Jump liquidity risk and its impact on CVaR

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Abstract

Purpose – The aim is to study jump liquidity risk and its impact on risk measures: value at risk (VaR) and conditional VaR (CVaR).

Design/methodology/approach – The liquidity discount factor is modelled with mean revision jump diffusion processes and the liquidity risk is integrated in the framework of VaR and CVaR.

Findings – The standard VaR, CVaR, and the liquidity adjusted VaR can seriously underestimate the potential loss over a short holding period for rare jump liquidity events. A better risk measure is the liquidity adjusted CVaR which gives a more realistic loss estimation in the presence of the liquidity risk. An efficient Monte Carlo method is also suggested to find approximate VaR and CVaR of all percentiles with one set of samples from the loss distribution, which applies to portfolios of securities as well as single securities.

Originality/value – The paper offers plausible stochastic processes to model liquidity risk.

Keywords Monte Carlo methods, Risk analysis, Liquidity

Paper type Research paper

1. Introduction

The liquidity drying up was the prevailing trigger element of some high profile failures in the recent past (e.g., the fall of long-term capital management). These highly leveraged hedge funds had great difficulty in raising cash to meet margin calls by unwinding positions in markets where liquidity had almost disappeared. Dunbar (1998) explains that:

Portfolios are usually marked to market at the middle of the bid-offer spread, and many hedge funds used models that incorporate this assumption. In late August, there was only one realistic value for the portfolio: the bid price. Amid such massive sell-offs, only the first seller obtains a reasonable price for its security; the rest lose a fortune by having to pay a liquidity premium if they want a sale. […] Models should be revised to include bid-offer behaviour.

The liquidity risk can be conceptually divided into an exogenous component and an endogenous component, the former depends on the general market condition and the latter relates the specific position of a trader. (Lawrence and Robinson, 1996) assert that ignoring totally the liquidity risk can provoke an underestimate of the market risk up to 30 percent. Despite the wide recognition of the importance of the liquidity risk, there is no universal agreement on the definition of liquidity. In the academic literature, the liquidity is usually defined in terms of the bid-ask spread and/or the transaction cost whereas in the practitioner literature the illiquidity is often viewed as the inability of buying and selling securities (at any price). The four major properties of the liquidity are the following Black (1971): the immediacy of the transaction, the tightness of the...
the resiliency of the market, and the depth of the market. The concept of liquidity can be summarized as the ability for traders to execute large trades rapidly at a price close to current market price. The liquidity risk refers to the loss stemming from costs of liquidating a position.

To manage the liquidity risk, a good risk measure is needed to account for the impact of the liquidity shock on tradable securities and portfolios. Value at risk (VaR) is the most popular market risk measure used in practice, which estimates the potential loss of a financial instrument at a certain level of probability and for a given period of time. However, VaR ignores the liquidity component and can seriously underestimate the potential loss if the loss distribution is fat-tailed. This is because VaR takes the value of the least loss among all possible losses if an event of a given probability does occur. To overcome the underestimation of the potential loss, VaR is often adjusted in an ad hoc fashion either by lengthening the holding period or by magnifying the VaR calculated with the desired holding period. A different risk measure that addresses the shortcoming of VaR is the conditional VaR (CVaR). Unlike VaR in predicting the potential loss, CVaR uses the average loss among all possible losses, which provides a more realistic loss estimation if an unexpected "bad" event occurs in a fat-tailed environment. CVaR is also a coherent risk measure whereas VaR is not (Artzner et al., 1999).

It is often difficult to compute directly the VaR and CVaR from their definitions as VaR requires to solve a nonlinear equation and CVaR to integrate over the tail distribution, especially when the closed-form expression of the loss distribution is unknown or is too complicated. Rockafellar and Uryasev (2002) suggest a viable method for the computation of VaR and CVaR by formulating a convex optimization problem in which the minimum value is the CVaR and the left end point of the minimum solution set gives the VaR. The resulting optimization problem can be easily solved in two steps by first generating samples of the loss distribution and then solving a large linear programming (LP) problem which gives the approximate VaR and CVaR. The same formulation can also be used to find the minimum CVaR portfolio.

We focus in this paper on the loss of the realized value (bid-price) of a tradable security, we define the bid-price as the product of the mid-price and the liquidity discount factor, both follow some stochastic processes. To highlight the key point and simplify the discussion, we assume that the mid-price follows a geometric Brownian motion (GBM) process, the liquidity discount factor follows a mean-reversion jump diffusion process, and two processes are independent of each other.

