Convergence in the semimartingale topology and constrained portfolios

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Abstract Consider an $\mathbb{R}^d$-valued semimartingale $S$ and a sequence of $\mathbb{R}^d$-valued $S$-integrable predictable processes $H^n$ valued in some closed convex set $\mathcal{K} \subset \mathbb{R}^d$, containing the origin. Suppose that the real-valued sequence $H^n \cdot S$ converges to $X$ in the semimartingale topology. We would like to know whether we may write $X = H^0 \cdot S$ for some $\mathbb{R}^d$-valued, $S$-integrable process $H^0$ valued in $\mathcal{K}$? This question is of crucial importance when looking at superreplication under constraints. The paper considers a generalization of the above problem to $\mathcal{K} = \mathcal{K}(\omega, t)$ possibly time dependent and random.
1 Introduction

The Émery distance between two real-valued semimartingales $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ is defined by

$$d(X, Y) = \sup_{|J| \leq 1} \left( \sum_{n \geq 1} 2^{-n} \mathbb{E}[1 \wedge |(J \cdot (X - Y))_n|] \right)$$

where $(J \cdot X)_t := \int_0^t J_u dX_u$, and the supremum is taken over all predictable processes $J$ bounded by 1. Émery [1] shows that with respect to this metric the space of semimartingales is complete. For a given $\mathbb{R}^d$-valued semimartingale $S$ we write $L(S)$ for the space of $\mathbb{R}^d$-valued, $S$-integrable, predictable processes $H$ and $L_{a \text{loc}}(S)$ for those processes in $L(S)$ for which $H \cdot S$ is locally bounded from below. By construction $H \cdot S$ is a real-valued semimartingale being the vector stochastic integral of an $\mathbb{R}^d$-valued process $H$ with respect to the $\mathbb{R}^d$-valued semimartingale $S$; see Jacod and Shiryaev [5] Section III.6. We write $L(S)$ for the space of all equivalence classes in $L(S)$ with respect to the quasi-norm (in the sense of Yosida [18] Definition 1.2.2),

$$d_S(\mathcal{H}^1, \mathcal{H}^2) = d(\mathcal{H}^1 \cdot S, \mathcal{H}^2 \cdot S).$$

We define $L_{a \text{loc}}(S)$ analogously. Hence in $L(S)$ we identify all processes $\mathcal{H} \in L(S)$ that yield the same stochastic integral $\mathcal{H} \cdot S$. Mémin [10] Theorem V.4 shows that $L(S)$ is a complete topological vector space with respect to $d_S$. Equivalently, the space of stochastic integrals

$$\{H \cdot S \mid H \in L(S)\}$$

is closed in the semimartingale topology.

A natural question to ask is the following. Given a sequence $H^n \in L(S)$ and a process $H^0 \in L(S)$ with $d_S(H^n, H^0)$ converging to 0 and a closed convex set $\mathcal{K}$, where $\mathcal{K} \subset \mathbb{R}^d$ contains the origin, suppose that $H^n_t \in \mathcal{K}$ a.s. for all $t$ and $n \in \mathbb{N}$, can we deduce that $H^0_t \in \mathcal{K}$ a.s. for all $t$? More precisely, we assume that each class $H^n \in L(S)$ has a representative $\mathcal{H}^n \in L(S)$ with $\mathcal{H}^n_t \in \mathcal{K}$ a.s. for all $t$, and we ask whether the class $H^0$ admits a representative $\mathcal{H}^0$ such that $\mathcal{H}^0_t \in \mathcal{K}$ a.s. for all $t$. For brevity we write $H^0 \in L(S)$ and $H^0_t \in \mathcal{K}$ a.s. for all $t$, etc.

This question is closely related to finding a constrained optional decomposition for a given process. Form the set of $\mathcal{K}$-valued integrands

$$\mathcal{J} := \{H \in L_{a \text{loc}}(S) \mid H_t \in \mathcal{K} \text{ a.s. for all } t\}$$

(where we have slightly abused notation as pointed out above). This defines a family of semimartingales via

$$\mathcal{S} := \{H \cdot S \mid H \in \mathcal{J}\}.$$  \hfill (1)

Föllmer and Kramkov [2] characterize those locally bounded below processes $Z$ which may be written as
for some increasing nonnegative optional process \( C \) and some \( H \in \mathcal{H} \). In mathematical finance \( S \) is the discounted asset price process and \( Z \) is typically related to some contingent claim. The existence of a decomposition (2) means that \( Z \) can be superreplicated by a \( \mathcal{H} \)-valued portfolio \( H \) with initial endowment \( Z_0 \). Karatzas and Žitković [7], in the context of utility maximization with consumption and random endowment, and Pham [11, 12], in the setting of utility maximization and shortfall risk minimization, apply [2] Theorem 4.1 to deduce the existence of a constrained optimal solution. A crucial condition on the set \( \mathcal{F} \) needed for [2] Theorem 4.1 to hold is the following:

**Assumption 1.1** ([2] Assumption 3.1). If \( H^n \cdot S \) is a sequence in \( \mathcal{F} \), uniformly bounded from below, which converges in the semimartingale topology to \( X \) then \( X \in \mathcal{F} \).

Consider the set \( \mathcal{F} \) defined by (1). Let us discuss whether it satisfies the above assumption. Suppose \( H^n \cdot S \) is a uniformly bounded below sequence converging in the semimartingale topology to \( X \). Since \( \{ H \cdot S \mid H \in L(S) \} \) is closed in the semimartingale topology and convergence in that topology implies uniform convergence on compacts in probability it follows that there exists \( H^0 \in L_{loc}^1(S) \) with \( X = H^0 \cdot S \). Thus we see that it is sufficient to check whether \( \mathcal{F} \) verifies the following:

**Assumption 1.2.** If \( H^n \cdot S \) is a sequence in \( \mathcal{F} \) which converges in the semimartingale topology to \( H^0 \cdot S \) then \( H^0 \cdot S \in \mathcal{F} \).

When \( \mathcal{H} \neq \mathbb{R}^d \) we are led to investigate whether we can find a representative \( \mathcal{H}^0 \) of the class \( H^0 \) which is \( \mathcal{H} \)-valued. This is precisely the problem considered in the main result of the present paper, Theorem 3.5. Note that we only require one, not every, representative of the limit class in \( L(S) \) to be \( \mathcal{H} \)-valued.

