Efficient Frontier of Utility and Conditional VaR

Harry Zheng*
Imperial College

Abstract. We study the efficient frontier problem of maximizing the expected utility of terminal wealth and minimizing the conditional VaR of the utility loss. We establish the existence of the optimal solution with the dual analysis and find the optimal value with the sequential penalty function and viscosity solution method.

Key words. utility maximization, CVaR minimization, efficient frontier, existence, state constraint, sequential penalty function.

AMS subject classifications. 90C46, 49L20

1 Introduction

There has been extensive research in continuous-time utility maximization, see Karatzas and Shreve (1998), Korn (1997) for an excellent introduction to the topic. Two standard methods for utility maximization are the martingale method and the stochastic control method. The first one is used to show the existence of the optimal solution with the convex duality analysis and the martingale representation theorem, see Karatzas, et al. (1987, 1991), Kramkov and Schachermayer (1999, 2003), and many recent papers on this subject. The second one is used to find the optimal value and the optimal solution with the dynamic programming (DP) principle and the HJB equation, see Touzi (2002), Fleming and Soner (2006). The utility function is usually assumed to be strictly increasing, strictly concave, continuously differentiable, plus some other conditions.

*Department of Mathematics, Imperial College, London SW7 2BZ, UK. h.zheng@imperial.ac.uk. This research was partially supported with the grant 1331 of the European Science Foundation and the grant 4544 of the London Mathematical Society.

In practice one often needs to find the optimal tradeoff between return and risk. This is the fundamental idea of the Markowitz's mean variance efficient frontier theory which has been extended to continuous time portfolio selection problems with bankruptcy prohibition and other type of constraints, see, for example, Bielecki et al. (2005). When one considers other types of risk measures, such as shortfall risk, the resulting objective function is often nondifferentiable and one has to use subgradients instead of gradients, and has to add some other conditions to ensure standard calculus rules, such as the sum rule, still hold true. To solve these nonsmooth utility maximization problems one has to either modify the problem such that results for smooth utility maximization can be applied or attack the nonsmooth utility functions directly. The first approach is used in Deelstra, Pham, and Touzi (2001). These authors use the quadratic inf-convolution method to study the nonsmooth utility maximization problem in an incomplete unconstrained market. They construct a sequence of approximating smooth utility maximization problems and then take the limit to drive the optimal solution. The second approach is used in Cvitanic (2000) for a hedging problem in an incomplete constrained market. The author directly applies the subdifferential calculus rule to derive the optimal solution which is the subgradient of a conjugate function at the optimal dual solution.

In this paper we extend the idea of efficient frontier from minimizing the variance to minimizing the Conditional Value-at-Risk. CVaR measures the average tail loss of an investment and has desirable theoretical properties such as coherency and provides realistic loss estimate for highly skewed, fat-tailed loss distribution. We model the efficient frontier problem of utility maximization and CVaR minimization in an incomplete constrained market. The formulation leads naturally to a two stage optimization problem: one parametric nonsmooth utility maximization and one scalar concave maximization. We follow Cvitanic (2000) with major changes to show the existence of the optimal solution with convex duality analysis and find the optimal value with the sequential penalty function and viscosity solution technique.

The paper is organized as follows: Section 2 presents a motivating example and formulates the efficient frontier model and the two-stage optimization problem. Section 3 proves the existence of the optimal solutions for the primal and dual problems and characterizes their duality relation. Section 4 applies the sequential penalty function method to solve the efficient fron-

tier problem and illustrates numerically that the inclusion of CVaR in the objective function reduces the potential loss in case a rare event happens. Section 5 reviews some standard results of stochastic control theory and fixes an error in the proof of the DP principle in Fleming and Soner (2006).

2 Model

Assume a financial market consists of one bank account and one stock. Assume the bank account is equal to 1 (we can always achieve this by simply taking it as a numeraire). The discounted stock price process S is modelled by

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t), \quad S(0) = s > 0$$

where α is a nonnegative constant representing the excess return of stock, σ a constant representing the stock volatility, and $W(t)$ the standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$, endowed with a filtration $\{\mathcal{F}_t\}$ which is the P-augmentation of the filtration $\mathcal{F}_t^W = \sigma(W(s) : 0 \leq s \leq t)$ generated by W for $0 \leq t \leq T$. The discounted wealth process X satisfies the stochastic differential equation

$$dX(t) = \alpha\pi(t)X(t)dt + \sigma\pi(t)X(t)dW(t) - d\kappa(t), \quad X(0) = x \tag{1}$$

where $\pi(t)$ is the percentage of the wealth invested in the stock at time t and $\kappa(t)$ is the cumulative consumption (or cash surplus) up to time t. Denote by $X^{x,\pi,\kappa}$ the strong solution to (1).

Denote by $\mathcal{A}(x)$ the set of pairs (π, κ) that satisfy the following conditions: π is progressively measurable and satisfies $\pi(t) \in K$ a.s. for $t \in [0, T]$, where K is a closed convex set that contains 0, κ is nonnegative, nondecreasing, progressively measurable with RCLL paths and with $\kappa(0) = 0$ and $\kappa(T) < \infty$, $X^{x,\pi,\kappa}(t) \geq 0$ a.s. for $t \in [0, T]$. A process pair (π, κ) is called admissible if $(\pi, \kappa) \in \mathcal{A}(x)$. Note that $0 \in K$ implies that doing nothing ($(\pi, \kappa) = (0, 0)$) is an admissible trading strategy.

To motivate the model formulation we first discuss the standard terminal utility maximization problem:

$$\max_{(\pi,\kappa)\in\mathcal{A}(x)} E[g(X^{x,\pi,\kappa}(T))] \text{ subject to (1)} \tag{2}$$

where E is the expectation operator and g is a utility function which is strictly increasing, strictly concave, and continuously differentiable.

We outline a few key steps in deriving the optimal solution for (2). Denote by $V(t,x)$ the value function of (2) for $0 \le t \le T$ and $x \ge 0$. Then V satisfies the HJB equation

$$V_t + \max_{\pi \in K, c \ge 0} \left[(\alpha \pi x - c) V_x + \frac{1}{2} \sigma^2 \pi^2 x^2 V_{xx} \right] = 0$$

with the terminal condition $V(T,x) = g(x)$. Here we have assumed $d\kappa(t) = c(t)dt$ with $c(t) \ge 0$ for all t. The maximum in the HJB equation is achieved at $c^* = 0$ and $\pi^* = -\frac{\alpha}{\sigma^2} \frac{V_x}{x V_{xx}}$, provided $V_x > 0$ and $\pi^* \in K$. Substituting c^* and π^* into the HJB equation leads to a nonlinear parabolic equation

$$V_t - \frac{\alpha^2}{2\sigma^2} \frac{V_x^2}{V_{xx}} = 0.$$

In general, one can only find the numerical solution to the PDE above. However, in case of g being a power (or log) utility function $g(x) = \frac{1}{\gamma} x^\gamma$ with $0 < \gamma < 1$ a constant, Merton (1971) derives the closed-form solution $V(t,x) = g(x) e^{\tilde{\alpha}(T-t)}$, where $\tilde{\alpha} = \frac{\alpha^2}{2\sigma^2} \frac{\gamma}{1-\gamma} > 0$. The verification theorem confirms that the optimal solution for (2) is $\kappa^*(t) = 0$ and $\pi^*(t) = \frac{\alpha}{\sigma^2(1-\gamma)}$, provided $\pi^*(t) \in K$, for $0 \le t \le T$. The optimal trading strategy is to consume nothing and to invest constant proportion of wealth in the stock. The terminal utility of the optimal wealth process is given by

$$g(X^{x,\pi^*,\kappa^*}(T)) = c e^{(\tilde{\alpha} - \frac{1}{2}\tilde{\sigma}^2)T + \tilde{\sigma} W(T)}$$

where $c = g(x)$ and $\tilde{\sigma} = \frac{\alpha}{\sigma} \frac{\gamma}{(1-\gamma)}$. The optimal expected terminal wealth is given by

$$E[g(X^{x,\pi^*,\kappa^*}(T))] = c e^{\tilde{\alpha} T}.$$

If one invests all wealth in the bank account, i.e., choose $\pi(t) = 0$ and $\kappa(t) = 0$ for $0 \le t \le T$, then $X^{x,0,0}(t) = x$ and one has the certain utility c at time T. The strategy (π^*, κ^*) performs better on average since $E[g(X^{x,\pi^*,\kappa^*}(T))] > c$, but the strategy $(\pi, \kappa) = (0,0)$ contain no risk. To measure the risk associated with any admissible strategy (π, κ) we define a random variable Z by

$$Z = c - g(X^{x,\pi,\kappa}(T)).$$

Z represents the terminal utility loss (or gain) if Z is positive (or negative). We are interested in the risk of utility loss. Two common risk measures are

