Contagion models a la carte: Which one to choose?

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Abstract. In this paper we propose a copula contagion mixture model for correlated default times. The model includes the well known factor, copula, and contagion models as its special cases. The key advantage of such a model is that we can study the interaction of different models and their pricing impact. Specifically, we model the default times of the underlying names in a reference portfolio to follow some contagion intensity processes with exponential decay coupled with copula dependence structure. We also model the default time of the counterparty and its dependence structure with the reference portfolio. The numerical tests show that the correlation and contagion have enormous joint impact on rates of CDO tranches and the corresponding CVAs are extremely high to compensate for the wrong way risk.

Keywords. copula contagion mixture model, exponential decay, basket CDS and CDO, counterparty risk, CVA.

AMS Subject Classification. Primary 60J75; Secondary 65C20, 91B28.

1 Introduction

The recent financial crisis has profound impact on the world financial systems. Many banks suffered heavy losses, some banks failed or had to be bailed out by the national governments. To avoid repeating the similar mistakes in the future many financial institutions are now active in credit risk modelling and management, especially in the area of counterparty risk and CVA (credit value adjustment). The purpose of this paper is to provide a unified correlation modelling framework for the reference portfolio and to discuss its pricing impact on portfolio credit derivatives in the presence of counterparty risk.

There are three main approaches to the correlation modelling in the portfolio credit risk literature: conditional independence, copula, and contagion. Factor models are most popular due to their semi-analytic tractability. Many effective algorithms have been developed to characterize the portfolio loss distribution, see Andersen et al. (2003), Hull and White (2004) for recursive exact methods, and Glasserman (2004), Zheng (2006) for analytic approximation methods. Factor models may underestimate the portfolio tail risk and economic capital, see Das et al. (2007). Copula models are also popular, especially the Gaussian copula. Some copulas (Archimedean and exponential) are good to model extreme tail events and simultaneous defaults, see Xu and Zheng (2009) for an application of exponential copulas in modelling portfolio asset price processes. Contagion models study the direct interaction of names in which the default intensity of one name may change upon defaults of other names and “infectious defaults” may develop, see Davis and Lo (2001), Jarrow and Yu (2001), Yu (2007). It is in general difficult to characterize the joint distribution of default times due to the looping dependence structure. For homogeneous portfolios there is a closed form formula for the density function of ordered default times, see Zheng and Jiang (2009). Monte Carlo
method is often used to price CDOs and basket CDSs no matter which correlation model is used and provides benchmark results to test efficiency and accuracy of analytic and numerical algorithms.

It is interesting to know which model one should choose in pricing portfolio credit derivatives. We know different models give different values. If one uses the Gaussian copula, the swap rate for the senior tranche of a CDO is low due to the thin tail distribution of the portfolio loss, on the other hand, if one uses the contagion model, the swap rate for the same senior tranche is much higher. However, one cannot simply say the contagion model is preferable to the Gaussian copula because it provides higher swap rates for senior tranche. It all depends on the underlying model assumptions. These correlation models are defined under different frameworks and are difficult to compare directly their pricing impact. It is therefore beneficial to have a unified model which covers all three known models as special cases. One may then extract the information of the interaction of these models and may give a more balanced view on which model one may choose for a specific application.

In this paper we suggest a general copula contagion mixture model which includes factor, copula, and contagion models as its special cases. The key advantage of such a model is that we can study the interaction of different models and their pricing impact. Specifically, we model the default times to follow some exponential decay contagion intensity processes coupled with some copula dependence structure. This is not a Markov process model and cannot be solved with the standard Kolmogorov equations or matrix exponentials, see Herbertsson and Rootzen (2008). Although there are analytic pricing formulas for some special cases, we choose to use the Monte Carlo method to price CDOs and basket CDSs, which is reliable, accurate and efficient with some optimized numerical procedure. We also model the dependence relation between the default time of the counterparty and those of the underlying names in a reference portfolio and show that there is an enormous joint impact of correlation and contagion on CVA due to the wrong way risk.