In [11, 12] it is shown that pointwise properties are preserved under the additional conditions that \( S \) is continuous and satisfies \( d[S,S]_t = \sigma_t dt \) for a matrix-valued process \( \sigma_t \), assumed positive definite a.s. for all \( t \). In this case positive definiteness implies that all components of the integrands converge pointwise a.s. for each \( t \) and therefore the closedness of \( \mathcal{H} \) gives that the limit is again in \( \mathcal{H} \). In incomplete markets \( \sigma_t \) is generally only positive semi-definite and one cannot argue in this way.

In [7] it is implicitly assumed that Assumption 1.2 is valid when \( \mathcal{H} \) is an arbitrary (fixed) closed convex cone \( \mathcal{H} \) and \( S \) is any \( \mathbb{R}^d \)-valued semimartingale satisfying an absence of arbitrage assumption. In Section 2 we give a counterexample to show that this is false in general. We show that (without imposing extra conditions on \( S \)) to obtain a positive answer to the question of whether the limit class of integrands admits a representative which is \( \mathcal{H} \)-valued, one must restrict \( \mathcal{H} \). In fact it is sufficient that \( \mathcal{H} \) be either a continuous or a polyhedral set.

The main contribution of this paper is to show that for these two choices of \( \mathcal{H} \) the sets \( \mathcal{F} \) defined by (1) satisfy Assumption 1.2 and [2] Theorem 4.1 may be applied. This covers many examples currently in the literature. In particular, as shown in Section 4, that of no-short-selling constraints and upper and lower bounds on the number of shares of each asset held, listed as examples in [2] without proof.
The layout of the paper is as follows: Section 2 contains two insightful examples, Section 3 provides the main result (Theorem 3.5), Section 4 gives some applications and Sections 5 and 6 contain the proof.

2 Motivating Examples

To illustrate some of the problems that can arise we give the following examples. They show that, without further conditions on $S$, for arbitrary closed convex sets it may not be possible to find an appropriate representative of the limiting class $H^0$. Note that in this section and throughout, all vectors are column vectors.

Example 1

Let $W$ denote a standard 1-dimensional Brownian motion and $\tau$ be the stopping time defined by

$$\tau := \inf \{ t \geq 0 \mid |W_t| = 1 \}.$$

We let $S := (W^\tau, 0)^\top$ so that we have

$$d\langle S, S \rangle_t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} I_{\{t \leq \tau\}} dt + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} I_{\{t > \tau\}} dt$$

$$:= C I_{\{t \leq \tau\}} dt + 0 I_{\{t > \tau\}} dt.$$

The matrix $C$ is not positive definite and the kernel of $C$ is given by $\text{Ker}(C) = \{ \mu(0, 1)^\top \mid \mu \in \mathbb{R} \}$. Set

$$\mathcal{H} := \left\{ (x, y)^\top \in (-1, \infty) \times \mathbb{R} \mid y \geq \frac{1}{x+1} - 1 \right\}.$$

This is a closed convex set containing the origin. We define, for $n \in \mathbb{N}$, a sequence of (constant) processes valued on the boundary of $\mathcal{H}$,

$$\mathcal{H}^n := (-1 + 1/n, n-1)^\top.$$

The $\mathcal{H}^n$ do not converge pointwise, their norms are unbounded. Observe that projecting $\mathcal{H}^n$ onto the orthogonal complement of $\text{Ker}(C)$ gives another representative of $H^n$ which we call $\hat{\mathcal{H}}^n$. Thus we have

$$\hat{\mathcal{H}}^n = (-1 + 1/n, 0)^\top,$$

$$\hat{\mathcal{H}}^n \cdot S = \mathcal{H}^n \cdot S, n \in \mathbb{N}.$$

If we define the (constant) process $\mathcal{H}^0 = (-1, 0)^\top$ then
\[ H^n \cdot S = (1 - 1/n)(H^0 \cdot S). \]

It then follows that \( H^n \cdot S \) converges in the semimartingale topology to \( H^0 \cdot S \).

We seek a representative of the class \( H^0 \) which is \( \mathcal{K} \)-valued. By construction, the stochastic integral of each \( \mathbb{R}^2 \)-valued predictable process, valued in \( \text{Ker}(C) \) on \( \{ t \leq \tau \} \) \( d\mathbb{P} \otimes dt \)-a.e., is zero. This implies that the equivalence class of the process \( 0 \in \mathcal{L}(S) \) consists up to \( d\mathbb{P} \otimes dt \)-a.e. equality of the processes \( G^1(0,1)^\top + G^2(1,0)^\top 1_{\{t>\tau\}} \), where \( G^1 \) and \( G^2 \) are some real-valued predictable processes. Since adding a representative of 0 to some element of \( \mathcal{L}(S) \) does not change its equivalence class, we obtain that the equivalence class \( H \) of any given \( H \in \mathcal{L}(S) \) is given up to \( d\mathbb{P} \otimes dt \)-a.e. equality by

\[
H = \{ H + G^1(0,1)^\top + G^2(1,0)^\top 1_{\{t>\tau\}} \mid G^1, G^2 \text{ real-valued predictable processes} \}.
\]

However, due to the vertical asymptote of \( \mathcal{K} \) at \( x = -1 \) we have

\[
\{(1,0)^\top + \mu(0,1)^\top \cap \mathcal{K} = \emptyset \text{ for all } \mu \in \mathbb{R}.}
\]

In particular, adding vectors valued in the kernel of \( C \) to \( H^0 \) will never give a \( \mathcal{K} \)-valued integrand and therefore one cannot find an appropriate representative \( \tilde{H}^0 \) of \( H^0 \). This illustrates that, without making further assumptions on \( S \), one cannot allow arbitrary closed convex sets. The crucial point here is that \( \mathcal{K} \) is not a continuous set. One may hope that by imposing some other restriction, for instance that \( \mathcal{K} \) is a cone, the closure property may still be proved. The following example shows that this is not the case.

