall-crossing formula as the parameter δ passes across the indicial roots of $N(P)$.

Theorem 4.2.4 (See [LMO85]). *If $\delta_1 < \alpha < \delta_2$ and α is the only indicial root of $N(P)$ between δ_1 and δ_2 which are not indicial roots, then*

$$\operatorname{Ind}(P,\delta_2) - \operatorname{Ind}(P,\delta_1) = \dim\{u \mid N(P)(x^\alpha u(y)) = 0\}.$$

Finally one may define the b-adjoint of a b-differential operator P by integration with respect to a b-volume form on M. If

$$P : x^\delta \mathcal{H}_b^m(M,E) \to x^\delta \mathcal{H}_b^{m-d}(M,E)$$

then

$$P^* : x^{-\delta} \mathcal{H}_b^m(M,E) \to x^{-\delta} \mathcal{H}_b^{m-d}(M,E)$$

and $\ker P^* = \operatorname{coker} P$, and $\operatorname{Ind}(P^*,-\delta) = -\operatorname{Ind}(P,\delta)$. This allows us to compute the index of an operator if we know its relationship to it's dual. For example, if P is b-self-adjoint, then we would have the following.

Theorem 4.2.5 (See [LMO85]). *If $P^* = P$ then*

$$2\operatorname{Ind}(P,\delta) = \operatorname{Ind}(P,\delta) - \operatorname{Ind}(P,-\delta) = \sum_{-\delta<\alpha<\delta} \dim\{u \mid N(P)(x^\alpha u(y)) = 0\}$$

where α runs over all indicial roots between $-\delta$ and δ.

For more details on the various constructions in b-geometry, including of the b-tangent bundle, b-metrics, b-integrals, and various other natural differential-geometric and analytic constructions, we refer to the seminal [Mel93], particularly Chapters 2, 4, and 5.

As it turns out, this setting will be adequate for studying our Dirac operators in the particular case where the mass is $s = 0$. This is the approach considered by Singer in the unpublished notes [Sin00], which we follow here.

Indeed, we see that the behaviours of $x_i \not{D}_{A,0}$ and $x_i \not{D}^*_{A,0}$ near the singularities and $x_\infty^{-1} \not{D}_{A,0}$ and $x_\infty^{-1} \not{D}^*_{A,0}$ near infinity are precisely the behaviours of b-operators (see (Eq. 4.16), (Eq. 4.17), (Eq. 4.18) and (Eq. 4.19)). Furthermore, one can easily see that they are b-elliptic. This means that they will be Fredholm operators

$$\not{D}_{A,0}, \not{D}_{A,0}^* : x^\mu x_\infty^\delta \mathcal{H}_b^m(M, \$ \otimes E) \to x^{\mu-1} x_\infty^{\delta+1} \mathcal{H}_b^{m-1}(M, \$ \otimes E). \qquad \text{(Eq. 4.20)}$$

In fact, we will think of μ as a kind of multi-index, in the sense that we have $x^{(\mu_1,\dots,\mu_n)} = \Pi x_i^{\mu_i}$ (and $\mu - 1$ refers to reducing each component by 1).

Let us then look at the normal operators. Consider, for and integer $\kappa > 0$ and a real number λ, the operator

$$N_\kappa^\lambda = \begin{pmatrix} i\left(\alpha + 1 + \frac{\lambda\kappa}{2}\right) & \not{D}^- \\ \not{D}^+ & -i\left(\alpha + 1 - \frac{\lambda\kappa}{2}\right) \end{pmatrix},$$

acting on sections of $\mathcal{O}(\kappa - 1) \oplus \mathcal{O}(\kappa + 1)$ of $\mathbb{CP}^1$.

N_k^1 and N_k^{-1} are the normal operators for $x_\infty^{-1}\not{D}_{A,0}$ and $x_\infty^{-1}\not{D}_{A,0}^*$, respectively, at the sphere at infinity, where k the sum of charges (and acting on sections of the form $x^\alpha u$). Analogously, taking κ to be k_i would give us the normal operators of $x_i\not{D}_{A,0}$ and $x_i\not{D}_{A,0}^*$. Therefore, we are interested in computing the indicial roots of these operators.