The paper is organized as follows: section 2 describes the copula contagion mixture model and the relation with the known models, section 3 applies the model to price CDOs and contagion on CVA due to the wrong way risk. It all depends on the underlying model assumptions. These correlation models are defined under different frameworks and are difficult to compare directly their pricing impact. It is preferable to the Gaussian copula because it provides higher swap rates for senior tranche. However, one cannot simply say the contagion model is the same senior tranche is much higher. However, one cannot simply say the contagion model is the best one. It is interesting to know which model one should choose in pricing portfolio credit derivatives. We know different models give different values. If one uses the Gaussian copula, the swap rate for the senior tranche of a CDO is low due to the thin tail distribution of the portfolio loss, on the other hand, if one uses the contagion model, the swap rate for the same senior tranche is much higher. However, one cannot simply say the contagion model is preferable to the Gaussian copula because it provides higher swap rates for senior tranche. It all depends on the underlying model assumptions. These correlation models are defined under different frameworks and are difficult to compare directly their pricing impact. It is therefore beneficial to have a unified model which covers all three known models as special cases. One may then extract the information of the interaction of these models and may give a more balanced view on which model one may choose for a specific application.

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2 The Model

Let \((\Omega, F, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a filtered probability space, where \(P\) is the martingale measure and \(\{\mathcal{F}_t\}_{t \geq 0}\) is the filtration satisfying the usual conditions. Let \(\tau_i\) be the default time of name \(i\), \(N_i(t) = 1_{\{\tau_i \leq t\}}\) the default indicator process of name \(i\), \(\mathcal{F}_t^i = \sigma(N_i(s) : s \leq t)\) the filtration generated by default process \(N_i\), \(i = 1, \ldots, n\), and \(\mathcal{F}_t = \mathcal{F}_t^1 \vee \ldots \vee \mathcal{F}_t^n\) the smallest \(\sigma\)-algebra needed to support \(\tau_1, \ldots, \tau_n\). Assume that \(\tau_i\) possesses a nonnegative \(\mathcal{F}_t\) predictable intensity process \(\lambda_i(t)\) satisfying \(\mathbb{E}[\int_0^t \lambda(s) ds] < \infty\) for all \(t\). Given \(\tau_j = t_j\), \(j \in J_k = \{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}\), satisfying \(0 = t_{j_0} < t_{j_1} < \ldots < t_{j_k}\) and \(\tau_i > t > t_{j_k}\) for \(i \not\in J_k\), the conditional hazard rate of \(\tau_i\) at time \(t\) is given by

\[
\lambda_i(t|t_{j_k}) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P(t < \tau_i \leq t + \Delta t|\tau_j = t_j, j \in J_k)
\]

where \(t_{j_k}\) is a short form for \((t_{j_1}, \ldots, t_{j_k})\).

In copula modelling of default times it is normally assumed that intensity processes are independent of default times of other names, i.e., \(\lambda_i(t|t_{j_k}) = \lambda_i(t)\) for all \(t\). The marginal distribution functions of default times \(\tau_i\) are given by \(F_i(t) = P(\tau_i \leq t) = \mathbb{E}[1 - \exp(-\int_0^t \lambda_i(s) ds)]\) (if \(\lambda_i\) are stochastic processes) and standard uniform variables \(F_i(\tau_i), i = 1, \ldots, n\), have a
joint distribution function $C$, a given copula. It is easy to generate default times $\tau_i$ with the Monte Carlo method. One can simply first generate standard uniform variables $U_i$ with copula $C$ and sample paths of $\lambda_i$ and then find the default times $\tau_i$ by

$$\tau_i = \inf \left\{ t > 0 : \int_0^t \lambda_i(s)ds \geq E_i \right\}$$

where $E_i = -\ln(1 - U_i)$, $i = 1, \ldots, n$, are correlated standard exponential variables. In particular, if $\lambda_i(t) = a_i$, then $\tau_i = E_i/a_i$.

There has been extensive research in literature on factor modelling of default times. These models are all special cases of copula modelling of default times. For example, the well-known Gaussian factor model is given by

$$X_i = \sqrt{\rho}Z + \sqrt{1 - \rho}Z_i$$

where $Z, Z_1, \ldots, Z_n$ are independent standard normal variables and $\rho$ is a nonnegative constant satisfying $\rho \leq 1$. $Z$ is often interpreted as a systematic factor and $Z_i$ idiosyncratic factors. If we set $U_i = \Phi(X_i)$, where $\Phi$ is the standard normal distribution function, then the distribution of $U_i$ is a Gaussian copula given by

$$C(u_1, \ldots, u_n) = \Phi_\Gamma(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n))$$

where $\Phi_\Gamma$ is the $n$-variate standard normal distribution function with correlation matrix $\Gamma$ that has diagonal elements 1 and all other elements $\rho$. Factor models are appealing from model interpretation and conditional independence point of view. The corresponding copulas may have some complex forms, but in general it is easy to generate correlated standard uniform variables due to the special structure of factor models. There is no need to treat them separately if we know how to generate standard uniform variables $U_i$ from a given copula $C$.