VaR and CVaR. Given a number $\beta \in (0,1)$ (close to 1) the VaR of Z at level β is defined by

$$\mathrm{VaR}_\beta = \inf\{z : P(Z \le z) \ge \beta\}$$

and the CVaR of Z at level β is defined by

$$\mathrm{CVaR}_\beta = E[Z | Z \ge \mathrm{VaR}_\beta].$$

It is often difficult to compute directly VaR and CVaR from their definitions as VaR requires to solve a nonlinear equation and CVaR to integrate over the tail loss distribution. Rockafellar and Uryasev (2000) suggest a viable method to compute VaR and CVaR by solving a convex minimization problem in which the minimum value is CVaR and the left end point of the minimum solution set gives VaR. Specifically, If Z is continuous then

$$\mathrm{CVaR}_\beta = \min_y [y + \delta E(Z-y)^+] \tag{3}$$

where $\delta = (1-\beta)^{-1}$. If y^* is the left endpoint of the minimum solution set, then $\mathrm{VaR}_\beta = y^*$.

We can now compute the VaR_β and CVaR_β associated with the strategies π^* and κ^*. Define $h(y) = y + \delta E(Z-y)^+$ and $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ and $\Phi(z) = \int_{-\infty}^z \phi(s)ds$. Then

$$\begin{aligned}
h(y) &= y + \delta E(c - g(X^{x,\pi^*,\kappa^*}(T)) - y)^+ \\
&= y + \delta E[(c - g(X^{x,\pi^*,\kappa^*}(T)) - y) 1_{c - g(X^{x,\pi^*,\kappa^*}(T)) - y \ge 0}] \\
&= y + \delta \int_{-\infty}^A (c - y - ce^{(\tilde{\alpha} - \frac{1}{2}\tilde{\sigma}^2)T + \tilde{\sigma}\sqrt{T}z})\phi(z)dz \\
&= y + \delta[(c-y)\Phi(A) - ce^{\tilde{\alpha}T}\Phi(A - \tilde{\sigma}\sqrt{T})]
\end{aligned}$$

where $A = \frac{1}{\tilde{\sigma}\sqrt{T}}[\ln(\frac{c-y}{c}) - (\tilde{\alpha} - \frac{1}{2}\tilde{\sigma}^2)T]$.

(3) implies that the minimum solution y^* is the stationary point of h, i.e., $0 = h'(y^*) = 1 - \delta\Phi(A^*)$, which results in $A^* = \Phi^{-1}(1/\delta) = \Phi^{-1}(1-\beta)$. We therefore have $\mathrm{VaR}_\beta = y^* = c - ce^{(\tilde{\alpha} - \frac{1}{2}\tilde{\sigma}^2)T + \tilde{\sigma}\sqrt{T}\Phi^{-1}(1-\beta)}$ and $\mathrm{CVaR}_\beta = h(y^*)$. As β tends to 1 both VaR_β and CVaR_β tend to c. (The utility loss Z here is bounded above by c since the utility function g is nonnegative. If g takes negative values then Z can be unbounded. For example, if $g(x) = \ln x$

then a similar discussion shows that as β tends to 1 both VaR_β and CVaR_β tend to ∞.)

This example illustrates the drawback of simply maximizing the utility function without considering the potential utility loss. A better objective function should involve both the utility maximization and the utility loss minimization. We may reconcile these two conflicting objectives by solving the following optimization problem

$$\sup_{(\pi,\kappa)\in\mathcal{A}(x)} (E[g(X(T))] - \lambda \mathrm{CVaR}_\beta) \text{ subject to (1)}, \tag{4}$$

where λ is a nonnegative parameter. $\lambda = 0$ corresponds to the maximization of the expected terminal utility while $\lambda \to \infty$ corresponds to the minimization of the CVaR of the utility loss. By letting λ change in $[0, \infty)$ we can derive the utility-CVaR efficient frontier in the same spirit as Markovitz's mean-variance efficient frontier.

Substituting (3) into (4) and exchanging the order of maximization we may determine the efficient frontier of the utility and the CVaR by first solving a parametric utility maximization problem

$$u(x, y) = \sup_{(\pi,\kappa)\in\mathcal{A}(x)} E[U(X(T), y)] \text{ subject to (1)}, \tag{5}$$

where

$$U(x, y) = g(x) - \lambda\delta(c - g(x) - y)^+ - \lambda y, \tag{6}$$

and then solving a scalar maximization problem

$$u(x) = \sup_y u(x, y), \tag{7}$$

provided that the joint optimization problem

$$\sup_{(\pi,\kappa)\in\mathcal{A}(x),y} E[U(X(T), y)] \text{ subject to (1)}$$

has a finite optimal value.

The existence of the optimal solution for (7) is straightforward because U is concave, $U(x, \cdot)$ is Lipschitz continuous, uniformly in x, and $U(x, y) \to -\infty$ as $|y| \to \infty$, which implies that $u(x, \cdot)$ is concave, Lipschitz continuous, and reaches the maximum in some bounded interval. We can find the optimal solution with the standard line search method, see Fletcher (1987).

We therefore focus on the existence of the optimal solution for (5) for fixed y in Section 3 and the method of finding the optimal solution in Section 4. To simplify the notation we suppress y when y is fixed and just write $U(x)$ instead of $U(x, y)$.

3 Existence of Optimal Solutions

Since there are portfolio process constraints we follow the idea of Cvitanic (2000) by first establishing the existence of the optimal solution for the dual problem and then showing the existence of the optimal solution for the original problem. We cannot apply the sum rule in L^1 as the utility function in (6) is only defined on the half real line and the interior point condition for the sum rule in L^1 is not satisfied, see Aubin and Ekeland (1984). Fortunately, utility function defined by (6) is strictly concave, which implies the conjugate function is continuously differentiable. We can then use convex analysis and monotone convergence theorem to construct the optimal terminal wealth.

The support function of the set $-K$ is defined by $\delta(v) := \sup_{\pi \in K}\{-\pi v\}$ with the effective domain $\tilde{K} := \{v : \delta(v) < \infty\}$. Note $\delta(v)$ is nonnegative since $0 \in K$. Denote by $\mathcal{D}$ the set of all bounded progressively measurable processes $\nu(\cdot)$ taking values in $\tilde{K}$ a.e. on $\Omega \times [0, T]$. The risk-premium process $\theta_\nu(\cdot)$ and the stochastic discount factor process $H_\nu(\cdot)$ are defined by $\theta_\nu(t) := (\nu(t) + \alpha)/\sigma$ and

$$H_\nu(t) := \exp\left(-\int_0^t \delta(\nu(s))ds - \int_0^t \theta_\nu(s)dW(s) - \frac{1}{2}\int_0^t \theta_\nu^2(s)ds\right)$$

for $0 \le t \le T$. Ito's rule implies

$$\begin{aligned} d(H_\nu(t)X(t)) &= -H_\nu(t)d\kappa(t) - (\delta(\nu(t)) + \pi(t)\nu(t))H_\nu(t)X(t)dt \\ &\quad + (\pi(t)\sigma(t) - \theta_\nu(t))H_\nu(t)X(t)dW(t) \end{aligned}$$

for all $\nu \in \mathcal{D}$. $H_\nu(\cdot)X(\cdot)$ is a nonnegative P-local supermartingale and hence a P supermartingale. We have

$$E[H_\nu(T)X(T)] \le x$$

for all admissible control process pairs $(\pi, \kappa) \in \mathcal{A}(x)$ and $\nu \in \mathcal{D}$. We assume that the utility function g is strictly increasing, strictly concave, continuously

differentiable, and satisfies

$$g(0) > -\infty,\ g(\infty) = \infty,\ g'(0) = \infty,\ g'(\infty) = 0, \text{ and} \tag{8}$$
$$g'(\beta y) \leq \gamma g'(y) \text{ for some } \beta \in (0,1),\ \gamma > 1 \text{ and for all } y > 0.$$