**Example 2**

Let \( W = (W^1,W^2,W^3)^\top \) be a 3-dimensional Brownian motion and set \( Y = \sigma \cdot W \) where

\[
\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}.
\]

The matrices \( \sigma^\top \) and \( C := \sigma \sigma^\top \) have nontrivial kernel spanned by \( w = \frac{1}{\sqrt{2}}(0,1,1)^\top \)

\[
\text{i.e., } \quad \text{Ker}(C) = \text{Ker}(\sigma \sigma^\top) = \mathbb{R}w = \left\{ \frac{1}{\sqrt{2}}(0,\mu,\mu)^\top \mid \mu \in \mathbb{R} \right\}.
\]

As in the previous example we obtain that the stochastic integral of each \( \mathbb{R}^3 \)-valued predictable process, valued in \( \text{Ker}(C) \) \( d\mathbb{P} \otimes dt \)-a.e., is zero and therefore that the equivalence class \( H \) of any given \( H \in \mathcal{L}(Y) \) is given up to \( d\mathbb{P} \otimes dt \)-a.e. equality by

\[
H = \{ H + \mathcal{G}w \mid \mathcal{G} \text{ a real-valued predictable process} \}.
\]
Let $\mathcal{K}$ be the closed convex cone 

$$\left\{(x,y,z)^\top \in \mathbb{R}^3 \mid x^2 + y^2 \leq z^2, \, z \geq 0\right\}.$$ 

Choose a sequence $(z_n)_{n \in \mathbb{N}}$ in $[1, \infty)$ tending to infinity and define the sequence of (constant) processes $(\mathcal{H}^n)_{n \in \mathbb{N}}$ by $\mathcal{H}^n = (1, \sqrt{z_n^2 - 1}, z_n)^\top$ for each $n \in \mathbb{N}$. Then each $\mathcal{H}^n$ is $\mathcal{K}$-valued and we obtain 

$$\mathbb{E}\left[\left(\left(\mathcal{H}^n \cdot Y\right)_t - W^1_t\right)^2\right] = \mathbb{E}\left[\langle \mathcal{H}^n \cdot Y - W^1, \mathcal{H}^n \cdot Y - W^1 \rangle_t\right]$$ 

$$= \int_0^t \left(\mathcal{H}^n_s - (1,0,0)^\top\right)^\top C_s \left(\mathcal{H}^n_s - (1,0,0)^\top\right) ds$$ 

$$= \int_0^t 2 \left[ \frac{2z^2}{z_n^2} \left( 1 - \sqrt{1 - \frac{1}{z_n^2}} \right) - 1 \right] ds$$ 

$$= \int_0^t 2 \left[ \frac{2z^2}{z_n^2} \left( 1 - \left\{ 1 - \frac{1}{2z^2} + O\left(\frac{1}{z_n^2}\right) \right\} \right) - 1 \right] ds$$ 

$$= t O\left(\frac{1}{z_n^2}\right). \quad (3)$$ 

Hence $\mathcal{H}^n \cdot Y$ converges to $W^1$ locally in $\mathcal{M}^{1,1}(\mathbb{P})$ (the space of $\mathbb{P}$-square-integrable 1-dimensional martingales) and thus by [10] Theorem IV.5 also in the semimartingale topology. However the (constant) process $(1,0,0)^\top$ having stochastic integral $(1,0,0)^\top \cdot Y = W^1$ is not $\mathcal{K}$-valued.

Recall that we identify processes in $\mathcal{L}(Y)$ yielding the same stochastic integral and therefore it would be sufficient to find one predictable process equivalent to $(1,0,0)^\top$ and valued in $\mathcal{K}$ a.s. for each $t$. As discussed above the equivalence class of $(1,0,0)^\top$ is, up to $d\mathbb{P} \otimes dt$-a.e. equality, 

$$\left\{(1,0,0)^\top + Gw \mid G \text{ a real-valued predictable process}\right\}.$$ 

For every fixed $t$ and $\omega$ these have the form 

$$\left(1, \frac{G_t(\omega)}{\sqrt{2}}, \frac{G_t(\omega)}{\sqrt{2}}\right)^\top.$$ 

It follows from the definition of $\mathcal{K}$ that there is no predictable process equivalent to $(1,0,0)^\top$ which is $\mathcal{K}$-valued. Defining the stopping time 

$$\tau := \inf \{t \geq 0 \mid \|W^1_t\|_{\mathbb{R}^3} = 1\}.$$ 

and setting $S = Y^\tau$ and defining $\mathcal{H}$ by (1) gives an example which does not satisfy Assumption 1.1. (For reference observe that we write $\|\cdot\|_{\mathbb{R}^d}$ for the Euclidean norm on $\mathbb{R}^d$). Thus we have a counterexample to the implicit claim in [7] as $S$ is a bounded martingale and therefore satisfies the no arbitrage assumption therein.
Exactly as in the previous example convergence of the stochastic integrals \( H_n \cdot S \) does not necessarily imply that the representatives \( H^n \) satisfying the constraints converge pointwise. Therefore one cannot argue using the pointwise closedness of \( \mathcal{H} \) to obtain that the limit is again valued in \( \mathcal{H} \). Indeed, in this case \( \|H^n\|_{\mathbb{R}^3} = \sqrt{2} \epsilon_n \) and the sequence of representatives actually diverges. Thus, for a general closed convex cone \( \mathcal{H} \), one cannot show that \( S \) is closed in the semimartingale topology.

However although the \( H^n \) need not converge pointwise, we can always find a related sequence of representatives \( \hat{H}_n \) that do. The issue then is that these need not be \( \mathcal{H} \)-valued anymore. To obtain the \( \hat{H}_n \), as in the previous example, we project onto the orthogonal complement of \( \text{Ker}(C) = \text{Ker}(\sigma \sigma^\top) \). The eigenvalue decomposition of \( C \) is given by \( C = P^\top DP \) with

\[
P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Define \( g^n := P H^n \) and recall from (3) that

\[
(H^n - (1,0,0)^\top) C (H^n - (1,0,0)^\top)
\] (4)

converges to 0 in \( dP \otimes dt \)-measure, hence \( dP \otimes dt \)-a.e. along a subsequence, also indexed by \( n \). Using the decomposition of \( C \) and writing \( g^{n,i} \) for the \( i \)-th component of \( g^n \), (4) is equivalent to

\[
(g^{n,1} - 1)^2 + 4(g^{n,2})^2 \rightarrow 0, \quad dP \otimes dt \text{-a.e.}
\]

Therefore, \( dP \otimes dt \)-a.e., \( g^{n,1} \) converges to 1 and \( g^{n,2} \) to 0. The vectors \( v_1 = (1,0,0)^\top \), \( v_2 = \frac{1}{\sqrt{2}} (0,1,-1)^\top \) together with \( w = \frac{1}{\sqrt{2}} (0,1,1)^\top \) form an orthonormal basis of \( \mathbb{R}^3 \). The decomposition of \( H^n \) with respect to \( v_1, v_2 \) and \( w \) is given by

\[
H^n = ((H^n)^\top v_1) v_1 + ((H^n)^\top v_2) v_2 + ((H^n)^\top w) w
= g^{n,1} v_1 + g^{n,2} v_2 + g^{n,3} w.
\]

If we now define \( \tilde{H}^n = g^{n,1} v_1 + g^{n,2} v_2 \) then we obtain that, for each \( n \in \mathbb{N} \), \( H^n - \tilde{H}^n \) is valued in \( \text{Ker}(C) \) \( dP \otimes dt \)-a.e. and therefore \( H^n \) and \( \tilde{H}^n \) are in the same equivalence class. Moreover, \( \tilde{H}^n \) converges to \( (1,0,0)^\top \) \( dP \otimes dt \)-a.e., and hence for all \( \omega \) and \( t \) if we set \( \tilde{H}^n \) equal to \( (1,0,0)^\top \) on the null set where this convergence does not hold.