In contagion modelling of default times the intensity processes $\lambda_i(t|t_{J_k})$ depend on default times of other names. The marginal distribution functions $F_i$ of default times $\tau_i$ cannot be simply expressed in terms of $\lambda_i(s)$ as information of other default times is needed to characterize the whole intensity process paths and there is a “looping” phenomenon. Although it is difficult to characterize the marginal and joint distributions of default times it is easy to generate default times with the total hazard construction method. One can first generate independent standard uniform variables $U_i$ and set $E_i = -\ln(1 - U_i)$, $i = 1, \ldots, n$, then find default times one by one as follows: To find the first default time $\tau_{j_1}$ and the corresponding name $j_1$, set

$$j_1 = \arg\min_{j=1,\ldots,n} \left\{ t_j > 0 : \int_0^{t_j} \lambda_j(s)ds \geq E_j \right\}$$

(1)

and $J_1 = \{j_1\}$, where $\lambda_j(s)$ are unconditional hazard rates of names $j$ at time $s$. To find the $k$th default time $\tau_{j_k}$ and the corresponding name $j_k$ for $k \geq 2$, set

$$j_k = \arg\min_{j \not\in J_{k-1}} \left\{ t_j > \tau_{j_{k-1}} : \int_0^{t_j} \lambda_j(s|t_{j_{k-1}})ds \geq E_j \right\}$$

(2)

and $J_k = J_{k-1} \cup \{j_k\}$, see Shaked and Shanthikumar (1987).

We suggest a copula contagion mixture model which covers both copula model and contagion model as special cases. Specifically, we assume that the default times are defined and generated by (1) and (2) using standard uniform variables $U_i$ that have a joint distribution function (copula) $C$ and intensity processes $\lambda_i$ that depend on default times of other names. This is a natural generalization of pure copula models and pure contagion models. If $C$ is
a product copula one recovers the pure contagion model and if \( \lambda_i \) are independent of other default times one recovers the pure copula model. The key advantage of this new mixture model is that, instead of studying three well known models in isolation, we can explore their interaction and their joint pricing impact on CDOs and basket CDSs. One can also specify the joint distribution of default times for the new model. To illustrate the point, assume that the number of names in the portfolio is \( n = 2 \) and the intensities of these two names are given by

\[
\lambda_1(t) = a_{10} + a_{12}e^{-d_{12}(t-\tau_2)}1_{\{\tau_2 \leq t\}} \quad \text{and} \quad \lambda_2(t) = a_{20} + a_{21}e^{-d_{21}(t-\tau_1)}1_{\{\tau_1 \leq t\}}
\]

where \( a_{ij} \) and \( d_{ij} \) are positive constants and \( 1_{\{\tau_i \leq t\}} \) are indicators. Assume standard uniform variables \( U_i \) have a copula \( C \). We can determine the joint distribution of default times \( \tau_i \) as follows: If \( \tau_1 < \tau_2 \) then \( E_1 = a_{10}\tau_1 \) and \( E_2 = a_{20}\tau_2 + \frac{a_{21}}{d_{21}}(1 - e^{-d_{21}(\tau_2-\tau_1)}) \). The Jacobi determinant of \( E_i \) in terms of \( \tau_i \) is given by

\[
c(\tau_1, \tau_2) = a_{10}(a_{20} + a_{21}e^{-d_{21}(\tau_2-\tau_1)})
\]

and that of \( U_i \) in terms of \( E_i \) is given by \( e^{-E_1-E_2} \). The joint density function of \( \tau_1 \) and \( \tau_2 \) in the region \( t_1 < t_2 \) is therefore given by

\[
f(t_1, t_2) = c(t_1, t_2)e^{-e_1-e_2} \frac{\partial^2 C}{\partial u_1 \partial u_2} (1 - e^{-e_1}, 1 - e^{-e_2}), \quad 0 < t_1 < t_2,
\]

where \( \frac{\partial^2 C}{\partial u_1 \partial u_2} (u_1, u_2) \) is the density function of \( C \) (if it exists) and

\[
e_1 = a_{10}t_1 \quad \text{and} \quad e_2 = a_{20}t_2 + \frac{a_{21}}{d_{21}}(1 - e^{-d_{21}(t_2-t_1)}).
\]