Note that $g(0) > -\infty$ excludes the log utility function. Let I be the inverse function of g'. Then I is strictly decreasing, $I(0) = \infty$, $I(\infty) = 0$, and $I(\beta y) \leq \gamma I(y)$ for all $y > 0$. Define $G : R^+ \to R$ by

$$G(w) = g(w) - \lambda y - \lambda\delta(c - g(w) - y)^+$$

where $c = g(x), \lambda > 0, \delta = 1/(1-\beta) > 1$ are constants. Then G is strictly increasing and strictly concave. Define the conjugate function of G by

$$\tilde{G}(v) = \sup_{w \geq 0}[G(w) - wv].$$

We have the following result.

Lemma 1 *$\tilde{G}$ is convex, decreasing, continuously differentiable, bounded from below, $\tilde{G}(0) = \infty$, $\tilde{G}(v) = G(h(v)) - vh(v)$, and $\tilde{G}'(v) = -h(v)$, where*

$$h(v) = \begin{cases} I(v) & 0 \leq v < g'(w^*) \\ w^* & g'(w^*) \leq v \leq (1+\lambda\delta)g'(w^*) \\ I(\frac{v}{1+\lambda\delta}) & v > (1+\lambda\delta)g'(w^*) \end{cases} \tag{9}$$

and w^* satisfies $g(w^*) = c - y$. Furthermore, $h(\beta v) \leq \gamma h(v)$ for $v > 0$.

Proof. It is obvious that $\tilde{G}$ is convex and decreasing. $\tilde{G}$ is also smooth since G is strictly concave (Rockafellar (1970), Theorem 26.3). Divide the interval $[0, \infty)$ into $[0, w^*]$ and (w^*, ∞).

In the first interval define $A(v) := \sup_{0 \leq w \leq w^*}[g(w) - \lambda y - \lambda\delta(c - g(w) - y) - wv]$. The stationary point w_1 is determined by solving the equation $g'(w) + \lambda\delta g'(w) - v = 0$, i.e., $w_1 = I(\frac{v}{1+\lambda\delta})$. The optimal solution to $A(v)$ is either w_1 if $0 \leq w_1 \leq w^*$ or w^* if $w_1 > w^*$.

In the second interval define $B(v) := \sup_{w \geq w^*}[g(w) - \lambda y - wv]$. The stationary point w_2 is determined by solving the equation $g'(w) - v = 0$, i.e., $w_2 = I(v)$. The optimal solution to $B(v)$ is either w_2 if $w_2 \geq w^*$ or w^* if $w_2 < w^*$.

Combining $A(v)$ and $B(v)$ together, we see that if $v < g'(w^*)$ then $w_2 = I(v) > w^*$ and the optimal solution to $\tilde{G}(v)$ is $I(v)$; if $v > (1+\lambda\delta)g'(w^*)$

then $w_1 = I(\frac{v}{1+\lambda\delta}) < w^*$ and the optimal solution to $\tilde{G}(v)$ is $I(\frac{v}{1+\lambda\delta})$; if $g'(w^*) \leq v \leq (1+\lambda\delta)g'(w^*)$ then $w_2 \leq w^* \leq w_1$ and the optimal solution to $\tilde{G}(v)$ is w^*.

It is easy to check that $\tilde{G}'(v) = -h(v)$, and $\tilde{G}(v) \geq \min\{g(0)-\lambda y, g(0)+\lambda\delta(g(0)-c+\beta y)\} > -\infty$ for all $v > 0$ by using the fact that $g(0) \leq g(I(v)) - vI(v)$ for all $v > 0$ and $\tilde{G}(0) = \sup_{w\geq 0} G(w) = \infty$. It is also straightforward, although a little tedious as there are six different cases, to check that $h(\beta v) \leq \gamma h(v)$ for $v > 0$ by the definition of h and the fact $I(\beta y) \leq \gamma I(y)$. $\square$

Define

$$\mathcal{H} = \{H \in L^1(\Omega, \mathcal{F}, P) : H \geq 0, a.s.,\ E[HX^{x,\pi,\kappa}(T)] \leq x, \forall(\pi,\kappa) \in \mathcal{A}(x)\}.$$

It is easy to see that $E(H) \leq 1$ for $H \in \mathcal{H}$ due to $(0,0) \in \mathcal{A}(x)$ for any $x > 0$ and $X^{x,0,0}(T) = x$. We have the following result.

Lemma 2 *Assume that*

$$\tilde{v}(z) := \inf_{H\in\mathcal{H}} E[\tilde{G}(zH)] < \infty \quad \textit{for } z > 0. \tag{10}$$

Then $\tilde{v}(z)$ is a convex, lower semi-continuous, proper function, and is bounded from below for $z > 0$, and $\tilde{v}(0) = \infty$.

Proof. To prove $\tilde{v}$ is convex we need to show that for $\alpha, \beta > 0$ satisfying $\alpha + \beta = 1$ and $z_1, z_2 \geq 0$ we have $\tilde{v}(\alpha z_1 + \beta z_2) \leq \alpha\tilde{v}(z_1) + \beta\tilde{v}(z_2)$. Since $\tilde{G}(0) = \infty$ (Lemma 1) we may assume $z_1, z_2 > 0$ (otherwise there is nothing to prove). For any fixed $\epsilon > 0$ there exist $H_i \in \mathcal{H}$ such that $E[\tilde{G}(z_i H_i)] < \tilde{v}(z_i) + \epsilon$. Define $H = (\alpha z_1 H_1 + \beta z_2 H_2)/(\alpha z_1 + \beta z_2)$. Then $H \in \mathcal{H}$ and

$$\begin{aligned} E[\tilde{G}((\alpha z_1 + \beta z_2)H)] &= E[\tilde{G}(\alpha z_1 H_1 + \beta z_2 H_2)] \\ &\leq \alpha E[\tilde{G}(z_1 H_1)] + \beta E[\tilde{G}(z_2 H_2)] \\ &< \alpha\tilde{v}(z_1) + \beta\tilde{v}(z_2) + \epsilon \end{aligned}$$

which shows $\tilde{v}$ is convex.

To prove $\tilde{v}$ is lower semi-continuous, we need to show that any sequence $z_n \to z$ implies $\tilde{v}(z) \leq \liminf_{n\to\infty} \tilde{v}(z_n)$. We may assume $\liminf_{n\to\infty} \tilde{v}(z_n) < \infty$ (otherwise there is nothing to prove) and $\bar{v} := \lim_{n\to\infty} \tilde{v}(z_n)$ exists and is finite (relabelling a subsequence if necessary). For any $\epsilon_n \downarrow 0$ there exists

a sequence $H_n \in \mathcal{H}$ such that $\tilde{v}(z_n) < E[\tilde{G}(z_n H_n)] \leq \tilde{v}(z_n) + \epsilon_n$. Therefore, $\lim_{n\to\infty} E[\tilde{G}(z_n H_n)] = \bar{v}$. The Komlos theorem implies that there exist $\hat{H} \in L^1(\Omega, \mathcal{F}, P)$ and a subsequence of $\{H_n\}$ such that (after relabelling)

$$G_n := \frac{1}{n}\sum_{i=1}^{n} H_i \to \hat{H},\ a.s.$$

We have $G_n \in \mathcal{H}$ and Fatou's lemma implies that $\hat{H} \in \mathcal{H}$.

