Interaction of credit and liquidity risks: Modelling and valuation

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Abstract

In this paper we discuss the interaction of default risk and liquidity risk on pricing financial contracts. We show that two risks are almost indistinguishable if the underlying contract has non-negative values; however, if it can take both positive and negative values then these two risks demand different risk premiums depending on their loss rates and distributions. We discuss a structural default model and a discrete time default model with exponentially distributed liquidity shocks. We show that short-term yield spreads are dominated by liquidity risk rather than credit risk. We suggest a two-stage procedure to calibrate the model with one scalar optimization problem and one linear programming problem.

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1. Introduction

Investors of financial securities face many types of risks. The main ones are market risk, credit risk, and liquidity risk. The market risk is the risk of adverse deviation of the security value due to changes of market conditions, such as changes in stock
market indexes, exchange rates, and interest rates. The credit risk is either the default risk or the credit migration risk. The former refers to that a counterparty may fail to comply with its obligations. The latter refers to that the credit spread may change due to changes in the credit quality of a counterparty over time. The migration risk is irrelevant if one takes the buy-hold policy, and the only credit risk in that case is the default risk. The liquidity risk is characterized by the inability of selling (or buying) an asset in a timely and cost-effective manner, which can be measured by bid–ask spreads. Much financial mathematics literature has been focused on identifying and quantifying these risks for the purposes of valuation and hedging.

There has been extensive research literature on market risk and its impact on pricing and hedging financial derivatives, see Duffie (2001) and Hull (2002) and numerous references cited within. The credit risk has attracted great attention in recent years due to increasing market uncertainty and vulnerability and some high profile defaults. Many credit risk models have been suggested for pricing and hedging credit risky securities. These models can be broadly classified as either structural models (Black and Scholes, 1973; Merton, 1974; Black and Cox, 1976; Longstaff and Schwartz, 1995) or intensity models (Jarrow and Turnbull, 1995; Duffie and Singleton, 1999).

The structural models assume that firm assets follow some log-normal processes and the default is triggered when firm assets fall below a threshold level, called a default barrier, so equity is a call option on assets. The approach has appealing economic interpretations and is easy to compute implied default probabilities, yield spreads, and recovery rates, however, it has difficulty in producing realistic yield spreads of short-term defaultable securities. Great efforts have been made to remove this limitation, for example, assets are modelled with jump-diffusion processes (Zhou, 2001), or default barriers are driven by unobservable random variables and are unknown until the default has happened (Finger, 2002; Duffie and Lando, 2001; Giesecke, 2003).

The intensity models typically assume that the default time indicator follows some point process that admits an intensity process. If the indicator is a Poisson process, the default time is exponential. If the indicator is a Cox process, i.e., conditional on the realization of intensity process, the indicator is an inhomogeneous Poisson process, the default time is conditionally exponential. The intensity approach uses the information of yield spreads from the market to extract default probabilities and is useful to compare prices of credit derivatives, however, it is unable to explain reasons of defaults or to provide views contrary to the market. Intensity models in general deal with aggregate yield spreads and do not separate different forms of risks (e.g., credit risks vs. liquidity risks).\(^1\) See Duffie and Singleton (2003) and Schonbucher (2003) for recent surveys on credit risk modelling, pricing and implementation.

In contrast to market and credit risks, the liquidity risk has been largely unexplored. There is no commonly agreed definition for liquidity measure yet. Some people use direct measures such as bid–ask spreads, trade sizes and frequencies. Others

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\(^1\) A recent paper by Jarrow (2001) suggests an intensity model that incorporates both credit and liquidity risks where the liquidity premium is interpreted as a convenience yield which can be positive or negative depending on whether the underlying defaultable security is in a shortage or in a glut.
use indirect measures such as issued amount, coupon, listed, age, missing prices, price volatility, number of contributors, and yield dispersion. (See Houweling et al. (2003) for an empirical analysis on these indirect liquidity measures.) Since it is difficult to define the liquidity measure precisely, yield spreads are often attributed to pure credit risk. Huang and Huang (2003) show that credit risk can only explain part of observed yield spreads in empirical analysis. Liquidity crisis can cause severe losses to market participants in a short period, and the markets may crash in the extreme case (Geanakoplos, 2003). Understanding liquidity risk and its relation with credit risk has paramount importance in pricing and hedging defaultable illiquid securities.

There has been some research on liquidity risk modelling and pricing. Longstaff (2001) applies bounded variation trading strategies to study the behaviour of liquidity-constrained investors and to derive the shadow cost of illiquidity. Subramanian and Jarrow (2001) model the liquidity discount as an optimal stochastic impulse control problem that maximizes the expected utility of consumption. Longstaff (1995) models the liquidity as the value of option to sell at most favourable price for a given time period and derives the upper bound for liquidity discounts. Janosi et al. (2002) suggest an empirical model to estimate the liquidity risk, which is modeled as a convenience yield and is determined by spot rate and market index. Ericsson and Renault (2001) apply a binomial model for defaultable illiquid bonds and give some qualitative properties of liquidity yield spreads, including dominance over credit yield spreads on short maturity bonds and downward-sloping term structures. However, it is difficult to apply their model to market data due to some unobservable cost parameters that are involved.

