Introduction to Higgs Bundles

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10th January 2020

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1 Motivation

Consider the problem of classifying vector bundles on a Riemann surface Σ_g of genus g.

1.1 Smoothly/Topologically:

Up to smooth or topological isomorphism, a complex vector bundle $E \to \Sigma_g$ is classified by its rank and degree. Recall

$$\deg(E) := \deg(\textstyle\bigwedge^{\operatorname{rk} E} E) = c_1(E)[\Sigma_g] \in \mathbb{Z},$$

or

$$\deg(E) = \int_{\Sigma_g} \frac{i}{2\pi} \operatorname{tr} F_\nabla$$

for any connection ∇ on E.

This is proved using the following interesting fact about vector bundles:

Theorem 1.1. *Let $E \to M$ be a real vector bundle over a manifold. Then if $\operatorname{rk} E > \dim M$, there is a real vector bundle E' of rank $\operatorname{rk} E' = \dim M$ such that*

$$E \cong E' \oplus \mathbf{1}_M^{\oplus(\operatorname{rk} E - \dim M)},$$

where $\mathbf{1}_M$ is the trivial rank one real vector bundle on M.

This theorem in essence says nothing interesting can happen to a vector bundle above the dimension of the base manifold. The number of non-trivial vector bundles is in some sense bounded by the dimension of the manifold. This theorem can be proved using some simple facts about homotopy and transversality of sections of vector bundles.

1.2 Holomorphically

If two holomorphic vector bundles $\mathcal{E}, \mathcal{F}$ over Σ_g are biholomorphic, then they are certainly diffeomorphic. So for any hope that $\mathcal{E}$ and $\mathcal{F}$ are biholomorphic, they must already be smoothly equivalent. That is, they must have equal rank and degree.

Thus the holomorphic classification of bundles happens in discrete steps: one by one for each possible pair (n, d) of rank and degree.

For $\operatorname{rk} \mathcal{E} = 1, \deg \mathcal{E} = 0$, the classification is well-known to be given by the *Jacobian*

$$\operatorname{Jac}_0(\Sigma_g) \cong \mathbb{C}^g / \mathbb{Z}^{2g}.$$

Here the subscript indicates we are taking the degree zero bundles. The Jacobian is a complex torus of complex dimension g.

When $\operatorname{rk} \mathcal{E} = 1$ and the degree is arbitrary, holomorphic bundles are classified by the sheaf cohomology group

$$\check{H}^1(\Sigma_g, \mathcal{O}^*) \cong \mathbb{Z} \times \operatorname{Jac}_0(\Sigma_g) =: \operatorname{Pic}(\Sigma_g),$$

commonly called the Picard variety of Σ_g. This variety is a disjoint union of $\mathbb{Z}$ copies of $\operatorname{Jac}_0(\Sigma_g)$, and one may pass between them by tensoring by a fixed line bundle of the correct degree. The degree d component of $\operatorname{Pic}(\Sigma_g)$ is denoted $\operatorname{Jac}_d(\Sigma_g)$.

What about when $\operatorname{rk} E > 1$?

The situation for line bundles indicates that it is too much to hope for for the classification to be discrete, as it was when considering smooth vector bundles. Instead one would like a *moduli space*, say $\mathcal{N}^g_{n,d}$, of holomorphic vector bundles of rank n, degree d, on a Riemann surface Σ_g. Notice this notation supresses the complex structure on Σ_g, only remembering the genus. It turns out that changing the complex structure does not change the topological type of $\mathcal{N}^g_{n,d}$, but only its own complex structure, so no more information is lost than there is when one writes Σ_g, which does not explicitly reference the complex structure of Σ_g.

Note that it is enough to just consider the possible holomorphic structures on a fixed smooth vector bundle $E \to \Sigma_g$ of rank n, degree d, because any other holomorphic bundle with such rank and degree must necessarily be smoothly isomorphic to E, so no information is lost.

2 Moduli of Stable Bundles

2.1 Dolbeault Operators

In order to define a moduli space, one needs a way of describing the set of all holomorphic structures. This can be done via algebraic geometry (Quot schemes) or differential geometry (Dolbeault operators). Each method has its own advantages and disadvantages, but we use the second, because it is easier to describe.

