On infectious model for dependent defaults

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Abstract. In this paper, we propose a general framework for modeling discrete-time default risk where default processes for all the entities are governed by predictable interacting default probabilities. We give a general formula for the joint distribution of two important random variables featuring the severity of the crisis: duration of a crisis (T) and severity of the defaults (WT). In particular, we present a two-sector Markovian infectious model, where the default probability is switching over time and depends on the current number of defaults of both sectors. The central idea of this model is that the causality of defaults of two sectors is in both directions, which enrich dynamics of the dependent default risk. The Bayesian Information Criterion (BIC) is adopted to compare the proposed model with the two-sector model in credit literature using real data. Numerical experiments are given to demonstrate that our proposed model is statistically better.

Keywords: Infectious models, correlated defaults, crisis duration

1. Introduction

Modeling dependent default risk has been a key issue in credit risk modeling. There are two important approaches to model the dependent default risk. The structural firm model has its origin in Merton \cite{merton1974optimal} and Black and Scholes \cite{black1973pricing}, which models the relationship between the firm’s asset value and the defaults. The reduced-form intensity-based, proposed by Jarrow and Turnbull \cite{jarrow1995determinants}, employ the Poisson jump processes to model the default event.

Copula function has been a very popular tool for modeling dependent risk. The idea of Copula is to transform the marginal variables to uniform variables by a simple transformation. After this is done, a n-dimensional function is used to model the dependence of the uniform variables, which is so called a Copula function. The Copula function enables one to deal with a multivariate distribution of uniform variables, without consideration of the original marginal variables. There are many useful Copula functions in finance, e.g., the Gaussian Copula, introduced by Li \cite{li1995modeling}, is widely used in risk modeling and financial assessment.

In addition, conditional independence model is also a commonly used model in credit risk modeling. Conditional on the systematical common factor, the loss random variables are independent. For example, the Bernoulli mixture model is adopted by CreditMetrics and KMV-model, while the Poisson mixture model is adopted by CreditRisk\textsuperscript{+}. In a recession, the default of a company is triggered by the underlying common risk factor and also by the related company’s defaults. The
contagion model is used to describe how the credit event of one company affects the other companies. Davis and Lo [9] introduce an infectious default model, where in a portfolio, a bond may be infected by defaults of other bonds or default directly. Jarrow and Yu [15] propose a reduced-form model to describe the defaultable bonds of different company, where the concept of counterparty risk is first introduced to the credit literature. Dong and Wang [10] show the impact of dependent jumps of the firm value and the default thresholds on the default probabilities.

Ching et al. [6] introduce an infectious default model based on the idea of Greenwood’s model considered in Daley and Gani [8]. This model aims at modeling the impact of default of a bond on the likelihood of defaults of other bonds. The original version of Greenwood’s model is a one-sector model. It is then extended to a two-sector model in Ching et al. [5]. Besides, the joint probability distribution function for the duration of a default crisis, (i.e., the default cycle), and the severity of defaults during the crisis period was also derived. Two concepts, namely, Crisis Value-at-Risk (CRVaR) and Crisis Expected Shortfall (CRES), are introduced and applied to assess the impact of a default crisis. The Greenwood’s model is also extended to a network of sectors in [5]. Gu et al. [12] propose a Markovian infectious model to describe the dependent relationship of default processes of credit securities based on [5,6], where the central idea is the concept of common shocks which is one of the major approaches to describe insurance risk. In recent years, Markov model is widely used in credit risk assessment. Although the Markov model does not use all the historical data, it can be seen from the literature [1,2,16,18] that it gives substantially good results. For example, in the literature [2], they consider a bottom-up Markovian copula model of portfolio credit risk.

If the number of defaults is small, other models in [20,21] and the theory in the book [22] can be applied. In literature, Mitra [21] proposed a new risk management framework and method which allows one to assess the risk of pension funds in terms of their value and provides a risk management framework for decision-making. It was proposed to modeling and managing pensions as European call options. If the correlation of default changes over time, one can refer to the method in [13]. They established a link between the dynamics of house price changes and the dynamics of default rates in the Gaussian copula framework by specifying a time series model for a common risk factor.

In this paper, we propose a general framework for modeling discrete-time default risk where default processes for all the entities are governed by predictable default probabilities. Existing literature [5,6,12] serve as our special cases. We give a general formula for the joint distribution of two important random variables featuring the severity of the crisis, i.e., the duration of the crisis and the severity of the defaults. In particular, we present a two-sector Markovian infectious model, where the default probability is switching over time and depends on the current number of defaults of both sectors. This model is a special case of our general framework and compared with the existing work, this can capture the causality of defaults from both direction. We adopt the maximum likelihood method to estimate the parameters and the Bayesian Information Criterion (BIC) to compare the proposed model with two-sector model considered in Ching et al. [6]. Experimental results show that our proposed model is statistically better (i.e., has a lower value of the BIC).