Since $H_n \geq 0$ a.s. for all n the series $\sum_{n=1}^{\infty} H_n$ is either finite or infinite. If it is finite on a set S of positive measure, then we must have $H_n \to 0$ as $n \to \infty$ on S, which implies $\tilde{G}(z_n H_n) \to \infty$ as $n \to \infty$ on S by Lemma 1. Since $\tilde{G}$ is bounded from below we conclude that $\bar{v} = \lim_{n\to\infty} E[\tilde{G}(z_n H_n)] = \infty$, a contradiction. Therefore, $\sum_{n=1}^{\infty} H_n = \infty$ almost surely. Define $\alpha_i^n = H_i/(\sum_{i=1}^{n} H_i)$, $i = 1, \ldots, n$. We have $0 \leq \alpha_i^n \leq 1$, $\sum_{i=1}^{n} \alpha_i^n = 1$, and $\alpha_i^n \to 0$ a.s. as $n \to \infty$ for any fixed i. Fix ω outside a P-null set and let $n \to \infty$ we have

$$\frac{1}{n}\sum_{i=1}^{n} z_i H_i = G_n \sum_{i=1}^{n} \alpha_i^n z_i \to z\hat{H}.$$

Finally, Fatou's lemma and the convexity of function $\tilde{G}$ imply that

$$\begin{aligned}
\tilde{v}(z) &\leq E[\tilde{G}(z\hat{H})] \\
&\leq \liminf_{n\to\infty} E[\tilde{G}(\frac{1}{n}\sum_{i=1}^{n} z_i H_i)] \\
&\leq \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} E[\tilde{G}(z_i H_i)] \\
&= \lim_{n\to\infty} E[\tilde{G}(z_n H_n)] \\
&= \liminf_{n\to\infty} \tilde{v}(z_n).
\end{aligned}$$

The other conclusions are straightforward. □

Lemma 3 *Assume $\tilde{v}(z) < \infty$ for $z > 0$. Then for every $x > 0$ there exist a $\hat{z} > 0$ and a $\hat{H} \in \mathcal{H}$ satisfying $\hat{H} > 0$ a.s. that solve the dual problem* $\inf_{z\geq 0, H\in\mathcal{H}} \Big(E[\tilde{G}(zH)] + xz\Big)$.

Proof. If $\tilde{v}(z)$ is finite, then there exist $H_n \in \mathcal{H}$ such that $\lim_{n\to\infty} E[\tilde{G}(zH_n)] = \tilde{v}(z)$. The Komlos theorem and Fatou's lemma imply that there exists a $\hat{H}_z \in \mathcal{H}$ such that $\tilde{v}(z) = E[\tilde{G}(z\hat{H}_z)]$.

Denote $\alpha(z) = \tilde{v}(z) + xz$. Lemma 2 implies $\alpha(z)$ is a convex, lower semi-continuous, proper function on $(0, \infty)$, and $\alpha(0) = \infty$ and $\alpha(\infty) = \infty$. We conclude $\alpha(z)$ achieves its minimum at a point $\hat{z} > 0$. The point $(\hat{z}, \hat{H}_{\hat{z}})$ then minimizes $E[\tilde{G}(zH)] + xz$ over $z \geq 0$ and $H \in \mathcal{H}$. Note $\hat{H} := \hat{H}_{\hat{z}} > 0$ a.s. because if $\hat{H} = 0$ on a set of positive measure, then $\tilde{G}(\hat{z}\hat{H}) = \infty$ on that set, which implies $\alpha(\hat{z}) = E[\tilde{G}(\hat{z}\hat{H})] + x\hat{z} = \infty$, a contradiction. □

To construct the optimal terminal wealth Cvitanic (2000) uses the following approach: Define the space $L := L^1(\Omega, \mathcal{F}_T, P) \times R$ with the norm $\|(Z, z)\| := E[|Z|] + |z|$, the subset $\mathcal{G} := \{(zH, z) \in L : z \geq 0, H \in \mathcal{H}\}$, and the functional $\tilde{U}(Z, z) := E[\tilde{G}(Z)] + xz$ for $(Z, z) \in L$. Then $(\hat{z}\hat{H}, \hat{z}) \in \mathrm{dom}\tilde{U} \cap \mathcal{G}$ minimizes $\tilde{U}(Z, z)$ over $(Z, z) \in \mathcal{G}$, which implies

$$0 \in \partial(\tilde{U} + \chi_{\mathcal{G}})(\hat{z}\hat{H}, \hat{z}) = \partial\tilde{U}(\hat{z}\hat{H}, \hat{z}) + N_{\mathcal{G}}(\hat{z}\hat{H}, \hat{z}), \tag{11}$$

where $\partial\tilde{U}(\hat{z}\hat{H}, \hat{z})$ is the subdifferential of $\tilde{U}$ at $(\hat{z}\hat{H}, \hat{z})$ and $N_{\mathcal{G}}(\hat{z}\hat{H}, \hat{z})$ is the normal cone of $\mathcal{G}$ at $(\hat{z}\hat{H}, \hat{z})$, then (11) implies that there exists a pair $(\hat{Y}, \hat{y}) \in L^*$ such that $(\hat{Y}, \hat{y}) \in N_{\mathcal{G}}(\hat{z}\hat{H}, \hat{z})$ and $(-\hat{Y}, -\hat{y}) \in \partial\tilde{U}(\hat{z}\hat{H}, \hat{z})$. It turns out that $\hat{Y}$ is the optimal terminal wealth. The success of this approach relies on the sum rule (11). The validity of (11) requires some regularity conditions to be satisfied. One well-known condition is the interior point condition (Aubin and Ekeland (1984)). The domain of the functional in Cvitanic (2000) is the whole space, which guarantees the interior point condition and the sum rule, but this is not the case in this paper. We therefore use a different method to construct the optimal terminal wealth.

Lemma 4 *Let $\hat{Y} = h(\hat{z}\hat{H})$. Then $E[\hat{Y}H] \leq E[\hat{Y}\hat{H}] = x$ for all $H \in \mathcal{H}$ and there exists a pair $(\hat{\pi}, \hat{\kappa}) \in \mathcal{A}(x)$ such that $X^{x,\hat{\pi},\hat{\kappa}}(T) = \hat{Y}$ a.s. and $\hat{Y}$ is the optimal terminal wealth.*

Proof. Note that $\tilde{G}'(\hat{z}\hat{H}) = -\hat{Y}$. From Lemma 3 we have

$$E[\tilde{G}(\hat{z}\hat{H})] + x\hat{z} \leq E[\tilde{G}(zH)] + xz, \qquad \forall z \geq 0, H \in \mathcal{H}. \tag{12}$$

We first show that $E[\hat{Y}\hat{H}] = x$. Let $H = \hat{H}$ and $z = \hat{z} + \epsilon$ in (12) we get

$$E\left[\frac{\tilde{G}(\hat{z}\hat{H} + \epsilon\hat{H}) - \tilde{G}(\hat{z}\hat{H})}{\epsilon}\right] \geq -x.$$

The difference quotient $(\tilde{G}(\hat{z}\hat{H} + \epsilon\hat{H}) - \tilde{G}(\hat{z}\hat{H}))/\epsilon$ decreases as $\epsilon \to 0$ (Rockafellar (1970), Theorem 23.1) and is bounded above by 0 from decreasing property of $\tilde{G}$. The Monotone Convergence Theorem implies $E[\hat{Y}\hat{H}] \leq x$.

