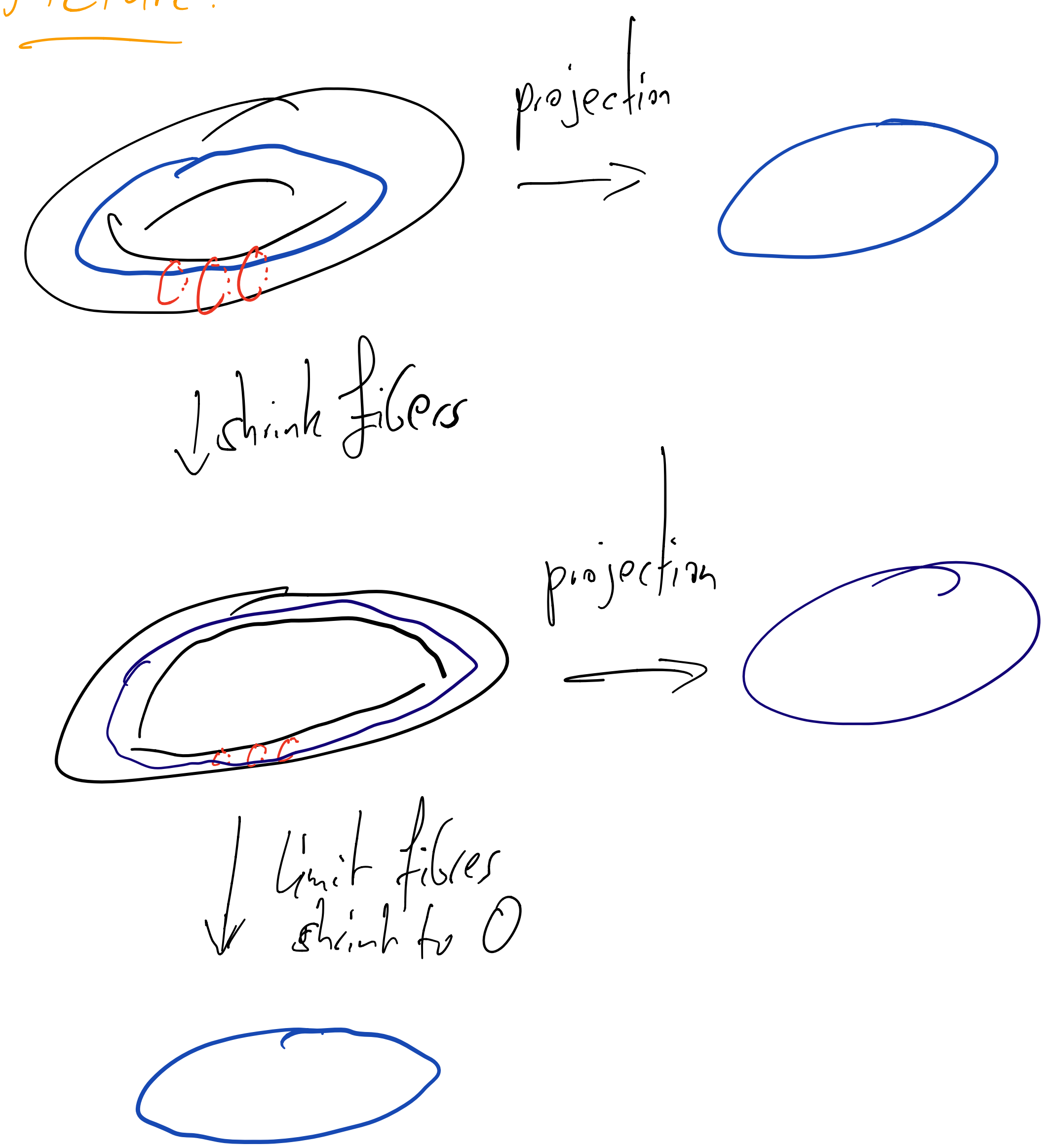
Adiabatic Limits

Technique in Riemannian geometry to study fibrations

Picture:



In practice: Let volume of base grow and keep fibers of ~~fixed~~ fixed size

I elliptic curve example

II general background about differential geometry on fibrations

III Leray spectral sequence

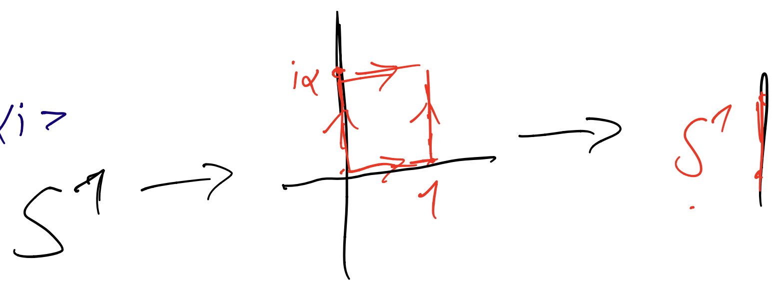
IV Constant scalar curvature Kähler metrics

I Elliptic curves

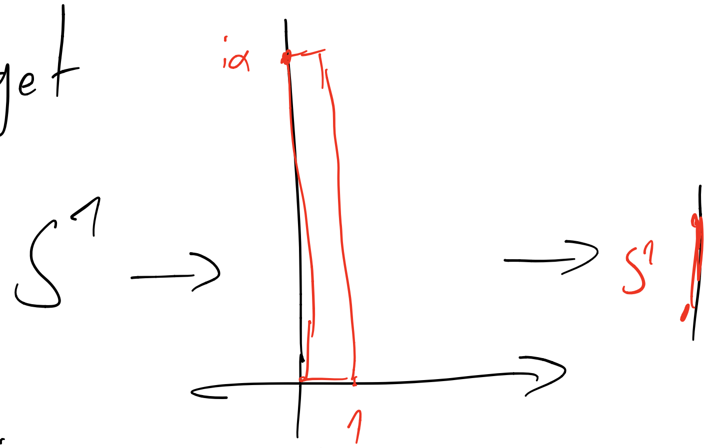
Source: Gross–Wilson "Large complex structure limits of K3-surfaces"

Let $\alpha \in \mathbb{R}_{>0}$, get an elliptic curve

1) $E_\alpha := \mathbb{C}/\langle 1, \alpha i \rangle$



For large α get



in Euclidian metric

$$\begin{array}{ccc} & & S^1 \\ & & \downarrow \\ \mathbb{C}/\langle 1,\alpha i\rangle & \xrightarrow{\alpha \to \infty} & \text{infinite cylinder} \\ & & \downarrow \\ & & \mathbb{R}^1 \end{array}$$

base changed

2) rescale metric as $g'_\alpha = \dfrac{g_{Eucl}}{\alpha}$

then $(E_\alpha, g'_\alpha) \overset{\text{isometric}}{\simeq} \left(\mathbb{C}/\langle \frac{1}{\alpha}, i\sqrt{\alpha} \rangle, g_{Eucl}\right)$

$(E_\alpha, g'_\alpha) \xrightarrow{\alpha \to \infty}$ straight line

fibers collapse to 0

3) rescale $g''_\alpha = \dfrac{g_{Eucl}}{\alpha^2}$

$(E_\alpha, g''_\alpha) \overset{\text{isometric}}{\simeq} \left(\mathbb{C}/\langle \frac{1}{\alpha^2}, i \rangle, g_{Eucl}\right)$

$(E_\alpha, g''_\alpha) \xrightarrow{\alpha \to \infty}$ circle of circumference 1

fibres collapse to points

Rem: Gross-Wilson do something similar with elliptically fibred K3's and this allows them to construct approximately Ricci flat metrics on these K3's

II Differential geometry on fibrations

Let's fix a fibration (smooth submersion)

$$\underset{\text{fibre}}{F} \to \underset{\text{total space}}{M} \xrightarrow{\pi} \underset{\text{base space}}{B}$$

The fibration has a vertical bundle

$$V := \ker(\pi_*) \subseteq TM$$

for any $p \in M$ naturally have $T_pF \simeq V_p$

But: There's no canonical complement to $V \subseteq TM$

A choice of a complementary subbundle $H \subseteq TM$
i.e. s.t. $H \oplus V = TM$ is called a connection

Pointwise, there's a canonical isomorphism

$$H_p \xrightarrow{\cong} T_{\pi(p)} B$$

Rem: This is analogous to Gauge theory but a bit more complicated. In gauge theory language this is a connection on the associated bundle

$$\underbrace{\mathrm{Diff}}_{\text{infinite dimensional}}(F) \longrightarrow \tilde{M} \longrightarrow B$$