In this paper, the default is modeled as an absorbing state. There are many research works that regard default as an absorbing state [4,7,23]. For example, they employ Copula theory to model the dependence across default rates in a credit card portfolio of a large UK bank and to estimate the likelihood of joint high default rates in [7]. And in [23], they focus on the predictability of sovereign debt crisis and propose a two-step procedure centered on the idea of a multidimensional distance-to-collapse point.

The remainder of the paper is structured as follows. Section 2 presents our general model framework. We derive a general formula for the joint probability distribution for the default cycle and the number of defaults during the crisis. We also discuss the limiting case. Section 3, we present a special case of our general model, namely the two-sector Markovian model and derive a recursive formula for the probability law of the two variables. We also outline the parameter estimation procedure. Section 4 presents the ideas of the CRVaR and the CRES. In Section 5, we present the results of empirical analysis using our proposed model. Finally, Section 6 concludes the paper.

2. The general model framework

Let \( T \) be the time index set \( \{1, 2, \ldots \} \) of our model. To model the uncertainty, let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a 1

1“Statistically better” means our purposed model has lower Bayesian information criterion value.
complete filtered probability space, where \( P \) is a real-
world probability and \( \{\mathcal{F}_t\}_{t \geq 0} \) is a filtration satisfying
the usual conditions (the right-continuity and \( P \)-
completeness). We consider \( n \) credit entities, where
each entity may default and the entity will stay at the
default state once it happens. For each \( i = 1, 2, \ldots, n \),
let \( \tau_i \) be the default time of name \( i \), which is a stop-
ing time with respect to the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \). Write
\( N_i(t) = 1_{[\tau_i \leq t]} \) the default indicator process and
\( \{\mathcal{F}_t\}_{t \geq 0} \) is the \( P \)-complete, natural filtration generated
by \( N_i \). For each \( t \geq 0 \), we write
\[
\mathcal{F}_t = \mathcal{F}_t^1 \vee \cdots \vee \mathcal{F}_t^n,
\]
where \( \mathcal{F}_t \) is the minimal \( \sigma \)-algebra containing infor-
mation about the processes \( \{N_i\}_{i \geq 1} \) up to and includ-
ing time \( t \). That is, \( \mathcal{F}_t \) contains information about the
common factor process and the defaults of the \( n \) credit
entities up to time \( t \). It represents the observed market
information up to time \( t \).

We assume that for each \( i = 1, 2, \ldots, n \), \( N_i \) pos-
sesses a nonnegative, \( \{\mathcal{F}_t\}_{t \geq 0} \)-predictable process \( p_i \)
satisfying
\[
E[N_i(t) \mid \mathcal{F}_{t-1}] = p_i(t), \quad t \geq 0.
\]

To determine the impact of a default crisis, we define
the duration of the default crisis \( (T) \), namely, the de-
fault cycle, and the severity of the defaults \( (W_T) \) dur-
ing the crisis period. We give a precise definition of the
default cycle as a stopping time:
\[
T := \inf\{t \in \mathcal{T} \mid W_t = W_{t-1} \}, \quad (2)
\]
where \( W_t \) represents the number of defaults over the
time duration \([1, t]\).

We let
\[
J(t) = (N_1(t), N_2(t), \ldots, N_n(t)).
\]
It can be verified that \( J(t) \) is a Markov chain with state
space \( S \) of size \( 2^n \). We let \( Q(t) \) denote the transition
matrix of Markov chain \( J \) from time \( t \) to \( t + 1 \) and
\( Q^*(t) \) the matrix that results from replacing the diag-
nal entries by 0 in \( Q(t) \).

**Proposition 1.** The joint distribution of \((T, W_T)\) is
given by
\[
P((T, W_T) = (t, w)) = \sum_{x \in \mathbb{N}, |x| = w} \bar{Q}(t - 2)(0, x) \cdot Q(t - 1)(x, x)
\]
for \( t \in \mathcal{T}, w \in \mathbb{N}, \) where
\[
\bar{Q}(t - 2) = \sum_{s=0}^{t-2} Q^*(s),
\]
\[
|w| = x^T x \text{ and } \theta = (0, \ldots, 0).
\]

The main idea of the proof is to sum up all the possi-
ble paths of the chain to stop at time \( t \) with \( w \) defaults.
However, the computation cost can be huge when \( n \) be-
comes large as the matrix size grows very quickly. In
Section 3, we shall consider a special case of practi-
cal value where the default probability of each name is
time-homogeneous and is assigned by some rules.