Motivated by these examples we now study those convex sets for which we can use such pointwise convergence to deduce the existence of a \( \mathcal{H} \)-valued representative of the limit class.
3 Main Results

We work on the filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\), which is assumed to satisfy the usual conditions. This space supports an \(\mathbb{R}^d\)-valued semimartingale \(S\), which one may think of as an asset price process. For notational simplicity the index \(t\) is valued in \([0,1]\), the extension to the case when \(t \in \mathbb{R}_+\) is straightforward. The predictable \(\sigma\)-field on \(\Omega \times [0,1]\) generated by all left-continuous adapted processes is denoted by \(\mathcal{P}\). For an \(\mathbb{R}^d\)-valued \(S\)-integrable predictable process \(H \in L(S)\) we write \(H \cdot S\) for the stochastic integral of \(H\) with respect to \(S\) and refer to [5] Section III.6 for the theory of stochastic integration of vector valued processes.

We consider possibly random and time dependent \(K = K(\omega, t)\) and for this we need the notion of a measurable multivalued mapping, taken from Rockafellar [13, 15] and Wagner [16]. Recently the idea to formulate constraints via measurable multivalued mappings has been used in Karatzas and Kardaras [6] to study the numéraire portfolio under convex constraints. Let \(\mathcal{T}\) be a set together with \(\mathcal{A}_{\mathcal{T}}\), a \(\sigma\)-field of subsets of \(\mathcal{T}\). We write \(2^{\mathbb{R}^d}\) for the power set of \(\mathbb{R}^d\).

Definition 3.1. A multivalued mapping \(F : \mathcal{T} \to 2^{\mathbb{R}^d}\) is called measurable if, for all closed subsets \(Q\) of \(\mathbb{R}^d\),

\[ F^{-1}(Q) := \{ t \in \mathcal{T} \mid F(t) \cap Q \neq \emptyset \} \in \mathcal{A}_{\mathcal{T}}. \]

When \(F(t)\) is a closed (convex) set for all \(t \in \mathcal{T}\) it is said to be closed (convex).

We say \(F\) is predictably measurable when \(\mathcal{T} = \Omega \times [0,1]\) and \(\mathcal{A}_{\mathcal{T}} = \mathcal{P}\). Motivated by the examples in Section 2 we place more restrictions on \(K\) in order to obtain a positive answer to the question posed in the introduction. The following definition is from Gale and Klee [3].

Definition 3.2. A convex set \(Q \subset \mathbb{R}^d\) is called continuous if it is closed and its support function

\[ \rho(u) = \sup_{q \in Q} (q^\top u) \]

is continuous for all vectors \(u \in \mathbb{R}^d\) with \(\|u\|_{\mathbb{R}^d} = 1\).

The set \(Q\) may be unbounded and hence we allow \(\rho\) to take the value \(+\infty\), with continuity at \(u\) for which \(\rho(u) = \infty\) defined in the usual way.

We can treat another type of \(K\), for this we use the definition of a polyhedral convex set, taken from Rockafellar [14]. By Theorem 19.1 therein this coincides with that of a finitely generated convex set.

Definition 3.3. A closed convex set \(Q \subset \mathbb{R}^d\) is called polyhedral if there exists \(m \in \mathbb{N}\), real numbers \(r_1, \ldots, r_m\) and vectors \(p_1, \ldots, p_m\) such that

\[ Q = \{ q \in \mathbb{R}^d \mid p_i^\top q \leq r_i \text{ for } 1 \leq i \leq m \}. \]

We make the following assumption throughout the rest of this paper.
**Assumption 3.4.** The multivalued mapping $\mathcal{K}$ is closed convex and predictably measurable with $0 \in \mathcal{K}(\cdot,t)$ a.s. for all $t$. For each $t$, $\mathcal{K}(\cdot,t)$ is a.s. either continuous or a polyhedral set.

With all the necessary preliminaries introduced we can now state our main result.

**Theorem 3.5.** Let $\mathcal{K}$ satisfy Assumption 3.4 and $H^n \in L(S)$ be a sequence of predictable processes with

(i) $H^n_t \in \mathcal{K}(\cdot,t)$ a.s. for all $t$ and $n \in \mathbb{N}$, i.e., there exists a representative $\mathcal{H}^n$ of each equivalence class $H^n$ such that $\mathcal{H}^n_t \in \mathcal{K}(\cdot,t)$ a.s. for all $t$.

(ii) $H^n \cdot S$ converges in the semimartingale topology to some semimartingale $X$.

Then there exists $H^0$ in $L(S)$ such that $X = H^0 \cdot S$ and $H^0_t \in \mathcal{K}(\cdot,t)$ a.s. for all $t$.

More precisely, there exists a representative $\mathcal{H}^0$ of the equivalence class $H^0$ such that $\mathcal{H}^0_t \in \mathcal{K}(\cdot,t)$ a.s. for all $t$.

Before we proceed to the proof we give some important situations where one can apply Theorem 3.5.

### 4 Applications

#### 4.1 Optional Decomposition Under Constraints

We suppose there exists $m \in \mathbb{N}$ and that for $1 \leq i \leq m$ there are predictable processes $I_i$ and $G_i$, valued in $\mathbb{R}^d$ and $\mathbb{R}_+$ respectively. We define a set of integrands $\mathcal{J}_1$ via

$$\mathcal{J}_1 := \left\{ H \in L^2_{\text{loc}}(S) \mid I_i^\top H \leq G_i \quad \text{for } 1 \leq i \leq m \right\}. \tag{5}$$

In the above all the inequalities are to be understood in the sense of Assumption 3.4. For example, in the above, we have that for $1 \leq i \leq m$,

$$I_i(\cdot,t)^\top H^i(\cdot,t) \leq G_i(\cdot,t) \quad \mathbb{P} \text{-a.s. for all } t.$$

By comparison with Definition 3.3 this is equivalent to saying that the integrand $H$ is valued in the closed convex polyhedral set

$$\mathcal{H}_1(\omega,t) := \left\{ k \in \mathbb{R}^d \mid (I_i(\omega,t))^\top k \leq G_i(\omega,t) \quad \text{for } 1 \leq i \leq m \right\}.$$

Contained within this framework are a very large class of examples. In particular no short selling as well as upper and lower bounds on the number of shares of each asset held; both given in [2] as examples to which their Theorem 4.1 applies. Define

$$\mathcal{J}_1 := \{ H \cdot S \mid H \in \mathcal{J}_1 \}.$$
As discussed in the introduction we need only show that $\mathcal{S}$ satisfies Assumption 1.2, more precisely that if $H^n \cdot S$ converges to $H^0 \cdot S$ then there exists an appropriate representative of the limiting class. This will follow from Theorem 3.5 once we show that the set $K_1(\omega, t)$ is predictably measurable.

