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Constrained nonsmooth utility maximization without quadratic inf convolution

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Abstract

We address a constrained utility maximization problem in an incomplete market for a utility function defined on the whole real line. We extend current research in two directions, firstly we allow for constraints on the portfolio process. Secondly we prove our results without relying on the technique of quadratic inf convolution, simplifying the proofs in this area.

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1. Introduction

In this paper we revisit the setting of [6]. For an incomplete market and a nonsmooth utility function U defined on the whole real line we study the problem

$$\sup_{\theta \in \Theta(S)} \mathbb{E}\left[U\left(X_T^{x,\theta} - B\right)\right].$$

Here B is a bounded contingency claim and $X^{x,\theta}$ represents the wealth process with initial capital x generated by portfolio θ. Our interest now lies in the following extension; the case when the portfolios are constrained to lie in a closed convex cone.

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For the case without constraints and with a smooth utility function the solution method is to approximate the utility function and look at the same problem on a bounded negative domain. However, when one attempts to solve this bounded domain problem for a nonsmooth utility function, the standard methods of proof, specifically the calculus of variations technique, cannot be made to apply. To circumvent this difficulty the idea of quadratic inf convolution was introduced. This method is mathematically very satisfying. However it always leads to very lengthy and technical proofs.

The contribution of the first part of this paper is to show, using a technique of [16] and similar to [5], that despite the presence of constraints, the dependence on quadratic inf convolution can be removed.

In the second part of this paper we focus on the case where the filtration is generated by a Brownian motion. There we show that the introduction of constraints changes the nature of the duality quite markedly. The contribution of this part is to show the existence of a constrained replicating portfolio for the optimal terminal wealth. This provides a natural generalization of the results of [14] to the whole real line.

As is to be expected we follow [6] very closely and try to keep identical notation wherever possible to aid comparison. The outline is as follows. In Sections 2 and 3 we give the model together with the assumptions. Section 4 contains the results and (5) and (6) contain the proofs.

2. Model formulation

The setup is very similar to [6] but we recall it here for the reader's convenience. There are a finite time horizon T and a market consisting of one bond, assumed constant, and d stocks, $S^1, \ldots, S^d$ modelled by a $(0, \infty)^d$-valued, locally bounded, semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ satisfying the usual conditions. At this stage the filtration is not assumed to be generated by a Brownian motion. We shall write L^∞ (L^0) for the set of a.s. bounded (a.s. finite) random variables with respect to $\mathbb{P}$. We consider the space of integrable random variables with respect to several measures and hence always specify the measure being used, e.g. $L^1(\mathbb{P})$. Throughout we write X for the process $(X_t)_{0\le t\le T}$.

Remark 2.1. It is well known that the condition of local boundedness is crucial for our approach to work, see Remark 3.1 of [6] and Remark 2.6 of [20] for further elaboration on this point.

Let $\mathcal{M}^e(S)$ ($\mathcal{M}^a(S)$) denote the set of equivalent[1] (respectively absolutely continuous) local martingale measures for S. The following assumption, of no arbitrage type, shall be required.

Assumption 2.2. $\mathcal{M}^e(S) \neq \emptyset$.

We take a set $\mathcal{K} \subset \mathbb{R}^d$ satisfying:

Assumption 2.3. $\mathcal{K}$ is a closed convex cone with $0 \in \mathcal{K}$. Moreover there exist $m \in \mathbb{N}$ and $k_1, \ldots, k_m$ in $\mathcal{K}$ with

$$\mathcal{K} = \left\{ \sum_{i=1}^{m} \lambda_i k_i : \lambda_i \ge 0 \right\}.$$

[1] For two measures $\mathbb{Q}$ and $\mathbb{P}$ we shall write $\mathbb{Q} \sim \mathbb{P}$ when they are equivalent and $\mathbb{Q} \ll \mathbb{P}$ when $\mathbb{Q}$ is only absolutely continuous with respect to $\mathbb{P}$.

Remark 2.4. This says that $\mathcal{K}$ is a polyhedral cone, see Goldman [11] and Goldman and Tucker [12] for further details. This property of $\mathcal{K}$ is crucial in allowing us to apply Theorem 4.1 of [10] to deduce the existence of a replicating strategy. One should note that this class of sets includes no short selling of the first m assets, $\mathcal{K} = \mathbb{R}^m_+ \times \mathbb{R}^{d-m}$, and hence contains some interesting examples.

Let us define the set of admissible trading strategies $\Theta(S)$. Every $\theta \in \Theta(S)$ must be an $\mathbb{R}^d$-valued predictable process, integrable with respect to S and valued in the closed convex cone $\mathcal{K} \subset \mathbb{R}^d$ which satisfies Assumption 2.3. Our agent starts with initial capital x and may choose, at each time t, to hold a number θ^i_t of shares of asset i with the condition that $\theta \in \Theta(S)$. We assume further that this portfolio is self-financing and hence the wealth process defined by initial capital x and admissible strategy θ evolves as follows

$$X^{x,\theta}_t = x + \int_0^t \theta_u \mathrm{d}S_u.$$

Following definition 1.2 in [20] we introduce

$$\mathcal{X}_b(x) := \left\{ X^{x,\theta} : \theta \in \Theta(S) \text{ and for some } b_\theta \in \mathbb{R}_+,\ X^{x,\theta}_t \geq -b_\theta \text{ for all } t \in [0,T] \right\}.$$

This is the set of wealth processes with admissible integrands that are uniformly bounded below. Let $\mathcal{X}_+(x)$ denote the subset of $\mathcal{X}_b(x)$ containing nonnegative admissible wealth processes. These are the sets that we shall focus on when considering the existence of a replicating portfolio. In our optimization problem however, we are interested in maximizing expected utility of terminal wealth. Hence we introduce the following related set of random variables.

$$\mathcal{X}^T_b(x) := \left\{ X^{x,\theta}_T : X^{x,\theta} \in \mathcal{X}_b(x) \right\}.$$

We define $\mathcal{X}^T_+(x)$ analogously.

Remark 2.5. The observant reader will notice that our definitions of $\mathcal{X}_b(x)$ and $\mathcal{X}_+(x)$ are slightly different from that of [6]. This is to allow us to form the set $\mathcal{P}^e(\mathcal{X}_b(x))$, introduced in Definition 3.6, and in fact makes no difference to the proofs of any of the theorems. Note also that for notational simplicity we shall omit the dependence of X on x and θ.

3. The utility maximization problem and its dual

The agent in our model has preferences modelled by a utility function U, increasing, concave and finitely valued on the whole real line. Note that we do not insist U is smooth or strictly concave. Our agent aims to maximize utility of terminal wealth subject to some liability $B \in L^\infty$. As in [20] and [6] we define the set $\mathcal{X}_U(x)$ of random variables $X \in L^0$ such that there exists a sequence $X^n_T \in \mathcal{X}^T_b(x)$ satisfying

$$U(X^n_T - B) \to U(X - B), \quad \text{in } L^1(\mathbb{P}).$$

Now we can formulate the utility maximization problem

$$V(x) := \sup_{X \in \mathcal{X}_U(x)} \mathbb{E}\left[U(X - B)\right].$$

The standard technique is to construct an appropriate dual problem and so we introduce the conjugate function

$$\tilde{U}(y) = \sup_{x\in\mathbb{R}}\{U(x) - xy\}.$$

This is known to be a convex function satisfying $\mathrm{dom}(\tilde{U}) \cap (-\infty, 0) = \emptyset$. We require:

Assumption 3.1.

$$\inf \bigcup_{x\in\mathbb{R}} \partial U(x) = 0, \qquad \sup \bigcup_{x\in\mathbb{R}} \partial U(x) = \infty.$$

Observe that in the smooth case these are the Inada conditions $U'(-\infty) = \infty$, $U'(\infty) = 0$. Without loss of generality we may assume $U(0) > 0$.