In what follows, we consider the simplest case that
the default probability for each name is a constant, i.e.,
\( p_i(t) = p_i \in (0, 1) \). The process \( W_t \) then becomes
a Markov chain, with transition probability matrix \( P \)
where \( P(i, j) = 0 \) if \( i > j \) and
\[
P(i, j) = \binom{n-i}{j-i} p_j (1-p)^{n-j}, \quad \text{if } i \leq j.
\]

We let \( P^* \) denote the matrix that results from replacing
the diagonal entries by 0 of \( P \). We can obtain the prob-
ability law of \((T, W_T)\) by summing up all the possible
paths for the chain to stop at time \( t \) with \( w \) defaults.

**Proposition 2.** The joint distribution of \((T, W_T)\) is
given by
\[
P((T, W_T) = (t, w)) = \tilde{P}(0, w) \cdot P(w, w)
\]
for \( t \in \mathcal{T}, w \in \mathbb{N}, \) where \( \tilde{P} = \sum_{j=0} \chi_j P^* \).

**3. The two-sector model**

In this section, we assume all the names are divided
into two sectors, namely Sector A and Sector B. To ap-
ply the concepts of default cycle and the severity of the
defaults to our proposed two-sector model, we write
\( W_A^1 \) and \( W_B^2 \) to represent the number of defaults in Sec-
tor A and Sector B, respectively, in \((0, t_1]\) and \((0, t_2]\). We denote

\[\]
To model the default probability, we define
\[ X := \{X_t\}_{t \in T} \quad \text{and} \quad Y := \{Y_t\}_{t \in T} \]
to denote two stochastic processes on \((\Omega, \mathcal{F}, P)\),
where \(X_t = (X^1_t, X^2_t)\) represent the numbers of surviving
bonds at \(t \in T\) in Sector A and Sector B, respectively,
and \(Y_t = (Y^1_t, Y^2_t)\) represent the numbers
of defaulted bonds at \(t \in T\) in Sector A and Sector B,
respectively, e.g., \(Y^1_t = W^1_t, i = 1, 2\). We assume that
the initial conditions are given as follow:
\[ X_0 = (x^1_0, x^2_0), \quad Y_0 = (y^1_0, y^2_0) \]
and
\[ x^1_0 + y^1_0 = n_1, \quad x^2_0 + y^2_0 = n_2, \]
where \(n_1, n_2\) represent the number of names in Sector A
and Sector B, respectively. Note that for each \(t \in T\),
the sum of the numbers of the defaulted bonds and the
surviving bonds at the time epoch \(t + 1\) must equal the
number of surviving bonds at time \(t\) in every sector,
i.e.,
\[ X^1_{t+1} + Y^1_{t+1} = X^1_t \quad \text{and} \quad X^2_{t+1} + Y^2_{t+1} = X^2_t. \]  
(3)
For each \(t \in T\), let \(\alpha_t\) and \(\beta_t\) be the probabilities
that the default of a surviving bond is infected by
the defaulted bonds at time \(t\) in Sector A and Sector B,
respectively. The joint probability distribution of
\(\{X_{t+1}, Y_{t+1}\}\) given \(\{X_t, Y_t\}\) is given by the following
Binomial probability:
\[
P(x_{t+1}, y_{t+1}) = P(\{X_{t+1}, Y_{t+1}\} = (x_{t+1}, y_{t+1}) | \\
(X_t, Y_t) = (x_t, y_t)) = \binom{x^1_t}{y^1_t} \binom{x^2_t}{y^2_t} (\alpha_t)^{y^1_t} (1 - \alpha_t)^{x^1_t - y^1_t} \times \binom{x^2_t}{y^2_t} (\beta_t)^{y^2_t} (1 - \beta_t)^{x^2_t - y^2_t}. \]  
(4)
We consider here the situation that the joint future
default probability depends on the current number of de-
faulted bonds in both industrial sectors. We assume
that
\[
\alpha_t = a(y_t) = \begin{cases} 
0 & \text{if } y^1_t = y^2_t = 0, \\
a_1 & \text{if } y^1_t > 0, y^2_t = 0, \\
(a_2^t) & \text{if } y^1_t = 0, y^2_t > 0, \\
(a_3^t) & \text{if } y^1_t > 0, y^2_t > 0
\end{cases}
\]  
(5)
and
\[
\beta_t = b(y_t) = \begin{cases} 
0 & \text{if } y^1_t = y^2_t = 0, \\
b_1 & \text{if } y^1_t = 0, y^2_t > 0, \\
b_2 & \text{if } y^1_t > 0, y^2_t = 0, \\
b_3 & \text{if } y^1_t > 0, y^2_t > 0
\end{cases}
\]  
(6)
where
\[
h_0(x, y) = \begin{cases} 
1 & \text{if } x = y = 0, \\
0 & \text{otherwise}
\end{cases}
\]  
\[
h_1(x, y) = \begin{cases} 
1 & \text{if } x > 0, y = 0, \\
0 & \text{otherwise}
\end{cases}
\]
and
\[
h_2(x, y) = \begin{cases} 
1 & \text{if } x = 0, y > 0, \\
0 & \text{otherwise}
\end{cases}
\]  
\[
h_3(x, y) = \begin{cases} 
1 & \text{if } x > 0, y > 0, \\
0 & \text{otherwise}
\end{cases}
\]
As it is shown in Eq. (3) and Eq. (4), one can see
that \(\{X_t, t = 0, 1, 2, \ldots\}\) is a second-order Markov
chain process. We remark that this two-sector model
provides a novel and flexible dependent structure for correlated defaults of two different industrial sectors.
First, an infectious default within one time period is
modeled by a Binomial distribution, which has been
widely used in modeling the spread of epidemics
whose situation seems similar to that of a financial
crisis. The causality of the infection is supposed to
be in both direction, i.e., a “looping default”. Sec-
ond, the process \((X_t, Y_t)\) has the Markov property, where the probabilistic structure of future states only depends on the current state. Third, conditioning on the current state \((X_t, Y_t)\), the future state of two sectors \((X_{t+1}, Y_{t+1})\) and \((X_{t+1}^2, Y_{t+1}^2)\) are stochastically independent. The step functions \(h_t(x, y)\) are used to describe the dependence of the default probabilities on the state of previous time epoch. This method provides a tractable and analytic solution for parameter estimation from empirical data.