The main contribution of this paper is that we suggest a new model that differentiates and integrates both default and liquidity risks. The liquidity measure used in the paper is the bid–ask spread. Our approach is similar in spirit to that of Hull (2002) in credit risk analysis, that is, for every defaultable and illiquid contract, there is an identical default-free and liquid contract whose value can be determined in an arbitrage free and complete economy, and the price difference is purely due to credit and liquidity risks. We do not attempt to provide an optimal control policy which maximizes some utility functions as in Longstaff (2001) and Subramanian and Jarrow (2001), instead we try to find the risk-neutral expected exposure when the distributions of the credit and liquidity risks are known. This approach enables us to use many models developed in credit risk analysis in pricing and hedging defaultable illiquid securities. In particular, we can combine structural models with intensity models to characterize default and liquidity risks. We attribute the short-term yield spread mainly to the liquidity risk premium instead of the credit risk premium.\(^2\)

\(^2\) This provides one explanation, among others, of the short-term yield spread. It basically says that the default is a predictable process. However, it is often argued that the default is not a predictable process. One evidence is that bond prices usually jump downwards upon default. This is true if we attribute yield spreads purely to credit risk. With liquidity risk in the place we may argue that upon default the demand for defaulted bonds evaporates and a heavy liquidity discount is needed to sell these defaulted bonds. In other words, the downward jump upon default is due to the liquidity premium and the default is then a predictable process, bar Enron accounting scandal.
can easily apply the model to value standard market instruments, and effectively calibrate the model to the market data with a two-stage optimization procedure.

The paper is organized as follows: Section 2 sets up the model for the valuation of risk-neutral expected credit and liquidity losses and the interaction of these two risks. Section 3 applies the model to standard market instruments, including bonds, FRAs, and interest rate swaps. Section 4 discusses distributions of default and liquidity processes and their impact on yield spreads of short-term maturity contracts. Section 5 calibrates the model with a two-stage optimization problem. Appendix A gives the proof of the main theorem.

2. The model

Assume there is a counterparty $B$ which issues a contract that is held by counterparty $A$, and we are interested in the exposure of $A$. Let $(\Omega, \mathcal{G}, P)$ be a risk-neutral probability space and $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ a filtration defined on $(\Omega, \mathcal{G}, P)$ with $\cup_{t \geq 0} \mathcal{F}_t \subseteq \mathcal{G}$. $\mathcal{F}_t$ represents the information generated by the observations of market variables (equity prices, interest rates, etc.) up to time $t$. Let $V_t$ be the value of a contract at time $t$ to party $A$ in an arbitrage free and complete economy. If the contract has the payoff $V_T$ at maturity $T$ then there exists a numeraire pair $(E, N)$ (Hunt and Kennedy, 2000), i.e., $E$ is the expectation operator and $N_t$ is the numeraire, such that the value $V_t$ at time $t \leq T$ is given by

$$V_t = N_tE(V_TN_T^{-1}|\mathcal{F}_t).$$

In this setting the only risk party $A$ faces is the market risk.

Let $\tau^1$ and $\tau^2$ be two non-negative random variables on $(\Omega, \mathcal{G}, P)$, which represents the arrival times of two events. Define two right-continuous processes $H^i_t = 1_{\{t \leq \tau^i\}}, i = 1, 2$, where $1_{\{t \leq \tau\}}$ is an indicator function which equals 0 for $t < \tau$ and 1 for $t \geq \tau$. Let $\mathcal{H} = (\mathcal{H}^i_t)_{t \geq 0}, i = 1, 2$, be two filtrations generated by processes $H^i_t$. The $\sigma$-fields $\mathcal{H}^i_t$ represent the information generated by the observations of $\tau^i$ up to time $t$ and $\tau^i$ are $\mathcal{H}^i$-stopping times. The arrival times $\tau^i$ may be driven by some other random variables (firm asset values, etc.), $\mathcal{H}^i_t$ then contain the information of these random variables.

2.1. Standing assumption

The random times $\tau^1$ and $\tau^2$ satisfy the following conditions: for any $T > 0$ and arbitrary $t_1, t_2 \in [0, T]$,

$$P(\tau^1 > t_1, \tau^2 > t_2 | \mathcal{F}_T) = P(\tau^1 > t_1 | \mathcal{F}_T)P(\tau^2 > t_2 | \mathcal{F}_T)$$

and

$$P(\tau^1 = \tau^2 | \mathcal{F}_T) = 0 \quad \text{a.s.}$$
(1) implies that $\tau^1$ and $\tau^2$ are independent given $\mathcal{F}_T$, which provides a simple but effective dependence structure of random variables and is used in CreditMetrics and other factor models (Schonbucher, 2000). (2) implies that $\tau^1$ and $\tau^2$ do not occur at the same time. (See Bielecki and Rutkowski (2002) for details on conditionally independent default times.)