To begin, notice that given any holomorphic vector bundle $\mathcal{E} \to X$ over a complex manifold X, there is an operator that picks out the holomorphic sections of $\mathcal{E}$. Namely, in a local holomorphic chart given by a frame e_α, any section may be written as

$$s|_{U_\alpha} = \sum s^i e_i$$

for some functions s^i. Then define

$$\bar{\partial}_{\mathcal{E}}(s|_{U_\alpha}) := \sum \bar{\partial}(s^i) \otimes e_i.$$

This section of $\Omega^{0,1}(\mathcal{E}U_\alpha|_)$ will vanish if and only if s is holomorphic on U_α. Furthermore, on an overlap of trivialisations this operator is well-defined: a section s will differ in two trivialisations by a system of holomorphic transition functions (because $\mathcal{E}$ is holomorphic), and so the term containing $\bar{\partial} g_{\alpha\beta} = 0$ will vanish.

Flipping this around, we get the following definition:

Definition 2.1. A *Dolbeault operator* on a smooth complex vector bundle $E \to X$ on a complex manifold X is a $\mathbb{C}$-linear operator

$$\bar{\partial}_{\mathcal{E}} : \Gamma(X, E) \to \Omega^{0,1}(X) \otimes \Gamma(X, E) =: \Omega^{0,1}(E)$$

such that $\bar{\partial}_{\mathcal{E}}^2 = 0$, and

$$\bar{\partial}_{\mathcal{E}}(fs) = \bar{\partial}f \otimes s + f\bar{\partial}_{\mathcal{E}}s$$

for any function f on X and section s of E.

By an application of the Newlander-Nirenberg theorem, a Dolbeault operator defines a unique holomorphic structure for which it is the operator for (as constructed above), and the holomorphic sections of this holomorphic vector bundle $\mathcal{E}$ with underlying smooth bundle E are precisely those $s \in \Gamma(X, E)$ such that $\bar{\partial}_{\mathcal{E}}s = 0$.

Let $\mathrm{Dol}(E)$ denote the set of all Dolbeault operators on a fixed smooth complex vector bundle $E \to \Sigma_g$. Then the set of holomorphic structures on E is in bijection with $\mathrm{Dol}(E)$. There is however redundancy in this: two Dolbeault operators may give isomorphic holomorphic structures.

Two Dolbeault operators are said to be equivalent if there is an automorphism of the smooth bundle E which conjugates one to the other. The group of automorphisms (the *gauge group*), is denoted $\mathscr{G}_{\mathbb{C}}$.

Then we have that holomorphic structures on E up to isomorphism are in bijection with

$$\mathrm{Dol}(E)/\mathscr{G}_{\mathbb{C}}.$$

What does this set look like? Dolbeault operators are like connections with only a $(0,1)$-part. In particular, $\mathrm{Dol}(E)$ is an affine space modelled on the infinite-dimensional vector space $\Omega^{0,1}(\mathrm{End}(E)) = \Omega^{0,1}(\Sigma_g) \otimes \Gamma(\Sigma_g, E)$. One may put a reasonable topology on this affine space, but the quotient $\mathrm{Dol}(E)/\mathscr{G}_{\mathbb{C}}$ is not Hausdorff! To get around this, we need to use geometric invariant theory (GIT).

2.2 The Moduli Space

Definition 2.2. A holomorphic bundle $\mathcal{E} \to \Sigma_g$ is *(semi-)stable* if for all $\mathcal{F} \subset \mathcal{E}$ proper non-zero holomorphic subbundles, we have

$$\frac{\deg \mathcal{F}}{\mathrm{rk}\, \mathcal{F}} < (\leq) \frac{\deg \mathcal{E}}{\mathrm{rk}\, \mathcal{E}}.$$

We write

$$\mu(\mathcal{E}) := \frac{\deg \mathcal{E}}{\mathrm{rk}\, \mathcal{E}}$$

for the quotient appearing here, which is called the *slope* of the vector bundle. Denote by $\mathrm{Dol}(E)^s$ and $\mathrm{Dol}(E)^{ss}$ the sets of stable an semi-stable holomorphic bundles inside $\mathrm{Dol}(E)$.