There are two main contributions in this paper. The first contribution is that it provides an explicit solution to the LP problem of Rockafellar and Uryasev (2002) with the following advantages:

1. approximate VaR and CVaR can be computed by simply generating samples of the loss distribution and no optimization is needed;
2. VaR and CVaR of any percentile can be computed with a given set of samples of the loss distribution;
3. it works for both a single security and a portfolio of securities as long as the joint distribution of security losses is known; and
4. it opens up other optimization methods (e.g., the augmented Lagrangian method) to find the minimum CVaR portfolio in supplement to the nonsmooth optimization method or the large-scaled LP method.
The second contribution is that it defines liquidity-adjusted VaR (LVaR) and liquidity-adjusted CVaR (LCVaR) for market risk of tradable securities when there exists jump liquidity risk. It shows that the conventional VaR and CVaR for the mid-price of a security can seriously underestimate the potential loss, especially over a short period such as one day, and can result in substantial loss if a “bad” rare event occurs. This partially explains the difficulty those hedge funds had to meet margin calls by unwinding the position when liquidity disappeared. The implication in risk management is that financial institutions should reserve sufficient liquid assets, much larger than what the conventional VaR and CVaR for the mid-price would have suggested, in their portfolios to withstand the potential large loss when a jump liquidity event strikes.

The paper is organized as follows: Section 2 reviews the convex optimization problem of Rockafellar and Uryasev (2002) for VaR and CVaR, and solves the resulting LP problem by the dual method and the Kuhn-Tucker conditions. It also discusses the way of finding the minimum CVaR portfolio. Section 3 models the liquidity discount factor with the mean reversion OU and CIR jump diffusion processes which seem to characterize well the general phenomenon of the liquidity risk: unpredictable sudden liquidity dry-up and gradual recovery afterwards. Section 4 compares LVaRs and LCVaRs of different models and parameters to see their effects for risk measures. Section 5 concludes and the appendix contains the proofs of theorems.

2. Computation of VaR and CVaR
Consider a real-valued random variable $L$ on a probability space $(\Omega, \mathcal{F}, P)$ that represents the loss of an investment over a fixed time horizon. Let $\alpha \in (0, 1)$ be fixed. Then the VaR of $L$ at level $\alpha$ is defined to be the smallest number $x$ such that the probability that the loss does not exceed is not less than $\alpha$, i.e.:

$$\text{VaR}_\alpha = \min\{x \in \mathbb{R} : P(L \leq x) \geq \alpha\}. \quad (1)$$

The CVaR at level $\alpha$ is defined to be the average loss given that the loss is at least VaR$_\alpha$, i.e.:

$$\text{CVaR}_\alpha = \text{mean of the } \alpha - \text{tail distribution of } L \quad (2)$$

where the $\alpha$-tail distribution $F_\alpha(x)$ is defined by:

$$F_\alpha(x) = \begin{cases} 0 & \text{for } x < \text{VaR}_\alpha \\ (P(L \leq x) - \alpha)/(1 - \alpha) & \text{for } x \geq \text{VaR}_\alpha. \end{cases}$$

Let $S_t$ be the discounted asset price at time $t$, following a GBM process:

$$dS_t = \sigma S_t dW_t, \quad (3)$$

where $\sigma$ is a constant asset volatility and $W_t$ a standard Brownian motion.

Remark. In general asset price $S_t$ is assumed to follow a GBM process with a drift $\mu$:

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$ 

The discounted asset price $\tilde{S}_t := e^{-rt}S_t$ satisfies SDE:
where $r$ is the risk-free interest rate. We can then apply the Girsanov theorem to change the probability measure $P$ to an equivalent probability measure $P^0$ such that:

$$d\tilde{S}_t = \sigma \tilde{S}_t dW^0_t$$

where $W^0_t$ is a standard Brownian motion under $P^0$ (Karatzas and Shreve, 1988). We therefore assume, without loss of generality, that discounted asset price $S_t$ has zero drift.

The loss of the discounted asset price at time is given by:

$$L = S_0 - S_T = S_0 (1 - e^{-(1/2)\sigma^2 T + \sigma W_T})$$

($L < 0$ represents a gain.) The distribution of $L$ is given by:

$$P(L \leq x) = \Phi \left( \frac{-\ln(1 - (x/S_0)) - (1/2)\sigma^2 T}{\sigma \sqrt{T}} \right),$$

if $x < S_0$ and $P(L \leq x) = 1$ if $x \geq S_0$ where $\Phi(x)$ is the standard normal cumulative distribution function. A simple calculation using (1) and (2) shows that:

$$\text{VaR}_\alpha = S_0 (1 - e^{-(1/2)\sigma^2 T - \alpha \sqrt{T} \Phi^{-1}(\alpha)}),$$

$$\text{CVaR}_\alpha = S_0 - \frac{1}{1-\alpha} S_0 \Phi(-\Phi^{-1}(\alpha) - \alpha \sqrt{T}),$$