Splitting forms: The splitting $H \oplus V = TM$ induces a splitting

$$\Lambda^k T^* M = \bigoplus_{p+q=k} \Lambda^p T^* B \otimes \Lambda^q T^* F$$

Prop: The exterior derivative decomposes as

$$d : \Omega^{p,q}M \longrightarrow \Omega^{p-1,q+2} \oplus \Omega^{p,q+1} \oplus \Omega^{p+1,q}$$

$$d = \underbrace{R_H}_{\text{curvature}} + \underbrace{d_f}_{\text{fibre derivative}} + \underbrace{d_H}_{\text{horizontal derivative}}$$

III Leray spectral sequence

For simplicity assume B is simply connected.

Quick recap of Hodge theory:

(N^n, g) riemannian manifold

- Hodge $*$-operator $* : \Lambda^k N \longrightarrow \Lambda^{n-k} N$
- codeirivative $d^* : \Lambda^k N \longrightarrow \Lambda^{k-1} N$

$$\begin{tikzcd} \Lambda^k N \arrow[r, "d^*"] \arrow[d, "*"] & \Lambda^{k-1} N \\ \Lambda^{n-k} N \arrow[r, "d"'] & \Lambda^{n-k+1} \arrow[u, "*"'] \end{tikzcd}$$

- Hodge-Laplacian $\Delta := dd^* + d^*d$

$$\Delta : \Lambda^k N \longrightarrow \Lambda^k N$$

Def: A form $\alpha \in \Omega^k N$ is called harmonic if one of the equivalent statements is true

i) $\Delta \alpha = 0$

ii) $d\alpha = 0$ and $d^*\alpha = 0$

Denote harmonic k-forms by $\mathcal{H}^k$

Theorem (Hodge) If N is compact, the map

$$\mathcal{H}^k \longrightarrow H^k_{dR} = \ker(d_k) / \operatorname{Im}(d_{k-1})$$

$$\omega \longmapsto [\omega]$$

is an isomorphism.

"Every deRham cohomology class has a unique harmonic representative"

$F \to M \to B$

Thm (Leray–Serre)

There is a spectral sequence

$$E_2^{p,q} := H^p(B, H^q(F)) \Rightarrow H^{p+q}(M)$$

should be harmonic forms on B with values in the harmonic forms on F

should be harmonic forms on M

Serre's proof is entirely based on algebraic topology and homological algebra.

The adiabatic limit technique allows to prove this result Hodge-theoretically, which is more geometric.

Hodge-Leray spectral sequence Fix $\delta > 0$ "adiabatic parameter"

Define a rescaled differential

$$d_\delta = d_H + \delta d_f + \delta^2 R_H$$

i.e. scale by δ for each fibre direction

$$d_\delta^* = d_H^* + \delta d_f^* + \delta^2 R_H^*$$

This gives a rescaled Laplacian $L_\delta = d_\delta d_\delta^* + d_\delta^* d_\delta$

Define

I'm cheating here, this should be Sobolev completed

$$E_k^p = \left\{ \omega \in \Omega^p \;\middle|\; \exists\, \omega_1, \dots, \omega_j \text{ s.t. } \begin{array}{l} d(\omega + \delta\omega_1 + \dots + \delta^j \omega_j) \in \delta^k \Omega^{p+1}[\delta] \\ d_\delta^*(\omega + \delta\omega_1 + \dots + \delta^j\omega_j) \in \delta^k \Omega^{p+1}[\delta] \end{array} \right\}$$

"Taylor expansion in δ"

formal polynomial ring in δ

Let's see $k = 0, 1, 2$

$$d_\delta(\omega + \delta\omega_1 + \delta^2\omega_2)$$
$$= d_H\omega + \delta(d_f\omega + d_H\omega_1) + \delta^2(R_H\omega + d_f\omega_1 + d_H\omega_2) + O(\delta^3)$$

$$\rightsquigarrow \quad E_0^p = \Omega^p, \; E_1^p = \{\omega \in \Omega^p \mid d_H\omega = 0 \;\; d_H^*\omega = 0\}$$

$$E_2^p = \left\{ \omega \in \Omega^p \;\middle|\; \exists\, \omega' \text{ s.t. } \begin{array}{ll} d_H\omega = 0, & d_f\omega = d_H\omega' \\ d_H^*\omega = 0 & d_f^*\omega = d_H^*\omega' \end{array} \right\}$$

E_1 are forms harmonic in horizontal direction

E_2 are harmonic in horizontal direction and have a restriction on their vertical Laplacian direction

The splitting $\Omega^* = \bigoplus \Omega^{p,q}$ gives a bigrading $E_k^{p,q}$

on on E_k^m. There further is a differential ∂ (which is slightly complicated) on E_k^*

Theorem: (Mazzeo-Melrose - Forman-Dai)

$(E_*^{p,q}, \partial)$ is a spectral sequence and is isomorphic to the standard Leray spectral sequence.