If \( C \) is a product copula then its density function is a constant 1. If \( C \) is a normal copula with correlation coefficient \( \rho \) then its density function is

\[
\frac{\partial^2 C}{\partial u_1 \partial u_2} (u_1, u_2) = \frac{\phi_2(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \rho)}{\phi(\Phi^{-1}(u_1))\phi(\Phi^{-1}(u_2))}
\]

where \( \phi_2 \) is the density function of a standard bivariate normal variable with correlation coefficient \( \rho \) and \( \phi \) is the density function of a standard normal variable. By symmetry one can write out the density function \( f(t_1, t_2) \) in the region \( t_1 > t_2 \) too. For a general portfolio with \( n \) names one can similarly derive the joint density function of default times with permutation. There are total \( n! \) possibilities. It is not practical to write them all out explicitly and the simulation is a more efficient way of generating joint default times.

We now impose some structure to the intensity processes. To simplify the notation and highlight the key point, we assume a homogeneous portfolio. The discussion is the same for general heterogeneous intensity processes except the expression is more complicated. We assume the intensity processes have the following structure

\[
\lambda_i(t) = a \left( 1 + \sum_{j=1, j \neq i}^{n} c e^{-d(t-\tau_j)}1_{\{\tau_j \leq t\}} \right), \quad i = 1, \ldots, n,
\]

where \( a, c, d \) are positive constants. (These parameters can be deterministic functions of \( t \) or even some stochastic processes, the discussion is essentially the same, see Zheng and Jiang (2009).) \( a \) is the unconditional default intensity, \( c \) is the contagion rate, and \( d \) is the exponential decay rate. When \( d = 0 \) we may introduce the default state space and use the
Markov Chain to study the joint distribution of default times. Apart from this extreme case the intensity processes (4) are non-Markov.

For homogeneous intensity processes (4) without exponential decay ($d = 0$) we can simplify the procedure in generating the ordered default times. This is because we only need to know the number of defaults at time $t$ but not the identities of names which have defaulted. We can generate the $k$th default times $\tau^k$, $k = 1, \ldots, n$, as follows:

**Step 1.** Generate correlated standard uniform variables $U_i$, $i = 1, \ldots, n$, from the copula $C$.

**Step 2.** Set $E_i = -\ln(1-U_i)$, $i = 1, \ldots, n$, to get correlated standard exponential variables and sort $E_i$ in increasing order to get their order statistics $E_i^*$ with $E_1^* < E_2^* < \ldots < E_n^*$.

**Step 3.** Find the $k$th default times $\tau^k$ by setting

$$\tau^1 = \frac{E_1^*}{a}, \quad \tau^k = \tau^{k-1} + \frac{E_k^* - E_{k-1}^*}{a(1 + (k-1)c)} \quad \text{for} \quad k = 2, \ldots, n. \quad (5)$$

For general homogeneous intensity processes (4) we cannot use (5) to generate the ordered default times. The computation is slightly more involved. Steps 1 to 2 are the same and so is the first default time $\tau^1$. Assume ordered default times $\tau^1, \ldots, \tau^{k-1}$ have already been generated for some $k \geq 2$. Now we want to generate $\tau^k$. Let $\tau^{k-1} \leq t \leq \tau^k$. The total hazard accumulated by name $k$ at time $t$ is

$$\int_0^t \lambda_k(s) ds = \sum_{j=1}^{k-1} \int_{\tau_{j-1}}^{\tau_j} a \left( 1 + \sum_{i=1}^{j-1} ce^{-d(s-\tau^i)} \right) ds + \int_{\tau_{k-1}}^{t} a \left( 1 + \sum_{i=1}^{k-1} ce^{-d(s-\tau^i)} \right) ds$$

where $\tau^0 = 0$ and $\sum_{i=1}^{0} = 0$ by convention. Simplifying the above expression we get

$$\int_0^t \lambda_k(s) ds = at + \frac{ac}{d} \sum_{i=1}^{k-1} \left( 1 - e^{-d(t-\tau^i)} \right).$$