In general, there are many factors which make the direct computation of VaR and CVaR with (1) and (2) difficult. For example, the closed form expression of the loss distribution is unknown, or equation (1) is highly nonlinear, etc. (Rockafellar and Uryasev, 2002) suggest a new way of computing VaR and CVaR by solving the following convex optimization problem:

$$\min_{x} F_\alpha(x) := x + \frac{1}{1-\alpha} E(L - x)^+, \quad (4)$$

where $x^+ = \max(x, 0)$. The VaR and CVaR are the optimal solution and optimal value of problem (4), given by:

$$\text{VaR}_\alpha = \text{left endpoint of } \arg\min_{x \in \mathbb{R}} F_\alpha(x)$$

and:

$$\text{CVaR}_\alpha = \min_{x \in \mathbb{R}} F_\alpha(x) = F_\alpha(\text{VaR}_\alpha).$$

Rockafellar and Uryasev (2002) suggest to solve (4) by first generating samples $M$ of random variable $L$ to approximate (4) by:

$$\min_{x \in \mathbb{R}} x + \frac{1}{1-\alpha} \frac{1}{M} \sum_{i=1}^{M} (L_i - x)^+$$

and then solving an equivalent LP problem:
\[
\min x + \frac{1}{1 - \alpha M} \sum_{i=1}^{M} z_i \quad \text{s.t.} \quad x + z_i \geq L_i \quad \text{and} \quad z_i \geq 0, \quad i = 1, \ldots, M. \tag{5}
\]

The optimal solution and the optimal value to problem (5) are the approximate VaR and CVaR as we have replaced the expectation by the sample average. These approximate VaR and CVaR tend to the exact VaR and CVaR as \( M \to \infty \). We investigate the computation of these approximate VaR and CVaR and their applications in liquidity risk analysis. From now on, the VaR and CVaR in the paper refer to these approximate VaR and CVaR, computed from (5).

We can solve problem (5) explicitly due to its special structure, and consequently we can get \( \text{VaR}_\alpha \) and \( \text{CVaR}_\alpha \) explicitly by simply sorting the samples.

**Theorem 1.** Let the \( M \) samples of loss random variable \( L \) be arranged in decreasing order \( L_1 \geq \cdots \geq L_M \). Let \( \alpha \in (0, 1) \) be a given percentile and \( N \) be the unique integer satisfying \( (1 - \alpha)M - 1 < N \leq (1 - \alpha)M \). Then the approximate VaR and CVaR are given by:

\[
\text{VaR}_\alpha = L_{N+1} \quad \text{and} \quad \text{CVaR}_\alpha = \gamma (L_1 + \cdots + L_N) + (1 - N\gamma) L_{N+1}
\]

where \( \gamma = (1/(1 - \alpha))(1/M) \). Furthermore, as \( M \to \infty \) the approximate VaR and CVaR tend to the exact VaR and CVaR defined in (1) and (2).

**Proof.** See Appendix.

Theorem 1 finds the approximate VaR and CVaR of all percentiles once a set of samples is generated and sorted, as the only difference with different \( \alpha \) is to choose different \( N \). When \( \alpha \) is close to 1 the number of samples should be sufficiently large to ensure a stable result. For example, if 100 samples are generated, then CVaR is the average of the first ten sorted samples and VaR is the eleventh sorted sample when \( \alpha = 0.9 \) whereas CVaR is the first sorted sample and VaR is the second sorted sample when \( \alpha = 0.99 \), which is bound to be unstable.

Theorem 1 can be used to find the approximate VaR and CVaR of a portfolio of securities, not necessarily a single security. Suppose there are \( n \) securities in a portfolio with \( L_i \) representing the loss of security \( i \). The loss of the portfolio is given by:

\[
L(w) = \sum_{i=1}^{n} w_i L_i
\]

where \( w_i \) are weights of securities in the portfolio. For fixed \( w \), we can find the VaR and CVaR of loss \( L(w) \) by first generating \( M \) samples of the joint distribution of \( (L_1, \ldots, L_n) \) say \( (L_{1k}, \ldots, L_{nk}) \) for \( k = 1, \ldots, M \) then sorting \( L_{k}(w) = \sum_{i=1}^{n} w_i L_{ik} \) into a decreasing sequence, say \( L_1(w) \geq \cdots \geq L_M(w) \), and finally applying Theorem 1 to get \( \text{VaR}_{\alpha}(w) \) and \( \text{CVaR}_{\alpha}(w) \), i.e.:

\[
\text{VaR}_{\alpha}(w) = \sum_{i=1}^{n} w_i L_{i,N+1} \quad \text{and} \quad \text{CVaR}_{\alpha}(w) = \sum_{i=1}^{n} w_i c_i(w), \tag{6}
\]

where \( c_i(w) = \gamma (L_{i1} + \cdots + L_{iN}) + (1 - N\gamma) L_{i,N+1} \).
As a byproduct we can also get the approximate marginal VaR and CVaR of $L(w)$ with respect to the weights $w$ under the mild condition that the sorted sequence $L_k(w)$ is strictly decreasing (as is typical). Then the perturbed loss sequence $L_k(w + \epsilon)$ keeps the same order as that of $L_k(w)$ if perturbation $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ has sufficiently small magnitude. We therefore have $c_i(w + \epsilon) = c_i(w)$ for all and:

$$\text{VaR}_\alpha(w + \epsilon) = \sum_{i=1}^{n} (w_i + \epsilon_i)L_{i,N+1}$$

and

$$\text{CVaR}_\alpha(w + \epsilon) = \sum_{i=1}^{n} (w_i + \epsilon_i)c_i(w).$$

This and (6) imply that the approximate gradients of VaR and CVaR are given by:

$$\nabla_w \text{VaR}_\alpha(w) = (L_{1,N+1}, \ldots, L_{n,N+1})^T,$$

$$\nabla_w \text{CVaR}_\alpha(w) = (c_1(w), \ldots, c_n(w))^T.$$
The discounted bid-price at time $T$ is given by $S_T X_T$ and the discounted loss of liquidating the asset at time $T$ is given by:

$$L = S_0 X_0 - S_T X_T.$$ 

The LVaR and the LCVaR of $L$ at level $\alpha$ are defined by (1) and (2), respectively. Since $S_T = S_0 e^{-(1/2)\sigma^2 T + \sigma W_T}$ we have by conditioning on $W_T$ that:

$$P(L \leq y) = \int_{-\infty}^{\infty} P \left( X_T \geq \left( X_0 - \frac{y}{S_0} \right) e^{(1/2)\sigma^2 T - \sigma \sqrt{T} z} \right) d\Phi(z).$$

Let $\tau_i$ be the $i$th jump time. Conditional on $j$ jumps over the interval $[0, T]$, i.e. $N_T = j$ the joint density function of $\tau_1, \ldots, \tau_j$ is given by:

$$f(u_1, \ldots, u_j | N_T = j) = j! T^{-j} 1_{\{0 < u_1 < \cdots < u_j < T\}}.$$ 

Let $X_{t,x}$ denote the strong solution to the SDE:

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dB_t$$

with the initial condition $X_0 = x$ and $F_t^{x,x}(y) = P(X_t^{x,x} \leq y)$ the corresponding distribution function. If there is no jump in the interval $[0, T]$ then the conditional distribution of $X_T$ is simply given by $P(X_T \leq y | N_T = 0) = F_T^{0,x}(y)$. We have by conditioning on the number of jumps $N_T$ that:

$$P(X_T \leq y) = \sum_{j=0}^{\infty} P(N_T = j) P(X_T \leq y | N_T = j)$$

and by conditioning on the jump times $\tau_1, \ldots, \tau_N$, that:

$$P(X_T \leq y | N_T = j) = j! T^{-j} \int_0^T \cdots \int_0^T P(X_T \leq y | \tau_i = u_i) du_j \cdots du_1$$

and by conditioning on the jump sizes $Y_1, \ldots, Y_N$ that:

$$P(X_T \leq y | \tau_i = u_i) = E_{\{Y_i\}_{i=1}^{j}} \left[ \int_R \cdots \int_R F_T^{u_{j-1} + \cdots + u_{j-1} + (1 + Y_{j-1}) x_{j-1}}(y) dF_{u_{j-1}}^{u_{j-1} + \cdots + u_{j-1} + (1 + Y_{j-1}) x_{j-1}}(x_j) \cdots dF_{u_1}^{0,x_0}(x_1) \right].$$

### OU jump diffusion process

Assume that $X_t$ follows a mean-reverting Ornstein-Uhlenbeck jump diffusion process (7) with $\mu(t, x) = k(x - \theta)$ and $\sigma(t, x) = \sigma$ where $k, \theta, \sigma$ are constants. It is known that the solution to (8) is given by:

$$X_{t,x}^{x,x} = xe^{-k(t-s)} + \theta(1 - e^{-k(t-s)}) + \sigma e^{-k(t-s)} \int_s^t e^{ku} dB_u,$$

and the distribution function of $X_t^{x,x}$ is given by:
We can express $X_T$ explicitly as follows.

**Theorem 2.** Let the number of jumps in interval $(0, T)$ be $N_T$ and jump times be $0 < \tau_1 < \cdots < \tau_{N_T} < T$ with $\tau_0 = 0$ and $\tau_{N_T} + 1 = T$. Then $X_T$ of the OU jump diffusion process is given by:

$$X_T = \mu_{N_T} + \bar{\sigma}e^{-kT} \sum_{n=1}^{N_T+1} U_{n,N_T} \int_{\tau_{n-1}}^{\tau_n} e^{ks} dB_s,$$

where $U_{n,j} = \prod_{i=j}^{1}(1 + Y_i)$ for $n \geq 1$ and $U_{n,j} = 1$ if $n > j$ by convention, and $u_j = U_{1,j}X_0 e^{-kT} + \bar{\sigma}e^{-kT} \sum_{n=1}^{j-1} U_{n,j}(e^{k\tau_n} - e^{k\tau_{n-1}})$. The conditional probability of (9) is equal to:

$$P(X_T \leq y|\tau_i = u_i, i = 1 \ldots, N_T) = E_{\{Y_i\}_{i=1}^{N_T}} \left[ \Phi \left( \frac{y - \mu_{N_T}}{\sigma_{N_T}} \right) \right]$$

where $\sigma^2_j = (\sigma^2/2k)e^{-2kT} \sum_{n=1}^{j+1} U_{n,j}^2(e^{2k\tau_n} - e^{2k\tau_{n-1}})$.