Recall that $\not{D}^+$ and $\not{D}^-$ are $2\overline{\partial}_\kappa$ and $2\overline{\partial}_\kappa^*$, the Dolbeault operator on the holomorphic bundle $\mathcal{O}(\kappa-1)$ and its dual (on $\mathcal{O}(\kappa+1) = \bigwedge^{0,1} \otimes \mathcal{O}(\kappa-1)$) on $S^2 = \mathbb{CP}^1$. If α is an indicial root of this operator, then we will be have u^+ and u^-, (smooth) sections of $\mathcal{O}(\kappa - 1)$ and $\mathcal{O}(\kappa + 1)$, respectively, so that

$$i\left(\alpha + 1 + \frac{\lambda\kappa}{2}\right) u^+ + 2\overline{\partial}^* u^- = 0, \qquad \text{(Eq. 4.21)}$$

$$-i\left(\alpha + 1 - \frac{\lambda\kappa}{2}\right) u^- + 2\overline{\partial} u^+ = 0. \qquad \text{(Eq. 4.22)}$$

We want to compute the dimension of the space of solutions of this system. In order to do so, we need to know the eigenvalues of the Laplacian $4\overline{\partial}_\kappa^*\overline{\partial}_\kappa$ on $\mathbb{CP}^1$.

Lemma 4.2.6. *Let $\overline{\partial}_\kappa$ denote the Dolbeault operator on $\mathcal{O}(\kappa) \otimes \$^+ \to \mathbb{CP}^1$ for $\kappa > 0$, and let ∇_κ denote the Chern connection. The eigenvalues of the Dolbeault Laplacian*

$$4\overline{\partial}_\kappa^*\overline{\partial}_\kappa$$

are given by

$$\{\ell(\ell + \kappa) \mid \ell \in \mathbb{Z}_{\geq 0}\}.$$

Proof. This follows from the Weitzenböck formula for the twisted Dirac operator on $\mathcal{O}(\kappa) \otimes \$^+$. Recall that $\$^+ \cong \mathcal{O}(-1)$. It is shown in [Kuw82, Thm. 5.1] that the eigenvalues of the Chern connection Laplacian on $\mathcal{O}(\kappa) \to \mathbb{CP}^1$ with the standard round metric are given by

$$\{p(p+1) - \frac{\kappa^2}{4} \mid p = \frac{|\kappa|}{2}, \frac{|\kappa|}{2} + 1, \dots\}.$$

Recall from Theorem 1.1.15 that the twisted Dirac operator on $\mathcal{O}(\kappa) \otimes \$$ is given by

$$\not{D} = \begin{pmatrix} 0 & 2\overline{\partial}_\kappa^* \\ \overline{\partial}_\kappa & 0 \end{pmatrix}.$$

The Weitzenböck formula Theorem 1.1.11 then says that

$$4\overline{\partial}_\kappa^* \overline{\partial}_\kappa = \nabla_\kappa^* \nabla_\kappa + \mathcal{R},$$

where

$$\mathcal{R} = \sum_{i<j} c(e_i)c(e_j)F_\kappa(e_i, e_j).$$

In this setting the bundle $\mathcal{O}(\kappa) \otimes \$^-$ with the Chern connection has curvature form $F_\kappa = -\frac{i(\kappa-1)}{2}d\text{vol}$, and in this orientation

$$c(e_1) = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad c(e_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus on the positive spinor bundle $\mathcal{O}(\kappa) \otimes \$^+$ we have

$$\mathcal{R} = -i \cdot -\frac{i(\kappa - 1)}{2} d\text{vol}(e_1, e_2) = -\frac{\kappa - 1}{2}.$$