Remark 3.2. Since U is concave we see that $\inf \partial U(x)$ and $\sup \partial U(x)$ are decreasing functions of x and hence the above assumption is equivalent to

$$\lim_{x\to\infty} \inf \partial U(x) = 0, \qquad \lim_{x\to-\infty} \sup \partial U(x) = \infty.$$

As identified in [15] we need conditions on the asymptotic elasticity (AE). It is known from [7] that we should put these on the dual function. Define

$$\mathrm{AE}_0(\tilde{U}) := \limsup_{y\to 0} \sup_{q\in\partial\tilde{U}(y)} \frac{|q|y}{\tilde{U}(y)}, \qquad \mathrm{AE}_\infty(\tilde{U}) := \limsup_{y\to\infty} \sup_{q\in\partial\tilde{U}(y)} \frac{|q|y}{\tilde{U}(y)}.$$

We shall need the following.

Assumption 3.3. $\mathrm{AE}_0(\tilde{U}) < \infty$, $\mathrm{AE}_\infty(\tilde{U}) < \infty$.

The following lemma taken from [6] will be used repeatedly and we state it for reference.

Lemma 3.4 (*Lemma 2.3 [6]*). *Let f be a positive, convex function with $\overline{\mathrm{dom}(f)} = \mathbb{R}_+$. Assume further that f is decreasing near 0, increasing near ∞ and satisfies the asymptotic elasticity conditions, $\mathrm{AE}_0(f) < \infty$ and $\mathrm{AE}_\infty(f) < \infty$. Then for all $0 < \mu_0 < \mu_1 < \infty$ there exists a constant C such that:*

(i) *$f(\mu y) \le Cf(y)$ for all $\mu \in [\mu_0, \mu_1]$ and $y > 0$;*
(ii) *$y|q| \le Cf(y)$ for all $y > 0$ and $q \in \partial f(y)$.*

The natural choice for a set of dual variables is

$$\mathcal{Y}_+(y) := \left\{ Y \in L^0_+ : \mathbb{E}[XY] \le xy \text{ for all } x \in \mathbb{R}_+ \text{ and } X \in \mathcal{X}^T_+(x) \right\}.$$

As first observed in [20] we may in fact use the set

$$\tilde{\mathcal{Y}}_+(y) := \{Y \in \mathcal{Y}_+(y) : \mathbb{E}[Y] = y\}.$$

This is a subset of $\mathcal{Y}_+(y)$ and is nonempty due to Assumption 2.2.

Remark 3.5. Let us briefly comment here on the observation of Remark 3.2 in [6] in our current setting. Due to the presence of constraints it is no longer possible to assert that for $y > 0$ and $Y \in \tilde{\mathcal{Y}}_+(y)$ the measure $\mathbb{Q} := \frac{Y}{y} \cdot \mathbb{P} \in \mathcal{M}^a(S)$. Indeed, the conscientious reader can easily construct 1-period examples where the optimal $\mathbb{Q}_* := \frac{Y_*}{y_*} \cdot \mathbb{P} \notin \mathcal{M}^a(S)$. In light of this phenomenon we introduce the following set of measures.

Definition 3.6.

$$\mathcal{P}^a(\mathcal{X}_b(x)) := \{\mathbb{Q} \ll \mathbb{P} : \text{ X is a } \mathbb{Q}\text{-supermartingale for all } X \in \mathcal{X}_b(x)\}$$
$$\mathcal{P}^e(\mathcal{X}_b(x)) := \left\{\mathbb{Q} \in \mathcal{P}^a(\mathcal{X}_b(x)) : \mathbb{Q} \sim \mathbb{P}\right\}.$$

Due to Proposition 3.3 in [1], $\mathcal{M}^e(S) \subset \mathcal{P}^e(\mathcal{X}_b(x))$. In fact, dependent upon $\mathcal{K}$, this can be strict. By identifying a measure $\mathbb{Q}$ with its Radon–Nikodym derivative we can view $\mathcal{P}^a(\mathcal{X}_b(x))$ as a subset of $\tilde{\mathcal{Y}}_+(1)$. We shall make use of this slight abuse of notation throughout the paper. Indeed we establish that for locally bounded S

$$\mathcal{P}^e(\mathcal{X}_b(x)) = \tilde{\mathcal{Y}}_+(1) \cap \{Y : Y > 0\}.$$

See the Appendix for details.

Returning to the previous discussion, our dual problem is now defined as

$$W(x) := \inf_{y>0} \inf_{Y \in \tilde{\mathcal{Y}}_+(y)} \mathbb{E}\left[\tilde{U}(Y) - YB + xy\right].$$

An elementary calculation shows that we have

$$V(x) \le W(x).$$

To avoid any degeneracy issues we shall make the following assumption.

Assumption 3.7. There exists $x \in \mathbb{R}$ such that $W(x) < \infty$.

Remark 3.8. Exactly as in Remark 3.3 of [6] this is equivalent to $W(x) < \infty$ for all $x \in \mathbb{R}$.

This concludes the discussion of preliminaries and we shall move on to the main results.

4. Main results

Theorem 4.1. *Suppose that Assumptions 2.2, 2.3, 3.1, 3.3 and 3.7 hold.*

(i) *There exist some $y_* \ge 0$ and an optimal $Y_* \in \tilde{\mathcal{Y}}_+(y_*)$ such that,*

$$W(x) = \mathbb{E}\left[\tilde{U}(Y_*) - Y_*B + xy_*\right].$$

If $y_* > 0$ then $\mathbb{Q}_* = \frac{Y_*}{y_*} \cdot \mathbb{P} \in \mathcal{P}^a(\mathcal{X}_b(x))$.

(ii) *There exists some $X_* \in \mathcal{X}_U(x)$, satisfying $\mathbb{E}[X_*Y_*] = xy_*$, such that*

$$X_* \in B - \partial\tilde{U}(Y_*) \quad \text{and} \quad V(x) = \mathbb{E}[U(X_* - B)].$$

(iii) $V(x) = W(x)$.

(iv) *If $Y_* > 0$ $\mathbb{P}$-a.s. and the filtration is generated by a Brownian motion then $X_* = X_T^{x,\theta_*}$ for some $\theta_* \in \Theta(S)$ where X^{x,θ_*} is a uniformly integrable martingale under the measure $\mathbb{Q}_* := \frac{Y_*}{y_*} \cdot \mathbb{P}$.*

Theorem 4.1 can be proved with the technique developed in [20], see [6] for the nonsmooth extension. One approximates the utility function U by functions U_n such that

$$U_n := U \quad \text{on } \mathrm{dom}(U_n) := (-2n, \infty) \text{ for } n \geq 2\|B\|_\infty.$$

Let $\tilde{U}_n$ denote the dual of the approximating U_n and observe that we have

$$U_n = U \quad \text{on } \mathrm{dom}(U_n) \quad \text{and} \quad \tilde{U}_n = \tilde{U} \quad \text{on } \partial U_n(\mathrm{dom}(U_n)).$$

Set $x_n := x + \frac{n}{2}$ and $B_n := B + \frac{n}{2}$. Before we introduce the two approximating primal and dual problems let us look at their appropriate domains. Define

$$\mathcal{C}(x) := \left\{ X \in L^0_+ - L^\infty : \mathbb{E}[XY] \leq xy \text{ for all } Y \in \mathcal{Y}_+(y) \right\}.$$

Now we set

$$V_n(x) := \sup_{X \in \mathcal{C}(x_n)} \mathbb{E}\left[U_n(X - B_n)\right], \qquad W_n(x) := \inf_{y>0} \inf_{Y \in \mathcal{Y}_+(y)} \mathbb{E}\left[\tilde{U}_n(Y) - YB_n + x_n y\right].$$

Remark 4.2. One may ask why we have defined the approximating problems for (x_n, B_n) instead of for (x, B)? As in [4] we want to show the convergence of some approximating solutions (y_n, Y_n) to a solution (y_*, Y_*) for $W(x)$, where $\mathbb{Q}_* := \frac{Y_*}{y_*} \cdot \mathbb{P} \in \mathcal{P}^a(\mathcal{X}_b(x))$. We want to force $\mathbb{E}[Y_n] - y_n$ to tend to 0. To see how the addition of $\frac{n}{2}$ achieves this we refer the reader to Lemma 5.4 in [6].