The mapping $\pi_i((\omega, t), x) := I_i(\omega, t)^\top x$ is continuous in $x$ and predictably measurable. Moreover the mapping $K'_i(\omega, t) := \{ k \in \mathbb{R}^d | k \leq G_i(\omega, t) \}$ is closed and predictably measurable. By definition we have

$$K_1(\omega, t) = \bigcap_{i=1}^m \left\{ k \in \mathbb{R}^d \mid \pi_i((\omega, t), k) \in K'_i(\omega, t) \right\}$$

and the result now follows from Definition 5.4 and Lemmas 5.3 and 5.5. We have shown that sets defined by (5) are valid examples to which the optional decomposition theorem under constraints applies.

### 4.2 Utility Maximization

In [7], within the framework of a utility maximization problem the authors propose the set

$$\mathcal{J}_2 := \{ H \in L^a_{\text{loc}}(S) \mid x + H \cdot S \geq 0 \text{ and } H_t \in K_2 \text{ a.s. for all } t \in [0, 1] \},$$

where $K_2$ is a closed convex cone in $\mathbb{R}^d$ containing the origin. The set $\mathcal{S}_2 := \{ x + H \cdot S \mid H \in \mathcal{J}_2 \}$ is the family of nonnegative wealth processes with initial capital $x$ and cone constraints on the investment strategy.

The existence of a solution to the utility maximization problem posed in [7] depends crucially on Proposition 2.13 therein, a dual characterization of superreplicable consumption processes. It is established by an application of [2] Theorem 4.1 to $\mathcal{J}_2$. It is not immediately clear from [7] that $\mathcal{S}_2$ satisfies Assumption 1.2, more specifically that a representative $\mathcal{H}^0$ of the limit class $H^0$ may be chosen to be $K_2$-valued. As illustrated by Example 2 in Section 2, this is not true for general closed convex cones.

However when $K_2$ is additionally assumed polyhedral, we can show the existence of a suitable representation. Indeed one has that $K_2$ is now a closed convex polyhedral cone containing 0 which is independent of $(\omega, t)$ and hence predictably measurable. Applying Theorem 3.5 now shows that Assumption 1.2 holds for $\mathcal{S}_2$.

One may also use Theorem 3.5 in utility maximization on the whole real line under cone constraints and we refer the reader to Westray and Zheng [17] for more details.
5 Measurable Selection

We review some results on stochastic processes, separation of convex sets and measurable selection. We then prove Lemma 5.9 which is crucial in establishing our main result, Theorem 3.5.

5.1 Stochastic Processes

We define all processes up to indistinguishability. We use the phrase “for all $t'$”, implicitly meaning “for all $t \in [0,1]$”. Throughout we write $X$ for the process $(X_t)_{t \in [0,1]}$. We reserve $n$ for sequences and $i$ for components of vectors in the sense that $X^n, i$ denotes the process formed from the $i$th-component of $X^n$.

The $\mathbb{R}^d$-valued semimartingale $S$ may be decomposed as $S = \tilde{M} + \tilde{A}$ where $\tilde{M}$ is a $\mathbb{P}$-local martingale and $\tilde{A}$ a process of finite variation. We write $\mathcal{M}^{d,d}(\mathbb{P})$ for the space of $d$-dimensional square-integrable martingales and $\mathcal{A}^{1,d}(\mathbb{P})$ for the space of $d$-dimensional predictable processes of integrable variation on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. These are turned into Banach spaces by equipping them with the norms

$$
\|\tilde{M}\|_{\mathcal{M}^{d,d}(\mathbb{P})} = \left( \mathbb{E} \left[ \sum_{i=1}^{d} (\tilde{M}_t^i, \tilde{M}_t^i)_1 \right] \right)^{\frac{1}{2}},
\|\tilde{A}\|_{\mathcal{A}^{1,d}(\mathbb{P})} = \mathbb{E} \left[ \sum_{i=1}^{d} \int_0^1 |d\tilde{A}_t^i| \right].
$$

Here $\tilde{M}_t^i$ is a $\mathbb{P}$-square-integrable martingale for $1 \leq i \leq d$ and in the above we have written $(\tilde{M}_t^i, \tilde{M}_t^j)$ for the predictable compensator of $[\tilde{M}_t^i, \tilde{M}_t^j]$ satisfying,

$$
\mathbb{E}[(\tilde{M}_t^i)^2] = \mathbb{E}[(\tilde{M}_t^i, \tilde{M}_t^i)_t] = \mathbb{E}[(\tilde{M}_t^i, \tilde{M}_t^j)_t] \text{ for all } t \text{ and } 1 \leq i \leq d.
$$

In addition, for each $i$, $|d\tilde{A}_t^i|$ denotes the differential of the total variation process $|\tilde{A}_t^i|$.

By [5] Proposition II.2.9 and II.2.29 there exist an increasing $\mathbb{Q}$-integrable, predictable process $V$, an $\mathbb{R}^d$-valued predictable process $B$ together with a predictable process $C$, taking values in the set of symmetric positive semi-definite $d \times d$ matrices, such that for $1 \leq i, j \leq d$

$$
A_t^i = B_t^i \cdot V \text{ and } (\tilde{M}_t^i, \tilde{M}_t^j) = C_{ij} \cdot V. \quad (6)
$$

By adding $t$ to $V$ and applying the Radon-Nikodym theorem, exactly as in the proof of [5] Proposition II.2.29, we may, without loss of generality, assume that $V$ is strictly increasing. There exist many processes $V$, $B$ and $C$ satisfying (6), but our results do not depend on the specific choice we make.