Putting this all together, and recalling that $\kappa > 0$, we see the eigenvalues of $4\overline{\partial}_\kappa^* \overline{\partial}_\kappa$ are

$$\left(\frac{|\kappa - 1|}{2} + \ell\right)\left(\frac{|\kappa - 1|}{2} + \ell + 1\right) - \frac{(\kappa - 1)^2}{4} - \frac{\kappa - 1}{2} = \ell(\ell + \kappa),$$

for $\ell \in \mathbb{Z}_{\geq 0}$. □

Remark 4.2.7. Notice that the first eigenvalue is always 0 for $\kappa > 0$, corresponding to the fact that $\mathcal{O}(\kappa - 1)$ has non-zero holomorphic sections (i.e. sections in the kernel of the Dolbeault Laplacian) for all such κ. If this computation was repeated with $\kappa \leq 0$ the first eigenvalue would be strictly positive, agreeing with the fact that $H^0(\mathbb{CP}^1, \mathcal{O}(\kappa - 1)) = 0$ for $\kappa \leq 0$.

We now return to the solutions of our system of equations. By applying $2\overline{\partial}^*$ to (Eq. 4.22) and substituting with (Eq. 4.21), we would get

$$4\overline{\partial}_\kappa^*\overline{\partial}_\kappa u = \left((\alpha+1)^2 - \left(\frac{\lambda\kappa}{2}\right)^2\right) u. \qquad \text{(Eq. 4.23)}$$

This means that, if the original system is to have any solution, we must have that $(\alpha+1)^2 - \left(\frac{\lambda\kappa}{2}\right)^2$ must be equal to some eigenvalue $\ell(\ell+\kappa)$, for some $\ell \in \mathbb{Z}_{\geq 0}$. In other words, $\alpha = -\sqrt{\left(\frac{\lambda\kappa}{2}\right)^2 + \ell(\ell+\kappa)} - 1$ or $\alpha = \sqrt{\left(\frac{\lambda\kappa}{2}\right)^2 + \ell(\ell+\kappa)} - 1$.

If $\ell \neq 0$, this corresponds to a non-zero eigenvalue, and a solution to (Eq. 4.23) will always give rise to solutions of the original system. Therefore, for all these values, α is an actual indicial root (the dimension of the space of solutions can be computed to be $\kappa + 2\ell$, but this will not be relevant to our case).

However, the case $\ell = 0$ is slightly more complicated, since we will not always be able to go back to a solution of the original system. Hence, let us look at this case more closely. Firstly, we see that this corresponds to the indicial roots $\alpha = \frac{\lambda\kappa}{2} - 1$ and $\alpha = -\frac{\lambda\kappa}{2} - 1$, so we can take both cases separately. Also, for now, let $\lambda \neq 0$.

Hence, let us first look at $\alpha = \frac{\lambda\kappa}{2} - 1$. In this case, the system becomes

$$\begin{aligned} 2\overline{\partial}^* u^- &= i\lambda\kappa u^+, \\ 2\overline{\partial} u^+ &= 0. \end{aligned}$$

By applying $2\overline{\partial}$ to the first equation, we get that $\overline{\partial}\overline{\partial}^* u^- = 0$, which implies that $u^- = 0$, given that it's a section of $\mathcal{O}(\kappa+1)$ (with $\kappa > 0$). Hence, we must also have $u^+ = 0$, so there are no non-trivial solutions. Therefore, this is not an actual indicial root.

If $\alpha = -\frac{\lambda\kappa}{2} - 1$, then the system becomes

$$\begin{aligned} 2\overline{\partial}^* u^- &= 0, \\ 2\overline{\partial} u^+ &= -i\lambda\kappa u^-. \end{aligned}$$

This implies that $u^- = 0$, so our solutions will precisely be given by holomorphic sections of $\mathcal{O}(\kappa - 1)$, of which there is precisely a κ-dimensional space.

If $\lambda = 0$, then the two previous cases are the same, and are simply

$$\begin{aligned} 2\overline{\partial}^* u^- &= 0, \\ 2\overline{\partial} u^+ &= 0, \end{aligned}$$

so again we have a κ-dimensional space of solutions.

