QUALITATIVE SENSITIVITY ANALYSIS IN MONOTROPIC PROGRAMMING

ANTOINE GAUTIER, FRIEDA GRANOT, AND HARRY ZHENG

Optimal selections are parameter-dependent optimal solutions of parametric optimization problems whose properties can be used in sensitivity analysis. Here we present a qualitative theory of sensitivity analysis for linearly-constrained convex separable (i.e., monotropic) parametric optimization problems. Three qualitative sensitivity analysis results previously derived for network flows are extended to monotropic problems: The Ripple and Smoothing Theorems give upper bounds on the magnitude of optimal-variable variations as a function of variations in the problem parameter(s), the theory of substitutes and complements provides necessary and sufficient conditions for optimal-variable changes to consistently have the same (or the opposite) sign(s) in two given variables, and the Monotonicity Theorem links changes in the value of the parameters to changes in optimal decision variables. We introduce a class of optimal selections for which these results hold, thereby answering a long-standing question due to Granot and Veinott (1985) with a simple and elegant method. Although a number of results are known to depend on the resolution of NP-complete problems, easily computable nonnetwork classes of monotropic problems such as unimodular systems of constraints emerge in the light of the present approach.

1. Introduction. In this paper we propose an approach to sensitivity analysis for monotropic problems, a term coined by R. T. Rockafellar (1984) for the optimization of a convex and separable function over a polyhedron. As an alternative to parametric programming (Rockafellar 1984, p. 568), we discuss qualitative sensitivity analysis (QSA) where, instead of exact values, bounds for comparative rates of change of model output versus parameters are obtained.

QSA has a number of distinctive features. First, the problem parameters can be real numbers (cost parameters, upper/lower bounds on variables, right-hand-sides, etc.) as well as vectors, intervals and more. Whenever possible, predictions are made regarding the direction and magnitude of change of individual decision variables. The statements made are valid for small as well as arbitrarily large variations of the problem parameters. Another attractive feature of QSA is that it can be performed prior to solving the optimization problem; in that sense it deserves to be called pre-optimal analysis.

Many practical applications of pre-optimal analysis using the qualitative approach presented in this paper are available. These include asset allocation management (Gautier and Granot 1992), currency management for multinational firms (Gautier, Granot, and Levi 1995), long term forest management and planning (Gautier and Granot 1995), inventory management (Gautier and Granot 1994a) and production-inventory systems (Ciurria 1989).

The aim of the paper is threefold. First, we extend previous results in QSA to the larger class of monotropic problems. We show in §3 the existence of parameter-dependent optimal solutions (called optimal selections) that are well-behaved in the sense that they possess a number of desirable properties. The extension presents no new theoretical challenge (for most proofs the reader is referred to earlier papers) but shows how broadly

Received January 16, 1995; revised February 16, 1997, June 18, 1997, November 19, 1997.


OR/MS subject classification. Primary: Programming/Convex.

Key words: Nonlinear, pre-optimal, sensitivity, qualitative analysis, monotropic, monotonicity, substitutes, complements, ripples.

695

0364-765X/98/2303/0695/$05.00

Copyright © 1998, Institute for Operations Research and the Management Sciences
these techniques can be used. In addition to being of independent interest, the results presented illustrate the use of elementary vectors as a powerful mathematical combinatorial tool. This, we believe, deserves to be pointed out since few authors have used this tool since Rockafellar’s 1967 paper.

Next we address the problem, left open by Granot and Veinott, Jr. (1985), of efficiently finding well-behaved selections when multiple optimal solutions exist—an existence result does not guarantee that a solution obtained by a given optimization algorithm will indeed possess the desired properties. This is an important issue when multiple solutions exist for more than a small number of “exceptional” parameter values, a situation which does occur in practice. For example, even for linear network flow problems situations arise where multiple solutions are the norm rather than the exception (for example, in the maximum flow problem, all flows but those on a minimum cut can have several values).

To address this problem Granot and Veinott, Jr. (1985) assign a perturbation parameter to every variable/cost pair that render the costs strictly convex and then taking the limits one perturbation at a time. In practice, their approach entails heavy computations as one would have to solve a large number of Monotropic Problems in order to obtain limits with reasonable accuracy, leading to prohibitive solution times for all but the simplest models. Moreover, the solutions they obtain may depend on the order of taking the limits, thus both the practical efficiency and the intuitive explanation of the optimal solution are lost. This problem is addressed and solved in §4.

Finally in §5 we present several examples of applications of QSA for Monotropic Problems which could not have been obtained before the derivation of the extensions in this paper. We illustrate how one can take advantage of size, structure, or both to obtain QSA results for several nonnetwork Monotropic Problems. Our examples include unimodular problems, multicommodity transportation, dual of network flows, and cyclic scheduling problems. The list is not restrictive, it only shows a range of available techniques.

Qualitative approaches to sensitivity analysis exist for monotropic problems on pure network flows (Granot and Veinott, Jr. 1985), multicommodity flows on suspension graphs (Ciurria, Granot, and Veinott, Jr. 1989), network flows with one additional constraint (Gautier and Granot 1994b), and network flows with gains (Gautier and Granot 1996). The main result obtained therein is the existence of an optimal selection with some important properties. These results share a commonality of form if not of content: The ripple theorem provides upper bounds on the absolute value of an optimum flow variation as a function of variations in the problem parameters. The theory of substitutes and complements presents necessary and sufficient conditions for changes in optimal values to consistently have the same (or the opposite) sign in two given arcs and the monotonicity theorem links changes in the value of the parameters to changes in optimal decision variables.