**Proof.** See Appendix. \hfill \Box

**CIR jump diffusion process.** Assume that $X_T$ follows a mean-reverting Cox-Ingersoll-Ross jump diffusion process (7) with $\mu(t, x) = -k(\theta - x)$ and $\sigma(t, x) = \bar{\sigma}\sqrt{x}$. It is known that there is no closed-form solution to (8) (Cox et al., 1985) and the distribution function of $X_T$ has a noncentral $\chi^2$ distribution:

$$F_{t,x}^r(y) = \chi^2 \left( \frac{4ky}{\sigma^2(1 - e^{-k(t-s)})}, \frac{4k\theta}{\sigma^2}, \frac{4kx}{\sigma^2(e^{k(t-s)} - 1)} \right),$$

where $\chi^2(x; n, \lambda)$ is the distribution function of a noncentral $\chi^2$ random variable with degrees of freedom and noncentral parameter $\lambda$.

**Remark.** The compensated GBM jump-diffusion process is given by (7) with $\mu(t, x) = -\beta x$ and $\sigma(t, x) = \bar{\sigma}x$ where $\beta = \tilde{E}(Y)$. Although it is simple and has a closed-form solution, it is not suitable for modelling the liquidity discount factor as $X_t$ increases (or decreases) at rate $-\beta \lambda$ (ignoring the diffusion effect) between jump times, which is not in line with the empirical observation that the bid-ask spread is stable and relatively flat in a normal market. To remove the obvious trend, one has to set the drift zero, but in doing so $X_t$ is no longer a martingale and is unlikely to move back to its original level after jumps, which is again at odds with the empirical observation that the liquidity discount factor tends to recover to the normal market level after market crash. Owing to these reasons we do not use the GBM jump diffusion process to model the liquidity discount factor process $X_t$.

Figure 1 displays sample paths of GBM, OU, and CIR jump diffusion processes. It is obvious that sample paths for the GBM jump diffusion process either have a clear trend between jumps if there is a compensator in the SDE or have no mean reversion after jumps if there is no compensator. Sample paths for OU and CIR jump diffusion processes are similar and both display the mean reversion property as expected.
Remark. In between jumps OU process $X_t$ is driven by Brownian motion and there is a positive probability that $X_t$ can be greater than 1 or less than 0, this is due to the nature of the Brownian motion, a well-known phenomenon for Gaussian interest rate models (Hull, 2000). To keep the liquidity discount factor process $X_t$ within the range of 0 and 1, we may use the reflected stochastic process. For example, the reflected OU process is modelled by:

$$dX_t = k(\theta - X_t)dt + \bar{\sigma}dW_t + dL_t - dU_t,$$

where both $L$ and $U$ are continuous nondecreasing processes with $L_0 = U_0 = 0$ and and increase only on the sets $\{t \in R^+ : X_t = 0\}$ and $\{t \in R^+ : Z_t = 1\}$. The reflected process $X_t$ is guaranteed to stay in between 0 and 1. The other possibility is to define the liquidity discount factor process as an exponential process $X_t = X_0 \exp(-Y_t)$ where $Y_t$ is a basic affine process:

$$dY_t = \bar{k}(\bar{\gamma} - Y_t)dt + \sigma \sqrt{Y_t}dB_t + \xi dN_t$$

and $N_t$ is a Poisson process with intensity $\lambda$ and $\xi$ is an exponential random variable with mean $\gamma$. All parameters $k, \bar{k}, \sigma, \lambda, \gamma$ are constants (Duffie and Singleton, 2003). Since $Y_t > 0$ a.s. the liquidity discount factor process $X_t$ takes values in range of 0 and 1. If there is a jump of size $\xi$ at time $t$ then the liquidity discount factor jumps downwards from $X_t^-$ to $X_t = e^{-\xi}X^t$.