Summarising, we have that the indicial roots of N_κ^λ are given by

$$\begin{gathered}
\dots \\
-\sqrt{\left(\frac{\lambda\kappa}{2}\right)^2 + 2(2+\kappa)} - 1 \\
-\sqrt{\left(\frac{\lambda\kappa}{2}\right)^2 + 1(1+\kappa)} - 1 \\
-\frac{\lambda\kappa}{2} - 1 \\
\sqrt{\left(\frac{\lambda\kappa}{2}\right)^2 + 1(1+\kappa)} - 1 \\
\sqrt{\left(\frac{\lambda\kappa}{2}\right)^2 + 2(2+\kappa)} - 1 \\
\dots
\end{gathered}$$

and the space of solutions corresponding to $-\frac{\lambda\kappa}{2} - 1$ has dimension κ.

Now, in order to find the index of our operators, we would like to use something like Theorem 4.2.5. The problem is that our operator $\not{D}_{A,0}^*$ is not self adjoint. However, we do know that its adjoint is $\not{D}_{A,0}$, which is very closely related. In particular, we can deform $\not{D}_{A,0}^* = \not{D}_A - \Phi_0$ to $\not{D}_{A,0} = \not{D}_A + \Phi_0$ through the operators $\not{D}_A + \lambda\Phi_0$, which have N_κ^λ as their normal operators (with the corresponding κ near each boundary component).

Similarly to (Eq. 4.20), we define the operators

$$D_{\lambda,(\mu,\delta)} := \not{D}_A + \lambda\Phi_0 : x^\mu x_\infty^\delta \mathcal{H}_b^m(M, \not{S} \otimes E) \to x^{\mu-1} x_\infty^{\delta+1} \mathcal{H}_b^{m-1}(M, \not{S} \otimes E),$$

which will be Fredholm except when μ and δ fall on the initial roots given above for the corresponding λ. The dual of such an operator $D_{\lambda,(\mu,\delta)}$ will be

$$D_{-\lambda,(-\mu+1,-\delta-1)} := \not{D}_A - \lambda\Phi_0 : x^{-\mu+1} x_\infty^{-\delta-1} \mathcal{H}_b^m(M, \not{S} \otimes E) \to x^{-\mu} x_\infty^{-\delta} \mathcal{H}_b^{m-1}(M, \not{S} \otimes E)$$

(we have simply changed the sign of λ to give the dual and exchanged the weights correspondingly). Hence,

$$\operatorname{Ind}(D_{\lambda,(\mu,\delta)}) + \operatorname{Ind}(D_{-\lambda,(-\mu+1,-\delta-1)}) = 0. \qquad \text{(Eq. 4.24)}$$

The key now is to relate these two indices using the discussion above. The result is the following.

Proposition 4.2.8. *Suppose that*

$$\mu = \left(\frac{k_1+1}{2} - \varepsilon, \dots, \frac{k_n+1}{2} - \varepsilon\right), \qquad \delta = \frac{k+1}{2} + \varepsilon,$$

for some small enough $\varepsilon > 0$. *Then,*

$$\operatorname{Ind}(D_{-1,(\mu,\delta)}) - \operatorname{Ind}(D_{1,(-\mu+1,-\delta-1)}) = 2k$$

(where $k, k_1, \dots, k_n$ *are the charges of* (A, Φ)*).*

Proof. The proof of this proposition is best explained through the diagrams presented in Figures 4.1 and 4.2. Indeed, given the weights of our spaces, that will determine which indicial roots fall into the domain near each boundary component. In the figures, we represent with a thick grey line the interval of possible indicial roots that fall into the domains. The indicial roots which actually exist (depending on λ) are represented by the curves and dots. The straight line in the middle represents the indicial root $-\frac{\lambda\kappa}{2} - 1$, which we know to correspond to a space of dimension κ.