2. Notation and preliminaries. Let I = {1, ..., m} and J = {1, ..., n} be two index sets, A = (a_{ij} : i \in I, j \in J) a real matrix, x = (x_j : j \in J) \in \mathbb{R}^n a vector of decision variables, and b = (b_i : i \in I) a (r.h.s.) vector. A parameter t = (t_j : j \in J) lies in a lattice denoted T. The parameter t_j associated with the decision variable x_j lies in the jth projection of T denoted T_j. The reader unfamiliar with lattices and lattice programming may envision T_j as being \mathbb{R} (range of values), intervals in \mathbb{R}, or \{0, 1\} for on/off type of parameters. For further possibilities, background on lattices and lattice programming, (see, e.g., Crawley and Dilworth 1973, and Granot and Veinott, Jr. 1985). The Parametric Monotropic Problem is

\begin{equation}
\begin{aligned}
c_{\min}(t) &= \min \{ c(x, t) : Ax = b \}
\end{aligned}
\end{equation}
where \( c(x, t) = \Sigma_j c_j(x_j, t_j) \) is a parametric, separable cost. We assume that (1) is both feasible and bounded \((-\infty < c_{\min}(t) < +\infty)\). Note that bounds on a variable \( x_i \) can easily be incorporated by setting \( c_i(x_i, t_i) = \infty \) for all forbidden values of the decision variable. \( X(t) \) denotes the set of optimal solutions to (1), and a function \( x(\cdot) \) such that \( x(t) \in X(t) \) for all \( t \) is called an optimal selection.

Let \( N(A) \) be the null space \( \{ x \in \mathbb{R}^n : Ax = 0 \} \). Any vector \( x \in N(A) \) has a support denoted by \( \sigma(x) = \{ j \in J : x_j \neq 0 \} \). Such a support is elementary if it is nonempty but does not properly include any other support of \( N(A) \). The elementary supports make up a finite collection of subsets of \( J \). The set \( E(A) \) of elementary vectors of \( N(A) \) (elementary vectors, for short) comprises vectors of \( N(A) \) whose supports are elementary (Rockafellar 1967, 1984). Since two elementary vectors with the same support must be proportional, each elementary support corresponds to a unique elementary vector up to a scalar multiple. A method for listing all elementary vectors is discussed in §5.1.

Right-hand-side changes. Formulation (1) can be adapted to allow for right-hand-side changes by treating the r.h.s. vector \( b \) as a parameter and introducing new variables \( y \) as follows

\[
c_{\min}(t, b) = \min \left\{ c(x, t) + \Sigma_i c_i'(y_i, b_i) : Ax - y = 0 \right\},
\]

where \( c_i'(y_i, b_i) = 0 \) if \( y_i = b_i \) and \( \infty \) otherwise. The constraint matrix is now \( [A - I] \).

Note on network flows. Previous work by Granot and Veinott, Jr. (1985) focused on the case of monotropic network flow problems (see Rockafellar 1984, Bertsekas 1995, and Ahuja, Magnanti, and Orlin 1993 for further references) where the constraint matrix \( A \) is a node-arc incidence matrix of a network. With such a constraint matrix, elementary vectors correspond to simple cycles. Conversely, the concepts presented in the present paper that use elementary vectors can be read in the context of network flows by replacing these by simple cycles. For instance, we use the result, due to Rockafellar (1967, 1984), that every nonzero vector of the null space can be expressed as a positive linear combination of conformal elementary vectors. (Two vectors \( u \) and \( v \) are conformal if \( u_j v_j > 0 \) \( \forall j \).) This result is an extension to general constraint matrices of the well-known decomposition of circulations in networks into conformal cycles, see, e.g., Berge (1973, p. 91) and Ahuja, Magnanti, and Orlin (1993, p. 81).

3. Qualitative properties of optimal selections. Changing the parameter(s) associated with one or more variables has effects that are bounded, that ripple through the whole set of variables, and whose direction can sometimes be predicted. Before stating three results in Theorems 1, 2, and 3 we define the following dependence and quantitative orders.

- For \( i, j, k \in J \), \( i \) is less dependent on \( j \) than \( k \) is (written \( i \preceq_j k \)) if

\[
\forall e \in E(A), \quad e_i e_j \neq 0 \iff e_k \neq 0.
\]

- For \( i, j, k \in J \), \( i \) is quantitatively less dependent on \( j \) than \( k \) is (written \( i \preceq^q_j k \)) if

\[
\forall e \in E(A), \quad e_i e_j \neq 0 \implies |e_i| \leq |e_k|.
\]

Both orders are partial orders, and \( i \preceq^q_j k \) implies \( i \preceq_j k \) but not the reverse. With a similar proof to the one given by Gautier and Granot (1994b) one can show that the Ripple Theorem for constrained network flows naturally extends to Monotropic Problems. This
result characterizes the change in an optimal solution resulting from a change in the parameter of a given variable.

