A class of approximate Greek weights

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Asset price dynamics

- Process $X = (X_t)_{t \geq 0}$ take values in $\mathbb{R}^d$, with dynamics described by the SDE

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + \gamma(X_t)\mathrm{d}W_t \,, \quad X_0 = x \in \mathbb{R}^d \,, \qquad (1)$$

where $W = (W_t)_{t \geq 0}$ is a Brownian motion in $\mathbb{R}^m$.

- Let $n \in \mathbb{N}^+$ be a positive integer and $T > 0$ a fixed time.
- Define a partition on the interval $[0, T]$ by

$$\pi := \{0 = t_0 < t_1 < \ldots < t_n = T\}.$$

Option Price and Greeks

- Let g be a function of process X at terminal time T.
- **Option price** $V(x)$, given the initial condition $X_0 = x$:

$$V(x) := \mathbb{E}\left[g(X_{t_n}) | X_0 = x\right] , \qquad \hat{V}^N(x) := \frac{1}{N} \sum_{j=1}^{N} g(\hat{X}_{t_n}^{(j)}).$$

- **Greeks** are sensitivities of an option price with respect to a parameter.
- To compute the sensitivity w.r.t. to x, use a central-difference

$$\Delta_{C,h} := \frac{V(x+h) - V(x-h)}{2h} .$$

Motivational Example: Bachelier Delta (Δ)

- Constant diffusion $\gamma > 0$, zero drift for forward SDE of price process X that satisfies (1):

$$\mathrm{d}X_t = \gamma \mathrm{d}W_t, \qquad X_0 = x.$$

- Cauchy problem:

$$L^{(0)} u_\cdot = 0, \qquad u_T = g(X_T),$$

where the differential operators are defined as

$$L^{(0)} := \partial_t + \frac{1}{2}\gamma^2 \partial_x^2, \qquad L^{(1)} := \gamma \partial_x.$$

- **Shorthand:** $u_0^{(0)} \equiv L^{(0)} u_0$, $u_0^{(1)*(0)} \equiv L^{(1)} L^{(0)} u_0$.

Delta using a $\mathcal{F}_h$-measurable weight

- The solution at time T can be written as

$$g(X_T) = u(T, X_T) = u(h, X_h) + \gamma \int_h^T \partial_x u_t \mathrm{d}W_t \ ,$$

and infer that $\mathbb{E}\left[g(X_T)\right] = \mathbb{E}\left[u_h\right]$, for $0 \leq h \leq T$.

- It follows:

$$\begin{aligned} \mathbb{E}\left[g(X_T)\frac{W_h}{h}\right] &= \mathbb{E}\left[\mathbb{E}[u_T\frac{W_h}{h}|\mathcal{F}_h]\right] \\ &= \mathbb{E}\left[\frac{W_h}{h}\mathbb{E}[u_T|\mathcal{F}_h]\right] \\ &= \mathbb{E}\left[u_h\frac{W_h}{h}\right] \\ &= \gamma\partial_x u_0 = L^{(1)}u_0. \end{aligned}$$

Monte Carlo approximation

- Rearranging yields a Delta of the form

$$\Delta := \partial_x u(0,x) = \mathbb{E}\left[g(X_T) \frac{W_h}{\gamma h} \right] ,$$

with an obvious MC scheme.

Aim

1. Find weights H_h such that for a general model for X:

$$\text{Greek} = \mathbb{E}[g(\hat{X}_T)H_h] + \mathcal{O}(h^m),$$

   where H_h is some $\mathcal{F}_h$-measurable weight.

2. Control MSE for convergence results of the Greek approximations.

Theoretical Coefficients H^{ψ}

- $\mathcal{B}^m_{[0,1]}$ as the set of bounded measurable functions $\psi : [0,1] \to \mathbb{R}$ such that
$$\int_0^1 \psi(s)\mathrm{d}s = 1 \text{ and if } m \in \mathbb{N}^+, \int_0^1 \psi(s)s^k\mathrm{d}s = 0, 1 \leq k \leq m.$$
- Using this family of functions, define the weights $H^{\psi}_{\cdot}$, which shall be used to approximate the Δ:

Definition (H^{ψ}_h-functionals)

Let $\psi \in \mathcal{B}^m_{[0,1]}$, and for $0 < h \leq T$, define $H^{\psi}_{t,h}$ as

$$H^{\psi}_{t,h} := \frac{1}{h}\int_{s=t}^{t+h} \psi\left(\frac{s-t}{h}\right)\mathrm{d}W_s,$$

and for shorthand $H^{\psi}_h := H^{\psi}_{0,h}$.