3.1. Default cycle and severity

In this subsection, we proceed to derive the joint probability distribution function for the duration of the default crisis \((T)\), namely, the default cycle, and the severity of the defaults \((W)\) during the crisis period. These two concepts are essential in determining the impact of a default crisis [1]. Under the two-sector Markovian model, we obtain

\[
T_1 := \inf\{t \in T \mid Y_t^1 = 0\} \quad \text{and} \quad T_2 := \inf\{t \in T \mid Y_t^2 = 0\}.
\]

To obtain the joint distribution of \((W_i^1, T_i)\) for \(i = 1, 2\), we assume that \((X_0, Y_0) = (x_0, y_0)\) with \(y_0 > 0\), \(y_0^2 > 0\). Let

\[
P_n(x_1, x_2, h) = P\{T_1 \geq n + 1, X_n^1 = x_1, X_n^2 = x_2, I_{Y_n^1 = 0} = h\}.
\]

The following lemma gives a recursive formulas for \(P_n(x_1, x_2, h)\) and the proof can be found in the Appendix.

**Lemma 1.**

\[
P_n(x_1, x_2, 0) = \sum_{x_1 > x_1} \left(\frac{s_1}{x_1}\right) [P_{n-1}(s_1, x_2, 0) \times (a_1)^{x_1-x_1} (1 - a_1)^{x_1} (1 - b_2)^{x_2} + P_{n-1}(s_1, x_2, 1) \times (a_3)^{x_1-x_1} (1 - a_3)^{x_1} (1 - b_3)^{x_2}]
\]

\[
= \sum_{x_1 > x_1} \sum_{x_2 > x_2} \left(\frac{s_1}{x_1}\right) [P_{n-1}(s_1, x_2, 0) \times (a_1)^{x_1-x_1} (1 - a_1)^{x_1} (1 - b_2)^{x_2} + P_{n-1}(s_1, x_2, 1) \times (a_3)^{x_1-x_1} (1 - a_3)^{x_1} (1 - b_3)^{x_2}]
\]

where the initial condition is given by

\[
P_0(x_1, x_2, h) = \begin{cases} 1, & (x_1, x_2, h) = (x_0^1, x_0^2, 1), \\ 0, & otherwise. \end{cases}
\]

By Lemma 1, we obtain the following proposition and its proof can be found in the Appendix.

**Proposition 3.** The joint distribution of \((T_1, W_1^1)\) is given by

\[
P\{(T_1, W_1^1) = (n, x)\}
\]

\[
= \sum_{x_2} P_{n-1}(x_1^0 - x, x_2, 0) (1 - a_1)^{x_1^0-x} + \sum_{x_2} P_{n-1}(x_1^0 - x, x_2, 1) (1 - a_3)^{x_1^0-x}.
\]

We remark that due to the symmetric property of the two sectors, the joint distribution \((W_2^1, T_2)\) shares a similar form of \((W_1^1, T_1)\).

3.2. Parameter estimation

In the two-sector model, there are eight parameters: \(a_0, a_1, a_2, a_3, b_0, b_1, b_2\) and \(b_3\). We employ the maximum likelihood method to estimate the parameters. Given the total bonds \(n_1, n_2\) and the observations of the number of defaulted bonds \(y_0^1, y_1^1, \ldots, y_n^1\) and \(y_0^2, y_1^2, \ldots, y_n^2\), where \(N\) denotes the period of observation time, the number of surviving bonds \(x_0^1, x_1^1, \ldots, x_N^1\) and \(x_0^2, x_1^2, \ldots, x_N^2\) are deterministic.