The technique of quadratic inf convolution is used in [6] to prove existence and uniqueness for these approximating problems and so in the following section we shall reprove the equivalent of Theorem 3.2 of [6] in the constrained case without this technique.

Remark 4.3. There is one final observation to be made in this section, an extension of Remark 3.5 in [6] to the constrained case. Observe that $\mathcal{P}^e(\mathcal{X}_b(x)) \subset \mathcal{Y}_+(1)$ and recall that $\mathcal{K}$ is a polyhedral cone. It follows that Assumption 3.1 of [10] holds and we can then apply Proposition 4.1 from [10] to get the following inclusion

$$\mathcal{C}(x) \subset \left\{ X \in L^0_+ - L^\infty : X \leq X^s_T \text{ for some } X^s_T \in \mathcal{X}^T_b(x) \right\}.$$

5. Utility functions with Bounded Negative Domain

Let us now focus on the approximate problems. We begin by restating Theorem 3.2 from [6].

Theorem 5.1. *Let β be a constant and consider a claim B with $\|B\| \leq \beta$. Let U be a nonconstant concave increasing function with*

$$\overline{\mathrm{dom}(U)} = [-2\beta, \infty), \qquad U(\infty) > 0, \qquad \overline{\mathrm{dom}(\tilde{U})} = \mathbb{R}_+.$$

Consider the optimization problems

$$V(x) := \sup_{X \in \mathcal{C}(x)} \mathbb{E}\left[U(X - B)\right], \qquad W(x) := \inf_{y>0} \inf_{Y \in \mathcal{Y}_+(y)} \mathbb{E}\left[\tilde{U}(Y) - YB + xy\right].$$

Suppose that Assumptions 2.2 and 2.3 hold, that U satisfies the asymptotic elasticity assumption $\mathrm{AE}_0(\tilde{U}) < \infty$ and that $W(x) < \infty$ for some $x > 0$.

(i) *There exist some* $y_* \geq 0$ *and an optimal* $Y_* \in \mathcal{Y}_+(y_*)$ *such that,*

$$W(x) = \mathbb{E}\left[\tilde{U}(Y_*) - Y_* B + x y_*\right].$$

(ii) *There exists some* $X_* \in \mathcal{C}(x)$, *satisfying* $\mathbb{E}[X_* Y_*] = x y_*$ *and* $X_* - B \geq -2\beta$ *such that*

$$X_* \in B - \partial\tilde{U}(Y_*) \quad \text{and} \quad V(x) = \mathbb{E}[U(X_* - B)].$$

Moreover, if $X_* \geq 0$ *then* $X_* \in \mathcal{X}_+^T(x)$.

(iii) $V(x) = W(x)$.

Remark 5.2. To clarify the situation further we remark that neither Assumption 3.1 nor the convention $U(0) > 0$ is needed for Theorem 5.1 to hold. As a result we may no longer have $\tilde{U} \geq 0$.

We now proceed to the proof, for ease of exposition we split it into a series of lemmas. Throughout the remainder of this section the basic assumptions of Theorem 5.1 will be assumed, so that the filtration is not necessarily generated by a Brownian motion.

Proceeding exactly as in Lemmas 6.2 and 6.3 of [6] one can prove:

Lemma 5.3. *There exists* $Y_* \in \mathcal{Y}_+(y_*)$ *such that* (y_*, Y_*) *are optimal for the dual problem.*

We want to apply the methodology developed in [16]. To this end, define the related dual problem.

$$\tilde{V}(y) := \inf_{Y \in \mathcal{Y}_+(y)} \mathbb{E}\left[\tilde{U}(Y) - YB\right].$$

Lemma 5.4. $\tilde{V}(y) < \infty$ *for all* $y > 0$.

Proof. The assumption $W(x) < \infty$ for some $x > 0$ implies that there exists a pair $(\tilde{y}, \tilde{Y})$ with $\tilde{Y} \in \mathcal{Y}_+(\tilde{y})$ and $\tilde{y} > 0$ such that

$$\mathbb{E}\left[\tilde{U}(\tilde{Y})\right] < \infty.$$

Observe that for all $y > 0$ $\frac{y}{\tilde{y}}\tilde{Y} \in \mathcal{Y}_+(y)$. Now we have

$$\begin{aligned}\tilde{V}(y) &\leq \mathbb{E}\left[\tilde{U}\left(\frac{y}{\tilde{y}}\tilde{Y}\right) - \frac{y}{\tilde{y}}\tilde{Y}B\right] \\ &\leq \mathbb{E}\left[\tilde{U}\left(\frac{y}{\tilde{y}}\tilde{Y}\right) - 2\beta\frac{y}{\tilde{y}}\tilde{Y}\right] + 3y\beta.\end{aligned}$$

Exactly as in [6] we have that $\tilde{U}(\cdot) - 2\beta\cdot$ is decreasing. Hence if $y \geq \tilde{y}$ we have immediately

$$\begin{aligned}\tilde{V}(y) &\leq \mathbb{E}\left[\tilde{U}\left(\tilde{Y}\right) - 2\beta\tilde{Y}\right] + 3y\beta \\ &\leq \mathbb{E}\left[\tilde{U}(\tilde{Y})\right] + 5\beta y < \infty.\end{aligned}$$

If $\tilde{y} \geq y$ then we use the asymptotic elasticity assumption, see Remark 6.1 in [6], which guarantees the existence of two constants, $\gamma > 0$ and $y_0 > 0$ such that

$$\tilde{U}(\mu y) - 2\beta\mu y \leq \mu^{-\gamma}\left(\tilde{U}(y) - 2\beta y\right) \quad \text{for all } \mu \in (0, 1] \text{ and } y \in (0, y_0].$$

Putting this together we see that

$$\begin{aligned}\tilde{V}(y) &\le \left(\frac{y}{\tilde{y}}\right)^{-\gamma} \mathbb{E}\left[\left(\tilde{U}(\tilde{Y}) - 2\beta\tilde{Y}\right)\mathbb{I}_{\{\tilde{Y}\le y_0\}}\right] + \tilde{U}\left(\frac{y}{\tilde{y}}y_0\right) - 2\beta\frac{y}{\tilde{y}}y_0 + 3\beta y \\ &\le C_0\mathbb{E}\left[|\tilde{U}(\tilde{Y})|\right] + C_1 < \infty\end{aligned}$$

for some constants C_0 and C_1. The statement of the lemma now follows. □

Let us turn to the existence of a solution for the primal problem.