Expansions using H^{ψ} for function v

- Let $m \geq 1$, for any $v(t,x)$ smooth enough and $\psi \in \mathcal{B}^{m-1}_{[0,1]}$, then for $\theta > 0$ we have the weak expansion

$$\mathbb{E}\left[H^{\psi}_{\theta} v(\theta, X^{0,x}_{\theta})\right] = v^{(1)}(0,x) + \theta v^{(1,0)}(0,x) + \ldots + \frac{\theta^{m-1}}{(m-1)!} v^{(1)*(0)_{m-1}}(0,x) + \mathcal{O}(\theta^m)\,. \quad (2)$$

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- Apply this to value function u satisfying $L^{(0)}u_{.} = u^{(0)}_{.} = 0$, to obtain

$$\mathbb{E}\left[H^{\psi}_{h} g(\hat{X}_{n})\right] = u^{(1)}(0,x) + \mathcal{O}(h^{m}).$$

Flavour of techniques

- Iterated Itô integrals, and weak Taylor schemes.
- Expansions introduced by [TT90].
- Choose weights for state-space Greeks.
- Refine H_h^{ψ} for higher order schemes.

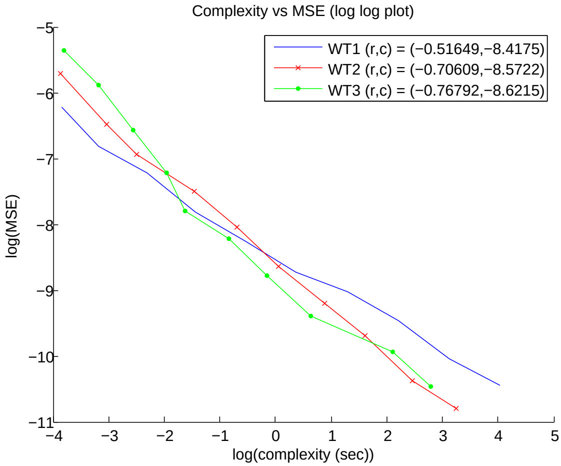
Higher order schemes

- Consider N simulations, and fix the step size to $h = 1/N^{\zeta}$.
- Approximate Δ, with $\mathbb{E}\left[g(\hat{X}_T^n)H_h^{\psi}\right]$.

r (Scheme)	Weight	ζ	MSE	Complexity	Slope
1 (Euler)	$\psi \equiv 1$	1/3	$\mathcal{O}(N^{-2/3})$	$\mathcal{O}(N^{4/3})$	−0.50
2 (WT2)	$\psi_{s,1},\ \psi_{p,1}$	1/5	$\mathcal{O}(N^{-4/5})$	$\mathcal{O}(N^{6/5})$	−0.66
3 (WT3)	$\psi_{s,2},\ \psi_{p,2}$	1/7	$\mathcal{O}(N^{-6/7})$	$\mathcal{O}(N^{8/7})$	−0.75

Table: Implementation and MSE for the Delta.

- $f(x) \equiv 0$, $\gamma(x) \equiv 1 + \sin^2(x)$, $g(x) \equiv \arctan(x)$.
- $(X_0, T) = (0.3, 1)$, $(\zeta_1, \zeta_2, \zeta_3) = (1/3, 1/5, 1/7)$.
- $\approx$ 20 seconds for WT3 vs $\approx$ 60 seconds for WT1!



Extrapolating schemes

- Show that $\mathbb{E}\left[H_h^{\psi} g(\hat{X}_T^n)\right] = u^{(1)}(0,x) + c_1 h + \mathcal{O}(h^2)$.
- Approximation $X^{n/2}$ is with a grid of stepsize $2h$.