**Theorem 1 (Ripple Effect).** Suppose \( t \) and \( t' \) differ only in one component \( j \), \( c_k(\cdot, t) \) is convex for each \( k \neq j \) and each \( t \in T \), \( x(t) \) is an element of \( X(t) \) and \( X(t') \) is nonempty. Then there exists an element \( x(t') \) of \( X(t') \) such that \( x(t') - x(t) \) is a conformal sum of elementary vectors each of whose support contains \( j \), and

- \( x(t') \) has the Ripple Property with respect to the dependence order:

\[
i \leq j \Rightarrow |x_i(t') - x_i(t)| \leq M_{ki}|x_k(t') - x_k(t)|
\]

where \( M_{ki} = \max \{|e_i| : e \in E(A), e_k = 1\} \) is a Ripple Multiplier;

- \( x(t') \) has the Ripple Property with respect to the quantitative order, with Ripple Multipliers of value one:

\[
i \leq j \Rightarrow |x_i(t') - x_i(t)| = |x_k(t') - x_k(t)|.
\]

**Remark 1 (Independent Variables).** If \( M_{ik} = 0 \) for two variables \( i \) and \( k \) (i.e., all elementary vectors \( e \) verify \( e_i e_k = 0 \)), then \( M_{ki} = 0 \) and a change in one variable's parameter never affects the optimal value of the other. In that sense, the two variables are independent.

To further Remark 1, notice that elementary supports are the circuits of the column matroid associated with the matrix \( A \), see Whitney (1935), Rockafellar (1967) as well as Rockafellar (1984, Chapter 10). One can therefore argue that the dependence relation by which two variables \( i \) and \( j \) are dependent when \( M_{ij} \neq 0 \) is an equivalence relation, whose equivalence classes are of interest in view of Remark 1; they are the components of the corresponding matroid and can be obtained using the following procedure.

**Components.** Suppose that the \( n \) columns of \( A \) are (re)-indexed so that the first \( m \) columns form a basis (if need be, rows of \( A \) have been removed to make it full row-rank). For all \( j = m + \ldots, n \) call \( e_j \) the unique solution of

\[
Ax = b, \quad x_j = 1, \quad x_k = 0 \quad (k \geq m, k \neq j)
\]

\((e_j \) can be computed from, say, a simplex tableau). Then \( e_j \) is an elementary vector; see, e.g., Rockafellar (1984, p. 457). The components then coincide with the connected components of the (undirected, bipartite) graph with vertices \( \{1, \ldots, n\} \) and edge set \( \{(i,j) : j \geq m, i \in \sigma(e_j)\} \); see, e.g., Oxley (1992, Lemma 10.2.8).

This can be seen as an extension of the case where \( A \) is the incidence matrix of a network (Granot and Veinott, Jr. 1985, biconnected components), a singly-constrained network (Gautier and Granot 1994b), or a generalized network (Gautier and Granot 1996).

**Remark 2 (Value of Multipliers).** \( M_{ij} \) may take on any positive value; for a trivial example take \( A = [\alpha, 1, \ldots, 1] \). A trivial argument on minimal supports will show that the only elementary vector with nonzero first and second entries is \((1, -\alpha, 0, \ldots, 0)\), up to a scalar multiple, and therefore \( M_{12} = |\alpha| \).

Monotonicity of optimal solutions is another important property of certain Monotropic Problems, and the Monotonicity and Smoothing Theorems for Network flows can be shown to extend to Monotropic Problems with some modifications. For example, in pure network flows, two arcs are called complements (resp., substitutes) if every simple cycle containing both arcs orient them in the same (resp., opposite) direction. We extend this...
definition to monotropic programs by calling two variables \( i \) and \( j \) complements (resp., substitutes) if \( e_i, e_j \leq 0 \) (resp., \( \leq 0 \)) for every elementary vector \( e \). The following definitions are required:

- Let \( t_j \vee t_j' \) and \( t_j \wedge t_j' \) denote, respectively, the least upper bound and greatest lower bound of \( t_j \) and \( t_j' \) in \( T_j \). The cost \( c_j(\cdot, \cdot) \) is subadditive if
  
  \[
  c_j(\max(x_j, x_j'), t_j \vee t_j') + c_j(\min(x_j, x_j'), t_j \wedge t_j') \leq c_j(x_j, t_j) + c_j(x_j', t_j')
  \]

  for all \( x_j, x_j' \) and parameters \( t_j, t_j' \).

- If the second argument of \( c_j \) is real (recall that the \( j \)th projection of \( T \) could be a set of reals, a vector space, etc.) then \( c_j' \) is defined by \( c_j'(x_j, t_j) = c_j(t_j - x_j, t_j) \). If both \( c_j \) and \( c_j' \) are subadditive then \( c_j \) is doubly subadditive.

- Two parameters \( t \) and \( t' \) are monotonically step-connected if there exists a finite sequence \( t^0 = t, t^1, \ldots, t^Q = t' \) in \( T \) such that \( t^q \) and \( t^{q-1} \) differ in at most one component for each \( q \) and \( t^q_j \) is monotone in \( q \) for each \( j \in J \).