The $\tau^k$ is determined by the relation $\int_0^{\tau^k} \lambda_k(s) ds = E_k^*$. Define

$$F_k(t) := at + \frac{ac}{d} \sum_{i=1}^{k-1} \left( 1 - e^{-d(t-\tau^i)} \right) - E_k^*.$$ 

Then $\tau^k$ is a root of nonlinear equation $F_k(t) = 0$. Since $F_k'(t) > 0$ and $F_k''(t) < 0$ function $F_k$ is strictly increasing and strictly concave. Observe also that from $F_{k-1}(\tau^{k-1}) = 0$ we have

$$F_k(\tau^{k-1}) = a\tau^{k-1} + \frac{ac}{d} \sum_{i=1}^{k-1} \left( 1 - e^{-d(\tau^{k-1}-\tau^i)} \right) - E_k^* = E_{k-1}^* - E_k^* < 0$$

and $F_k(\infty) = \infty$. There is a unique root of equation $F_k(t) = 0$ on the interval $[\tau^{k-1}, \infty)$. The special structure of function $F_k$ guarantees that the Newton algorithm with an initial iterating point $\tau^{k-1}$ converges quadratically to the root $\tau^k$. We can now summarize Sept 3 in the presence of exponential decay rate $d > 0$ as follows.

**Step 3'.** Set $\tau^1 = E_1^*/a$ and find the $k$th default time $\tau^k$ by solving numerically the equation $F_k(t) = 0$ with the Newton algorithm and the initial iterating point $\tau^{k-1}$ for $k = 2, \ldots, n$.

We now discuss the impact of exponential decay rate $d$ on ordered default times $\tau^k$. From $F_k'(t) = a + ac \sum_{i=1}^{k-1} e^{-d(t-\tau^i)}$ we know that $F_k(t)$ is a strictly decreasing function of $d$ for $t > \tau^{k-1}$. If $d = 0$ we have $F_k'(t) = a + ac(k-1)$ and $F_k$ is a linear function

$$F_k(t) = E_{k-1}^* - E_k^* + (a + ac(k-1))(t - \tau^{k-1}).$$
The \( k \)th default time \( \tau^k \) is given by (5) as expected. If \( d = \infty \) we have \( F'_k(t) = a \) and \( F_k \) is again a linear function
\[
F_k(t) = E^*_{k-1} - E^*_k + a(t - \tau^{k-1}).
\]
The \( k \)th default time is given by \( \tau^k = \tau^{k-1} + (E^*_k - E^*_{k-1})/a \), or equivalently, \( \tau^k = E^*_k/a \), which corresponds to the case when there is no contagion effect. For any other \( d \) the \( k \)th default time \( \tau^k \) lies between these two extreme cases. We conclude that the smaller the exponential decay rate, the stronger the contagion effect and the sooner the ordered default times, which makes CDO and basket CDS riskier and demands higher spreads.

### 3 Pricing Without Counterparty Risk

We can now value the basket CDS and CDO with the copula contagion mixture model. We assume homogeneous intensity processes (4) to simplify the computation, but the same method can be applied to general intensity processes. For both basket CDS and CDO we assume that \( T \) is the maturity of the contract, \( t_1 < t_2 \ldots < t_N \) are swap rate payment dates, \( t_0 = 0 \) is the initial time and \( t_N = T \) is the terminal time, \( R \) is the recovery rate, \( r \) is the riskless interest rate, and \( B(t) = e^{-rt} \) is the discount factor at time \( t \).

To price basket CDS we assume \( S_k \) is the annualized \( k \)th default swap rate. The expected value of the contingent leg at time 0 is equal to
\[
E[\{(1 - R)B(\tau^k)1_{\{\tau^k \leq T\}}]\]
and that of the fee leg with accrued interest is equal to
\[
S_kE\left[ \sum_{i=1}^N \left( (t_i - t_{i-1})B(t_i)1_{\{\tau^k > t_i\}} + (\tau^k - t_{i-1})B(\tau^k)1_{\{t_{i-1} < \tau^k \leq t_i\}} \right) \right].
\]
We can easily find \( S_k \) with the Monte Carlo method by generating ordered default times \( \tau^k \).