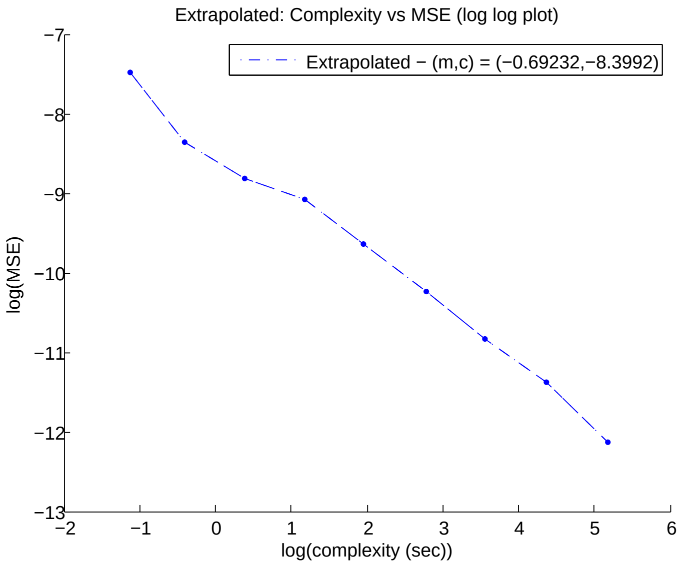
Theorem (Romberg extrapolation)

$$2\mathbb{E}\left[H_h^{\psi} g(\hat{X}_T^n)\right] - \mathbb{E}\left[H_{2h}^{\psi} g(\hat{X}_T^{n/2})\right] = L^{(1)} u(0,x) + O(h^2).$$

- Similar expansion for higher order Romberg extrapolation using better $\psi \in \mathcal{B}_{[0,1]}^m$ and weak Taylor expansions.

r (Scheme)	Weight	ζ	MSE	Complexity	Slope
1 (Euler)	$\psi \equiv 1$	1/5	$\mathcal{O}(N^{-4/5})$	$\mathcal{O}(N^{6/5})$	-0.66
2 (WT2)	$\psi_{s,1}$	1/7	$\mathcal{O}(N^{-6/7})$	$\mathcal{O}(N^{8/7})$	-0.75
3 (WT3)	$\psi_{s,2}$	1/9	$\mathcal{O}(N^{-8/9})$	$\mathcal{O}(N^{10/9})$	-0.80

Table: Implementation and MSE for the Delta, using extrapolation.


Extrapolated: Complexity vs MSE (log log plot)
Extrapolated − (m,c) = (−0.69232,−8.3992)
log(MSE)
log(complexity (sec))
−7
−8
−9
−10
−11
−12
−13
−2
−1
0
1
2
3
4
5
6

Heston Delta

- The Heston model can be represented with i.i.d. Brownian motions $W^{(1)} = (W_t^{(1)})_{t\geq 0}$ and $W^{(2)} = (W_t^{(2)})_{t\geq 0}$ as

$$\mathrm{d}\begin{pmatrix} S_t \\ X_t \end{pmatrix} = \begin{pmatrix} rS_t \\ \kappa(\theta - X_t) \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \sqrt{X_t}S_t & 0 \\ 0 & \xi\sqrt{X_t} \end{pmatrix} \begin{pmatrix} \mathrm{d}W_t^{(1)} \\ \mathrm{d}W_t^{(2)} \end{pmatrix}$$

where $(S_0, X_0) = (x, v)$.

- In general:

$$\Delta = \mathbb{E}\left[g(X_T)\frac{(H_h^{\psi_\cdot,m})^{(1)}}{x\sqrt{v}}\right] + \mathcal{O}(h^m),$$

where $(H_h^{\psi_\cdot,m})^{(1)}$ is an order m weight, defined using $W_\cdot^{(1)}$.

Explicit and drift-implicit schemes

- $(\kappa, \theta, \xi, r, x, v) = (1.15, 0.04, 0.2, 0, 100, 0.04)$.
- Mean reversion $\omega := 2\kappa\theta/\xi^2 = 2.3$.
- Call option with strike $K = 100$, and $T = 1$.

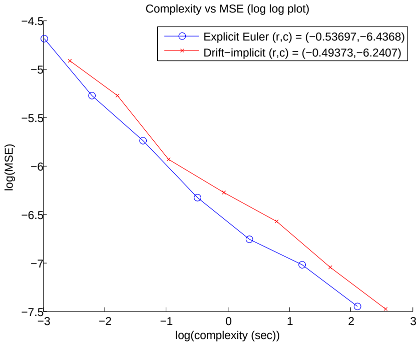


Figure: MSE for Δ vs Complexity, using 100 repeats, $\zeta = 1/3$.

Further work

Extending results to:

- Non-linear PDEs and higher order Greeks.
- Increase space dimension for sensitivities with respect to constant parameter.
- Related work (see [Cha13, CC14]).

Thank you for listening

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