Therefore, we see that in all the boundary components, as we deform our operator we are going to leave out a κ-dimensional space corresponding to the indicial roots. Therefore, the index will decrease by all of these contributions. The total contributions are $k + k_1 + \cdots + k_n = k + k = 2k$. □

Corollary 4.2.9. *The index of the Dirac operator*

$$\not{D}^*_{A,0} : x^\mu x_\infty^\delta \mathcal{H}_b^m(M, \not{S} \otimes E) \to x^{\mu-1} x_\infty^{\delta+1} \mathcal{H}_b^{m-1}(M, \not{S} \otimes E)$$

is k*, where* μ *and* δ *are like in Proposition 4.2.8.*

Proof. This is a direct consequence of Proposition 4.2.8 and (Eq. 4.24). □

4.2.2 sc-Geometry and the case $s > 0$

In addition to b-geometry, we will also be interested in the so-called scattering calculus, or *sc*-geometry, also pioneered by Melrose. See [Mel08, Ch. 7] or [Mel95] for more details on the constructions in this section, which we summarise.

The *sc*-geometry is constructed parallel to b-geometry as follows. Again let M be a manifold with boundary and x a boundary-defining function. Define the scattering *sc*-vector fields by $\operatorname{Vect}_{sc}(M) := x \operatorname{Vect}_b(M)$. That is, *sc*-vector fields are smooth linear combinations of

$$x^2 \frac{\partial}{\partial x}, x\frac{\partial}{\partial y^1}, \dots, x\frac{\partial}{\partial y^n}$$

near ∂M.

Remark 4.2.10. Let $r := \frac{1}{x}$. Then $-x^2\frac{\partial}{\partial x} = \frac{\partial}{\partial r}$ and $x\frac{\partial}{\partial y^i} = \frac{1}{r}\frac{\partial}{\partial y^i}$. This is the local model for a Euclidean coordinate system, as opposed to the b-geometry in which the boundary is viewed as a cylindrical end. In this sense sc-calculus is the method of analysis on manifolds with boundary by viewing the boundary as a Euclidean limit.

Just as in the b-geometry case, the sc-vector fields allow one to define sc-differential operators $\mathrm{Diff}^d_{sc}(M)$ of order d. Notice that although $\mathrm{Vect}_{sc}(M) = x\,\mathrm{Vect}_b(M)$, it is not the case that $\mathrm{Diff}^d_{sc}(M) = x\,\mathrm{Diff}^d_b(M)$. Indeed $x^2\frac{\partial}{\partial x} + 1 = x\left(x\frac{\partial}{\partial x} + \frac{1}{x}\right) \in \mathrm{Diff}^1_{sc}(M)$ but is clearly not in $x\,\mathrm{Diff}^1_b(M)$.

Again one may define the sc-Sobolev spaces $\mathcal{H}^m_{sc}$ and the sc-symbol $\sigma_{sc}(P)$ of an sc-differential operator by the same expressions as in the b-geometry case. The notion of sc-ellipticity is defined in the same way.

Again define the normal operator $N(P)$ of a scattering operator P by

$$N(P) := \sum_{i=0}^{d} p_i(0, y, \xi, \eta_1, \dots, \eta_n).$$

We say P is *fully elliptic* if $\sigma_{sc}(P)$ is invertible for $(\xi, \eta_j) \neq 0$ for all (x, y) and $N(P)$ is invertible for all (ξ, η_j). In this setting there is a divergence with b-geometry, and we have the following theorem.

Theorem 4.2.11 (See [Mel95, §6.5]). *If P is a fully elliptic sc-differential operator of order d, then*

$$P : x^\delta \mathcal{H}^m_{sc} \to x^\delta H^{m-d}_{sc}$$

is Fredholm for all m, δ and the index of P is independent of the choice of m, δ. Furthermore

$$\ker P \subset \bigcap_{\delta, m} x^\delta \mathcal{H}^m_{sc}.$$

Now, if we look back at the asymptotic behaviour of our Dirac operators near infinity ((Eq. 4.16) and (Eq. 4.17)), we see that, if $s > 0$, they behave precisely like fully elliptic sc-operators at the boundary component $\partial_\infty M$ (to see this more concretely, we need to consider the symbol of the Dolbeault Laplacian and check that the normal operator is always invertible).

Hence, whereas before we were only able to consider the case $s = 0$, now we can study the case $s > 0$, which is, in fact, the one that interests us, since it is here that we will want to define the Nahm data.