4. Numerical tests
To find numerical values of LVaR and LCVaR one may apply the results of Rockafellar and Uryasev (2002) to solve a convex optimization problem (4) with the Monte Carlo method. In fact, if $S_T^i$ and $X_T^i, i = 1, \ldots, M$ are samples of random variables $S_T$ and $X_T$ set $L_i = S_0X_0 - S_T^iX_T^i$ sort $L_i$ in decreasing order, and apply Theorem 1 to find LVaR$_\alpha$ and LCVaR$_\alpha$.
Since $S_T$ follows a GBM process, it is easy to generate $S_T$. To generate $X_T$, we need to know the distribution of $X_T$ which is known for the mean reversion OU and CIR processes. In each simulation run, we first generate jump times $\tau_i$ and jump sizes $Y_i$, $i = 1, \ldots, N_T$, in the interval $[0, T]$. If $X_t$ follows an OU jump diffusion process, then we generate further $N_T + 1$-independent standard normal random variables $Z_n$, $n = 1, \ldots, N_T + 1$, and compute $X_T$ by (12) with the Ito integral:

$$
\int_{\tau_{i-1}}^{\tau_i} e^{ks} dB_s = \sqrt{\frac{(e^{2k\tau_i} - e^{2k\tau_{i-1}})}{(2k)}} Z_n.
$$

If $X_t$ follows a CIR jump diffusion process, then we generate recursively further $N_T + 1$ noncentral $\chi^2$ variables $X_{\tau_{i+1}}$ from the noncentral $\chi^2$ distribution function (14) with $s = \tau_i$, $x = X_{\tau_i} (1 + Y_i)$ and $t = \tau_{i+1}$ for $i = 0, \ldots, N_T$ and finally set $X_T = X_{\tau_{N_T}} + 1 -$. (Here, we denote $X_{\tau_0} (1 + Y_0) = X_0$ and $\tau_0 = 0$ and $\tau_{N_T} + 1 = T$.) Noncentral $\chi^2$ random variables can be generated with the algorithm discussed in Glasserman (2003).

Table I lists the values of LVaR and LCVaR with the OU and CIR jump diffusion processes. The parameters used represent a market in which the liquidity premium is small and stable (mean reversion level close to 1 and volatility close to 0), the liquidity dry up event is rare (once every five years on average), potential liquidity loss is severe (20-50 percent of asset value), the holding period is two weeks. The number of simulations is $= 100,000$.

Table I clearly shows the following outcomes:

- The OU and CIR jump diffusion processes produce very similar values for LVaR and LCVaR, which implies that one can essentially use either of these two models to compute liquidity adjusted risk measures.
- LCVaR is much larger than LVaR at 0.99 percentile, which implies that LVaR can still seriously underestimate the potential loss even after the jump liquidity risk is included, LCVaR is a more realistic potential loss indicator.
- LVaR and LCVaR are close at 0.999 percentile, which implies these two risk measures produce similar results at the extreme tail part of the loss distribution.

Table II shows that jump intensity $\lambda$ affects greatly the values of LVaR and LCVaR. When there are no jumps ($\lambda = 0$) LVaR and LCVaR are close to VaR and CVaR, the difference is mainly due to the CIR mean reversion diffusion process for the liquidity

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<th>CVaR</th>
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Table I.
LVaR and LCVaR for OU and CIR jump diffusion processes

Notes: Data: $S_0 = 100$, $\sigma = 0.2$, $X_0 = 1$, $k = 1$, $\theta = 0.98$, $\sigma = 0.02$, $\lambda = 0.2$, $T = 0.04$, $Y_i \sim U[-0.5, -0.2]$
discount factor. When $\lambda$ increases, both LVaR and LCVaR increase at different speed. For example, at 0.99 percentile, when $\lambda = 0.2$, LVaR is increased by 23 percent over the standard VaR, while LCVaR is increased by 194 percent over the standard CVaR, and the ratio of LCVaR to LVaR is about 2.7. This implies that the traditional VaR and CVaR are inappropriate risk measures in the presence of jump liquidity risk, and that one should take cautious views on the loss suggested by the LVaR as it may seriously underestimate the potential average loss for rare jump liquidity events.

Table III shows that as the holding period $T$ increases both LVaR and LCVaR increase and LVaR gives a good indication of the average loss. When the holding period $T$ is very short (e.g., one day) the LVaR, VaR, and CVaR all suggest a small loss. However, LCVaR points out a much larger loss. At 0.999 percentile, the ratio of LCVaR to LVaR is 6.0, which implies that if one manages the risk with the liquid asset suggested by VaR/CVaR/LVaR, one is possibly unable to withstand the potential severe loss. This sheds some light to the cause of the fall of the LTCM which had great difficulty in raising sufficient cash in a short spell of time to meet margin calls by liquidating the asset in a market where the liquidity essentially disappeared.

We have also tested cases for different mean-reversion rate $k$, mean-reversion level $\theta$, and volatility $\sigma$. We find that LVaR and LCVaR are not very sensitive to changes of these parameters. This is because over a short period (two weeks) the change caused by diffusion part of the liquidity discount factor process is small, but if there is a jump liquidity event, then there is little time to recover and the loss is likely to be substantial. On the other hand, LVaR and LCVaR are sensitive to the magnitude of the jump size.