Let $H = \hat{H}$ and $z = \hat{z}(1-\epsilon)$ in (12) we get

$$E\left[\frac{\tilde{G}((1-\epsilon)\hat{z}\hat{H}) - \tilde{G}(\hat{z}\hat{H})}{\epsilon}\right] \geq x\hat{z}.$$

The difference quotient $(\tilde{G}(\hat{z}\hat{H} - \epsilon\hat{z}\hat{H}) - \tilde{G}(\hat{z}\hat{H}))/\epsilon$ decreases as $\epsilon \to 0$. Let $0 < \epsilon < 1-\beta$. Since $\tilde{G}$ is convex $\tilde{G}'$ is increasing and

$$\begin{aligned}
\tilde{G}(\hat{z}\hat{H}) &\geq \tilde{G}(\hat{z}\hat{H} - \epsilon\hat{z}\hat{H}) + \tilde{G}'(\hat{z}\hat{H} - \epsilon\hat{z}\hat{H})\epsilon\hat{z}\hat{H} \\
&\geq \tilde{G}(\hat{z}\hat{H} - \epsilon\hat{z}\hat{H}) + \tilde{G}'(\beta\hat{z}\hat{H})\epsilon\hat{z}\hat{H} \\
&\geq \tilde{G}(\hat{z}\hat{H} - \epsilon\hat{z}\hat{H}) + \gamma\tilde{G}'(\hat{z}\hat{H})\epsilon\hat{z}\hat{H}
\end{aligned}$$

which shows that

$$\frac{\tilde{G}(\hat{z}\hat{H} - \epsilon\hat{z}\hat{H}) - \tilde{G}(\hat{z}\hat{H})}{\epsilon} \leq \gamma\hat{z}\hat{Y}\hat{H}.$$

The Monotone Convergence Theorem implies $E[\hat{Y}\hat{H}] \geq x$.

We now prove $E[\hat{Y}H] \leq E[\hat{Y}\hat{H}]$ for all $H \in \mathcal{H}$. For $H \in \mathcal{H}$ let $z = \hat{z}$ and $H = (1-\epsilon)\hat{H} + \epsilon H \in \mathcal{H}$ in (12) we get

$$E\left[\frac{\tilde{G}(\hat{z}\hat{H} + \epsilon\hat{z}(H - \hat{H})) - \tilde{G}(\hat{z}\hat{H})}{\epsilon}\right] \geq 0.$$

The difference quotient $(\tilde{G}(\hat{z}\hat{H} + \epsilon\hat{z}(H - \hat{H})) - \tilde{G}(\hat{z}\hat{H}))/\epsilon$ decreases as $\epsilon \to 0$. Define a set $A = \{H < \hat{H}\}$. Then on $A^c = \{H \geq \hat{H}\}$ the difference quotient is bounded above by 0. Let $0 < \epsilon < 1-\beta$ we have on A that

$$\begin{aligned}
\tilde{G}(\hat{z}\hat{H}) &\geq \tilde{G}(\hat{z}\hat{H} + \epsilon\hat{z}(H - \hat{H})) + \tilde{G}'(\hat{z}\hat{H} + \epsilon\hat{z}(H - \hat{H}))\epsilon\hat{z}(\hat{H} - H) \\
&\geq \tilde{G}(\hat{z}\hat{H} + \epsilon\hat{z}(H - \hat{H})) + \tilde{G}'(\hat{z}\hat{H} - \epsilon\hat{z}\hat{H})\epsilon\hat{z}(\hat{H} - H) \\
&\geq \tilde{G}(\hat{z}\hat{H} + \epsilon\hat{z}(H - \hat{H})) + \gamma\tilde{G}'(\hat{z}\hat{H})\epsilon\hat{z}(\hat{H} - H)
\end{aligned}$$

which shows that

$$\frac{\tilde{G}(\hat{z}\hat{H} + \epsilon\hat{z}(H - \hat{H})) - \tilde{G}(\hat{z}\hat{H})}{\epsilon} \leq \gamma\hat{Y}\hat{z}(\hat{H} - H)\chi_A \leq \gamma\hat{Y}\hat{z}\hat{H}\chi_A \leq \gamma\hat{z}\hat{Y}\hat{H}.$$

The Monotone Convergence Theorem implies $E[\hat{Y}H] \leq E[\hat{Y}\hat{H}]$. Since $\mathcal{H}_{\mathcal{D}} \subset \mathcal{H}$ we have

$$\sup_{\nu \in \mathcal{D}} E[\hat{Y}H_\nu(T)] \leq E[\hat{Y}\hat{H}] = x.$$

Theorem 3.2 of Cvitanic (2000) confirms that there exists a pair $(\hat{\pi}, \hat{\kappa}) \in \mathcal{A}(x)$ to hedge $\hat{Y}$, i.e., $X^{x,\hat{\pi},\hat{\kappa}}(T) = \hat{Y}$ a.s. $\square$

We can now summarize the existence results for (5) as follows.

Theorem 1 *Assume that (8) and (10) hold and $x > 0$. Then there exists an optimal trading strategy $(\hat{\pi}, \hat{\kappa}) \in \mathcal{A}(x)$ for (5). The optimal terminal wealth is characterized by $X^{x,\hat{\pi},\hat{\kappa})}(T) = h(\hat{z}\hat{H})$ a.s., where $(\hat{z}, \hat{H})$ is the optimal solution for the dual problem defined in Lemma 3 and h is defined in (9).*

4 Numerical Viscosity Solutions

In this section we assume throughout that there exists an optimal solution for (5) such that $\hat{\kappa} \equiv 0$, i.e., we can achieve the maximum without the consumption. (The equivalent condition which ensures $\hat{\kappa} \equiv 0$ is that there exists a $\hat{\nu} \in \mathcal{D}$ such that $E[\hat{Y} H_{\hat{\nu}}(T)] = x$, see Cvitanic and Karatzas (1993).) We apply the stochastic control theory to solve (5). To this end we write the state constraint $X(t) \geq 0$ a.s. for $0 \leq t \leq T$ equivalently as

$$E\Big[\int_0^T \max(-X(t), 0)dt\Big] = 0 \tag{13}$$

and reformulate (5) as

$$V(x) = \inf_{\pi} E[\Psi(X(T))] \text{ subject to (1) and (13)} \tag{14}$$

where $\Psi(x) = -g(x) + \lambda(y + \delta(c - g(x) - y)^+$ and $V(x) = -u(x)$. Oksendal (1998) (see also Bielecki et al. (2005)) suggests the Lagrange multiplier method to solve (14): for every constant μ consider the unconstrained stochastic control problem

$$V_\mu(x) = \inf_{\pi} E\Big[\Psi(X(T)) + \mu \int_0^T L(X(t))dt\Big] \text{ subject to (1)}$$

where $L(x) = \max(-x, 0)$. Suppose that there exists an optimal control π_μ for every μ and that there exists a π_{μ_0} for some μ_0 such that (13) is satisfied, then π_{μ_0} is the optimal control for (14) and $V(x) = V_{\mu_0}(x)$. This is an appealing way of solving the stochastic control problem with equality constraints if we can manage to find these optimal controls. However, it is difficult to find the required π_μ and π_{μ_0} for (14) as functions L, Ψ are not

differentiable and the uniform parabolic condition for (1) is not satisfied (see Fleming and Soner (2006), IV(3.5)).

The sequential penalty function technique plays an important part in nonlinear programming. It solves a constrained optimization problem by transforming it into a sequence of unconstrained optimization problems and then taking the limit, see Fletcher (1987). We use this technique to study (14). Let $\{\mu_n\}$ be an increasing sequence of positive numbers satisfying $\mu_n \to \infty$ as $n \to \infty$. Consider a sequence of minimization problems:

$$\inf_{\pi} E\Big[\Psi(X(T)) + \mu_n \int_0^T L(X(t))dt\Big] \text{ subject to (1).} \tag{15}$$

We have the following result.

Theorem 2 *Let $V_n(x)$, $n = 1, 2, \ldots$, be optimal values for (15) and let $V(x)$ be finite. Then*

(a) $\{V_n(x)\}$ is an increasing sequence and is bounded above by $V(x)$.