We are only interested in the representation of a real-valued semimartingale $X$, which is the limit of a sequence of stochastic integrals $H^n \cdot S$ converging in the semimartingale topology. Hence we can, as in the proof of [10] Theorem V.4, switch
to an equivalent probability measure $Q$ and find a subsequence, also indexed by $n$, such that $S = M + A \in \mathcal{M}^0, S \in \mathcal{M}^1, H^0 \cdot S \in \mathcal{M}^2, H^1 \cdot S \in \mathcal{M}^3$ and $H^0 \cdot S$ converges to $X$ in $\mathcal{M}^2, \mathcal{M}^3$ and $H^1 \cdot S$ converges to $L$ in $\mathcal{M}^3, \mathcal{M}^4$. It then follows that $X$ is given by $H^0 \cdot S$ for some $H^0 \in L^2(M, \mathcal{Q}) \cap L^1(A, \mathcal{Q})$.

Let us explain the previous notation. $L^2(M, \mathcal{Q})$ is the set of $M$-integrable predictable processes $\mathcal{H}$ such that

$$
\mathbb{E}_\mathcal{Q}(\langle \mathcal{H} \cdot M, \mathcal{H} \cdot M \rangle_1) = \mathbb{E}_\mathcal{Q} \left[ \sum_{i,j=1}^d (\langle \mathcal{H}^i \mathcal{H}^j \rangle) \right] < \infty.
$$

The set of equivalence classes in $L^2(M, \mathcal{Q})$ with respect to the relation

$$
\mathcal{H}^1 \sim \mathcal{H}^2 \text{ iff } \mathbb{E}_\mathcal{Q} \left[ (\langle \mathcal{H}^1 - \mathcal{H}^2 \rangle \cdot M, (\mathcal{H}^1 - \mathcal{H}^2) \cdot M \rangle_1 \right] = 0
$$

is then denoted $L^2(M, \mathcal{Q})$. For the set of $A$-integrable predictable processes $\mathcal{H}$ such that

$$
\mathbb{E}_\mathcal{Q} \left[ \int_0^1 |d(\mathcal{H} \cdot A)| \right] < \infty
$$

we write $L^1(A, \mathcal{Q})$. As in the martingale case $L^1(A, \mathcal{Q})$ is then the set of equivalence classes in $L^1(A, \mathcal{Q})$ with respect to the relation

$$
\mathcal{H}^1 \sim \mathcal{H}^2 \text{ iff } \mathbb{E}_\mathcal{Q} \left[ \int_0^1 |d(\mathcal{H}^1 - \mathcal{H}^2) \cdot A)| \right] = 0.
$$

It follows that a predictable process $\mathcal{H}$ in $L^2(M, \mathcal{Q}) \cap L^1(A, \mathcal{Q})$ has stochastic integral $\mathcal{H} \cdot S = 0$ if and only if

$$
\mathbb{E}_\mathcal{Q} \left[ \langle \mathcal{H} \cdot M, \mathcal{H} \cdot M \rangle_1 \right] + \mathbb{E}_\mathcal{Q} \left[ \int_0^1 |d(\mathcal{H} \cdot A)| \right] = 0.
$$

Using (6) this may be equivalently written as

$$
\mathbb{E}_\mathcal{Q} \left[ \int_0^1 \mathcal{H}_t^\top C_i \mathcal{H}_t dV_t \right] + \mathbb{E}_\mathcal{Q} \left[ \int_0^1 \mathcal{H}_t^\top B_t dV_t \right] = 0.
$$

Therefore the equivalence class of 0 in $L^2(M, \mathcal{Q}) \cap L^1(A, \mathcal{Q})$ consists of those predictable processes which are valued in $\text{Ker}(B) \cap \text{Ker}(C) dQ \otimes dV$-a.e. If we add processes which are equivalent to 0 we do not change the equivalence class. Hence there exist different predictable processes in $L^2(M, \mathcal{Q}) \cap L^1(A, \mathcal{Q})$, not $dQ \otimes dV$-a.e. equal, which have the same stochastic integral whenever $\text{Ker}(B) \cap \text{Ker}(C)$ is nontrivial.

Recall that the stochastic integral is invariant under a change to an equivalent probability measure so that any representation we derive under $\mathcal{Q}$ also holds under $\mathbb{P}$.
5.2 Separation Theorems

Here we collect separation theorems which are used in the proof of Theorem 3.5 and motivate Assumption 3.4. First we consider continuous sets, as in Definition 3.2, and refer to [3] for further discussion.

Given two compact convex subsets it is known that their sum and the convex hull of their union are again compact and convex. If in addition they are disjoint then there exists a linear functional which strongly separates them.

In [3] it is shown that these three properties hold for a wider class of sets. This is the class of continuous sets and contains convex compact sets as a proper subclass. An example of a continuous set which is not compact is the area enclosed by a parabola, e.g. \{ (x, y) \in \mathbb{R}^2 \mid y \geq x^2 \}. The key theorem on continuous sets we use is the following.

**Theorem 5.1** (Klee [8] Theorem 2). Let \( Q_1, Q_2 \) be disjoint nonempty convex subsets of \( \mathbb{R}^d \). If \( Q_1 \) is continuous and \( Q_2 \) is closed then they can be strongly separated. That is to say there exist \( \phi \in \mathbb{R}^d, a \in \mathbb{R} \) and \( \delta > 0 \) such that

\[
\phi^\top q_1 \geq a + \delta > a \geq \phi^\top q_2, \quad \text{for all } q_1 \in Q_1, q_2 \in Q_2.
\]

There are however many sets which are not continuous. In fact any cone with a ray in its boundary is not continuous. In Section 4 it is shown that the case when \( \mathcal{X} \) is a cone is of interest, thus we want to find a restriction on the type of convex sets which allows us to prove our result and includes some interesting examples. The class we consider is polyhedral sets, see Definition 3.3, and we refer to [4] and [14] for further details and properties. We are particularly interested in separation theorems for these sets and the important result is the following.

**Theorem 5.2** (Rockafellar [14] Corollary 19.3.3). Let \( Q_1, Q_2 \) be disjoint nonempty polyhedral convex subsets of \( \mathbb{R}^d \). Then they can be strongly separated; there exist \( \phi \in \mathbb{R}^d, a \in \mathbb{R} \) and \( \delta > 0 \) such that

\[
\phi^\top q_1 \geq a + \delta > a \geq \phi^\top q_2, \quad \text{for all } q_1 \in Q_1, q_2 \in Q_2.
\]

5.3 Measurable Selection

We recall some results from measurable selection and refer the reader to [13, 15, 16] for more details. The setup is as in Definition 3.1, we have a set \( \mathcal{F} \) together with a \( \sigma \)-field of subsets of \( \mathcal{F} \). We begin with a result on the intersection of measurable mappings.