Out operator, however, is not a b-operator or an sc-operator, since it behaves differently at different components of the boundary. However, we can consider a mix of these two kinds of geometry in order to study this case.

We start by noting that all the concepts of b- and sc-geometry (vector fields, ellipticity, Sobolev spaces, etc.) only differ from the usual ones in the behaviour near the boundary. That is, in the interior all the notions coincide. Therefore, we can consider consider something which we could call bsc-geometry, which would be the exact analogous, but imposing b-behaviour near some components of the boundary, and sc behaviour on other components. In the setting of the previous section, we would like to consider bsc-geometry in the sense that we impose b-behaviour near $\partial_i M$ for all i but sc-behaviour near $\partial_\infty M$. In this sense, our Dirac operator is a fully elliptic bsc-operator between Sobolev spaces with the appropriate weights, and therefore we would expect it to be Fredholm.

Now, we could try to compute the index using a similar technique to the one we used before. The problem, however, arises when we deform the operators, since at an intermediate point they will cease to be fully elliptic (in the scattering sense). The contribution from the singularities would be the same as before, but we would need a wall-crossing formula for the situation where an sc-operator ceases to be fully elliptic. If this gave us another contribution of k, then we would have our result.

Another way to tackle this would be by considering the following principle: we have deduced from the previous section that, if we have no mass, then the index is not affected by whether there are several singularities or whether all the charge is at a single point. On the other hand, the addition of mass affects only the behaviour towards infinity (since, recall, the behaviour near the singularities was dictated by $x_i D^*_{A,s}$, where multiplying by x_i made the mass become irrelevant). Therefore, if we had the case of a monopole with mass, but with all charge concentrated at a point, changing this to the general situation where we have multiple singularities should not have an impact on the index. This is precisely the case discussed in Remark 4.2.1.

4.2.3 Limit of Nahm Data

In order to determine the behaviour of the Nahm data near the limits, we would need to have a model, similar to the case of SU(2) monopoles (cf. (Eq. 4.9)). In analogy with the previous case, we would expect that the limit as $s \to 0$ be dictated by the behaviour near of the solutions for large r, and we would expect a similar formula. In the limit $s \to \infty$, however, the limit would be determined by the behaviour of the solutions near the singularities, where we would also expect solutions of the same form. In what follows we will study what these limits would be given these possible models. Note, in particular, that these model solutions are consistent with the weights defined in the previous sections for the domains of the Dirac operator.

As $s \to \infty$:

Let

$$\psi_{i,j} := (\chi(r_i) e^{-r_i s} r_i^{\frac{k_i - 2}{2}} f_{i,j}, 0),$$

where $f_{i,j} \in H^0(\mathbb{CP}^1, \mathcal{O}(k_i - 1))$ for $j = 1, \ldots, k_i$, give orthonormal bases. This would be the local model for a solution of $\not{D}^*_{A,s}\psi = 0$ near a singularity p_i. Here χ is a cut off function which is identically 1 near zero and 0 away from zero. In principle, the solution of $\not{D}^*_{A,s}\psi = 0$ near the singularities should be a small deformation of a sum

$$\psi := \sum_{i=1}^{n} a_{i,j} \psi_{i,j}.$$

Note that, if the cut off function is chose appropriately, then this would give, in fact, an (approximate) orthonormal decomposition.

Now, we would want to look at $\langle ix_\alpha \psi_{i,j}, \psi_{i',j'} \rangle$. For $i \neq i'$ this is zero, so we suppose that $i = i'$. We have

$$\langle ix_\alpha \psi_{i,j}, \psi_{i,j'} \rangle = \int_{\mathbb{R}^3} \chi(r_i)^2 e^{-2r_i s} r_i^{k_i - 2} i x_\alpha \langle f_{i,j}, f_{i,j'} \rangle,$$

where we integrate using the appropriate measure. Now, we observe that, as s becomes arbitrarily big, the exponential factor will cause an arbitrarily big proportion of the integral to be concentrated around the point p_i. Therefore, for sufficiently big s, this would be arbitrarily close to

$$\int_{\mathbb{R}^3} \chi(r_i)^2 e^{-2r_i s} r_i^{k_i - 2} i(p_i)_\alpha \langle f_{i,j}, f_{i,j'} \rangle = i(p_i)_\alpha \langle \psi_{i,j}, \psi_{i,j'} \rangle.$$

Therefore, one concludes that, in the limit, $\pi(ix_\alpha \psi_{i,j})$ behaves like $i(p_i)_\alpha \psi_{i,j}$.