Then geometric invariant theory says that $\mathrm{Dol}(E)^{ss} /\!\!/ \mathscr{G}_{\mathbb{C}}$ and $\mathrm{Dol}(E)^s/\mathscr{G}_{\mathbb{C}}$ are Hausdorff (and in fact much more). Here the double slash means there is something more going on in this quotient (one has to identify *S-equivalence classes of semi-stable bundles*, and in fact this is necessary: there is no reasonable way of making $\mathrm{Dol}(E)^{ss}/\mathscr{G}_{\mathbb{C}}$ Hausdorff in the strictly semi-stable case).

Problem 2.3. When $(n,d) = 1$, (i.e. rank and degree of E are coprime) then semi-stable $\Leftrightarrow$ stable.

Definition 2.4. The *moduli space of stable holomorphic vector bundles of rank n and degree d over a Riemann surface Σ_g* is

$$\mathcal{N}_{n,d}^g := \mathrm{Dol}(E)^s/\mathscr{G}_{\mathbb{C}}.$$

Theorem 2.5 (Mumford, Narasimhan-Seshadri, Atiyah-Bott, Ramanan, others..)**.** *When $(n,d) = 1$, $\mathcal{N}_{n,d}^g$ is a non-singular, projective complex algebraic variety, and a fine moduli space for the classification problem we are considering (i.e. there is a universal bundle over $\mathcal{N}_{n,d} \times \Sigma_g$ which restricts to each given holomorphic vector bundle $\mathcal{E}$ on each slice $\{[\mathcal{E}]\} \times \Sigma_g$).*

Theorem 2.6 (Narasimhan-Seshadri '65, Donaldson '82)**.** *The following three spaces are isomorphic:*

1. $\mathcal{N}_{n,d}^g$
2. *Moduli space of projectively flat irreducible connections on the underlying smooth bundle $E \to \Sigma_g$*
3. *The* character variety $\mathrm{Hom}_d^{irr}(\hat{\pi}_1(\Sigma_g), \mathrm{U}(n))/U(n)$ *classifying irreducible projective unitary representations of the fundamental group of Σ_g (of a certain type).*

The equivalence (1) $\Leftrightarrow$ (3) was the original theorem of Narasimhan-Seshadri, and uses algebraic geometry and representation theory. The equivalence (2) $\Leftrightarrow$ (3) is given by taking the holonomy of the connection, and in the other direction by constructing the associated bundle to the universal $\pi_1(\Sigma_g)$-bundle over Σ_g given by its universal cover. The equivalence (1) $\Leftrightarrow$ (2) was proven by Donaldson, who used gauge theory techniques.

It was proven by Goldman that the representation variety is symplectic, and by Atiyah-Bott that the moduli space of flat connections is symplectic (this structure is called the *Atiyah-Bott symplectic form*). These two symplectic structures agree. Since $\mathcal{N}_{n,d}^g$ is naturally a complex manifold, it turns out that through the Narasimhan-Seshadri theorem it is a compact Kähler manifold (at least when $(n,d) = 1$).

The dimension of moduli space is given by

$$\dim_{\mathbb{R}} \mathcal{N}_{n,d}^g = 2 + 2n^2(g-1).$$

Remark 2.7. In the case $n = 1, d = 0$ one should expect to recover the Jacobian variety of the Riemann surface Σ_g, which classifies holomorphic line bundles of degree 0. This is indeed the case, and it is even possible to see how the infinite dimensional quotients here can be reduced to the quotient

$$\mathrm{Jac}_0(\Sigma_g) \cong H^1(\Sigma_g, \mathcal{O})/H^1(\Sigma_g, \mathbb{Z}).$$

For more details see the notes of Goldman-Xia on Higgs bundles of rank one.

Remark 2.8. By taking the map induced by tensoring with a line bundle of fixed degree d_0, one obtains an isomorphism between the moduli spaces $\mathcal{N}^g_{n,d}$ and $\mathcal{N}^g_{n,d+nd_0}$. In particular the moduli space for fixed Σ_g depends only on the pair $(n, d \mod n)$!