The proof of Theorem 2 is similar to that of Granot and Veinott, Jr. (1985, Theorem 10), and that of Theorem 3 follows by combining the proof of Theorem 13 in Gautier and Granot (1994b) and Theorem 2. In the statement of Theorem 2 the hypothesis of lower-semicontinuity (l.s.c.) of a real-valued convex function, say \( f \), on \( \mathbb{R}^n \) refers to the property that all its level sets \( \{x : f(x) < a\}, (a \in \mathbb{R}) \), are closed.

**Theorem 2 (Monotone Optimal Selections).** Suppose that \( c_j(\cdot, t_j) \) is convex and l.s.c. for each \( t_j \in T_j \), \( c_j \) is subadditive for \( j \in J \) and \( X(t) \) is both nonempty and bounded for all \( t \in T \). Then there exists an optimal selection \( x(t) = (x_k(t)) \) with the Ripple Property such that \( x_i(t) \) is nondecreasing (resp., nonincreasing) in \( t_j \) whenever \( i \) and \( j \) are complements (resp., substitutes).

**Theorem 3 (Smooth Optimal Selections).** Suppose that \( T \subseteq \mathbb{R}^n \), \( c(\cdot, t) \) is convex and l.s.c. for each \( t \in T \), \( c_j \) is doubly subadditive and \( X(t) \) is both nonempty and bounded for all \( t \in T \). Then there exists an optimal selection \( x(\cdot) \) with the Ripple Property such that \( x_i(t) \) and \( M_{ij}x_j - x_i(t) \) (resp., \( -M_{ij}x_j - x_i(t) \)) are nondecreasing (resp., nonincreasing) in \( t_j \) whenever \( i \) and \( j \) are complements (resp., substitutes). Moreover,

\[
\|x(t') - x(t)\|_\infty \leq \max_{j \neq j} M_{ij}\|t' - t\|_1,
\]

for all monotonically step-connected \( t \) and \( t' \) in \( T \).

**Remark 3 (Need for convexity).** Without the convexity assumption, one may construct examples in which the conclusions of Theorems 1, 2, and 3 do not hold; see Gautier, Granot, and Zheng (1995).

We end this section with a definition. Given an instance of (1), an optimal selection \( x(t) \) is well-behaved if it possesses the ripple, monotonicity and smoothing properties, provided the respective hypotheses of Theorems 1, 2, and 3 hold.

**4. Minimal optimal selections.** Theorems 1, 2, and 3 are existence theorems which by themselves provide no means of obtaining a well-behaved optimal selection. If the optimal selection is unique, as for example when \( c(\cdot, t) \) is strictly convex for all \( t \), then it clearly is well behaved. One could hope that only "exceptional" values of the parameter lead to multiple optimal solutions and forsake QSA for these values. On top of the theoretical vacuum left by such an approach, serious problems would arise with monotropic problems with a bias toward multiple solutions, see Chiang and Chu (1996).
In this section we present a systematic procedure to select well-behaved optimal solutions called minimal optimal selections.

Let \( v \) be some positive, separable, continuous, strictly convex and l.s.c. function of \( x \in \mathbb{R}^n \). Further assume that \( v \) is coercive over \( \mathbb{R}^n \), that is, that \( \lim_{k \to \infty} v(x^k) = \infty \) for every sequence \( \{x^k\} \) of elements of \( \mathbb{R}^n \) such that \( \|x^k\| \to \infty \) for some norm \( \|\cdot\| \), see, e.g., Bertsekas (1995, p. 539). While the choice of \( v \) is postponed until §4.4, the expression \( v(x) = \sum_j x_j^2 \) can be thought of as a likely candidate. The minimal optimal selection with respect to \( v \) is

\[
(2) \quad x(t) = \operatorname{argmin}\{v(x) : x \in X(t)\}.
\]

Notice that \( x(t) \) is well defined: the existence of a minimum follows Weierstrass' Theorem—see, e.g., Bertsekas (1995, p. 540)—for \( X(t) \) is closed, nonempty and \( v \) is coercive, and its uniqueness follows from the strict convexity of \( v \). The computation of \( x(t) \) is addressed in §4.3.

To motivate the remainder of this section, we now state an important theorem whose proof is given in §4.2.

**Theorem 4 (Qualitative Properties of Minimal Optimal Selections).** If for all \( j \) the function \( c_j(\cdot, t_j) \) is convex, l.s.c., and \( c_j(\cdot, \cdot) \) is subadditive, then \( x(t) \) is well behaved.

### 4.1. Unique solutions of perturbed problems.

Since the selection problem may only occur when \( c(\cdot, t) \) is not strictly convex, it is natural to examine perturbations of the original problem (1). To that end, replace the cost function therein by the perturbed cost \( c(x, t, \epsilon) = c(x, t) + \epsilon v(x), \) where \( \epsilon > 0 \). We claim that the perturbed problem

\[
(3) \quad \min \{c(x, t, \epsilon) : Ax = b\}
\]

has a unique optimal solution. Indeed, since \( c(\cdot, t) \) is bounded from below by \( c_{\min}(t) \) over \( \{x : Ax = b\} \) and \( v \) is coercive, then \( c(\cdot, t, \epsilon) \) is coercive over \( \{x : Ax = b\} \) as well. The existence of an optimal solution for (3) then follows from Weierstrass' Theorem and its uniqueness from the strict convexity of \( c(\cdot, t, \epsilon) \). Call this solution \( x(t, \epsilon) \), that is

\[
(4) \quad x(t, \epsilon) = \operatorname{argmin}\{c(x, t, \epsilon) : Ax = b\}.
\]

The following lemma addresses the limiting behaviour of \( x(\epsilon, t) \) and sets the stage for the proof of Theorem 4.