(b) If there exist optimal solutions $\{\pi_n\}$ for (15) then $\{E[\int_0^T L(X_n(t))dt]\}$ is a decreasing sequence and converges to 0, and $\{E[\Psi(X_n(T))]\}$ is an increasing sequence, where $X_n(\cdot)$, $n = 1, 2, \ldots$, are solutions to (1) with controls π_n.

(c) If for some $\mu_0 > 0$ there exists optimal solution π_{μ_0} for (15) such that $E[\int_0^T L(X_{\mu_0}(t))dt] = 0$, then π_{μ_0} is the optimal solution for (4) and $V_n(x) = V(x)$ for all n such that $\mu_n \geq \mu_0$.

Proof. (a) Let $m < n$ then $\mu_m < \mu_n$. For every $\epsilon > 0$ there exist π_i, $i = m, n$, such that

$$E\Big[\Psi(X_i(T)) + \mu_i \int_0^T L(X_i(t))dt\Big] < V_i(x) + \epsilon$$

for $i = m, n$. Therefore

$$\begin{aligned} V_m(x) &\leq E\Big[\Psi(X_n(T)) + \mu_m \int_0^T L(X_n(t))dt\Big] \\ &\leq E\Big[\Psi(X_n(T)) + \mu_n \int_0^T L(X_n(t))dt\Big] \\ &\leq V_n(x) + \epsilon \end{aligned}$$

where X_i, $i = m, n$, are the solutions to (1) with controls $\pi_i(\cdot)$. Since $\epsilon > 0$ is arbitrary, we have $V_m(x) \leq V_n(x)$. Furthermore, for any π such that the corresponding solution X to (1) satisfies the nonnegative constraint (**??**), then

$$V_n(x) \leq E\Psi(X(T))$$

which implies $V_n(x) \leq V(x)$.

(b) Let π_i, $i = m, n$, be optimal solution to (15). Then

$$E\Big[\Psi(X_m(T)) + \mu_m \int_0^T L(X_m(t))dt\Big] \leq E\Big[\Psi(X_n(T)) + \mu_m \int_0^T L(X_n(t))dt\Big] \tag{16}$$

and

$$E\Big[\Psi(X_n(T)) + \mu_n \int_0^T L(X_n(t))dt\Big] \leq E\Big[\Psi(X_m(T)) + \mu_n \int_0^T L(X_m(t))dt\Big] \tag{17}$$

Adding (16) and (17) together we get

$$0 \leq (\mu_n - \mu_m)E\Big[\int_0^T (L(X_m(t)) - L(X_n(t)))dt\Big].$$

Therefore $E\int_0^T L(X_n(t))dt \leq E\int_0^T L(X_m(t))dt$. Substituting this into (16) we get $E[\Psi(X_m(T))] \leq E[\Psi(X_n(T))]$. Finally, since $\{E[\Psi(X_n(T))]\}$ is increasing, $V_n(x) \leq V(x)$, and $\mu_n \to \infty$, we must have $E[\int_0^T L(X_n(t))dt] \to 0$ as $n \to \infty$.

(c) If for some $\mu_0 > 0$ there exists optimal solution π_{μ_0} for (15) such that $E[\int_0^T L(X_{\mu_0}(t))dt] = 0$, then the state process X_{μ_0} satisfies (1) and the nonnegative constraint. We have

$$V(x) \leq E[\Psi(X_{\mu_0}(T))] = V_{\mu_0}(x) \leq V(x).$$

The last inequality is from (a). Therefore π_{μ_0} is the optimal solution for (4). Furthermore,

$$V(x) = V_{\mu_0}(x) \leq V_n(x) \leq V(x)$$

for all n such that $\mu_n \geq \mu_0$. Therefore, $V_n(x) = V(x)$ for n sufficiently large. □

Theorem 2(a) shows that (15) is well-defined. Theorem 2(b) implies that π_n is almost optimal for n sufficiently large and $E\Psi(X_n(T))$ is likely to

be a good approximation to $V(x)$. Theorem 2(c) confirms that if for some μ_0 we have π_{μ_0} solves (4) then the sequence $\{V_n(x)\}$ converges to $V(x)$, in other words, if the method suggested by Oksendal (1998) works then the sequential penalty function method works too, but not necessarily the other way around. The other advantage is that we can bypass the steps of finding the optimal control in studying the value function and instead we can use directly the viscosity solution method.

The sequential penalty function method can be used to deal with other constraints. For example, if there is a constraint on the traded dollar value of the stock, i.e., $u^- \leq \pi(t)X(t) \leq u^+$ a.s. for $t > 0$, then we can define

$$L(x,\pi) = \max(-x,0) + \max(\pi x - u^+, 0) + \max(u^- - \pi x, 0)$$

and study the constraint $E[\int_0^T L(X(t),\pi(t))dt] = 0$ instead.

To compute $V_n(x)$ define the value function

$$V_n(t,x) = \inf_\pi E_{tx}\Big[\Psi(X(T)) + \mu_n \int_t^T L(X(t))dt\Big]$$

where E_{tx} is the conditional expectation operator given $X(t) = x$. Theorem 4 shows that V_n is the viscosity solution to the HJB equation

$$-V_t + \sup_{\pi\in K}\Big(- f(x,\pi)V_x - \frac{1}{2}\sigma^2\pi^2 V_{xx} - \mu_n L(x)\Big) = 0 \tag{18}$$

with the terminal condition $V(T,x) = \Psi(x)$ for all $x \in R$, where $f(x,\pi) = (r + (\alpha - r)\pi)x$.

To solve (18) numerically we need the uniqueness of the solution which is not clear here as functions f, L, Ψ are unbounded and Theorem 4 cannot be applied. Given $\rho > 0$ consider the stochastic control problem

$$V_n^\rho(t,x) = \inf_\pi E_{tx}\Big[\Psi_\rho(X_\rho(T)) + \mu_n \int_t^T L_\rho(X_\rho(s))ds\Big]$$

where X_ρ is the solution to the SDE

$$dX_\rho(s) = f_\rho(X_\rho(s),\pi(s))ds + \sigma\pi(s)dW(s), \quad X_\rho(t) = x$$

and $f_\rho(x,\pi)$ is defined by

$$f_\rho(x,\pi) = f(x,\pi)1_{\{|x|\leq\rho\}} + f(\rho|x|^{-1}x,\pi)1_{\{|x|>\rho\}}$$

β	λ	Utility	CVaR	β	λ	Utility	CVaR
	0	0.058	0.39		0	0.058	0.52
0.99	1	0.051	0.05	0.999	1	0.051	0.05
	100	0.050	0.00		100	0.050	0.00

Table 1: The comparison of the optimal utility and CVaR with different parameters β and λ. The data used are $r = 0.05, \alpha = 0.1, \sigma = 0.4, T = 1, x = 1$.

and Ψ_ρ, L_ρ are defined similarly. Then $f_\rho, \Psi_\rho, L_\rho$ satisfy all the conditions of Theorem 4 (see Lemma 5 in Section 5) which implies that the value function $V_\rho(t,x)$ is the unique bounded viscosity solution of the HJB equation and can be computed with the explicit finite difference method (see Fleming and Soner (2006), Chapter IX). Furthermore, we have $V_n^\rho(t,x)$ converges to $V_n(t,x)$ uniformly on compact sets as $\rho \to \infty$ (see the proof of Theorem 3 in Section 6). We may therefore find the numerical value $V_n(t,x)$ by first computing $V_n^\rho(t,x)$ and then taking the limit.

We solve numerically a simple efficient frontier problem with the utility function $g(x) = \ln x$. Define a new variable $w(t) = \ln X(t)$. Then $w(\cdot)$ satisfies the equation $dw(t) = (r + (\alpha - r)\pi - \frac{1}{2}\sigma^2\pi^2)dt + \sigma\pi dW(t)$ with the initial condition $w(0) = \ln x$. Assume the control set $K = [0,1]$ (no borrowing and no short-selling). Since the state constraint $X(t) = e^{w(t)} > 0$ is always satisfied, there is no need to add a state constraint on $w(t)$. The value function $V(0,x) = \inf_y \inf_\pi E[\tilde{\Psi}(w(T), y)]$ with the objective function $\tilde{\Psi}(w,y) = -w + \lambda y + \lambda\delta(c - w - y)^+$. We can derive the optimal value for fixed β and λ with the explicit finite difference method and the line search method in optimization.