One can throw in a fourth equivalent characterisation of the moduli space. Namely, if $\bar{\partial}_{\mathcal{E}}$ is a stable vector bundle, then by the theorem there is a Hermitian metric on E such that the associated Chern connection ∇ is projectively flat. Concretely this means

$$\star F_\nabla = -2\pi i \mu(E)$$

where we have chosen a Riemannian metric on Σ_g and normalised so that $\mathrm{Vol}(\Sigma_g) = 1$. In other words $F_\nabla = -2\pi i \mu(E) \mathbf{1}_E \otimes \omega$ where ω is the associated Kähler form on Σ_g to the Riemannian metric. In particular notice that

$$d_\nabla \star F_\nabla = 0,$$

that is, ∇ solves the *Yang-Mills equations*! The Narasimhan-Seshadri theorem can be rephrased as saying the moduli space of stable vector bundles is isomorphic to the moduli space of solutions to the Yang-Mills equations!.

Remark 2.9. All of this above was formal, in so far as to actually set up the problem in this form, one should complete the spaces $\mathrm{Dol}(E)$ and $\mathscr{G}_\mathbb{C}$ to Banach manifolds and Banach Lie groups, and prove some hard analytical details. In fact Atiyah-Bott prove that every L^2-equivalence class of connections contains a smooth representative, and if two such smooth connections are gauge-equivalent by an L^2-gauge transformation, then they are equivalent by a smooth gauge transformation. The upshot of this is that the quotient spaces will be isomorphic, and we don't have to care and can just formally write everything down as we have done!

Question: What if we replace $\mathrm{U}(n)$ in the Narasimhan-Seshadri theorem with $\mathrm{GL}(n, \mathbb{C})$? Or equivalently, if we replace "projectively flat unitary connections" with "projectively flat connections"?

Answer:

There is a general principle in geometry that has appeared since the work of Donaldson on the Narasimhan-Seshadri theorem:

Stable algebraic objects correspond to extremal objects in differential geometry

Remark 2.10. There are now many instances of this principle being realised. The most concrete example is the theorem above, but arguably the first example is the uniformization theorem: every smooth curve (Deligne-Mumford showed that smooth curves are stable) admits a metric of constant scalar curvature either $+1, 0$, or -1. These metrics are extremal in that they are Kähler-Einstein. The Narasimhan-Seshadri has also been generalised significantly (in at least one direction) to the *Hitchin-Kobayashi correspondence* (or Donaldson-Uhlenbeck-Yau theorem), which says that (poly)stable holomorphic vector bundles over compact Kähler manifolds (in fact work of Li-Yau allows you to remove Kähler) correspond to Hermitian Yang-Mills connections.

Another instance of the principle is for manifolds themselves: a theorem of Chen-Donaldson-Sun from 2012 says that a Fano manifold is Kähler-Einstein if and only if it is *K-polystable*, and it is conjectured (the Yau-Tian-Donaldson conjecture) that a compact Kähler manifold admits a constant scalar curvature metric if and only if it is K-polystable.

In our case the principle indicates there should be an algebraic object and a stability condition so that stable objects correspond to these connections.

The answer of course is **Higgs bundles**.

3 Higgs Bundles

Definition 3.1 (Higgs bundle)**.** A *Higgs bundle* is a pair

$$(\mathcal{E}, \Phi)$$

where $\mathcal{E} \to \Sigma_g$ is a holomorphic vector bundle and

$$\Phi : \mathcal{E} \to \mathcal{E} \otimes K$$

is an $\operatorname{End}\mathcal{E}$-valued holomorphic $(1,0)$-form on Σ_g. That is,

$$\Phi \in H^0(\Sigma_g, \operatorname{End}\mathcal{E} \otimes K).$$

These bundles were first investigated by Hitchin in 1987 (although the term Higgs bundle and their pure algebraic definition was probably coined by Carlos Simpson a few years later). The field Φ is called the Higgs field of the Higgs bundle, and is so named by analogy with the Higgs field in physics: an additional scalar field "coupled" to other particle fields (i.e. there are terms in the Lagrangian of the standard model involving both this scalar field and other fields) in such a way that it imbues particles with mass. Any time someone adds on an auxilliary field that is coupled to the original data in geometry, they call it a Higgs field (see: magnetic monopoles/the Bogomolny equation).