**Lemma 5.** \( \lim_{\epsilon \to 0^+} x(\epsilon, t) = x(t) \).

**Proof.** We start by showing that the set \( \mathcal{A}(t) = \{x(t, \epsilon) : \epsilon > 0\} \) is bounded. To that end we will show that \( \mathcal{A}(t) \subseteq \{x : v(x) \leq v(x(t))\} \), one of the level sets of \( v \), and thus a bounded set. By (4), the definition of \( c(x, t, \epsilon) \), and the fact that \( x(t) \in X(t) \) we have

\[
(5) \quad c(x(\epsilon, t), t) + \epsilon v(x(\epsilon, t)) \leq c(x(t), t) + \epsilon v(x(t))
\]

and

\[
(6) \quad c(x(t), t) \leq c(x(\epsilon, t), t),
\]

implying

\[
v(x(\epsilon, t)) \leq v(x(t)),
\]

establishing our first claim.
Now, by the Bolzano-Weierstrass Theorem, the bounded set $A(t)$ has at least one cluster point. We complete the proof of the theorem by showing that only one cluster point exists, and that it is equal to $x(t)$. To that end, let $y$ be such a cluster point, that is, $\lim_{k \to \infty} x(\epsilon_k, t) = y$ for some sequence of positive reals $\{\epsilon_k\}$ that goes to 0. First notice that by (4) and (5), $Ax(\epsilon_k, t) = b$ and $v(x(\epsilon_k, t)) \leq v(x(t))$ for all $k$. Since $v$ is continuous, the limiting process guarantees that $Ay = b$ and $v(y) \leq v(x(t))$. Lastly we show that $y \in X(t)$. Indeed, $Ay = b$ and

$$c(y, t) \leq \liminf_{k \to \infty} c(x(\epsilon_k, t), t)$$

for $c$ is l.s.c.

$$\leq \liminf_{k \to \infty} [c(x(\epsilon_k, t), t) + \epsilon_k v(x(\epsilon_k, t))] \quad \text{adding a positive term}$$

$$= \liminf_{k \to \infty} \min\{c(x(t), t) : Ax = b\} \quad \text{by the definition of } x(\epsilon_k, t)$$

$$\leq \liminf_{k \to \infty} c(x(t), t, \epsilon_k) \quad \text{for } x(t) \text{ is feasible for (3)}$$

$$= \liminf_{k \to \infty} [c(x(t), t) + \epsilon_k v(x(t)))]$$

$$= c(x(t), t) = c_{\min}(t) \quad \text{for } v(x(t)) < \infty.$$
A. GAUTIER, F. GRANOT, AND H. ZHENG

- λ)c_j(u_j(t_j), t_j), and suppose that for some λ this inequality is strict. From the definition of X_j(t_j) we may find three points x^I(t), x^u(t), and x^X(t) in X(t) whose jth components are, respectively, l_j(t_j), u_j(t_j) and λl_j(t_j) + (1 - λ)u_j(t_j). Since c(x, t) = Σ_j c_j(x_j, t_j) then by the convexity of all cost functions, we have c(x^I(t)) < λc(x^I(t)) + (1 - λ)c(x^u(t)), contradicting the fact that all three points are optimal. Now, let c_j^0(t_j) and c_j^1(t_j) be, respectively, the intercept and the slope of c_j(·, t_j) on (l_j(t_j), u_j(t_j)) if X_j(t_j) is not a singleton, and both be 0 otherwise. Then from the above c(·, t) coincides with the linear function c^*(x, t) = Σ_j c_j^0(t_j) + Σ_j c_j^1(t_j)x_j whenever x ∈ B(t), and the proof follows.

A direct consequence of Lemma 6 is that (2) is equivalent to

\[ x(t) = \text{argmin}\{v(x) : Ax = b, \ l_j(t) \leq x_j \leq u_j(t) \ \forall j\} \]

which apparently requires the knowledge of l_j(t) and u_j(t) for all j. This necessity can be avoided, though, as follows. Let x*(t) ∈ X(t) be any optimal solution. Assume without loss of generality that c_{min}(t) = 0 and c_j(x_j*(t), t_j) = 0 for all j. For any index j, check

- whether or not c_j(·, t_j) is linear to the left of x_j*(t), and if so let [l_j(t), x_j*(t)] be the largest interval to the left of x_j*(t) on which linearity holds and s_j^L(t) the corresponding slope (l_j(t) may be -∞), and
- whether or not c_j(·, t_j) is linear to the right of x_j*(t), and if so let [x_j*(t), u_j(t)] be the largest interval to the right of x_j*(t) on which linearity holds and s_j^R(t) the corresponding slope (u_j(t) may be +∞).