Lemma 5.5. *For $x > 0$ there exists $X_* \in \mathcal{C}(x)$ such that $X_* - B \ge -2\beta$ and*

$$V(x) = \mathbb{E}[U(X_* - B)].$$

Proof. Let $x > 0$ and $(X_n)_n \in \mathcal{C}(x)$ be a maximizing sequence for V. Under assumption $U(x) = -\infty$ for $x < -2\beta$ and we have the chain of inequalities

$$V(x) \ge U(x - \beta) > -\infty.$$

Hence we have for sufficiently large n

$$-\infty < V(x) - 1 \le \mathbb{E}\left[U\left(X_n - B\right)\right].$$

This allows us to conclude that for sufficiently large n we must have $X_n - B \ge -2\beta$ a.s. We pass to a subsequence to ensure that this uniform lower boundedness holds for all n. It is clear that the negative parts have bounded convex hull, i.e. the set

$$\operatorname{conv}\left\{(X_n - B)^- : n \ge 1\right\}$$

is bounded. We may now apply Lemma A.1 from [8], also noting Remark 2 there, to deduce the existence of a convex combination

$$\overline{X}_n - B := \sum_{k\ge n} \lambda^{n,k}\left(X_k - B\right),$$

with $\overline{X}_n \in \mathcal{C}(x)$ for all n which converges a.s. to $X_* - B$, satisfying

$$X_* - B \ge -2\beta.$$

We know $\overline{X}_n - B \ge -2\beta$. Hence $\overline{X}_n \ge -3\beta$. This gives the bound $\left(\overline{X}_n\right)^- Y \le 3\beta Y$ for all $Y \in \mathcal{Y}_+(y)$. Thus an application of Fatou's lemma shows that $X_* \in \mathcal{C}(x)$. It follows that

$$\lim_{n\to\infty} \mathbb{E}\left[U(\overline{X}_n - B)\right] = V(x).$$

Fatou's lemma implies that we have

$$\liminf_{n\to\infty} \mathbb{E}\left[U(\overline{X}_n - B)^-\right] \ge \mathbb{E}\left[U(X_* - B)^-\right].$$

Thus to complete the proof we need only show the uniform integrability of $\{U(\overline{X}_n - B)^+ : n \ge 1\}$, compare Lemma 1 in [16]. Suppose this is not the case, then we may find a sequence of mutually disjoint sets $(A_n)_n$ and $\alpha > 0$ such that

$$\mathbb{E}\left[U\left(\overline{X}_n - B\right)^+ \mathbb{I}_{A_n}\right] \ge \alpha.$$

Let $x_0 := \inf\{x > 0 : U(x - 3\beta) > 0\}$ and observe that this is finite.

Define, for $n \in \mathbb{N}$, the sequence

$$\hat{X}_n := x_0 + \sum_{k=1}^{n} \overline{X}_k \mathbb{I}_{A_k}.$$

We need to show that each $\hat{X}_n \in \mathcal{C}(x)$ for some $x > 0$. Since $\overline{X}_n - B \geq -2\beta$ we deduce that $\overline{X}_n + 3\beta \geq 0$, so that for $Y \in \mathcal{Y}_+(y)$

$$\begin{aligned}\mathbb{E}\left[\hat{X}_n Y\right] &\leq x_0 y + \sum_{k=1}^{n} \mathbb{E}\left[\left(\overline{X}_k + 3\beta\right) \mathbb{I}_{A_k} Y\right] \\ &\leq (x_0 + 3\beta n) y + \sum_{k=1}^{n} \mathbb{E}\left[\overline{X}_k Y\right] \\ &\leq (x_0 + n(3\beta + x)) y,\end{aligned}$$

which gives $\hat{X}_n \in \mathcal{C}(x_0 + n(3\beta + x))$. Now we have

$$\mathbb{E}\left[U\left(\hat{X}_n - B\right)\right] \geq \sum_{k=1}^{n} \mathbb{E}\left[U\left(\overline{X}_k - B\right)^+ \mathbb{I}_{A_k}\right] \geq n\alpha.$$

It is easy to show that for sufficiently large x and all $y > 0$

$$0 \leq U(x - \beta) \leq V(x) \leq \tilde{V}(y) + xy.$$

Hence the result of Lemma 5.4 implies that

$$\lim_{x \to \infty} \frac{V(x)}{x} = 0.$$

However, putting together our estimates we deduce

$$\begin{aligned}\limsup_{z \to \infty} \frac{V(z)}{z} &\geq \limsup_{n \to \infty} \frac{\mathbb{E}\left[U\left(\hat{X}_n - B\right)\right]}{x_0 + n(x + 3\beta)} \\ &\geq \limsup_{n \to \infty} \frac{n\alpha}{x_0 + n(x + 3\beta)} = \frac{\alpha}{x + 3\beta} > 0.\end{aligned}$$

We have our contradiction and the proof is complete. □

The final lemma shows the conjugacy relations between $\tilde{V}(y)$ and $V(x)$.

Lemma 5.6. *For each $y > 0$ we have*

$$\tilde{V}(y) = \sup_{x > 0} \{V(x) - xy\}.$$

Proof. We apply the method of Lemma 3.4 in [15] with a few minor changes and some extra details. For a given random variable H bounded below define the function

$$\psi(H) := \sup_{Y \in \mathcal{Y}_+(1)} \mathbb{E}[HY].$$

Observe that $H \in \mathcal{C}(\psi(H))$ and $x = \psi(H)$ is the smallest x such that $H \in \mathcal{C}(x)$. For $n > 0$ introduce the set

$$\mathcal{B}_n := \{H : -3\beta \leq H \leq n\}.$$

It is the ball of radius n in L^∞ together with the 3β bounded negative elements. This is weak star compact by Alaoglu's theorem and since $\mathcal{Y}_+(y)$ is a closed convex subset of $L^1(\mathbb{P})$ we may apply the minimax theorem to deduce

$$\sup_{H\in\mathcal{B}_n}\inf_{Y\in\mathcal{Y}_+(y)}\mathbb{E}\left[U(H-B)-HY\right]=\inf_{Y\in\mathcal{Y}_+(y)}\sup_{H\in\mathcal{B}_n}\mathbb{E}\left[U(H-B)-HY\right].$$

Next we want to show that

$$\lim_{n\to\infty}\sup_{H\in\mathcal{B}_n}\inf_{Y\in\mathcal{Y}_+(y)}\mathbb{E}[U(H-B)-HY]=\sup_{x>0}\{V(x)-xy\}.$$

Recall that $\mathcal{B}_n\subseteq\mathcal{B}_{n+1}\subseteq\cdots$. So choose H_ϵ ϵ-optimal for the left-hand side and n sufficiently large such that $H_\epsilon\in\mathcal{B}_n$. Set $x_\epsilon=\psi(H_\epsilon)$ and we have

$$\begin{aligned}\lim_{n\to\infty}\sup_{H\in\mathcal{B}_n}\inf_{Y\in\mathcal{Y}_+(y)}\mathbb{E}[U(H-B)-HY]-\epsilon &\le \inf_{Y\in\mathcal{Y}_+(y)}\mathbb{E}[U(H_\epsilon-B)-H_\epsilon Y]\\ &=\mathbb{E}[U(H_\epsilon-B)]-x_\epsilon y\\ &\le\sup_{x>0}\sup_{H\in\mathcal{C}(x)}\{\mathbb{E}[U(H-B)-xy]\}\\ &=\sup_{x>0}\{V(x)-xy\}.\end{aligned}$$

Conversely, pick $H_\epsilon\in\mathcal{C}(x_\epsilon)$ such that

$$\mathbb{E}[U(H_\epsilon-B)]-x_\epsilon y\ge\sup_{x>0}\sup_{H\in\mathcal{C}(x)}\mathbb{E}[U(H-B)-xy]-\frac{\epsilon}{2}.$$

Since we have the trivial inequality

$$\sup_{x>0}\sup_{H\in\mathcal{C}(x)}\mathbb{E}[U(H-B)-xy]-\frac{\epsilon}{2}\ge U(x-\beta)-xy-\frac{\epsilon}{2}>-\infty.$$

We may conclude that $H_\epsilon-B\ge-2\beta$ and hence $H_\epsilon\ge-3\beta$. This implies that, for sufficiently large m, $H_\epsilon\wedge m\in\mathcal{B}_m$ and for all $Y\in\mathcal{Y}_+(y)$

$$\mathbb{E}[(H_\epsilon\wedge m)Y]\le\mathbb{E}[H_\epsilon Y]\le x_\epsilon y.$$