To price CDO we assume \( k_l, l = 0, \ldots, M - 1 \), are attachment points of tranches \( l \) with \( 0 = k_0 < k_1 < \ldots < k_M = 1 \), \( \Delta k_l = k_l - k_{l-1} \) are tranche sizes for \( l = 1, \ldots, M \), the cumulative percentage portfolio loss at time \( t \) is given by
\[
L(t) = \sum_{k=1}^{\infty} \frac{k}{n} 1_{\{\tau^k \leq t < \tau^{k+1}\}}
\]
with \( \tau^0 = 0 \) and \( \tau^{n+1} = \infty \), the cumulative tranche \( l \) loss at time \( t \) is given by
\[
L_l(t) = (L(t) - k_{l-1})1_{\{k_{l-1} \leq L(t) \leq k_l\}} + \Delta k_l 1_{\{L(t) > k_l\}}.
\]
Assume \( S_l \) is the swap rate of tranche \( l \). The expected value of the contingent leg for tranche \( l \) loss at time 0 is given by (note \( L_l(0) = 0 \))
\[
E\left[ \sum_{i=1}^N B(t_i)(L_l(t_i) - L_l(t_{i-1})) \right]
\]
and that of the fee leg for tranche \( l \) is
\[
S_lE\left[ \sum_{i=1}^N (t_i - t_{i-1})B(t_i)(\Delta k_l - L_l(t_i)) \right].
\]
We can again easily find \( S_l \) with the Monte Carlo method.
We have done some numerical tests with the Gaussian copula contagion mixture model and the homogeneous intensity processes (4). The data used are as follows: number of names $n = 40$, riskless interest rate $r = 0.05$, time to maturity $T = 3$, number of payments $N = 6$ with equally spaced time intervals, unconditional intensity rate $a = 0.01$, recovery rate $R = 0.5$, exponential decay rate $d = 0$, and number of simulations is 1 million.

Table 1 lists CDO rates computed with $a = 0.01$ and different $c$ and $\rho$. We can see that swap rates increase if $c$ increases, which is expected as higher $c$ causes higher contagion and more defaults. $c = 0$ corresponds to the Gaussian factor model. As $\rho$ increases swap rates for equity tranche decrease while those for mezzanine and senior tranches increase, a well known fact. $\rho = 0$ corresponds to the pure contagion model (or the product copula contagion mixture model) and we see $c$ has huge impact on swap rates for mezzanine and senior tranches. When both $c$ and $\rho$ are positive, we see swap rates for senior tranche are greater than those with the pure contagion model ($\rho = 0$) and the pure factor model ($c = 0$). Note that swap rates for mezzanine tranche decrease as $\rho$ increases when $c = 3$. This is because when $c = 3$ the default intensity increases quickly for surviving names and many more names are likely to default, which makes the mezzanine tranche behave increasingly like the equity tranche. For the same reason the senior tranche behaves increasingly like the mezzanine tranche and its swap rates increase and then decrease as $\rho$ increases.

The unconditional intensity rate $a$ is fixed at value 0.01 in Table 1. If there is no contagion ($c = 0$) the distribution of individual default times is unaffected by correlation coefficient $\rho$ and the swap rate for single name CDS is a constant. However, when there is contagion effect ($c > 0$) the distribution of individual default times is influenced by both $c$ and $\rho$. Intuitively, as $c$ increases a name is more likely to default and the swap rate for single name CDS shall increase too. This indicates that the comparison in Table 1 has not been performed in the same setting, i.e., with different $c$ and $\rho$, the distribution of individual default times should be kept the same. Due to the looping behaviour it is difficult to characterize the distribution of individual default times, we instead ask the single name CDS rates to be kept the same for different $c$ and $\rho$ by adjusting $a$. We can then see the effects of $c$ and $\rho$ on pricing of...
Table 3: CDO rates with same single name CDS rates.