Performing this operation simultaneously for all the model solutions demonstrates that the Nahm data T_α will, when represented in the basis described above, have the form

$$T_\alpha = i \operatorname{diag}(\underbrace{(p_1)_\alpha, \ldots, (p_1)_\alpha}_{k_1 \text{times}}, \ldots, \underbrace{(p_n)_\alpha, \ldots, (p_n)_\alpha}_{k_n \text{times}}).$$

As $s \to 0$:

The limit when s approaches 0 would be expected to be analogous to the behaviour of the SU(2) monopoles as $t \to \pm 1$, with the same computations providing the irreducible representation.

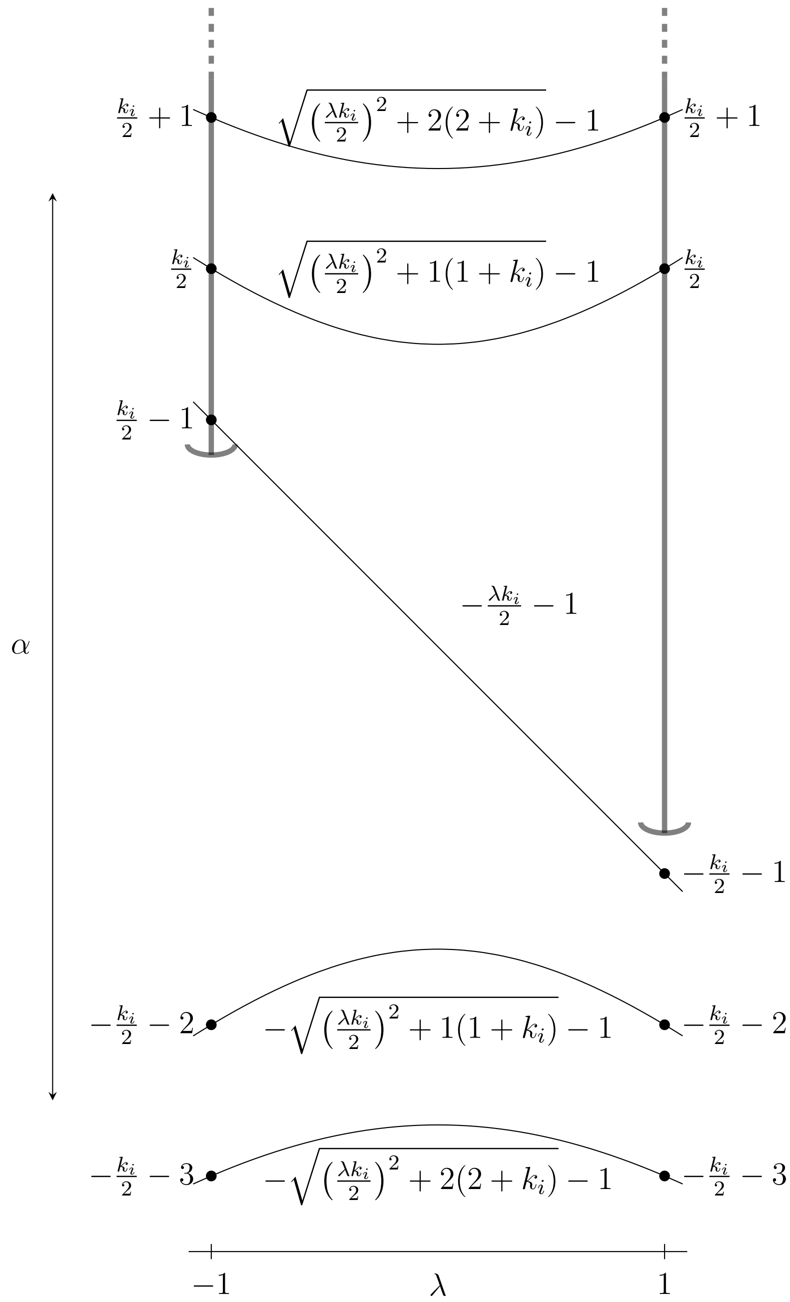

Figure 4.1: Operators near $\partial_i M$

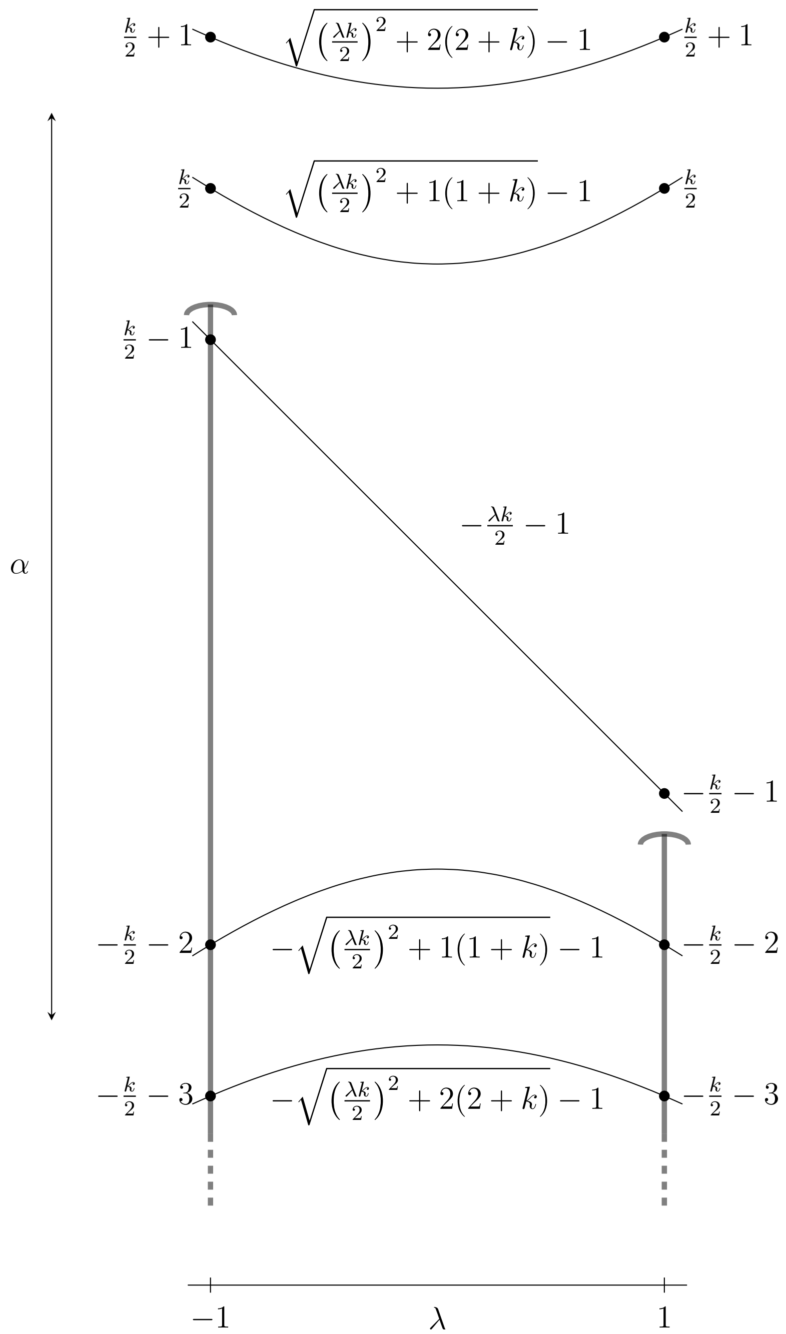

Figure 4.2: Operators near $\partial_\infty M$

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