Hence $H_\epsilon\wedge m\in\mathcal{B}_m\cap\mathcal{C}(x_\epsilon)$. The monotone convergence theorem now gives

$$\mathbb{E}[U(H_\epsilon\wedge m-B)]\to\mathbb{E}[U(H_\epsilon-B)]\quad\text{as } m\to\infty.$$

Next let m be sufficiently large so that

$$\begin{aligned}\mathbb{E}[U(H_\epsilon\wedge m-B)-x_\epsilon y]&\ge\mathbb{E}[U(H_\epsilon-B)-x_\epsilon y]-\frac{\epsilon}{2}\\ &\ge\sup_{x>0}\sup_{H\in\mathcal{C}(x)}\mathbb{E}[U(H-B)-xy]-\epsilon,\end{aligned}$$

where for the final inequality we have used the definition of H_ϵ. Recalling that

$$x_\epsilon y\ge\sup_{Y\in\mathcal{Y}_+(y)}\mathbb{E}[(H_\epsilon\wedge m)Y]$$

and substituting yields

$$\inf_{Y\in\mathcal{Y}_+(y)}\mathbb{E}[U(H_\epsilon\wedge m-B)-(H_\epsilon\wedge m)Y]\ge\sup_{x>0}\sup_{H\in\mathcal{C}(x)}\mathbb{E}[U(H-B)-xy]-\epsilon.$$

This gives the converse inequality. On the other hand we have

$$\inf_{Y\in\mathcal{Y}_+(y)} \sup_{H\in\mathcal{B}_n} \mathbb{E}\left[U(H-B)-HY\right] = \inf_{Y\in\mathcal{Y}_+(y)} \mathbb{E}\left[\tilde{U}_n(Y)-YB\right] := \tilde{V}_n(y),$$

where,

$$\tilde{U}_n(y) := \sup_{-2\beta\le x\le n+\beta} \{U(x)-xy\}.$$

It remains to show that

$$\lim_{n\to\infty} \tilde{V}_n(y) = \tilde{V}(y).$$

We clearly have $\tilde{V}_n(y) \le \tilde{V}(y)$. Pick a sequence $(Y_n)_n$ such that

$$\lim_{n\to\infty} \mathbb{E}\left[\tilde{U}_n(Y_n)-Y_nB\right] = \lim_{n\to\infty} \tilde{V}_n(y).$$

$(Y_n)_n$ is bounded in $L^1(\mathbb{P})$. So an application of Komlos' theorem gives us a convex subsequence $(\hat{Y}_n)_n$ converging to some Y_* in $\mathcal{Y}_+(y)$. We claim that $\{(\tilde{U}_n(\hat{Y}_n)-\hat{Y}_nB)^- : n\ge 1\}$ is uniformly integrable. If this is the case by convexity and Fatou's lemma

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left[\tilde{U}_n\left(Y_n\right)-Y_nB\right] &\ge \liminf_{n\to\infty} \mathbb{E}\left[\tilde{U}_n\left(\hat{Y}_n\right)-\hat{Y}_nB\right]\\ &\ge \mathbb{E}\left[\tilde{U}\left(Y_*\right)-Y_*B\right] \ge \tilde{V}(y).\end{aligned}$$

The statement of the lemma then follows. Finally we must check the uniform integrability claim. If D^+ denotes right derivative we have

$$\begin{aligned}\tilde{U}_n(\hat{Y}_n)-\hat{Y}_nB &= \left(\tilde{U}(\hat{Y}_n)-\hat{Y}_nB\right)\mathbb{I}_{\{\hat{Y}_n\ge D^+U(n+\beta)\}}\\ &\quad+\left(U(n+\beta)-(n+\beta)\hat{Y}_n-\hat{Y}_nB\right)\mathbb{I}_{\{\hat{Y}_n<D^+U(n+\beta)\}}.\end{aligned} \tag{1}$$

It was shown in the proof of Lemmas 6.2 and 6.3 of [6] that the first term on the right-hand side of (1) is uniformly integrable below. For the second term note that we have the estimate

$$\begin{aligned}&\left(U(n+\beta)-(n+\beta)\hat{Y}_n-\hat{Y}_nB\right)\mathbb{I}_{\{\hat{Y}_n<D^+U(n+\beta)\}}\\ &\quad\ge -\left(U(n+\beta)-(n+2\beta)D^+U(n+\beta)\right)^-.\end{aligned}$$

Pick a constant $a>0$. Using the subgradient inequality

$$U(n+\beta)-U(a) \ge D^+U(n+\beta)\,(n+\beta-a)\,.$$

This is easily seen to imply the estimate

$$U(n+\beta)-(n+2\beta)D^+U(n+\beta) \ge U(a)-(a+\beta)D^+U(n+\beta).$$

Recall that U is concave and increasing so that $D^+U(n+\beta)$ is positive and decreasing in n. Thus, for sufficiently large n, $D^+U(n+\beta)\le D^+(a)$ and is therefore bounded. Thus the second term in (1) is bounded below for sufficiently large n which proves the claim. □

Corollary 5.7. *For $x>0$ we have $V(x)=W(x)$.*

Proof. The standard conjugacy relations tell us that given the result of Lemma 5.6 we have

$$V(x) = \inf_{y>0}\{\tilde{V}(y) + xy\}.$$

Moreover it is easy to see from the definitions of $\tilde{V}$ and W that we have

$$W(x) = \inf_{y>0}\{\tilde{V}(y) + xy\}.$$

The result, and item (iii) of Theorem 5.1, follow. □

All that remains is to complete the proof of Theorem 5.1(ii). The key result is that for conjugate functions $f(x)$ and $g(y)$

$$f(x) = g(\hat{y}) + x\hat{y} \Longleftrightarrow \hat{y} \in \partial f(x).$$

Lemma 5.8. *Let (y_*, Y_*) and X_* be as in Lemmas 5.3 and 5.5. Then,*

(i) $\mathbb{E}[X_*Y_*] = xy_*$.
(ii) $X_* \in B - \partial\tilde{U}(Y_*)$.
(iii) *if* $X_* \geq 0$ *then* $X_* \in \mathcal{X}_+(x)$.

Proof. (i) Since $X_* \in \mathcal{C}(x)$ and $Y_* \in \mathcal{Y}_+(y_*)$ we have $\mathbb{E}[X_*Y_*] \leq xy_*$. Suppose for a contradiction that it is strict, now we have

$$\begin{aligned} V(x) = \mathbb{E}[U(X_* - B)] &\leq \mathbb{E}\left[\tilde{U}(Y_*) + Y_*(X_* - B)\right] \\ &< \mathbb{E}\left[\tilde{U}(Y_*) - Y_*B\right] + xy_* = W(x). \end{aligned}$$

This is the required contradiction.

(ii) Suppose again for a contradiction that there exists a set of positive measure A, such that we have $X_* - B \notin -\partial\tilde{U}(Y_*)$ on A. This gives, using item (i),

$$V(x) = \mathbb{E}[U(X_* - B)] < \mathbb{E}[\tilde{U}(Y_*) - Y_*B] + xy_* = W(x),$$

another contradiction.