Note that it is reasonable to assume that such a test for linearity is possible, since the cost function is often given analytically on intervals. If, however, there is no analytical description of the costs, then line search ought to be used in order to determine the above intervals. Numerical errors become legitimate concerns, but not greater ones than in the solution of the original problem.

Given these intervals, partition the index set J into

\[ J_1 = \{ j \in J : \bar{l}_j(t) = x_j^*(t) = \bar{u}_j(t) \}, \quad J_2 = \{ j \in J : \bar{l}_j(t) < x_j^*(t) = \bar{u}_j(t) \}, \]

\[ J_3 = \{ j \in J : \bar{l}_j(t) = x_j^*(t) < \bar{u}_j(t) \}, \quad J_4 = \{ j \in J : \bar{l}_j(t) < x_j^*(t) < \bar{u}_j(t) \}. \]

It follows from Lemma 6 that

- j ∈ J_1 ⇒ (L1): X_j(t) = {x_j^*(t)} and c_j^1(t_j) = 0;
- j ∈ J_2 ⇒ (L1) or (L2): X_j(t) = [l_j(t), x_j^*(t)] and c_j^1(t_j) = s_j^L(t);
- j ∈ J_3 ⇒ (L1) or (L3): X_j(t) = [x_j^*(t), \bar{u}_j(t)] and c_j^1(t_j) = s_j^R(t);
- j ∈ J_4 ⇒ either (L1), (L2) or (L3) holds.

By definition, c_j(·, t_j) coincides on [\bar{l}_j(t), \bar{u}_j(t)] with a piecewise linear function with at most two pieces. Moreover, X(t) ⊆ \{ x : \bar{l}_j(t) ≤ x ≤ \bar{u}_j(t) \ \forall j \} by (6), thus (2) is equivalent to

(7) \[ x(t) = \text{argmin}\{v(x) : Ax = b, \ c(x, t) = 0, \ \bar{l}_j(t) ≤ x ≤ \bar{u}_j(t) \ \forall j\}. \]

Now, since c_j(·, t_j) is convex and c_j(x_j^*(t), t_j) = 0,

\[ c_j(x_j, t_j) = \max\{s_j^L(t)(x_j - x_j^*(t)), s_j^R(t)(x_j - x_j^*(t))\} \]

\[ = \min\{y_j \in \mathbb{R} : y_j \geq s_j^L(t)(x_j - x_j^*(t)), y_j \geq s_j^R(t)(x_j - x_j^*(t))\} \]

whenever j ∈ J_4 and \bar{l}_j(t) ≤ x_j ≤ \bar{u}_j(t). Therefore (7) is again equivalent to
The minimal optimal solution thus solves uniquely problem (8), whose constraints are linear. With an adequate choice of \( v \), problem (8) can be solved with standard tools.

### 4.4. Choice of \( v \)
As defined in (2), \( x(t) \) is the optimal solution of the original Monotropic Problem which minimizes \( v \). When \( X(t) \) is not a singleton for all \( t \), the choice of \( v \) may lead to various optimal selections. In practice, it may be dictated by practical concerns, including the computability of the minimal optimal selection in (8). Some options are discussed now.

For example, \( v(x) = x^T x \) yields the optimal solution which is closest to the origin. The optimal solution closest to some “ideal” or “target” point, say \( x^0 \), will be obtained by setting \( v(x) = (x - x^0)^T (x - x^0) \). More generally, \( v(x) = (x - x^0)^T Q (x - x^0) \) where \( Q \) is diagonal and positive definite leads to easily solvable quadratic instances of (8) to obtain \( x(t) \).

If one wishes to observe the effect of changes to an already existing optimal solution to (1), say \( x^0 \in X(t^0) \), of the parameter the function \( v \) should be such that \( x(t^0) = x^0 \). This is clearly achieved for any \( v \) that attains a global minimum at \( x^0 \), such as, for example, \( v(x) = (x - x^0)^T (x - x^0) \) or \( v(x) = (x - x^0)^T Q (x - x^0) \) (where \( Q \) is as above).

### 5. Computations and applications of monotropic programming
The statements of Theorems 1, 2, and 3 suggest that QSA for Monotropic Problems is based on the ability to determine elementary vectors, less dependent relations as well as complements, substitutes, and so on. The class of Monotropic Problems includes that of network flows with a single additional constraint for which deciding whether three given arcs \( i, j, k \) verify \( i <_k j \) can be done in linear time but (i) computing the coefficient \( M_{i k} \) in the case of \( i <_k j \), (ii) deciding whether the relation \( i <_k j \) holds, and (iii) deciding whether two arcs are complements, substitutes, or neither, are \( \mathcal{NP} \)-complete decision problems, see Gautier and Granot (1994b). This complexity result naturally carries on to the class of Monotropic Problems, indicating that there is no significant shortcut to the exploration of an exponential number of elementary vectors.

We illustrate in the remainder of this section how to take advantage of size, structure, or both to obtain QSA results for several nonnetwork Monotropic Problems.

#### 5.1. Small problems
Small \( \mathcal{NP} \)-complete problems can often be solved by complete enumeration. Here, enumerating all elementary vectors is realistic for small dimensions of \( A \). If the \( n \) columns of \( A \) are (re)-indexed so that the first \( m \) columns form a basis, then the unique solution of

\[
Ax = b, \quad x_{m+1} = 1, \quad x_{m+2} = \cdots = x_n = 0
\]

is an elementary vector—see Rockafellar (1984, p. 457). In this fashion the set of all elementary vectors may be obtained by exploring all bases of \( N(A) \). This set can then be
used to test whether two variables are substitutes, complements or neither, and to obtain ripple multipliers.