The following proposition gives analytical expressions for the maximum likelihood estimates of the model parameters.

**Proposition 4.** For \(i = 0, 1, 2, 3\),

\[
\hat{a}_i = \frac{\sum_{j=0}^{N-1} y_{j+1}^i h_j(y_j^1, y_j^2)}{\sum_{j=0}^{N-1} x_j^i h_j(y_j^1, y_j^2)} \quad \text{and} \quad \hat{b}_i = \frac{\sum_{j=0}^{N-1} y_{j+1}^i h_j(y_j^1, y_j^2)}{\sum_{j=0}^{N-1} x_j^i h_j(y_j^1, y_j^2)}.
\]
Proof. We prove the expression for \( \hat{a}_0 \) here and the proof for the others are similar. The likelihood function

\[
L(a, b \mid x_0, x_1, \ldots, x_N, y_0, y_1, \ldots, y_N) = f(x_0, x_1, \ldots, x_N, y_0, y_1, \ldots, y_N \mid a, b)
\]

\[
= \left(\frac{1}{a(y_0)}\right)^{h_0(y_0, y_0)} \left(\frac{1}{1 - a(y_1)}\right)^{h_1(y_1, y_1) a(y_1)} \left(\frac{1}{1 - b(y_1)}\right)^{h_2(y_1, y_1) b(y_1)} \left(\frac{1}{1 - a(y_2)}\right)^{h_3(y_2, y_2) a(y_2)} \left(\frac{1}{1 - b(y_2)}\right)^{h_4(y_2, y_2) b(y_2)} \cdots
\]

\[
\times \left(\frac{1}{1 - a(y_N-1)}\right)^{h_{N-1}(y_{N-1}, y_{N-1}) a(y_{N-1})} \left(\frac{1}{1 - b(y_N-1)}\right)^{h_N(y_N-1, y_N-1) b(y_N-1)} \left(\frac{1}{1 - a(y_N)}\right)^{h_N(y_N, y_N)} \left(\frac{1}{1 - b(y_N)}\right)^{h_N(y_N, y_N)}
\]

Then by solving

\[
\begin{align*}
\frac{\partial \ln L(a, b \mid x_0, x_1, \ldots, x_N, y_0, y_1, \ldots, y_N \mid a, b)}{\partial a_0} &= 0,
\end{align*}
\]

we have

\[
- \sum_{t=0}^{N-1} \frac{x_t^1}{1 - a(y_t)} h_0(y_t^1, y_t^2) + \sum_{t=0}^{N-1} \frac{x_t^2}{1 - b(y_t)} h_0(y_t^1, y_t^2) = 0.
\]

Since for any \( t \),

\[
\frac{1}{1 - a(y_t)} = \sum_{i=0}^{3} \frac{h_i(y_t^1, y_t^2)}{a_i},
\]

\[
\frac{1}{1 - b(y_t)} = \sum_{i=0}^{3} \frac{h_i(y_t^1, y_t^2)}{a_i},
\]

we have

\[
0 = - \sum_{t=0}^{N-1} \sum_{i=0}^{3} x_{t+1}^1 h_0(y_t^1, y_t^2) h_i(y_t^1, y_t^2) + \sum_{t=0}^{N-1} \sum_{i=0}^{3} x_{t+1}^2 h_0(y_t^1, y_t^2) h_i(y_t^1, y_t^2)
\]

Thus we obtain

\[
\hat{a}_0 = \frac{\sum_{t=0}^{N-1} x_{t+1}^1 h_0(y_t^1, y_t^2)}{\sum_{t=0}^{N-1} x_{t+1}^2 h_0(y_t^1, y_t^2)}.
\]

\[\square\]

4. Crisis VaR and crisis ES

In this section, we give a brief introduction to the concepts of the CRVaR and the CRES in Ching et al. [5,6]. Then we present the evaluation of the CRVaR and the CRES using the proposed models. The CRVaR and the CRES are measures for the duration and the severity of a default crisis. Let

\[
L(T, W_T) = \min\{T \in \Omega \mid P(L(T, W_T) > l) = \beta\}
\]

be a real-valued function \( L(T, W_T)(\omega) \) of \( T \) and \( W_T \). We then suppose that for a fixed \( \omega \in \Omega \),

\[
T(\omega) = t, \quad W_t(\omega) = w, \quad \text{and} \quad L(t, w)(\omega) = l(t, w) \in \mathbb{R}.
\]

That is, the loss from the default crisis is \( l(t, w) \) when the duration of default crisis \( T = t \) and the number of defaulted bonds in the crisis \( W = w \). We write \( L(T, W_T) \) for the space of all loss functions \( L(T, W_T)(\omega) \) generated by \( T \) and \( W_T \).