(iii) Under the present assumptions, $X_* \geq 0$ and $X_* \in \mathcal{C}(x)$. Thus we have

$$\mathbb{E}[X_*Y] \leq xy \quad \text{for all } Y \in \mathcal{Y}_+(y),\ y > 0.$$

Since $\mathcal{P}^e(\mathcal{X}_b(x)) \subset \mathcal{Y}_+(1)$ this immediately implies

$$\sup_{\mathbb{Q}\in\mathcal{P}^e(\mathcal{X}_b(x))} \mathbb{E}_{\mathbb{Q}}[X_*] \leq x.$$

An application of Proposition 4.1 from [10] provides the existence of $\hat{X} \in \mathcal{X}_+(x)$ such that $\hat{X}_T \geq X_*$. Using the definition of $\mathcal{Y}_+(y)$ we have

$$\mathbb{E}\left[\hat{X}_T Y\right] \leq xy \quad \text{for all } Y \in \mathcal{Y}_+(y),\ y > 0.$$

Now put $y = y_*, Y = Y_*$ in the above, recalling that $\mathbb{E}[X_*Y_*] = xy_*$, to deduce $\mathbb{E}[(\hat{X}_T - X_*)Y_*] \leq 0$. Hence we have immediately that $\hat{X}_T = X_*$ on $Y_* > 0$. We may now proceed exactly as in Lemma 6.6 of [6]. □

This completes the proof of (i)–(iii) in Theorem 5.1. As we alluded to earlier, the idea is to use this result to establish (i)–(iv) of Theorem 4.1. Despite the presence of constraints we can use exactly the same technique as in [6] to prove (i)–(iii) and to avoid needless repetition we do not comment further on this. However, as remarked earlier, in the case of item (iv) the situation is quite different and to allow full discussion we give the proof of this result in the following section.

6. Existence of a constrained replicating strategy

The proof of Theorem 4.1(iv) rests upon conclusions drawn from the extension of the method of [6] to the constrained case as well as some technical lemmas. We isolate these for the reader before undertaking the proof. Throughout this section we shall work under the assumptions of Theorem 4.1 and use the notation therein. We write $\theta \cdot S$ for the stochastic integral of θ with respect to S and remind the reader that X denotes the process $(X_t)_{0\le t\le T}$ and similarly X^τ denotes $(X_{t\wedge\tau})_{0\le t\le T}$. In addition, the utility function is assumed to satisfy $U(0) > 0$ so that $\tilde{U} \ge 0$. We shall also suppose that the optimal $Y_* > 0$ and that the filtration is generated by a Brownian motion. The next definition is common in the literature see for example [2,3,17].

Definition 6.1. A measure $\mathbb{Q} \ll \mathbb{P}$ is said to have finite generalized entropy or FGE if

$$\mathbb{E}\left[\tilde{U}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right] < \infty.$$

Lemma 6.2. *If we set* $\mathbb{Q}_* := \frac{Y_*}{y_*} \cdot \mathbb{P}$ *then there exists a sequence of processes* $X^n = x + (\theta^n \cdot S)$ *such that:*

(i) $\theta_t^n \in \mathcal{K}$ *a.s. for all* $t \in [0, T]$,
(ii) $X^n \in \mathcal{X}_b(x)$,
(iii) $X_T^n \to X_*$ *in* $L^1(\mathbb{Q}_*)$,
(iv) $\left(X_T^n\right)^-$ *is uniformly integrable under all measures* $\mathbb{Q} \in \mathcal{P}^a(\mathcal{X}_b(x))$ *with* FGE.

Proof. Properties (i)–(iii) are simply a restatement of Corollary 5.1 and Lemma 5.6 in [6]. For (iv), let $\mathbb{Q} \in \mathcal{P}^a(\mathcal{X}_b(x))$ have FGE. Since the functions U and $\tilde{U}$ are conjugate we have

$$(X_T^n - B)\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \ge U(X_T^n - B) - \tilde{U}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right). \tag{2}$$

Corollary 5.1 in [6] implies that $\left(U(X_T^n - B)\right)_n$ is uniformly integrable. This, together with the FGE property of $\mathbb{Q}$ and the boundedness of B, completes the proof. □

As discussed before we identify a measure $\mathbb{Q} \in \mathcal{P}^a(\mathcal{X}_b(x))$ with its Radon–Nikodym derivative and then we have:

Lemma 6.3. $\mathcal{P}^e(\mathcal{X}_b(x)) = \tilde{\mathcal{Y}}_+(1) \cap \{Y : Y > 0\}$.

Proof. This result has a rather lengthy proof and so to aid brevity we include it as an Appendix. □

Lemma 6.4. *For all $\mathbb{Q} \in \mathcal{P}^e(\mathcal{X}_b(x))$ there exists a sequence $(\mathbb{Q}_n)_n \subset \mathcal{P}^e(\mathcal{X}_b(x))$ with* FGE *for all n such that*

$$\frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{P}} \to \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \quad \text{in } L^1(\mathbb{P}).$$

Proof. We know that under the current assumptions $Y_* > 0$ and hence $\mathbb{Q}_* \sim \mathbb{P}$. An application of Lemma 6.3 tells us that $\mathbb{Q}_* \in \mathcal{P}^e(\mathcal{X}_b(x))$ and it trivially has FGE. Let $Z^{\mathbb{Q}}$ denote the Radon–Nikodym derivative process of $\mathbb{Q} \in \mathcal{P}^e(\mathcal{X}_b(x))$. Consider an arbitrary $\mathbb{Q} \in \mathcal{P}^e(\mathcal{X}_b(x))$ and define the following sequence of stopping times

$$\tau_n := \inf\left\{t > 0 : Z_t^{\mathbb{Q}_*} \notin \left[\frac{1}{n}, n\right] \text{ or } Z_t^{\mathbb{Q}} \notin \left[\frac{1}{n}, n\right]\right\} \wedge T.$$

It is clear that $\tau_n \uparrow T$ a.s.

Since both $Z^{\mathbb{Q}_*}$ and $Z^{\mathbb{Q}}$ are $\mathbb{P}$-uniformly integrable martingales and, under assumption, the filtration is generated by Brownian motion we apply Theorem V.3.4 from [19] to get continuous versions of $Z^{\mathbb{Q}}$ and $Z^{\mathbb{Q}*}$. Define the sequence of measures

$$\frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{P}} := Z_T^{\mathbb{Q}}\mathbb{I}_{\{\tau_n = T\}} + \frac{Z_{\tau_n}^{\mathbb{Q}}}{Z_{\tau_n}^{\mathbb{Q}_*}} Z_T^{\mathbb{Q}_*}\mathbb{I}_{\{\tau_n < T\}}.$$

Exactly as in Lemma 2 of [21] one can show that $\mathbb{Q}_n \in \mathcal{P}^e(\mathcal{X}_b(x))$ for all n. To complete the proof we need only show the FGE property and the $L^1(\mathbb{P})$ convergence. Observe that $\left(\frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{P}}\right)_n$ is a sequence of nonnegative random variables converging a.s. whose expectations are equal to 1 for all n. Thus by Theorem 10.3 of [22], $\frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{P}} \to \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ in $L^1(\mathbb{P})$. Finally we have

$$\begin{aligned}
\mathbb{E}\left[\tilde{U}\left(\frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{P}}\right)\right] &= \mathbb{E}\left[\tilde{U}\left(Z_T^{\mathbb{Q}}\mathbb{I}_{\{\tau_n = T\}} + \frac{Z_{\tau_n}^{\mathbb{Q}}}{Z_{\tau_n}^{\mathbb{Q}_*}} Z_T^{\mathbb{Q}_*}\mathbb{I}_{\{\tau_n < T\}}\right)\right] \\
&\le \mathbb{E}\left[\tilde{U}\left(Z_T^{\mathbb{Q}}\right)\right] + \mathbb{E}\left[\tilde{U}\left(\frac{Z_{\tau_n}^{\mathbb{Q}}}{Z_{\tau_n}^{\mathbb{Q}_*}} Z_T^{\mathbb{Q}_*}\right)\right] \\
&\le \tilde{U}\left(\frac{1}{n}\right) + \tilde{U}(n) + C_n\mathbb{E}\left[\tilde{U}(Z_T^{\mathbb{Q}_*})\right] < \infty
\end{aligned}$$

for some constant C_n. In the final inequality we have used the boundedness of $\frac{Z_{\tau_n}^{\mathbb{Q}}}{Z_{\tau_n}^{\mathbb{Q}_*}}$ together with Lemma 3.4. □

We are now in a position to prove Theorem 4.1(iv). The reason for the increased difficulty of proof compared with [6,20] is twofold. Firstly, as described in Remark 3.5, we cannot assert that $\mathbb{Q}_* \in \mathcal{M}^e(S)$ so that Theorem D of [9] does not apply. Secondly, even if we could somehow manipulate the processes to apply Theorem D, the candidate optimal integrand constructed seems to be unrelated to the approximating θ^n. Thus it appears to be very difficult to verify that it is indeed valued in the constraint set $\mathcal{K}$.