If we want to make a moduli space, then we should rephrase the definition in terms of Dolbeault operators. A Higgs bundle is given by a pair

$$(\bar{\partial}_{\mathcal{E}}, \Phi)$$

such that $\bar{\partial}_{\mathcal{E}}(\Phi) = 0$, this condition implying that Φ is holomorphic. Our analogue of $\mathrm{Dol}(E)$ will be

$$\mathcal{B} := \{(\bar{\partial}_{\mathcal{E}}, \Phi) \in \mathrm{Dol}(E) \times \Omega^{1,0}(\mathrm{End}(E)) \mid \bar{\partial}_{\mathcal{E}}(\Phi) = 0\}.$$

If everything is set up right then $\mathcal{B}$ is an infinite-dimensional orbifold, and there is an action of $\mathscr{G}_{\mathbb{C}}$ on $\mathcal{B}$ by conjugation on both the Dolbeault operator and the Higgs field:

$$g \cdot (\bar{\partial}_{\mathcal{E}}, \Phi) = (g\bar{\partial}_{\mathcal{E}}g^{-1}, g\Phi g^{-1}).$$

Definition 3.2. Call a Higgs bundle $(\mathcal{E}, \Phi)$ *(semi-)stable* if

$$\mu(\mathcal{F}) < (\leq)\, \mu(\mathcal{E})$$

for all proper, non-trivial, **Φ-invariant** subbundles $\mathcal{F} \subset \mathcal{E}$.

Example 3.3.

1. If $\mathrm{rk}\,\mathcal{E} = 1$, then $(\mathcal{E}, \Phi)$ is stable for any $\Phi \in H^0(\Sigma_g, K)$, a holomorphic 1-form on Σ_g.
2. If $\mathcal{E}$ is (semi-)stable, then $(\mathcal{E}, \Phi)$ is (semi-)stable for any Φ.
3. Let $g > 1$ and fix a square root $K^{1/2}$ of the canonical bundle $K \to \Sigma_g$. Define a holomorphic vector bundle by

$$\mathcal{E} := K^{1/2} \oplus K^{-1/2}.$$

Then $\mu(\mathcal{E}) = 0$. The endomorphisms of $\mathcal{E}$ split under the direct sum, and one piece is $\mathrm{Hom}(K^{1/2}, K^{-1/2}) \cong K^{-1}$. Thus $\mathrm{Hom}(K^{1/2}, K^{-1/2}) \otimes K \cong \mathcal{O}$, which has the constant section 1. Thus we can take

$$\Phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(i.e. just take Φ to be zero on all other components). The only Φ-invariant subbundle is $K^{-1/2}$, which has degree $1 - g < 0$ for $g > 1$, and so $(\mathcal{E}, \Phi)$ is a stable Higgs bundle! Notice that $\mu(K^{1/2}) > 0$ so in fact $\mathcal{E}$ is not stable as a holomorphic vector bundle, but $K^{1/2}$ is not Φ-invariant so does not destabilise $(\mathcal{E}, \Phi)$.

4. If $(\mathcal{E}, \Phi)$ is (semi-stable), then so is $(\mathcal{E}, t\Phi)$ for any $t \in \mathbb{C}^*$. This will give us a $\mathbb{C}^*$ action on the moduli space of Higgs bundles.

Definition 3.4. Define the *moduli space of stable Higgs bundles of rank* n*, degree* d *on a Riemann surface* Σ_g by

$$\mathcal{M}_{n,d}^g := \mathcal{B}^s / \mathscr{G}_{\mathbb{C}},$$

where $\mathcal{B}^s$ denotes the subset of $\mathcal{B}$ consisting of stable Higgs bundles.