Table 1 lists the corresponding optimal utility and CVaR. We observe the following phenomenon: 1. If $\lambda = 0$ (maximizing the utility function only) then the optimal utility (the exact value is 0.0578125) is independent of β, and CVaR increases as β increases as expected, and the ratio of the CVaR to the utility is large and increasing, which implies the potential average loss is great if the rare event of given percentile does happen. 2. If $\lambda = 1$ (maximizing the utility function and minimizing the CVaR) then the optimal utility value depends on β and the utility value is reduced, however, the CVaR is significantly reduced, and the ratio of the CVaR to the utility is less than one, which implies the potential average loss is controlled and

bounded at the cost of moderately reduced utility. 3. If $\lambda = 100$, a high penalty associated with CVaR, then the optimal strategy is to invest only in savings account and nothing in stock .

5 Appendix: Dynamic Programming Equations

We review in this section the DP principle, the HJB equation, and the viscosity solution method for general stochastic optimal control problems. The main results are discussed in detail in Fleming and Soner (2006) and Touzi (2002). We fix an error in the proof of Theorem IV.7.1 of Fleming and Soner (2006) and extend their results to a more general setting.

The fixed finite time horizon stochastic optimal control problem is to minimize

$$J(t,x,u) := E_{tx}\Big[\int_t^T L(s,x(s),u(s))ds + \Psi(x(T))\Big] \tag{19}$$

over all admissible control process $u(\cdot)$, where $x(\cdot)$ is the solution to the controlled diffusion process

$$dx(s) = f(s,x(s),u(s))ds + \sigma(s,x(s),u(s))dW(s), \quad s > t \tag{20}$$

with initial condition $x(t) = x$, E_{tx} is the conditional expectation operator given $x(t) = x$, L is the running cost, and Ψ is the terminal cost. The value function is defined by

$$V(t,x) := \inf_u J(t,x,u) \tag{21}$$

The standard assumptions. For some positive constants K and k, and for all $(t,x) \in Q := (0,T) \times R^n$ and $u \in U$

(a) U is compact,

(b) $g = f, \sigma$ is continuous on $\bar{Q} \times U$, $g(t,\cdot,u)$ is Lipschitz uniformly with respect to t, u, i.e., $|g(t,x,u) - g(t,y,u)| \leq K|x-y|$, and $|g(t,0,u)| \leq K$,

(c) $g = L, \Psi$ is continuous on $\bar{Q} \times U$ and satisfies the polynomial growth condition $|g(t,x,u)| \leq K(1+|x|^k)$.

Remark. The main differences of the standard assumptions here with those of Fleming and Soner (2006) are that σ is not required to be bounded and Ψ is not required to be uniformly continuous and bounded.

Theorem 3 *(Fleming and Soner (2006), Theorem IV.7.1) Assume the standard assumptions. Then the value function V defined by (21) is continuous on Q and satisfies the DP principle: for every $(t,x) \in Q$, every admissible control $u(\cdot)$, and $\{\mathcal{F}_s\}$-stopping time θ,*

$$V(t,x) \leq E_{tx}\Big[\int_t^\theta L(s,x(s),u(s))ds + V(\theta,x(\theta))\Big].$$

For every $\eta > 0$, there exists an admissible control $u(\cdot)$ such that

$$V(t,x) + \eta \geq E_{tx}\Big[\int_t^\theta L(s,x(s),u(s))ds + V(\theta,x(\theta))\Big]$$

for every $\{\mathcal{F}_s\}$-stopping time θ.

Remark. In the proof of Theorem IV.7.1, Fleming and Soner (2006), f is approximated by $f_\rho(t,x,u) = \alpha_\rho(x)f(t,x,u)$, where $\alpha_\rho \in C^\infty(R^n)$ satisfying $0 \leq \alpha_\rho(x) \leq 1$, $|D\alpha_\rho(x)| \leq 2$, and $\alpha_\rho(x) = 1$ for $|x| \leq \rho$, $\alpha_\rho(x) = 0$ for $|x| \geq \rho + 1$. It is claimed that if $f(t,\cdot,u)$ is Lipschitz continuous, then $f_\rho(t,\cdot,u)$ is Lipschitz continuous and the Lipschitz constant is independent of ρ. This is in general not true unless f is bounded. For example, let $f(x) = x$, then $f_\rho(x) = x\alpha(x)$, the mean value theorem says $-\rho = f_\rho(\rho + 1) - f_\rho(\rho) = f'_\rho(\xi_\rho)$ for some $\xi_\rho \in (\rho, \rho+1)$, which implies $|f'_\rho(\xi_\rho)| = \rho \to \infty$ as $\rho \to \infty$, i.e., the Lipschitz constant of f_ρ depends on ρ and is unbounded as $\rho \to \infty$. The constant C in Lemma IV.6.1 therefore depends on ρ, which implies $|V_\rho(t,x) - V(t,x)| \leq C(\rho)\rho^{-1/2}(1+|x|^{2k+1})^{1/2}$. It is not obvious that $V_\rho(t,x) \to V(t,x)$ as $\rho \to \infty$. Also note that in the proof of Theorem IV.7.1 σ is not approximated, which forces σ to be bounded, and Ψ is approximated with the standard smoothing method, which requires Ψ to be bounded and uniformly continuous.

Outline of the proof of Theorem 3. For $g = f, \sigma, L, \Psi$ define

$$g_\rho(t,x,v) = g(t,x,v)1_{\{|x|\leq\rho\}} + g(t,\rho|x|^{-1}x,v)1_{\{|x|>\rho\}}. \tag{22}$$

Denote by x_ρ the solution to (20) with f, σ replaced by f_ρ, σ_ρ. Since $f_\rho(t,\cdot,v)$ and $\sigma_\rho(t,\cdot,v)$ are Lipschitz with Lipschitz constant $2K$ (Lemma 5(a)) we

have

$$E_{tx}\|x_\rho(\cdot)\|^m \le B_m(1+|x|^m) \tag{23}$$

for all m and B_m are constants independent of ρ. Define J_ρ the objective function (19) with L, Ψ replaced by L_ρ, Ψ_ρ, and V_ρ the corresponding value function. Then the standard assumptions (c), Lemma 5(c), and (23) imply that for some constant C (independent of ρ)

$$|V_\rho(t,x) - V(t,x)| \le C\rho^{-1}(1+|x|^{2k+1})^{1/2}$$

which implies that $V_\rho \to V$ uniformly on compact sets as $\rho \to \infty$. The rest of the proof follows essentially that of Theorem IV.7.1, see Fleming and Soner (2006), page 171-182, for details. □

The following lemma is needed in the proof of Theorem 3.

Lemma 5 *Let g be a function on R^n. For $\rho > 0$ define*

$$g_\rho(x) = g(x)1_{\{|x|\le\rho\}} + g(\rho|x|^{-1}x)1_{\{|x|>\rho\}}.$$

The following statements are true.

(a) If g is Lipschitz with Lipschitz constant K, then g_ρ is Lipschitz with Lipschitz constant $2K$.

(b) If g is continuous, then g_ρ is uniformly continuous.

(c) If $|g(x)| \le K(1+|x|^k)$ for some constants K, k, then $|g_\rho(x)| \le K(1+|x|^k)$ and $|g_\rho(x)| \le K(1+\rho^k)$ for all x.