Proof of Theorem 4.1 Item (iv). Define the $\mathbb{Q}_*$-uniformly integrable martingale

$$X_t^* := \mathbb{E}_{\mathbb{Q}_*}\left[X_* | \mathcal{F}_t\right].$$

Observe that this gives us the relationship $X_T^* = X_*$. By Theorem V.3.4 of [19], we may take a continuous version of X^*. In particular we may assert that it is locally bounded below. Hence we

may find a sequence of stopping times $(\tau_m^1)_m$ such that $\tau_m^1 \uparrow T$ a.s. and $(X^*)^{\tau_m^1} \geq -m$. Exactly as in Step 10 of [20] we pass to a subsequence such that

$$M := \left[\sup_n X_T^n\right]^- \leq W_0$$

for some random variable $W_0 \in L^1(\mathbb{Q}_*)$. Define the nonnegative $\mathbb{Q}_*$-martingale $M_t := \mathbb{E}_{\mathbb{Q}_*}[M|\mathcal{F}_t]$, continuous due to Theorem V.3.4 of [19], together with the sequence of stopping times

$$\tau_m := \inf\{t : M_t \geq m\} \wedge \tau_m^1.$$

We have that $\tau_m \uparrow T$ a.s. and

$$(X^n)^{\tau_m} \geq -m \quad \text{for all } n.$$

We may proceed exactly as in Step 10 of [20] and pass to a sequence of convex combinations of processes

$$\tilde{X}^n \in \operatorname{conv}\left(X^n, X^{n+1}, \ldots\right)$$

such that $\tilde{X}_{\tau_m}^n$ converges a.s., for all m, to a random variable denoted X_{τ_m}. Since, for all $\mathbb{Q} \in \mathcal{P}^a(\mathcal{X}_b(x))$ and n, X^n is a $\mathbb{Q}$-supermartingale, we have

$$\tilde{X}_{\tau_m}^n \geq \mathbb{E}_{\mathbb{Q}}\left[\tilde{X}_T^n|\mathcal{F}_{\tau_m}\right].$$

If $\mathbb{Q} = \mathbb{Q}_*$ Lemma 6.2(iii) implies that $(\tilde{X}_T^n)_n$ is $\mathbb{Q}_*$-uniformly integrable and hence applying Fatou's lemma in the above equation we deduce

$$X_{\tau_m} \geq \mathbb{E}_{\mathbb{Q}_*}\left[X_*|\mathcal{F}_{\tau_m}\right] = X_{\tau_m}^*. \tag{3}$$

It can be shown that $((\tilde{X}_{\tau_m}^n)^-)_n$ inherits $\mathbb{Q}$-uniform integrability from $\left((X_T^n)^-\right)_n$ for all $\mathbb{Q}$ with FGE. Since $\tilde{X}^n$ is a $\mathbb{Q}$-supermartingale for all $\mathbb{Q} \in \mathcal{P}^a(\mathcal{X}_b(x))$ Doob's optional sampling theorem implies $x \geq \mathbb{E}_{\mathbb{Q}}[\tilde{X}_{\tau_m}^n]$. An application of Fatou's lemma now gives

$$x \geq \mathbb{E}_{\mathbb{Q}}\left[X_{\tau_m}\right] \quad \text{for all } m, \mathbb{Q} \in \mathcal{P}^a(\mathcal{X}_b(x)) \text{ with FGE.} \tag{4}$$

If we choose $\mathbb{Q} = \mathbb{Q}_*$ and combine it with (3) we see

$$x \geq \mathbb{E}_{\mathbb{Q}_*}\left[X_{\tau_m}\right] \geq \mathbb{E}_{\mathbb{Q}_*}\left[X_{\tau_m}^*\right] = x,$$

so that $X_{\tau_m}^* = X_{\tau_m}$ a.s. We want to apply Proposition 4.1 from [10]. For this we need to show that for all m

$$x \geq \mathbb{E}_{\mathbb{Q}}\left[X_{\tau_m}\right] \quad \text{for all } \mathbb{Q} \in \mathcal{P}^e(\mathcal{X}_b(x)). \tag{5}$$

It is here we use our density result, Lemma 6.4. Let $\mathbb{Q} \in \mathcal{P}^e(\mathcal{X}_b(x))$ and choose a sequence $(\mathbb{Q}_n)_n \subset \mathcal{P}^e(\mathcal{X}_b(x))$ with FGE such that the densities converge in $L^1(\mathbb{P})$. From (4), $x \geq \mathbb{E}_{\mathbb{Q}_n}\left[X_{\tau_m}^*\right]$ and we have the inequality

$$\left(\frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{P}} X_{\tau_m}^*\right)^- \leq m\frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{P}}.$$

Thus an application of Fatou's lemma gives us (5). The random variable $X^*_{\tau_m} + m$ is nonnegative and $\mathcal{F}_T$-measurable. Thus we may apply Lemma A.1 from [10] to deduce that for each m the process U^m defined by,

$$U^m_t := \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{P}^e(\mathcal{X}_b(x))} \mathbb{E}_{\mathbb{Q}}\left[X^*_{\tau_m}|\mathcal{F}_t\right]$$

is a $\mathbb{Q}$-supermartingale under all $\mathbb{Q} \in \mathcal{P}^e(\mathcal{X}_b(x))$. Theorem 4.1 in [10] guarantees the existence of a constrained ($\mathcal{K}$-valued) $\tilde{\theta}^m$ with $x+(\tilde{\theta}^m \cdot S)$ uniformly bounded from below and an increasing nonnegative optional process C with $C_0 = 0$ such that

$$U^m = x + (\tilde{\theta}^m \cdot S) - C.$$

Since $\mathbb{Q}_* \in \mathcal{P}^e(\mathcal{X}_b(x))$ we deduce

$$\begin{aligned} x \geq \mathbb{E}_{\mathbb{Q}_*}\left[x + (\tilde{\theta}^m \cdot S)_{\tau_m}\right] &\geq \mathbb{E}_{\mathbb{Q}_*}\left[x + (\tilde{\theta}^m \cdot S)_{\tau_m} - C_{\tau_m}\right] \\ &\geq \mathbb{E}_{\mathbb{Q}_*}\left[X^*_{\tau_m}\right] = x. \end{aligned}$$

It follows that $C^{\tau_m} = 0$ and $(X^*)^{\tau_m} = x + (\tilde{\theta}^m \cdot S)^{\tau_m}$. Repeating this procedure provides the existence of integrands supported on $[0, \tau_m]$ such that

$$(X^*)^{\tau_m} = x + (\tilde{\theta}^m \cdot S)^{\tau_m}.$$

It is clear that, with the convention $\tau_0 = 0$, we have the formula

$$(X^*)^{\tau_m} = x + \left(\left(\sum_{i=1}^{m} \tilde{\theta}^i \mathbb{I}_{(\tau_{i-1},\tau_i]}\right) \cdot S\right).$$

Define a candidate optimal θ by

$$\theta := \sum_{i=1}^{\infty} \tilde{\theta}^i \mathbb{I}_{(\tau_{i-1},\tau_i]}.$$

This is an S-integrable predictable integrand and is valued in $\mathcal{K}$. If we set $\hat{X} := x + (\theta \cdot S)$ then we have

$$\hat{X}_{t\wedge\tau_m} = X^*_{t\wedge\tau_m} \quad \text{for } t \in [0, T].$$

Letting m go to infinity shows that if we put $\theta^* = \theta$ we have the required representation property. □

There is the following interesting related result.