5. Conclusion
We have suggested in this paper some plausible stochastic processes to model liquidity risk and discussed their impact on VaR and CVaR. We have shown that VaR, CVaR,

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Notes: Data: $S_0 = 100, \sigma = 0.2, X_0 = 1, k = 1, \theta = 0.98, \sigma = 0.02, T = 0.04, Y_i \sim U[-0.5, -0.2]$
and LVaR can seriously underestimate the potential loss over a short holding period for rare jump liquidity events. This has significant implication for short-term risk management, i.e. one should keep a much larger liquid asset reserve than the suggested VaR value to withstand the potential severe loss if a rare “bad” event does happen. The LTCM’s fall is a recent example to illustrate such a need. A better risk measure is the LCVaR which gives a more realistic loss estimation in the presence of the liquidity risk.

We have also suggested a simple and fast Monte Carlo method to compute approximate VaR and CVaR without having to solve nonlinear equations and to integrate tail expectations. The only work involved is to generate and sort samples of the loss distribution, which is sufficient to find VaR and CVaR of all percentiles, and their marginal values for a portfolio of securities.

Many questions remain to be answered, especially in the area of calibration, empirical studies, and correlation modelling. For example, how to calibrate the jump liquidity risk process to the market data? how good are these models in explaining market liquidity-crunch and crash behaviour? What is the correlation of the liquidity risk with other market risks such as credit risk? We will focus on these questions in our future work.

References

Further reading
Appendix: Proofs

Proof of Theorem 1. The dual problem for (5) is:

$$\max \, L_1 y_1 + \cdots + L_M y_M \quad \text{s.t.} \quad y_1 + y_2 + \cdots + y_M = 1, \quad 0 \leq y_i \leq \gamma, \quad i = 1, \ldots, M.$$  \hspace{1cm} (A1)

The choice of integer ensures that $N_\gamma \leq 1$ and $(N + 1)\gamma > 1$. Since $L_1 \geq \cdots \geq L_M$ we know the optimal solution to (A1) is:

$$y_1^* = y_2^* = \cdots = y_N^* = \gamma, \quad y_{N+1}^* = 1 - N_\gamma, \quad y_{N+2}^* = y_{N+3}^* = \cdots = y_M^* = 0.$$  \hspace{1cm} (A2)

The Lagrangian function for (A1) is given by:

$$L = -\sum_{i=1}^M L_i y_i + x \left( \sum_{i=1}^M y_i - 1 \right) + \sum_{i=1}^M z_i (y_i - \gamma) - \sum_{i=1}^M \mu_i y_i,$$

where $x, z_i, \mu_i, i = 1, \ldots, M$ are Lagrange multipliers. The optimal solution to the dual problem (A1) is characterized by the following Kuhn-Tucker conditions:

$$-L_i + x + z_i - \mu_i = 0,$$  \hspace{1cm} (A3)

$$z_i (y_i - \gamma) = 0,$$  \hspace{1cm} (A4)

$$\mu_i y_i = 0,$$  \hspace{1cm} (A5)

$$z_i \geq 0,$$  \hspace{1cm} (A6)

$$\mu_i \geq 0,$$  \hspace{1cm} (A7)

for $i = 1, \ldots, M$. Since the optimal solution to the dual problem is (A2) we can find the optimal Lagrange multipliers $x, z_i$ and $\mu_i, i = 1, \ldots, M$ from the Kuhn-Tucker conditions as follows:

- For $i = 1, \ldots, N$
  $$y_i^* = \gamma \Rightarrow \mu_i = 0 \Rightarrow -L_i + x + z_i = 0.$$  

- For $i = N + 2, \ldots, M$
  $$y_i^* = 0 \Rightarrow z_i = 0.$$  

- For $i = N + 1$
  $$y_{N+1}^* = 1 - N_\gamma \Rightarrow z_{N+1} = 0.$$  

- If $N_\gamma < 1$ then
  $$y_{N+1}^* > 0 \Rightarrow \mu_{N+1} = 0 \Rightarrow -L_{N+1} + x + z_{N+1} = 0 \Rightarrow x = L_{N+1}.$$  

and

- If $N_\gamma = 0$ then
  $$y_{N+1}^* = 0 \Rightarrow x + z_{N+1} = L_{N+1} + \mu_{N+1} \Rightarrow x = L_{N+1}.$$  

Since (a) and (A6) imply $y_i = L_i - x \geq 0, i = 1, \ldots, N$, must also satisfy $x \leq L_N$ as $\{L_i\}$ is a non-increasing sequence. The optimal solution to the primal problem (5) is not unique: $x^* \in [L_{N+1}, L_N], z_i^* = L_i - x^*, i = 1, \ldots, N$ and $z_i^* = 0, i = N + 1, \ldots, M.$
Rockafellar and Uryasev (2002) show that VaR is equal to the left endpoint of the optimal solution set, which implies VaR$_a = L_{N+1}$ whether the primal problem has a unique solution or not, and CVaR$_a = (1 - N_a) L_{N+1} + (\gamma L_1 + \cdots + L_N)$ is the corresponding optimal value. □