The CRVaR with probability level \( \beta \) under \( P \) is then defined as a functional \( V_\beta(\cdot) : L(T, W_T) \rightarrow \mathbb{R} \) such that for each \( L(T, W_T) \in L(T, W_T) \),

\[
V_\beta(L(T, W_T)) := \inf\{l \in \mathbb{R} | P(L(T, W_T) > l) \leq \beta\}. \tag{7}
\]

In the language of statistics, \( V_\beta(L(T, W_T)) \) is the generalized \( \beta \)-quantile of the distribution of the loss variable \( L(T, W_T) \) under \( P \). Since the loss from the default crisis \( L(T, W_T) \) is completely determined when \( T \) and \( W_T \) are given, \( P(L(T, W_T) > l) \) is completely determined by the joint p.d.f. of \( W_T \) and \( T \).

The CRES with probability level \( \beta \) under \( P \) is also defined as a functional \( E_\beta(\cdot) : L(T, W_T) \rightarrow \mathbb{R} \) such...
that for each \( L(T, W_T) \in L(T, W_T) \),

\[
E_\beta \left( L(T, W_T) \right) := E_P \left[ L(T, W_T) \mid L(T, W_T) \geq V_\beta \left( L(T, W_T) \right) \right].
\] (8)

In other words, \( E_\beta \left( L(T, W_T) \right) \) is the average of the loss from the default crisis when the loss exceeds the CRVaR of the default crisis with probability level \( \beta \) under \( P \).

5. Numerical experiments

In this section, we present the empirical results of the proposed two-sector model using real default data extracted from the figures in Giampieri et al. [11], where we adopt the estimation methods and techniques presented in the previous section.

The default data comes from four different sectors. They include consumer/service sector, energy and natural resources sector, leisure time/media sector and transportation sector. Table 1 shows the default data taken from Giampieri et al. [11]. From the table, the proportions of defaults for Consumer, Energy, Media and Transport are 24.1%, 16.9%, 20.5% and 21.0%, respectively. The default probabilities of all four sectors are significantly greater than zero. This means that the default risk of each of the four sectors is substantial.

We then construct the infectious disease model using these real data. The asterisk "*" in the table indicates the pair of sectors which has the largest correlation. Figure 1 gives the partner relations among the sectors using correlation. Later in this section, we shall give the results for BIC to support the matched pair presented in Fig. 1.

The estimation results for proposed infectious model and two-sector model studied in Ching et al. [5] are presented in Table 3.

To compare the proposed infectious model with the two-sector model in Ching et al. [5], we consider the Bayesian information criterion (BIC), which is also named as Schwarz criterion. The formula for the BIC is given by

\[
\text{BIC} = -2 \log(L) + k \log(m),
\]

where \( m \) is the number of observation data, \( k \) is the number of free parameters to be estimated, and \( L \) is the maximized value of the likelihood function for the estimated model. Given any two estimated models, the smaller the value of BIC is, the better the model will be. Table 4 presents the value of the BIC for the proposed model and the two-sector model in Ching et al. [5]. We remark that for all the four sectors, the proposed model with lower value of BIC is statistically better.

To compare the matched pairs in Fig. 1 with other matched pairs for the proposed model, we also adopt the BIC. Since the models of different matched pairs have the same number of parameters and length of data set, to compare their BIC is equivalent to compare their

\begin{table}[h]
\centering
\caption{Correlations of the sectors}
\begin{tabular}{cccc}
\hline
\text{Consumer} & Energy & Media & Transport \\
\hline
Consumer & -- & 0.0224 & 0.6013* & 0.3487 \\
Energy & 0.0224 & -- & 0.1258* & 0.1045 \\
Media & 0.6013* & 0.1258 & -- & 0.3708 \\
Transport & 0.3487 & 0.1045 & 0.3708* & -- \\
\hline
\end{tabular}
\end{table}
log-likelihood ratio, Table 5 presents the log-likelihood ratios for the matched pairs in Fig. 1 against other matched pairs. We remark that all the log-likelihood ratios are positive which supports the matched pairs in Fig. 1 for the proposed model.

Our proposed model aims at modeling causality of defaults in both direction. From the pair up results, one may find that the relation is not necessarily symmetric. This relation is only found symmetric for the sectors media and consumer, which means the causality of defaults from both direction is more reasonable for the media and consumer sector.

We provide a scatter plot to depict the correlation of defaults in the matched sectors. A simulation of defaults in matched sectors in our proposed model is also conducted. Figure 2 presents the number of surviving bonds in the matched sectors of empirical data and simulation.

To apply the two measures CRVaR and CRES in the proposed model, we consider some hypothetical values for the loss. The loss $L(W_T, T)$, for each $T = 1, 2, \ldots, T_0$ and $W_0 = 1, \ldots, W_0$, are as in Eq. (9). Then we present the value of CRVaR and CRES for the proposed model as well as the two-sector model Ching et al. [5] in Table 6. And the loss distribution are presented in Fig. 3.