**Lemma 5.3** (Rockafellar [13] Corollary 1.3). Let \( \{ F_i \mid i \in I \} \) be a countable collection of closed measurable multivalued mappings from \( \mathcal{F} \) to \( 2^{\mathbb{R}^d} \). Then the multivalued mapping
\[ F : t \rightarrow \bigcap_{i \in I} F_i(t) \]

is measurable.

The following is taken from Rockafellar [15].

**Definition 5.4.** A mapping \( \pi(t,x) : T \times \mathbb{R}^d \rightarrow \mathbb{R}^m \) is called Carathéodory if \( \pi(t,x) \) is measurable with respect to \( t \) and continuous with respect to \( x \).

We now give our main lemma on measurability.

**Lemma 5.5** (Rockafellar [15] Corollary 1Q). Let \( F' : T \rightarrow 2^{\mathbb{R}^m} \) be a closed measurable multivalued mapping and \( \pi : T \times \mathbb{R}^d \rightarrow \mathbb{R}^m \) be a Carathéodory mapping. Then \( F : T \rightarrow 2^{\mathbb{R}^d} \) given by

\[ F(t) = \{ x \in \mathbb{R}^d \mid \pi(t,x) \in F'(t) \} \]

is a closed measurable multivalued mapping.

**Definition 5.6.** A measurable selector for \( F \) is a measurable function \( f : T \rightarrow \mathbb{R}^d \) such that \( f(t) \in F(t) \) for all \( t \in T \).

In Section 6 we are interested in finding such a measurable selector. The major theorem in this area is the following.

**Theorem 5.7** (Kuratowski and Ryll-Nardzewski [9]). Let \( F : T \rightarrow 2^{\mathbb{R}^d} \) be a measurable multivalued mapping such that \( F(t) \) is a nonempty closed set for all \( t \in T \). Then there exists a measurable selector for \( F \).

**Remark 5.8.** As in [13], if \( F : T \rightarrow 2^{\mathbb{R}^d} \) is a closed measurable multivalued mapping, then the set

\[ T_0 = \{ t \in T \mid F(t) \neq \emptyset \} \]

is measurable. The restriction of \( F \) to \( T_0 \) is then a measurable multivalued mapping \( F_0 : T_0 \rightarrow 2^{\mathbb{R}^d} \) of the type to which Theorem 5.7 applies.

The setting in which we apply these results is with \( T = \Omega \times [0,1] \) and \( \mathcal{F}_{\mathcal{P}} = \mathcal{P} \), the predictable \( \sigma \)-field on \( \Omega \times [0,1] \). As described after Definition 3.1, any multivalued mapping \( F \) measurable with respect to \( \mathcal{P} \) is called predictably measurable. This is to emphasize the fact that any measurable selector of \( F \) is a predictable process. For a predictable process \( \mathcal{H} \) define the multivalued mapping

\[ F_{\mathcal{H}}(\omega, t) = \{ \mathcal{H}_i(\omega) + \text{Ker}(B_i(\omega)) \cap \text{Ker}(C_i(\omega)) \} \cap \mathcal{H}(\omega, t), \]

where \( B \) and \( C \) are from (6). Note that in the above for a vector \( q \in \mathbb{R}^d \) we set

\[ \text{Ker}(q) := \{ p \in \mathbb{R}^d \mid q^\top p = 0 \}. \]

The following result is the principal one of this section.

**Lemma 5.9.** The multivalued mapping \( F_{\mathcal{H}} \) is closed and predictably measurable.
Proof. $F_H$ is closed as the intersection of two closed sets. The multivalued mapping $\mathcal{K}(\omega,t)$ is predictably measurable by Assumption 3.4. Since we have

$$F_H(\omega,t) = \{ H_t(\omega) + \text{Ker}(B_t(\omega)) \} \cap \mathcal{K}(\omega,t),$$

by Lemma 5.3 we only need to show the measurability of $F_1(\omega,t) := \{ H_t(\omega) + \text{Ker}(B_t(\omega)) \}$ and $F_2(\omega,t) := \{ H_t(\omega) + \text{Ker}(C_t(\omega)) \}$.

Define the Carathéodory mappings

$$\pi_1((\omega,t),x) = (B_t(\omega))^\top (x - H_t(\omega)),$$
$$\pi_2((\omega,t),x) = C_t(\omega)(x - H_t(\omega)).$$

A calculation shows that

$$F_1(\omega,t) = \left\{ x \in \mathbb{R}^d \mid \pi_1((\omega,t),x) \in \{0\} \right\},$$
$$F_2(\omega,t) = \left\{ x \in \mathbb{R}^d \mid \pi_2((\omega,t),x) \in \{0\} \right\}.$$

Thus $F_1$ and $F_2$ are the preimages of the closed predictably measurable multivalued mapping $F' \equiv \{0\}$ ($0$ taken in $\mathbb{R}$ and $\mathbb{R}^d$ respectively). Thus by Lemma 5.5 they are predictably measurable and the proof is complete.

6 Proof of Theorem 3.5

The main difficulty (as in Section 2) comes from the fact that the pointwise constraints need only be satisfied for one representative $\mathcal{H}^n$ within the equivalence class $H^n$. However, we only assume convergence of the equivalence classes and this does not necessarily imply pointwise convergence of the representatives $\mathcal{H}^n$ which satisfy the constraints.

Proof (Proof of Theorem 3.5). Let $H^n_t$ be in $\mathcal{K}(\cdot,t)$ a.s. for all $t$ and $n \in \mathbb{N}$. Suppose that the sequence $H^n_t \cdot S$ converges in the semimartingale topology to $X$. As discussed in Section 5.1 we can find a measure $Q$ equivalent to $P$ and a subsequence, also indexed by $n$, such that $S = M + A \in \mathcal{M}^{2,1}(Q) \oplus \mathcal{A}^{1,1}(Q)$ and

$$(H^n - H^0_t) \cdot S \rightarrow 0 \text{ in } \mathcal{M}^{2,1}(Q) \oplus \mathcal{A}^{1,1}(Q).$$