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Table 3 lists CDO rates with different $c$ and $\rho$ but with the same single name CDS rate $S(a, c, \rho) = S$. We can see that contagion still has significant impact on mezzanine and senior tranches, but with much smaller magnitude. For fixed $c$ all rates behave similarly to those in Table 1 and the impact of $\rho$ increases (percentage change) for senior tranches. For fixed $\rho$ the equity tranche rate decreases as $c$ increases and the impact of $c$ decreases (percentage change) for senior tranches. The senior tranche rate always increases whenever $c$ or $\rho$ increases, which implies that correlation and contagion always have adverse impact on the senior tranche while their impact on equity and mezzanine tranches are mixed. It seems that the Gaussian factor model is capable of producing different CDO spreads (high or low) of different tranches as long as one is prepared to use a sufficiently large correlation coefficient $\rho$. If that seems unrealistic then a combination of correlation $\rho$ and contagion $c$ can produce different CDO spreads.

Table 4 lists basket CDS and CDO rates with different exponential decay rate $d$. The data used are $a = 0.01$, $c = 3$, $\rho = 0.5$, and all other data same as those in Table 1. $d = 0$ corresponds to the Gaussian copula contagion mixture model without decay and $d = \infty$ to the case $c = 0$ (no contagion effect). As $d$ increases basket CDS and CDO rates decrease. The exponential decay has much greater impact to the $k$th default rates for larger $k$ than for smaller $k$. The same phenomenon is observed for CDO rates, that is, the exponential decay has much greater impact to senior tranche rates than to junior tranche rates. Basket CDS and CDO rates are highly sensitive to the exponential decay rate $d$, which requires an accurate estimation of $d$ in calibration if one is to use it in pricing. (Table 4 is produced with fixed $a$ as in Table 1. The resulting single name CDS rates are not constant for different $d$.

The author thanks the anonymous referee for suggesting the test for different $\rho$ and $c$ while keeping single name CDS rates constant to preserve the same spread.
If one wants to compare the results as in Table 3, then one needs to adjust $a$ as in Table 2.

## 4 Pricing With Counterparty Risk

Assume there is a counterparty risk and the intensity process of default time $\tau^B$ of the counterparty is given by

$$\lambda_B(t) = a_B \left( 1 + \sum_{i=1}^{n} c_B 1\{\tau^i \leq t\} \right)$$

where $a_B$ is the unconditional default intensity, $c_B$ the contagion rate, and $\tau^i$ the $i$th default time of the reference portfolio with homogeneous intensity processes (4). Note that the hazard rate process $\lambda_B$ of the counterparty is influenced by defaults of names in the reference portfolio, but not vice versa. This follows the observation in Leung and Kwok (2005) and Yu (2007) that the contagion of the counterparty on underlying names does not affect CDS pricing.

To price basket CDS we only need to compute the expected value of the contingent leg and the fee leg at time 0, given respectively by

$$\mathbb{E}\left[(1-R)B(\tau^k)1\{\tau^k \leq T, \tau^B > \tau^k\}\right]$$

and

$$S_k \mathbb{E}\left[\sum_{i=1}^{N} \left( (t_i - t_i-1)B(t_i)1\{\tau^k > t_i, \tau^B > t_i\} + (\tau^k - t_i-1)B(\tau^k)1\{t_i-1 < \tau^k \leq t_i, \tau^B > \tau^k\} \right)\right].$$

Similarly to price CDO tranche $l$ we only need to compute the expected value of the contingent leg and the fee leg at time 0, given respectively by

$$\mathbb{E}\left[\sum_{i=1}^{N} B(t_i)(L_l(t_i) - L_l(t_i-1))1\{\tau^B > t_i\}\right]$$

and

$$S_l \mathbb{E}\left[\sum_{i=1}^{N} (t_i - t_i-1)B(t_i)(\Delta k_l - L_l(t_i))1\{\tau^B > t_i\}\right].$$

We can easily find swap rates by generating ordered default times $\tau^k$ and counterparty default time $\tau^B$. 