Proof. (a) Let $\bar{B}(\rho) = \{x : |x| \le \rho\}$. We need to discuss 3 cases.
Case 1: $x, y \in \bar{B}(\rho)$. Then

$$|g_\rho(x) - g_\rho(y)| = |g(x) - g(y)| \le K|x-y|.$$

Case 2: $x, y \notin \bar{B}(\rho)$. Then

$$\begin{aligned}
|g_\rho(x) - g_\rho(y)| &= \Big|g(\rho|x|^{-1}x) - g(\rho|y|^{-1}y)\Big| \\
&\le K\Big|\rho|x|^{-1}x - \rho|y|^{-1}y\Big| \\
&= K\rho|x|^{-1}|y|^{-1}\Big|x|y| - y|x|\Big| \\
&\le K|y|^{-1}\Big|(x-y)|y| - y(|x|-|y|)\Big| \\
&\le K|y|^{-1}\Big(|x-y||y| + |y|\Big||x|-|y|\Big|\Big) \\
&\le 2K|x-y|
\end{aligned}$$

Case 3: $x \in \bar{B}(\rho)$ and $y \notin \bar{B}(\rho)$. Then

$$|g_\rho(x) - g_\rho(y)| = \Big|g(x) - g(\rho|y|^{-1}y)\Big| \le K\Big|x - \rho|y|^{-1}y\Big|.$$

Since $y - \rho|y|^{-1}y$ is a normal vector of $\bar{B}(\rho)$ at point $\rho|y|^{-1}y$ we have

$$\Big(y - \rho|y|^{-1}y\Big) \cdot \Big(z - \rho|y|^{-1}y\Big) \le 0$$

for all $z \in \bar{B}(\rho)$, in particular, $(\rho|y|^{-1}y - y) \cdot (x - \rho|y|^{-1}y) \ge 0$. Therefore

$$\begin{aligned} |x-y|^2 &= \Big|x - \rho|y|^{-1}y\Big|^2 + \Big|\rho|y|^{-1}y - y\Big|^2 + 2\Big(\rho|y|^{-1}y - y\Big) \cdot \Big(x - \rho|y|^{-1}y\Big) \\ &\ge \Big|x - \rho|y|^{-1}y\Big|^2. \end{aligned}$$

We have $|g_\rho(x) - g_\rho(y)| \le K|x-y|$.

(b) The discussion in (a) shows the relations $|\rho|x|^{-1}x - \rho|y|^{-1}y| \le 2|x-y|$ if $x, y \notin \bar{B}(\rho)$ and $|x - \rho|y|^{-1}y| \le |x-y|$ if $x \in \bar{B}(\rho)$ and $y \notin \bar{B}(\rho)$. If g is continuous, then g is uniformly continuous on $\bar{B}(\rho)$. The definition of g_ρ and the above inequalities imply that g_ρ is uniformly continuous on R^n.

(c) If $|x| \le \rho$ then $|g_\rho(x)| = |g(x)| \le K(1 + |x|^k)$. If $|x| \ge \rho$ then $|g_\rho(x)| = |g(\rho|x|^{-1}x)| \le K(1 + \rho^k)$. $\square$

The HJB equation for value function V is given by

$$-V_t(t,x) + \mathcal{H}(t,x,V_x(t,x),V_{xx}(t,x)) = 0, \quad \forall (t,x) \in Q \tag{24}$$

where V_t, V_x, V_{xx} are partial derivatives of V with respect to t and x, and $\mathcal{H}$ is defined by

$$\mathcal{H}(t,x,p,A) = \max_{u \in U}\Big(- f(t,x,u) \cdot p - \frac{1}{2}\mathrm{Tr}(a(t,x,u)A) - L(t,x,u)\Big) \tag{25}$$

where $p \in R^n$, A is a $n \times n$ symmetric matrix, $a(t,x,u) = \sigma(t,x,u)\sigma^t(t,x,u)$, and σ^t is the transpose of σ. In general, value function V is not differentiable and is not the solution of (24) in the classical sense, however, it can be shown that V is the solution of (24) in the viscosity sense.

Definition. let V be continuous on $\bar{Q}$. V is a *viscosity subsolution* of (24) in Q if for each $\phi \in C^\infty(Q)$,

$$-\frac{\partial}{\partial t}\phi(t_0,x_0) + \mathcal{H}(t_0,x_0,D_x\phi(t_0,x_0),D_x^2\phi(t_0,x_0)) \le 0$$

at every $(t_0, x_0) \in Q$ which is a local maximum point of $V - \phi$ in Q. V is a *viscosity supersolution* of (24) in Q if for each $\phi \in C^\infty(Q)$,

$$-\frac{\partial}{\partial t}\phi(t_0, x_0) + \mathcal{H}(t_0, x_0, D_x\phi(t_0, x_0), D_x^2\phi(t_0, x_0)) \geq 0$$

at every $(t_0, x_0) \in Q$ which is a local minimum point of $V - \phi$ in Q. V is a *viscosity solution* of (24) in Q if it is both a viscosity subsolution and a viscosity supersolution.

Theorem 4 *Assume the standard assumptions. Then the value function V defined by (21) is a continuous viscosity solution of (24) in Q. Furthermore, if functions f, σ, L, Ψ and their derivatives (with respect to t and x) are bounded in addition to the standard assumptions, then the value function V is the unique bounded viscosity solution of (24) in Q.*

Proof of Theorem 4. One can easily verify that under the standard assumptions the function $\mathcal{H}$ defined by (25) is continuous, which confirms the first part of Theorem 4, see Touzi (2002). The second part of Theorem 4 is proved in Fleming and Soner (2006) with a comparison theorem (Theorem V.9.1) for bounded viscosity solutions. □

6 References

1. Aubin, J. and Ekeland, I., *Applied Nonlinear Analysis*, Wiley, 1984.

2. Bielecki, T.R., Jin, H., Pliska, S.R., and Zhou, X.Y., Continuous-time mean-variance portfolio selection with bankruptcy prohibition, *Mathematical Finance* 15, 213–244, 2005.

3. Cvitanic, J., Minimizing expected loss of hedging in incomplete and constrained markets, *SIAM J. Control Optimization* 38, 1050–1066, 2000.

4. Cvitanic, J. and Karatzas, I., Hedging contingent claims with constrained portfolios, *Annals Applied Probability* 3, 652–681, 1993.

5. Deelstra, G., Pham, H., and Touzi, N., Dual formulation of the utility maximization problem under transaction costs, *Annals Applied Probability* 11, 1353–1383, 2001.

6. Fleming, W.H. and Soner, H.M., *Controlled Markov Processes and Viscosity Solutions*, Springer, 2006.

7. Fletcher, R., *Practical Methods of Optimization*, John Wiley, 1987.

8. Karatzas, I., Lehoczky, J.P. and Shreve, S.E., Optimal portfolio and consumption decisions for a "small investor" on a finite horizon, *SIAM J. Control Optim.* 25, 1557–1586, 1987.

9. Karatzas, I., Lehoczky, J.P., Shreve, S.E. and Xu, G.L., Martingale and duality methods for utility maximization in an incomplete market, *SIAM J. Control Optim.* 29, 702–730, 1991.

10. Karatzas, I. and Shreve, S.E., *Methods of Mathematical Finance*, Springer, 1998.

11. Korn, R., *Optimal Portfolios*, World Scientific, 1997.

12. Kramkov, D. and Schachermayer, W., The asymptotic elasticity of utility functions and optimal investment in incomplete markets, *Annals Applied Probability* 9, 904–950, 1999.

13. Kramkov, D. and Schachermayer, W., Necessary and sufficient conditions in the problem of optimal investment in incomplete markets, *Annals Applied Probability* 13, 1504–1516, 2003.

14. Merton, R.C., Optimum consumption and portfolio rules in a continuous time model, *J. Economic Theory* 3, 373–413, 1971.

15. Oksendal, B., *Stochastic Differential Equations*, Springer, 1998.

16. Rockafellar, R.T., *Convex Analysis*, Princeton University Press, 1970.

17. Rockafellar, R.T. and Uryasev, S., Optimization of conditional value-at-risk. *Journal of Risk*, 2, 21–41, 2000.

18. Touzi, N., *Stochastic Control Problems, Viscosity Solutions, and Application to Finance*, Lecture Note, 2002.