Proof of Theorem 2. We first use the induction method to prove (12). In fact, when $N_T = 0$, i.e. there is no jump in interval $[0, T]$, then (12) is the same as (10). Now assume that (12) is correct for $N_T = j - 1 (j \geq 1)$ and we only need to show that (12) is correct for $N_T = j$ too. Since there is no jump between the $j$th jump time $\tau_j$ and the terminal time the solution $X_T$ is given by:

$$X_T = X_{\tau_j} e^{-k(T-\tau_j)} + \theta (1 - e^{-k(T-\tau_j)}) + \tilde{\sigma} e^{-kT} \int_{\tau_j}^T e^{ks} dB_s.$$  \hspace{1cm} (A8)

On the other hand, since $\tau_j$ is a jump time and there are only $j - 1$ jumps in interval $[0, \tau_j)$ the induction assumption implies:

$$X_{\tau_j} = (1 + Y_j) X_{\tau_{j-1}} = (1 + Y_j) \left( \mu_{j-1} + \tilde{\sigma} e^{-k\tau_j} \sum_{n=1}^{j} U_{n,j-1} \int_{\tau_{n-1}}^{\tau_n} e^{ks} dB_s \right)$$

$$= U_{1,j} X_0 e^{-k\tau_j} + \theta e^{-k\tau_j} \sum_{n=1}^{j} U_{n,j} (e^{k\tau_n} - e^{k\tau_{n-1}}) + \tilde{\sigma} e^{-k\tau_j} \sum_{n=1}^{j} U_{n,j} \int_{\tau_{n-1}}^{\tau_n} e^{ks} dB_s.$$  \hspace{1cm} (A9)

Substituting $X_{\tau_j}$ into (A8) we see that (12) holds true for $N_T = j$.

Given jump sizes $Y_1, \ldots, Y_j$ $X_T$ defined in (12) is a normal variable with mean $\mu_j$ and variance $\sigma^2$. Here, we have used the independent increment property of a Brownian motion and the Ito isometry property. The conditional probability (9) is therefore given by (13).

We can also prove (13) by substituting (11) directly into (9). First, note that if $Z_1 \sim N(\mu_1, \sigma_1^2)$ and $Z_2 \sim N(0, \sigma_2^2)$ are two independent normal variables, then $c Z_1 + Z_2 (c$ is a constant) is a normal variable with mean $c \mu_1$ and variance $c^2 \sigma_1^2 + \sigma_2^2$, and:

$$P(c Z_1 + Z_2 \leq y) = \Phi \left( \frac{y - c \mu_1}{\sqrt{c^2 \sigma_1^2 + \sigma_2^2}} \right).$$

On the other hand, by conditioning on $Z_1$ we have:

$$P(c Z_1 + Z_2 \leq y) = \int_{-\infty}^{\infty} \Phi \left( \frac{y - cz}{\sigma_2} \right) \Phi \left( \frac{z - \mu_1}{\sigma_1} \right).$$

Therefore, the following relation holds:

$$\int_{-\infty}^{\infty} \Phi \left( \frac{y - cz}{\sigma_2} \right) d\Phi \left( \frac{z - \mu_1}{\sigma_1} \right) = \Phi \left( \frac{y - c \mu_1}{\sqrt{c^2 \sigma_1^2 + \sigma_2^2}} \right).$$  \hspace{1cm} (A9)

With the expression (11) for $F_{T,T}^x(y)$ and the relation (A9) we can get:

$$\int_{R} F_{T,T}^{u_j,(1+Y_{j})x_j}(y) dF_{u_j-1,(1+Y_{j-1})x_{j-1}}(x_j)$$

$$= \Phi \left( \frac{y - e^{-k(T-u_{j-1})} U_{j-1,j-1} x_{j-1} - \theta e^{-kT} \sum_{n=1}^{j+1} U_{n,j} (e^{2u_{n-1}} - e^{2u_n})}{\sqrt{\sigma^2 e^{-2kT} \sum_{n=1}^{j+1} U_{n,j}^2 (e^{2u_{n-1}} - e^{2u_n})/(2k)}} \right) \right)$$
Repeating the same argument, also noting $u_0 = 0, x_0 = X_0$, and $Y_0 = 0$, we get:

$$
\int_R \cdots \int_R \int \frac{1}{Z_0 \cdots Z_{j+1}} \cdot \prod_{i=1}^{j} \Phi\left(\frac{y_{ij} - \mu_j}{\sigma_j}\right) \cdot dF_{X_0}^{0,X_0}(x_1) = \Phi\left(\frac{y_{1j} - \mu_1}{\sigma_1}\right).
$$

Jump liquidity risk

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