Theorem 3.5 (Hitchin '87, Simpson '90s, Donaldson-Corlette)**.** *The following spaces are isomorphic:*

1. $\mathcal{M}_{n,d}^g$
2. *The moduli space of irreducible projectively flat connections on* $E \to \Sigma_g$
3. *The character variety* $\operatorname{Hom}_d^{irr}(\hat{\pi}_1(\Sigma_g), \operatorname{GL}(n,\mathbb{C}))/\operatorname{GL}(n,\mathbb{C})$ *classifying complex representations of the fundamental group (of a certain type).*

This theorem is sometimes called the *non-Abelian Hodge theorem* for reasons that we will explain later. Hitchin proved (1) $\Leftrightarrow$ (2), in the case of $n = 2, d = 1$ (and fixed determinant bundle). Work of Donaldson-Corlette on harmonic representations gives (2) $\Leftrightarrow$ (3), and Carlos Simpson proved the rest of the theorem for any (n, d).

Remark 3.6. The Narasimhan-Seshadri theorem can be recovered by taking $\Phi = 0$ in this theorem. This theorem has also been significantly generalised, primarily by Carlos Simpson, to the case of Higgs bundles over any compact Kähler manifold (where one also requires the condition $\Phi \wedge \Phi = 0$ in general). This theorem is usually known as the *non-Abelian Hodge theorem*, but I prefer the name *Narasimhan-Seshadri-Hitchin-Kobayashi-Donaldson-Uhlenbeck-Yau-Simpson-Corlette theorem.*

What is $\mathcal{M}_{n,d}^g$ like? Again in the situation where $(n, d) = 1$, $\mathcal{M}_{n,d}^g$ is a non-singular quasi-projective complex algebraic variety. Notice how before $\mathcal{N}_{n,d}^g$ became Kähler by using a symplectic structure coming from the representation space. In this setting the representation space is itself a complex manifold, so $\mathcal{M}_{n,d}^g$ has two different complex structures, I, J say! In fact, one can show these are distinct, and further that $IJ = -JI$. That is, if one writes $K = IJ$, then the triple (I, J, K) turns $\mathcal{M}_{n,d}^g$ into a *hyper-Kähler manifold*! The moduli space of projectively flat connections also has a complex structure, but it turns out this is the same as J, the structure coming from the representations.

The dimension is $\dim_{\mathbb{R}} \mathcal{M}_{n,d}^g = 4 + 4n^2(g-1)$, which is twice the dimension fo $\mathcal{N}_{n,d}^g$. This is not a coincidence! Given a stable vector bundle $\mathcal{E}$, standard moduli space voodoo tells you that the tangent space to $\mathcal{N}^g_{n,d}$ at $[\mathcal{E}]$ is given by the cohomology group $H^1(\Sigma_g, \mathrm{End}(\mathcal{E}))$. The Serre duality theorem tells you that

$$T^*_{[\mathcal{E}]}\mathcal{N}^g_{n,d} \cong H^1(\Sigma_g, \mathrm{End}\,\mathcal{E})^* \cong H^0(\Sigma_g, \mathrm{End}(\mathcal{E}) \otimes K),$$

which is precisely the space of compatible Higgs fields for $\mathcal{E}$! Namely, there is an inclusion

$$T^*\mathcal{N}^g_{n,d} \subset \mathcal{M}^g_{n,d}$$

of the cotangent bundle to the stable bundle moduli space into the Higgs bnudle moduli space! This subset is open and dense, and the complement has (in most cases) complex codimension ≥ 2. In particular $\mathcal{M}^g_{n,d}$ has dimension twice that of $\mathcal{N}^g_{n,d}$.

Remark 3.7. Notice that the representation variety in the Higgs case is an affine variety! In particular the different complex structures on $\mathcal{M}^g_{n,d}$ are very different: one is (essentially) the cotangent bundle of a projective variety, whereas the other is a Stein manifold.