Corollary 6.5. *If $\mathbb{Q}_* \sim \mathbb{P}$ then the optimal wealth process X^* is a $\mathbb{Q}$-supermartingale for all $\mathbb{Q} \in \mathcal{P}^a(\mathcal{X}_b(x))$ with* FGE.

Proof. Let $0 \leq s \leq t \leq T$, using the notation and results of Lemma 6.2 we know that X^n is a supermartingale under all $\mathbb{Q} \in \mathcal{P}^a(\mathcal{X}_b(x))$ with FGE. Hence we have

$$\mathbb{E}_{\mathbb{Q}}\left[X^n_t|\mathcal{F}_s\right] \leq X^n_s. \tag{6}$$

If we proceed exactly as in the previous proof we can show that there exist convex combinations, $\tilde{X}^n_t$, $\tilde{X}^n_s$, uniformly integrable below under all $\mathbb{Q} \in \mathcal{P}^a(\mathcal{X}_b(x))$ with FGE and convergent to X^*_t, X^*_s respectively. The result now follows from applying Fatou's lemma in (6). □

Remark 6.6. A careful inspection of the proof above shows that this result holds even when the filtration is not generated by a Brownian motion.

Remark 6.7. Corollary 6.5 extends the result of [21] to the constrained case and offers a simpler proof without the need for dynamic primal and dual problems. Importantly we do not use the representation property for X^* to prove that it is a supermartingale and instead deduce it from the approximating X^n. One can also see that it is the lower uniform integrability which is crucial for this approach to work and (2) shows how this relates to the FGE property of $\mathbb{Q}$.

Corollary 6.5 shows exactly the two properties we would need to prove our result in a more general setting. Firstly one would want to show that Theorem 4.1 of [10] extends to the case of nonlocally bounded below processes. Secondly one would want to show that X^* is a local supermartingale under all $\mathbb{Q} \in \mathcal{P}^e(\mathcal{X}_b(x))$ not just those with FGE. These are highly nontrivial questions in the case of constraints and we leave them for further research.

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Appendix. Proof of Lemma 6.3

Identifying a measure $\mathbb{Q}$ with its Radon–Nikodym derivative we now prove that, for locally bounded S, $\mathcal{P}^e(\mathcal{X}_b(x)) = \tilde{\mathcal{Y}}_+(1) \cap \{Y : Y > 0\}$. Let us first give a preparatory lemma.

Lemma A.1. *Let S be locally bounded. Then for all X in $\mathcal{X}_b(x)$ there exists a sequence of processes X^n, uniformly bounded below in n, bounded above for each n and such that $X^n \to X$ uniformly on compacts in probability, (UCP).*

Proof. We proceed similarly to Lemma 5.1 in [13]. Let $X \in \mathcal{X}_b(x)$. By definition there exists some constrained θ such that $X = x + (\theta \cdot S)$. We know S is locally bounded. So let $(\tau_n^1)_n$ be a sequence of stopping times such that $\tau_n \uparrow T$ a.s. and $|S^{\tau_n^1}| \le K_n$ for a sequence of constants K_n. Set $\theta^n := \theta \mathbb{I}_{\{|\theta| \le n\}} \mathbb{I}_{[0,\tau_n^1]}$. Using the dominated convergence theorem, Theorem IV.32 from [18], we know $(\theta^n \cdot S) \to (\theta \cdot S)$ in UCP. Define the stopping times

$$\tau_n^2 := \inf\left\{ t : \sup_{0 \le u \le t} \left| \left((\theta^n - \theta) \cdot S \right)_u \right| > 1 \right\} \wedge T.$$

Since $(\theta^n \cdot S) \to (\theta \cdot S)$ in UCP we have that $\tau_n^2 \uparrow T$ a.s. Next set $\tilde{\theta}^n := \theta^n \mathbb{I}_{[0,\tau_n^2]}$, and another application of the dominated convergence theorem gives that $(\tilde{\theta}^n \cdot S) \to (\theta \cdot S)$ in UCP. Finally put $\tau_n^3 := \inf\{t : |X_t| \ge n\}$ and $\hat{\theta}^n := \tilde{\theta}^n \mathbb{I}_{[0,\tau_n^3]}$. Observe that the jumps in $(\hat{\theta}^n \cdot S)$ are either zero or the jumps of $(\theta \cdot S)$. Thus if K is the lower bound for X we have, for $t \in [0, T]$

$$K - 1 \le x + (\hat{\theta}^n \cdot S)_t \le n + 1 + 2nK_n.$$

Setting $X^n := x + (\hat{\theta}^n \cdot S)$ gives a sequence with the required properties. □

Lemma A.2. $\mathcal{P}^e(\mathcal{X}_b(x)) = \tilde{\mathcal{Y}}_+(1) \cap \{Y : Y > 0\}$.

Proof. Let us first assume that X is bounded above for all $X \in \mathcal{X}_b(x)$. Since $\mathcal{X}_+(x) \subset \mathcal{X}_b(x)$ it is clear that $\mathcal{P}^e(\mathcal{X}_b(x)) \subset \tilde{\mathcal{Y}}_+(1)$. Moreover, as $\mathbb{Q} \sim \mathbb{P}$ for all $\mathbb{Q} \in \mathcal{P}^e(\mathcal{X}_b(x))$ we have

$$\mathcal{P}^e(\mathcal{X}_b(x)) \subset \tilde{\mathcal{Y}}_+(1) \cap \{Y : Y > 0\}.$$

Next choose $X \in \mathcal{X}_b(x)$. Under our present assumption X is bounded. Hence we may find a constant C such that for any $t \geq s$ and $A \in \mathcal{F}_s$

$$C + \mathbb{I}_A (X_t - X_s) \in \mathcal{X}_+^T(C).$$

If $Y > 0$ and $Y \in \tilde{\mathcal{Y}}_+(1)$, let $\mathbb{Q} \sim \mathbb{P}$ be defined by $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = Y$. Since $Y \in \tilde{\mathcal{Y}}_+(1)$ we have

$$\mathbb{E}\left[Y\left(C + \mathbb{I}_A (X_t - X_s)\right)\right] \leq C.$$

Equivalently, $\mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_A (X_t - X_s)\right] \leq 0$ for all $A \in \mathcal{F}_s$, $0 \leq s \leq t \leq T$. This clearly implies that X is a $\mathbb{Q}$-supermartingale. Since X was arbitrary we deduce

$$\tilde{\mathcal{Y}}_+(1) \cap \{Y : Y > 0\} \subset \mathcal{P}^e(\mathcal{X}_b(x)).$$

Now suppose that X is not bounded above. Recall from Lemma A.1 that there exists a sequence $X^n \in \mathcal{X}_b(x)$, bounded from above, such that X^n is uniformly bounded from below in n and converges UCP to X. For each X^n we are in the case above and we see that for any $t \geq s$

$$X_s^n \geq \mathbb{E}_{\mathbb{Q}}\left[X_t^n | \mathcal{F}_s\right].$$

The uniform lower bound permits the use of Fatou's lemma and hence we deduce

$$X_s \geq \mathbb{E}_{\mathbb{Q}}\left[X_t | \mathcal{F}_s\right] \quad \text{for all } 0 \leq s \leq t \leq T.$$

This completes the proof of the lemma. □

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