From Table 6, we see that for all of the four sectors, the existing two-sector model underestimates both the CRES and CRVaR. This reflects that failure to incorporate the contagion effect described in our proposed model leads to an underestimation of credit risk. This has an important consequence for credit risk management, such as inadequate capital charges for credit portfolios. Indeed, the loss distribution implied by the proposed model has a much fatter tail than that arising from the existing two-sector model. This explains why the proposed model provides more prudent estimates for the risk measures than the existing two-sector model via incorporating contagion. We also remark that the contagion model including the causality of defaults in both direction (i.e., looping defaults), has a significant impact on the loss distribution.

<table>
<thead>
<tr>
<th>Sector A:</th>
<th>Consumer</th>
<th>Energy</th>
<th>Media</th>
<th>Transport</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sector B:</td>
<td>Media</td>
<td>Media</td>
<td>Media</td>
<td>Media</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>0.0007</td>
<td>0.0004</td>
<td>0.0005</td>
<td>0.0013</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>0.0018</td>
<td>0.0033</td>
<td>0.0005</td>
<td>0.0012</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>0.0013</td>
<td>0.0018</td>
<td>0.0017</td>
<td>0.0026</td>
</tr>
<tr>
<td>$\alpha_3$</td>
<td>0.0049</td>
<td>0.0032</td>
<td>0.0042</td>
<td>0.0052</td>
</tr>
</tbody>
</table>

The value of BIC for proposed model and two-sector model [5]

<table>
<thead>
<tr>
<th>Sector A:</th>
<th>Consumer</th>
<th>Energy</th>
<th>Media</th>
<th>Transport</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sector B:</td>
<td>Media</td>
<td>Media</td>
<td>Media</td>
<td>Media</td>
</tr>
<tr>
<td>BIC (proposed model)</td>
<td>419.0813</td>
<td>215.4654</td>
<td>301.2534</td>
<td>2.1287</td>
</tr>
<tr>
<td>BIC (two-sector model)</td>
<td>434.6700</td>
<td>231.8225</td>
<td>321.0501</td>
<td>2.1460</td>
</tr>
</tbody>
</table>

The value of BIC for matched pairs in Fig. 1 and other matched pairs

<table>
<thead>
<tr>
<th>Sector A:</th>
<th>Consumer</th>
<th>Energy</th>
<th>Media</th>
<th>Transport</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sector B:</td>
<td>Media</td>
<td>Media</td>
<td>Media</td>
<td>Media</td>
</tr>
<tr>
<td>Log-likelihood ratio</td>
<td>12.2717</td>
<td>12.6559</td>
<td>14.3757</td>
<td>4.4860</td>
</tr>
<tr>
<td>Other matched pairs</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6. Concluding remarks

In this paper, we propose a general model framework for discrete-time default risk where default processes for all the entities are governed by predictable default probabilities. Existing literature [5,6,12] serve as our special cases. We give a general formula for the joint distribution of two important random variables featuring the severity of the crisis, i.e., the duration of crisis ($T$) and severity of the defaults ($W_T$). We propose a two-sector Markovian infectious model as a special case of the general framework. The proposed model incorporated two important features of credit contagion, namely, the chain reactions of defaults and the bi-lateral causality of defaults between sectors.
two industrial sectors. We capture the chain reactions of defaults by postulating that the future default probability switches over time according to the current number of defaults of two industrial sectors. The bi-lateral causality of defaults means that defaults in one sector are caused by defaults in another sector, and vice versa. This bi-lateral causality of defaults enriches the dependent structures of credit risk model. We provide an efficient estimation method of the proposed model based on the maximum likelihood estimation. Two important risk measures, namely, the CRVaR and the CRES, are evaluated under the proposed model.

We also conduct empirical studies on the credit risk models using real default data. We adopted the BIC to compare the proposed model with the existing two-sector model proposed in Ching et al. [5]. The numerical results reveal that the proposed two-sector model outperforms empirically the existing model. By comparing the risk measures evaluated from the proposed model and those evaluated from the existing two-sector...
model, we find that failure to incorporate the contagion effect described in the proposed model leads to an underestimation of risk measures. This provides some evidence to support the proposed model.

One possible topic for future research is to incorporate the impact of the number of defaults on the likelihood of future defaults via a different parametrization of the future default probability. In the current paper, we assume that the joint future default probability switches over time depending on the region where the current number of defaults falls in. Four parameters, namely, \( a_0, a_1, a_2 \) and \( a_3 \) were involved. To provide a more parsimonious way to incorporate the current number of defaults on the joint future default probability, one may consider the following parametrization for the default probability:

\[
\alpha_i = a_0 + a_1 y_i^1 + a_2 y_i^2,
\]

where \( y_i^1 \) and \( y_i^2 \) are the current numbers of defaults in the two industrial sectors. Using this parametrization, we can reduce the number of parameters by one and accounts for more information of the current number of defaults when evaluating the future default probability.