Example 3.8. In the rank 1 case, there are no Higgs bundles that aren't arising from stable vector bundles (of course everything is stable), so there is an isomorphism

$$\mathcal{M}^g_{1,d} \cong T^*\mathcal{N}^g_{1,d} \cong T^*\,\mathrm{Jac}_d(\Sigma_g) \cong (\mathbb{C}^g/(\mathbb{Z}^{2g})) \times \mathbb{R}^{2g}.$$

In this case the complex structures and geometric structures on $\mathcal{M}^g_{1,d}$ are all very explicit (see the notes of Goldman-Xia on Higgs bundles of rank one). In this case the theorem for Higgs bundles says

$$T^*\,\mathrm{Jac}_0(\Sigma_g) \cong (\mathbb{C}^*)^{2g},$$

but only as smooth manifolds! These spaces have different explicit complex structures, and one may verify they satisfy $IJ = -JI$!

What else is known about these moduli spaces?

Theorem 3.9 (Hitchin)**.** *The Betti polynomial of $\hat{\mathcal{M}}^2_{2,1}$ (the hat means fixed holomorphic structure on the determinant of $\mathcal{E}$) is*

$$P_t(\hat{\mathcal{M}}^2_{2,1}) = 1 + t^2 + 4t^3 + 2t^4 + 34t^5 + 2t^6,$$

and $\dim_{\mathbb{R}} \hat{\mathcal{M}}^2_{2,1} = 12$.

We remark that all the things said about the Higgs moduli space (and indeed stable bundle moduli space) hold when we fix a determinant line bundle. In some sense this is a simpler space (when you don't fix a line bundle, there is some redundancy in the moduli space coming from an action of the Jacobian $\mathrm{Jac}_0(\Sigma_g)$ which you can get rid of, and the "interesting parts" of the moduli space come from the space where you have fixed the determinant).

Remark 3.10. Notice that whilst the dimension of $\hat{\mathcal{M}}^2_{2,1}$ is 12, it only has Betti numbers up to dimension 6. This phenomenon always occurs, and is a result of the remark we made earlier that one of the complex structures on $\mathcal{M}^g_{n,d}$ is as an affine variety (or Stein manifold). It is a remarkable theorem that every Stein manifold of complex dimension n actually has the homotopy type of a space of *real* dimension n, and therefore only has Betti numbers up to half of its real dimension! Even though $\mathcal{M}^g_{n,d}$ is not affine with respect to its regular construction, its hyper-Kähler structure causes this phenomenon!

Through work of Atiyah-Bott, Harder-Narasimhan, and Newstead, the Betti numbers of $\hat{\mathcal{N}}^g_{n,d}$ are known for all (n, d, g). In fact this was first proved by Harder-Narasimhan in 1975 by counting the number of points in $\hat{\mathcal{N}}^g_{n,d}$ over finite fields and applying the Weil conjectures (which had only just been proven by Deligne). Atiyah and Bott used a very clever infinite-dimensional equivariant Morse-Bott theory to arrive at the same Betti numbers (essentially: the norm-squared of the curvature $f(\nabla) = \|F_\nabla\|^2$ is an equivariantly perfect Morse-Bott function for the action of the gauge group $\mathscr{G}$).

In contrast, not much is known about the Betti numbers of the Higgs moduli space in general, and it is only computed up to (I believe) $n = 4$. There is even less known about the cohomology ring structure, which is only known up to $n = 3$ (but probably in much more generality for the stable bundle moduli space).

4 Applications

4.1 Hitchin Integrable System

Since $\Phi \in H^0(\Sigma_g, \mathrm{End}(\mathcal{E}) \otimes K)$,

$$\frac{1}{k}\operatorname{tr}(\Phi^k) \in H^0(K^k).$$

Define a map

$$h : \mathcal{M}^g_{n,d} \to \bigoplus_{k=1}^n H^0(\Sigma_g, K^k) =: \mathcal{A}.$$

Theorem 4.1 (Hitchin)**.**

$$\dim \mathcal{A} = \frac{1}{2}\dim \mathcal{M}^g_{n,d} = 2 + 2n^2(g-1),$$

and h is a completely integrable Hamiltonian system.

Such completely integrable systems are generally quite rare. In 2010 Ngõ used the Hitchin integrable system over finite fields to prove the Fundamental Lemma in the Langlands program, for which he was awarded the Fields medal!