5.2. Unimodular problems. An integral matrix $A$ is unimodular if it is integral and every full rank square submatrix has a determinant of 0, 1 or $-1$ (see, e.g., Truemper 1978, and Schrijver 1986), a property that can be tested for in polynomial time (Schrijver 1986, p. 303). The class of unimodular matrices strictly includes the class of network incidence matrices as well as its superset of totally unimodular matrices. The following theorem shows the importance of unimodularity in QSA.

**Theorem 7.** If $A$ is unimodular, then the elementary vectors all have entries 0, ±1 (up to a scalar multiple) and $M_{ik} = 1$ for all pairs of dependent variables.

**Proof.** Suppose without loss of generality that $i = 1$ and let $e$ be an elementary vector, scaled so that $e_1 = -1$. Since $Ae = 0$, the columns $\{A_i : i \in \sigma(e)\}$ are linearly dependent ($A_i$ denotes the $i$th column of $A$). Now, $Ae = 0$ can be rewritten as $A_1 = \Sigma e_i A_i$ where $P = \sigma(e) \setminus \{1\}$. Moreover, the minimal character of $\sigma(e)$ indicates that $\{A_i : i \in P\}$ is independent and can therefore be expanded into a basis of $A$, say $B$. The expression $A_1 = \Sigma e_i A_i$ then translates into $A_i = Bw$, where $w$ is a vector with $w_i = e_i$ if $i \in P$, and $w_i = 0$ otherwise. Since $A$ is unimodular, both $A_1$ and $B^{-1}$ are 0/±1-valued, and since $w = B^{-1}A_1$, so are $w$ and $e$. The second statement follows then from the definition of $M_{ik}$. □

5.3. Multicommodity transportation flows. A new class of problems that can be approached with the present method is that of minimum cost multicommodity transportation problem, where the underlying graph is bipartite with $q$ source nodes and $r$ sink nodes is shared by $K$ commodities (an arc can handle several different kinds of flows simultaneously, see, e.g., Ahuja, Magnanti, and Orlin (1993) or Evans, Jarvis, and Duke (1977). Multicommodity flow transportation problems do not possess unimodular constraint matrices unless either $q \leq 2$ or $r \leq 2$ as shown by Evans, Jarvis, and Duke (1977).

Previous work on QSA was able to cover a number of combinations of $[q, r, K]$; these are given in Table 1. For problems with different combinations, one can explore the set of elementary vectors one at a time, as illustrated in the example below.

First, let $x(i, j, l)$ denote the flow of commodity $l$ from source node $i$ to sink node $j$, the commodity $l = 0$ representing slack variables on arc capacities. The constraints of the Monotropic Problem are the offer at source nodes ($\Sigma_j x(i, j, l) = b_i^l$), demand at sink nodes ($\Sigma_i x(i, j, l) = b_j^l$) and arc-capacities ($\Sigma_l x(i, j, l) = u_{ij}$). Thus an elementary vector $e$ of the null space should verify

$$\sum_j e(i, j, l) = 0 \quad \forall i, l, \quad \sum_i e(i, j, l) = 0 \quad \forall j, l,$$

and $$\sum_i e(i, j, l) = 0 \quad \forall i, j.$$

**TABLE 1.** Combinations of $[q, r, K]$

<table>
<thead>
<tr>
<th>Combination</th>
<th>Note</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[q, r, 1]$</td>
<td>one commodity</td>
<td>Granot and Veinott, Jr. (1985)</td>
</tr>
<tr>
<td>$[1, r, K]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$[q, 1, K]$</td>
<td>trivial solution</td>
<td></td>
</tr>
<tr>
<td>$[q, 2, K]$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Consider now the \([3, 3, K]\) combinations. When \(K = 2\), a simple argument on minimal supports would show that all elementary vectors are obtained from the prototypes \(e^1, \ldots, e^4\) in Figure 1 by permutations of sources, sinks and/or commodities. (In Figure 1 commodities are represented on different copies of the graph. For example, \(e^1(1, 1, 1) = 1\), \(e^1(1, 2, 1) = -1\), \(e^1(i, j, 0) = 0\), \(e^5(2, 2, 3) = -2\), etc.). When \(K \geq 3\), these elementary vectors still exist but new ones appear, such as \(e^5\). Inspecting these elementary vectors yields the patterns of variables that are substitutes (S), complements (C) or neither (−) given in Table 2. (Note that the negative results extend easily to all cases where \(q, r \geq 4\).) For example, \(e^k(2, 2, 1) \cdot e^k(2, 2, 2) = -1 < 0\) for \(k = 1, 2, 3\) and =0 for \(k = 4\), therefore commodities 1 and 2 on arc \((2, 2)\) are substitutes when \(K = 2\) (and so are all pairs of different commodities on a same arc). However, when \(K = 3\), the elementary vector \(e^5\) exists and \(e^k(2, 2, 1) \cdot e^k(2, 2, 2) = 1 > 0\), these pairs of variables are no longer substitutes or complements. Similar arguments are used to generate Table 2.