### Acknowledgements

The authors would like to thank the anonymous referee and the editor for their helpful and constructive comments. This research work was supported by Research Grants Council of Hong Kong under Grant Number 17301214.

### Appendix

#### A.1. Proof of Lemma 1

By the law of total probability and Markov property,

\[
P_n(x_1, x_2, 0) = P\{T_1 \geq n + 1, X_n^1 = x_1, X_n^2 = x_2, I_{[Y_2 > 0]} = 0\}
\]

\[
= \sum_{s_1, x_2, h} P_n(s_1, x_2, h) \times P\{Y_1^2 > 0, X_1^1 = x_1, X_{n-1}^2 = x_2, I_{[Y_2 > 0]} = 0\}
\]

\[
= \sum_{s_1, x_2, h} P_n(s_1, x_2, h) \times P\{Y_1^2 > 0, X_1^1 = x_1, X_{n-1}^2 = x_2, I_{[Y_2 > 0]} = 1\}
\]

\[
\times P\{T_1 \geq n + 1, X_n^1 = x_1, X_n^2 = x_2, I_{[Y_2 > 0]} = 0\}
\]

\[
= \sum_{s_1, x_2, h} P_n(s_1, x_2, h) \times P\{Y_1^2 > 0, X_1^1 = x_1, X_{n-1}^2 = x_2, I_{[Y_2 > 0]} = 1\}
\]

\[
\times (a_1 y_1^1 - x_1) (1 - a_1) y_1^1 (1 - b_2) y_2^1
\]

\[
+ P_n(s_1, x_2, 1) \times (a_3 y_2^2 - x_2) (1 - a_3) y_2^2 (1 - b_3) y_3^2.
\]

Similarly, we have

\[
P_n(x_1, x_2, 1)
\]

\[
= P\{T_1 \geq n + 1, X_n^1 = x_1, X_n^2 = x_2, I_{[Y_2 > 0]} = 1\}
\]
\[
= \sum_{s_1 > x_1, s_2 > x_2} \sum_{h=0}^1 P\{T_1 \geq n, X_{n-1}^1 = s_1, X_n^2 = s_2, I_{Y_{n-1} > 0}^1 = h\}
\times P\{T_1 \geq n + 1, X_n^1 = x_1, X_{n-1}^2 = s_2, I_{Y_n > 0}^2 = 1 \mid T_1 \geq n, X_{n-1}^1 = s_1, X_n^2 = s_2, I_{Y_n^1 > 0}^2 = 1\}
\]

Fig. 3. Loss distribution for proposed model and two-sector model Ching et al. [5].
1. \[ X_{n-1}^2 = s^2, I_{\{Y_{n-1}^2 > 0\}} = h \]

\[ = \sum_{s_1 \times s_2 \times s_3 \times s_4} P_{n-1}(s_1, s_2, h) \times P \left\{ Y_{n-1}^2 > 0, X_{n-1}^2 = s_1 \right\} \times P \left\{ X_{n-1}^2 = s_2, I_{\{Y_{n-1}^2 > 0\}} = h \right\} \]

\[ = \sum_{s_1 \times s_2 \times s_3 \times s_4} \left( \frac{s_1}{s_2} \right) \left( \frac{s_2}{s_2} \right) \left( \frac{p_{n-1}(s_1, s_2, 0)}{p_{n-1}(s_1, s_2, 0)} \right) \times (a_1)^{s_1-s_3} (1-a_1)^{s_2} (b_2)^{s_3-s_2} (1-b_2)^{s_2} + P_{n-1}(s_1, s_2, 1) \times (a_3)^{s_1-s_3} (1-a_3)^{s_2} (b_3)^{s_3-s_2} (1-b_3)^{s_2} \]

A.2. Proof of Proposition 3

\[ P \left\{ T_1, W_{T_1}^1 \right\} = (n, x) \]

\[ = P \left\{ T_1 \geq n, Y_1^0 = 0, X_1^0 = x^0 - x \right\} \]

\[ = \sum_{x_2} \sum_{h=0,1} \left( \sum_{x_2} P_{n-1}(x_0^1 - x_2, h) \times P \left\{ Y_{n-1}^0 = 0, X_{n-1}^0 = x^0 - x \right\} \times P \left\{ Y_{n-1}^2 > 0, X_{n-1}^2 = x^0 - x \right\} \right) \]

\[ = \sum_{x_2} P_{n-1}(x_0^1 - x_2, x_0^1 - x_2, 0) \times (1-a_1)^{s_1-s_3} \times (1-a_3)^{s_2} (b_3)^{s_3-s_2} (1-b_3)^{s_2} \]

References

[1] T. Bielecki, A. Cousin and S. Crépey, Dynamic hedging of portfolio credit risk in a Markov copula model (Previous ti-