<table>
<thead>
<tr>
<th>$k$</th>
<th>$d = 0$</th>
<th>$d = 1$</th>
<th>$d = 10$</th>
<th>$d = 100$</th>
<th>$d = \infty$</th>
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<tr>
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<td>0.1153</td>
<td>0.1153</td>
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<td>0.1153</td>
</tr>
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</tr>
<tr>
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<td>0.0000</td>
</tr>
<tr>
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<td>0.0001</td>
<td>0.0000</td>
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</tr>
<tr>
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<td>0.1323</td>
<td>0.0810</td>
<td>0.0696</td>
<td>0.0682</td>
</tr>
<tr>
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<td>0.0127</td>
<td>0.0048</td>
<td>0.0042</td>
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<tr>
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<td>0.0328</td>
<td>0.0012</td>
<td>0.0002</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

Table 4: Basket CDS and CDO rates with exponential decay.
Table 5 lists the swap rates of all tranches with counterparty risk. We use the Gaussian copula contagion mixture model for underlying reference names and the data \( a_B = 0.005 \) and \( c_B = c \). The counterparty has very low default intensity \( a_B \) but is exposed to the contagion risk from defaults in the reference portfolio. The other data used are the same as those in Table 3, which guarantees single name CDS rates are kept constant for different \( c \) and \( \rho \) by adjusting \( a \). It can be observed that the equity tranche is least affected by the counterparty risk whereas the senior tranche is most affected. This is expected as the counterparty is much more likely to default due to the contagion effect from defaults of names in equity and mezzanine tranches, and therefore the senior tranche investors require higher compensation for increased counterparty risk.

Table 6 lists the ratio of CDO rates with counterparty risk to those without counterparty risk. We observe that \( D(c, \rho) \) is a decreasing function of \( c \) and \( \rho \), which implies that the higher the contagion \( c \) and the correlation \( \rho \) the deeper the discount factor \( D(c, \rho) \). (When \( \rho = 0 \) and \( c = 0 \) the rates for mezzanine and senior tranches are extremely small, which makes the ratio highly sensitive and less reliable with Monte Carlo even with 1 million simulation runs.) We also note that \( D(c, \rho) \) decreases much faster for senior tranches than for equity and mezzanine tranches. This implies that contagion and correlation will have much greater pricing impact on senior tranches in the presence of counterparty risk. We
can quantify this with CVA, defined by

\[ \text{CVA}(c, \rho) = S(c, \rho)P(c, \rho) \]

where \( P(c, \rho) = 1 - D(c, \rho) \) is the percentage of the riskless (no counterparty risk) CDO rate used to compensate for the counterparty risk. We can see that with \( c = 0 \) or \( \rho = 0 \) the corresponding \( P(c, \rho) \) is relatively small except for senior tranches which have values \( P(3, 0) = 0.2297 \) and \( P(0, 0.9) = 0.3527 \). However, when both \( c \) and \( \rho \) are positive, i.e., with correlation to systemic risk factor and contagion among names in the reference portfolio and with contagion of the reference portfolio to the counterparty, \( P(c, \rho) \) becomes significant to all tranches, even equity tranche. When \( c = 3 \) and \( \rho = 0.9 \), \( P(3, 0.9) = 0.5607 \) for equity tranche, 0.8466 for mezzanine tranche and 0.9695 for senior tranche. The correlation and contagion have enormous impact on senior tranche rate and the corresponding CVA is extremely high to compensate for the wrong way counterparty risk.

5 Conclusions

In this paper we have suggested a copula contagion mixture model with exponential decay which unifies the factor, copula and contagion models. The key advantage is that one can investigate the interaction of these models and their joint pricing impact on basket CDS and CDO. The ordered default times can be easily generated with the Monte Carlo method. We have done some numerical tests with the Gaussian copula contagion mixture model with different contagion and correlation coefficients and with fixed and adjusted unconditional default intensity rates. We have found that when single name CDS rates are kept constant the level and speed of contagion impact is much reduced for senior tranches and that the Gaussian factor model can produce required spread for senior tranches if the correlation to common risk factor is sufficiently high. We have also compared the pricing results when there is counterparty risk which subjects to the contagion risk from defaulted names in a reference portfolio. We have found that contagion and correlation have huge impact to tranches, especially senior tranche. The resulting CVA is extremely high due to the wrong way risk. Our conclusion is that one has to be cautious in pricing portfolio credit derivatives when a particular model is used as different models may greatly influence the portfolio loss distribution and can significantly affect the pricing and hedging. No single model is best for all purposes. Stress testing, netting, collateral and other risk control measures should be in place to withstand the potential loss due to the wrong choice of a model.

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References.