5.4. Transpose of node-arc incidence matrices. A number of problems such as the Minimum Cut, Reorder Intervals in multistage production-distribution systems and Repair Kit problems can be formulated as in (1) where \(A^1\) is the node-arc incidence matrix of

| TABLE 2. Substitutes and Complements for \([3, 3, K]\) Combinations |
|--------------------------------|-----------------|--------------------------------|
| \(K = 2\)                     | \(K \geq 3\)    |
| \begin{tabular}{l|l|l|l|l|l} \hline
<table>
<thead>
<tr>
<th>Same Commodity</th>
<th>Different Commodity</th>
<th>Same Commodity</th>
<th>Different Commodity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Same Arc</td>
<td>C</td>
<td>C</td>
<td>S</td>
</tr>
<tr>
<td>Arcs That Share One Node</td>
<td>S</td>
<td>S</td>
<td>C</td>
</tr>
<tr>
<td>Arcs That Share No Node</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>\end{tabular}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>----------------</td>
<td>-------------------</td>
<td>----------------</td>
<td>-------------------</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
some directed graph, say \( G \) (these are not network flow problems). An arc \((i, j)\) of \( G \) 
corresponds to a column of \( A^t \) which gives rise to a constraint of the form \( x_i - x_j = 0 \) in 
the system \( Ax = 0 \) that defines the null space. Thus, there exists exactly one elementary 
vector per connected component of \( G \) and we may conclude that two variables are comple-
ments with ripple multipliers are 1 if they correspond to connected nodes, and inde-
pendent otherwise.

5.5. Cyclic matrices. Define the cyclic matrix \( A_{r,n} \) \((1 < r < n)\) as the square \( n \times n \) 
matrix whose first row is composed of \( r \) ones followed by \( n - r \) zeroes, and subsequent 
rows are obtained by rotating to the right the entries of the preceding row in a wraparound 
fashion (i.e., considering that the first column is next to the last one). For example,

\[
A_{2,3} = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}, \quad A_{2,4} = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1
\end{bmatrix}, \quad \text{and} \quad A_{3,4} = \begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1
\end{bmatrix}.
\]

Clearly, cyclic matrices are not network incidence matrices, nor are they all unimodular, 
for example \( \det(A_{2,3}) = 2 \). They appear in cyclic staffing problems involving the optimal 
scheduling of resources to meet demands where both resource availability and demand 
profile are cyclic. For surveys of cyclic staffing problems, see Baker (1976) and Bartholdi, 
III, Orlin, and Ratliff (1980).

To find all elementary vectors, we must first make sure that the matrix \( A \) is singular 
\((\det(A) = 0)\), otherwise the null space is reduced to \( N(A) = \{0\} \). A useful property of 
circular matrices is that

\[
\det(A_{r,m+r}) = \det(A_{r,m}) \quad \text{when} \quad m > r.
\]

(10) can be shown by defining the \((m + r) \times (m + r)\) matrix

\[
P = \begin{bmatrix}
I_r & 0 \\
X & I_m
\end{bmatrix}
\]

where \( X = \begin{bmatrix}
0_{(m-r+1)\times 1} & 0_{(m-r+1)\times (r-1)} \\
-I_{r-1} & I_{r-1}
\end{bmatrix}\),

and noticing that \( \det(PA_{r,m+r}) = \det(A_{r,m}) \) while \( \det(P) = 1 \).

If the number of 1's in each column is \( r = 2 \), then by (10) \( \det(A_{2,n}) = \det(A_{2,3}) = 2 \) 
when \( n \) is odd, implying that (1) has a unique solution. When \( n \) is even, \( \det(A_{2,n}) = 0 \) 
and solving \( A_{2,n}x = 0 \) clearly leads to a unique elementary vector \((1, -1, \ldots, 1, -1)\). Thus, all ripple multipliers are 1, and two variables \( i \) and \( j \) are comple-
ments if \((i - j)\) is even and substitutes otherwise.

When \( r = 3 \), then \( \det(A_{3,n}) = 0 \) only if \( n \) is a multiple of 3 (this by (10) and the fact 
that \( \det(A_{3,6}) = 0 \) and \( \det(A_{3,4}) = \det(A_{3,5}) = 3 \)). In this case one finds three elementary 
 vectors \((1, -1, 0, \ldots, 1, -1, 0), (1, 0, -1, \ldots, 1, 0, -1) \) and \((0, 1, -1, \ldots, 0, 1, -1)\). 
Two variables \( i \) and \( j \) are therefore complements if \((i - j)\) is a multiple of 3, substitutes 
otherwise, and all ripple multipliers are 1. The number of elementary vectors, and thus 
the complexity of the task of obtaining QSA elements increases with \( r \).

Acknowledgments. The authors would like to express their appreciation to the referees 
for their constructive suggestions and patient reading of earlier versions of this paper, 
leading to significant improvements in the exposition of this paper. They are also indebted 
to Bob Bixby, Paul Seymour, and James Oxley for valuable assistance with matroid issues.
This work was supported in part by NSERC grants OGP0121627, OGP5-8-3998, and FCAR 98-NC-0939.

References