IV constant scalar curvature metrics

Setup: $F^1 \longrightarrow X^2 \longrightarrow \Sigma^1$

a holomorphic fibration of a compact complex surface X.

- assume $g(F) \geq 2$ $\rightsquigarrow$ F admits a hyperbolic metric of constant scalar curvature -1
- $g(\Sigma) \geq 1$

Recall: If ω is a Kähler form on X, then the scalar curvature $\mathrm{Scal}(\omega_R)$ is given

by $\rho \wedge \omega = \mathrm{Scal}(\omega)\, \omega^2$

Ricci form $\quad \rho(X,Y) = \mathrm{Ric}(IX,Y)$ Ricci tensor

complex structure

Note that $\mathrm{Scal}(\omega)$ is a smooth function on X

Goal: Find metric with $\mathrm{Scal}(\omega) = \text{constant}$

Let $V \subseteq TX$ be the vertical bundle

Define for $r>0$ a cohomology class

$$\kappa_r := -c_1(V) - r\,\pi^* c_1(\Sigma)$$

fiber direction

scale parameter

base direction

Thm (Fine) For large r the class κ_r contains a constant scalar curvature metric.

Let's have a brief outline of this result and why the r large is needed.

i) Construct approximate solutions $\omega_{r,n}$ such that

$$\mathrm{Scal}(\omega_{r,n}) = -1 + \sum_{i=1}^{n} c_i r^{-i} + O(r^{-n-1})$$

for constants c_i

ii) Setup analytic machinery to solve the (nonlinear) PDE

$$S : C^\infty(X,\mathbb{R}) \longrightarrow C^\infty(X,\mathbb{R})$$
$$\phi \longmapsto \mathrm{Scal}(\omega_{r,n} + i\partial\bar\partial\phi)$$

$$S(\phi) \overset{!}{=} \text{constant}$$

iii) Do sophisticated analysis to use the (quantitative) inverse function theorem in the style
"If there's a solution close to a constant one, then there's a constant one"

iv) Choosing n, r large enough yields a solution close to a constant one
$\overset{\text{iii)}}{\Longrightarrow}$ constant solution $\square$

Finding the first order solution:

Want to construct ω_r with $\mathrm{Scal}(\omega_r) = -1 + O(r^{-1})$

Step 1: Each fiber has a unique hyperbolic metric of constant scalar curvature -1 $\Rightarrow$ yields hermitian metric on vertical bundle V

$\rightsquigarrow$ defines a unique Chern connection ∇ on V

$\rightsquigarrow$ the curvature of ∇ gives a $(1,1)$-form

$$\omega_0 = -i F_\nabla$$

Fact: ω_0 restricts to hyperbolic Kähler forms on fibres

ω_0 further defines a connection H on the fibration via

$$H_x = \{ u \in T_x X \mid \omega_0(u,v) = 0 \quad \forall v \in V_x \}$$

Choose a Kähler form ω_Σ on base Σ

(we'll make a better choice later)

Pointwise can write

$$\omega_0 = \underbrace{\omega_\sigma}_{\substack{\text{purely}\\ \text{vertical}\\ \text{(hyperbolic Kähler form)}}} \oplus \overset{\text{some function}}{\theta} \underbrace{\omega_\Sigma}_{\substack{\text{purely}\\ \text{horizontal}}}$$

Hence for $r >-\inf \theta$ the form

$$\omega_r = \omega_0 + r\pi^* \omega_\Sigma$$

is a Kähler form!

This is one instance where we need r large enough

Prop: $\mathrm{Scal}(\omega_r) = -1 + r^{-1}(\mathrm{Scal}(\omega_\Sigma) - \theta + \Delta_V(\theta)) + O(r^{-2})$

Then continue with finding a good ω_Σ