From the proof of [10] Theorem V.4 we may pass to a subsequence, also indexed by $n$ and find representatives $\mathcal{H}^n$ and $\mathcal{H}^0$ of the corresponding equivalence classes $H^n$ and $H^0$ such that $\mathcal{H}^n(\omega)$ converges to $\mathcal{H}^0(\omega)$ for all $(\omega,t)$. For each $n \in \mathbb{N}$ the stochastic integrals of $\mathcal{H}^n \cdot S$ and $\mathcal{H}^n \cdot S$ coincide and thus their difference
\[
\mathcal{H}^n - \mathcal{H}^n \text{ is valued in } \text{Ker}(B) \cap \text{Ker}(C) \ dQ \otimes dV \text{-a.e. Consider the predictable and } dQ \otimes dV \text{-null set}
\]

\[
\Lambda := \bigcup_{n=1}^{\infty} \{ (\omega, t) \in \Omega \times [0, 1] \mid (\mathcal{H}^n(\omega) - \mathcal{H}^n(\omega)) \notin \text{Ker}(B_t(\omega)) \cap \text{Ker}(C_t(\omega)) \}.
\]

We set \(\mathcal{H}^n\) and \(\mathcal{H}^n\), for \(n \in \mathbb{N}\), as well as \(\mathcal{H}^0\), to be zero on \(\Lambda\). This does not change the stochastic integrals with respect to \(S\) and now, in addition, \((\mathcal{H}^n - \mathcal{H}^n)\) is valued in \(\text{Ker}(B) \cap \text{Ker}(C)\) for all \((\omega, t)\) and \(n \in \mathbb{N}\). Since \(0 \in \mathcal{H}(\cdot, t)\) a.s. for all \(t\) the \(\mathcal{H}^n\) remain in \(\mathcal{H}(\cdot, t)\) a.s. for all \(t\) and for all \(n \in \mathbb{N}\). Observe also that \(\mathcal{H}^n\) now converges pointwise to \(\mathcal{H}^0\) for all \((\omega, t)\). Define the multivalued mapping

\[
F_{\mathcal{H}^0}(\omega, t) = \{ \mathcal{H}^0(\omega) + \text{Ker}(B_t(\omega)) \cap \text{Ker}(C_t(\omega)) \} \cap \mathcal{H}(\omega, t),
\] (7)

which is closed and predictably measurable by Lemma 5.9. We want to make \(\mathcal{H}^0\) valued in \(\mathcal{H}(\cdot, t)\) a.s. without altering the stochastic integral. We must therefore add a predictable process valued in \(\text{Ker}(B_t(\omega)) \cap \text{Ker}(C_t(\omega))\) to get back into \(\mathcal{H}(\omega, t)\), which motivates the choice of \(F_{\mathcal{H}^0}\).

We now want to apply Theorem 5.7 to find a measurable selector for \(F_{\mathcal{H}^0}\). In particular we must check that the mapping \(F_{\mathcal{H}^0}\) defined by (7) is nonempty.

Fix \((\omega, t)\) such that for all \(n \in \mathbb{N}\), \(\mathcal{H}^n(\omega)\) is in \(\mathcal{H}(\omega, t)\) and \(\mathcal{H}(\omega, t)\) is either a continuous or a polyhedral set. Suppose for a contradiction that \(F_{\mathcal{H}^0}(\omega, t) = \emptyset\). By Theorem 5.1 or Theorem 5.2 the sets \(\mathcal{H}(\omega, t)\) and

\[
\{ \mathcal{H}^0(\omega) + \text{Ker}(B_t(\omega)) \cap \text{Ker}(C_t(\omega)) \}
\]

may be strongly separated. In particular there exist \(\phi \in \mathbb{R}^d\), \(a \in \mathbb{R}\) and \(\delta > 0\) such that for all \(k \in \mathcal{H}(\omega, t)\) and \(q \in \text{Ker}(B_t(\omega)) \cap \text{Ker}(C_t(\omega))\)

\[
\phi^\top k \geq a + \delta > a \geq \phi^\top (\mathcal{H}^0(\omega) + q).
\] (8)

Since, for each \(n \in \mathbb{N}\), \(\mathcal{H}^n(\omega) \in \mathcal{H}(\omega, t)\) and \(\mathcal{H}^n(\omega) - \mathcal{H}^n(\omega)\) is an element of \(\text{Ker}(B_t(\omega)) \cap \text{Ker}(C_t(\omega))\), (8) implies that for all \(n \in \mathbb{N}\),

\[
\phi^\top \mathcal{H}^n(\omega) \geq a + \delta > a \geq \phi^\top (\mathcal{H}^0(\omega) + \mathcal{H}^n(\omega) - \mathcal{H}^n(\omega)).
\]

It then follows that, for all \(n \in \mathbb{N}\),

\[
\phi^\top (\mathcal{H}^n(\omega) - \mathcal{H}^0(\omega)) \geq \delta.
\]

However we now have a contradiction as \(\mathcal{H}^n(\omega)\) converges to \(\mathcal{H}^0(\omega)\). Recall the definition of \(F_{\mathcal{H}^0}\) from (7) and define the set

\[
\Gamma := \{ (\omega, t) \in \Omega \times [0, 1] \mid F_{\mathcal{H}^0}(\omega, t) \neq \emptyset \}.
\]
By the above reasoning we have that $F^\hat{H}_0(\omega,t) \neq \emptyset$ for those $(\omega,t) \in \Omega \times [0,1]$ such that $\mathcal{H}^n(\omega)$ is in $\mathcal{K}(\omega,t)$ for all $n \in \mathbb{N}$ and $\mathcal{K}(\omega,t)$ is either a continuous or a polyhedral set. Since these conditions hold $\mathbb{P}$-a.s. for all $t$, and hence $\mathbb{Q}$-a.s. for all $t$, we can find, for each $t$, a $\mathbb{Q}$-null set $\Lambda^1_t$ such that

$$\left\{ (\omega,t) \in \Omega \times [0,1] \mid \omega \in (\Lambda^1_t)^c \right\} \subset \Gamma.$$ 

Exactly as in Remark 5.8 the restriction of $F^\hat{H}_0$ to $\Gamma$ is a closed, nonempty, predictably measurable, multivalued mapping to which Theorem 5.7 applies. Thus we get a measurable selector $\mathcal{I}$, a predictable process defined on $\Gamma$, with $\mathcal{I}(\omega) \in \mathcal{K}(\cdot,t)$. We now construct a representative $\mathcal{H}^0$ of $H^0$ which is in $\mathcal{K}(\cdot,t)$ $\mathbb{Q}$-a.s. for all $t$ by setting

$$\mathcal{H}^0 := \mathcal{H}^0_{\mathbb{Q}|}\Gamma + \mathcal{I}$$

The stochastic integral is invariant under a change to an equivalent probability measure, i.e., $H^0 \cdot S$ is the same under $\mathbb{P}$ as under $\mathbb{Q}$. In particular $H^0$ is valued in $\mathcal{K}(\cdot,t)$ $\mathbb{P}$-a.s. for all $t$ and this completes the proof.

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