4.2 Mirror Symmetry

It is possible to define $\mathcal{M}(G)$, the moduli space of G-Higgs bundles for arbitrary Lie groups $G = \mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C}), \mathrm{PGL}(n,\mathbb{C}), \ldots$ (where GL is the regular situation and SL is the fixed determinant situation). Essentially this is done by considering holomorphic principal G-bundles (in fact one can even define this notion for real compact Lie groups).

The Hitchin system $h : \mathcal{M}(G) \to \mathcal{A}$ is generally regular, and $h^{-1}(p)$ is an Abelian variety for $p \in \mathcal{A}^{reg}$ (in fact, there is a special curve called the *spectral curve* such that $h^{-1}(p)$ is the Jacobian of this curve).

For $\mathcal{M}(\mathrm{PGL}(n,\mathbb{C}))$ and $\mathcal{M}(\mathrm{SL}(n,\mathbb{C}))$, $\dim \mathcal{A}$ is the same and there exists a diagram such that $h_1^{-1}(p)^{\vee} \cong h_2^{-1}(p)$ for $p \in \mathcal{A}^{reg}$. That is, the Abelian varieties

$$\begin{tikzcd}
\mathcal{M}(\mathrm{PGL}(n,\mathbb{C})) \arrow[dr, "h_1"] & & \mathcal{M}(\mathrm{SL}(n,\mathbb{C})) \arrow[dl, "h_2"] \\
& \mathcal{A} &
\end{tikzcd}$$

over a point are dual.

This is an example of an SYZ fibration in Mirror symmetry, and it is expected this works for any pair G, ${}^LG$ where ${}^LG$ is the Langlands dual group.

Hitchin has proven this for $G = \mathrm{Sp}(n)$, ${}^LG = \mathrm{SO}(2n+1)$ and $G =^L G = G_2$!

It turns out that these mirror manifolds do not neccesarily satisfy precisely the Hodge diamond mirror relations that mirror symmetry predicts, but do satisfy an alternative set of relations between Hodge numbers. This is work of Hausel and Thaddeus.

4.3 Physics

$\mathcal{N}^g_{n,d}$ and $\mathcal{M}^g_{n,d}$ are the configuration spaces of Chern-Simons gauge theory for $\mathrm{U}(n)$, $\mathrm{GL}(n,\mathbb{C})$ in $2+1$ dimensions, for the three-manifolds $\Sigma_g \times [0,1]$.

In 2007, Kapustin-Witten used $\mathcal{M}^g_{n,d}$ to describe S-duality in string theory, and mathematically this translates to a "geometric Langlands correspondence."

4.4 Non-Abelian Hodge Theorem

The Hodge theorem states that

$$H^1(\Sigma_g, \mathbb{C}) = H^{1,0}(\Sigma_g) \oplus H^{0,1}(\Sigma_g) = H^1(\Sigma_g, \mathcal{O}) \oplus H^0(\Sigma_g, K).$$

And it is straightforward to see

$$H^1(\Sigma_g, \mathbb{C}) \cong H^1(\pi_1(\Sigma_g), \mathbb{C}) = \mathrm{Hom}(\pi_1(\Sigma_g), \mathbb{C}).$$

The non-Abelian Hodge theorem in higher dimensions takes in a representation of the fundamental group modulo conjugation, i.e. an element of

$$H^1(\Sigma_g, \mathrm{GL}(n, \mathbb{C})) = H^1(\pi_1(\Sigma_g), \mathrm{GL}(n, \mathbb{C})) = \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{GL}(n, \mathbb{C}))/\, \mathrm{GL}(n, \mathbb{C}),$$

and produces a holomorphic vector bundle and a Higgs field, i.e. an element of

$$H^1(\Sigma_g, \mathcal{GL}(n, \mathbb{C})) \oplus H^0(\Sigma_g, \mathcal{GL}(n, \mathbb{C}) \otimes K)$$

where we are taking the sheaf of holomorphic sections of $\mathrm{GL}(n, \mathbb{C})$. Since these groups are non-Abelian the corresponding cohomology groups are not actually groups, but only pointed sets (with distinguished point corresponding to zero). This is where the name "non-Abelian Hodge theorem" comes from.

Obviously this is only formally true, and not literally an isomorphism of cohomology sets.

References

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