Let $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ be an enlarged filtration with $\mathcal{G} \subset \mathcal{G}_t$, where $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}^1_t \vee \mathcal{H}^2_t$ is the $\sigma$-field generated by $\mathcal{F}_t$, $\mathcal{H}^1_t$, and $\mathcal{H}^2_t$. The filtration $\mathcal{F}_t$ contains the market information that is needed to value the contract in an arbitrage free and complete economy (e.g., $\mathcal{F}_t$ contains the information of risk free interest rates if the contract is a bond). The filtration $\mathcal{G}_t$ contains the market information that is needed to value the contract in an arbitrage free and incomplete economy (e.g., $\mathcal{G}_t$ contains the information of risk free interest rates, yield spreads, bid–ask spreads, asset values, transaction costs, etc., if the contract is a bond).

Assume now that counterparty $B$ is not default-free and the contract is an illiquid instrument. Counterparty $A$ is now subject to two additional risks: the credit risk (loss due to default of $B$) and the liquidity risk (loss due to high premium in liquidating the position). Let $\tau^1$ be the default time of $B$ and $\tau^2$ the liquidating time of $A$. Let $\tau^1$ and $\tau^2$ satisfy the standing assumption.

We want to know the expected exposure (loss) of $A$ at time $t \leq T$. Denote $\tau$ the first arrival time of two events, i.e., $\tau = \tau^1 \wedge \tau^2 = \min(\tau^1, \tau^2)$. The standing assumption implies that the conditional survival probability of $\tau$ is given by $P(\tau > u|\mathcal{F}_u) = P(\tau^1 > u|\mathcal{F}_u)P(\tau^2 > u|\mathcal{F}_u)$ for $u \in [0, T]$. If $\tau > T$ then neither $A$ closes the position nor $B$ defaults before the maturity of the contract. The exposure of $A$ is zero. If $t < \tau \leq T$ then either $B$ defaults or $A$ closes the position during the time interval $(t, T]$. The exposure of $A$ can be characterized by two cases:

Case 1: $\tau = \tau^1$, that is, $B$ defaults before $A$ closes the position. If $V_\tau$ is negative (the contract has positive value to $B$) then the contract is to be sold to a third party and there is no change to $A$’s position. The exposure of $A$ is zero. On the other hand, if $V_\tau$ is positive (the contract has negative value to $B$) then $B$ defaults the contract and the exposure of $A$ is $\delta^1 V_\tau$ where $\delta^1$ is a write-down rate. The exposure of $A$ at time $\tau$ is $\delta^1 V^+_{\tau}$ where $x^+ = \max(x, 0)$.

Case 2: $\tau = \tau^2$, that is, $A$ closes the position before $B$ defaults. If $V_\tau$ is positive, then due to the illiquidity of the contract, $A$ only gets the value $(1 - \delta^2) V_\tau$ in closing the position and the loss is $\delta^2 V_\tau$, where $\delta^2$ is a percentage transaction cost. On the other hand, if $V_\tau$ is negative, then $A$ has to pay $(1 + \delta^2) (-V_\tau)$ to close the position and the loss is $\delta^2 (-V_\tau)$. Assume there is also a fixed transaction cost $\delta^3$ to close the position. The exposure of $A$ at time $\tau$ is $\delta^2 |V_\tau| + \delta^3$.

We have the following result. (The proof is in Appendix A.)

**Theorem 1.** Let random times $\tau^1$ and $\tau^2$ satisfy the standing assumption and let $\delta^i$, $i = 1, 2, 3$, be exogenous non-negative constants. The risk-neutral expected exposure $L^0_t$ of a defaultable illiquid contract at time $t$ can be decomposed as $L^0_t = L^1_t + L^2_t$, where
\[ L_t^1 = \frac{1}{(1 - F_t^1)(1 - F_t^2)} N_t E \left( \int_{(t,T]} \delta^1 V_u^+ N_u^{-1} (1 - F_u^2) \, dF_u^1 | \mathcal{F}_t \right) \]

is the expected credit exposure and

\[ L_t^2 = \frac{1}{(1 - F_t^1)(1 - F_t^2)} N_t E \left( \int_{(t,T]} (\delta^2 |V_u| + \delta^3) N_u^{-1} (1 - F_u^2) \, dF_u^2 | \mathcal{F}_t \right) \]

is the expected liquidity exposure and \( F_{u,i} = P(\tau^i \leq u | \mathcal{F}_u) \) are the conditional distributions of random times \( \tau^i \), \( i = 1, 2 \). The value of the defaulatable illiquid contract at time \( t \) is equal to \( V_t - L_t^0 \). Furthermore, if \( \mathcal{H}^1, \mathcal{H}^2, \mathcal{F} \) are independent, then \( L_t^0 \) can be simplified to

\[ L_t^0 = \int_{(t,T]} V_t^1(u)(1 - F_t^2(u)) \, dF_t^1(u) + \int_{(t,T]} V_t^2(u)(1 - F_t^1(u)) \, dF_t^2(u), \tag{3} \]

where \( V_t^1(u) = N_t E(\delta^1 V_u^+ N_u^{-1} | \mathcal{F}_t) \), \( V_t^2(u) = N_t E((\delta^2 |V_u| + \delta^3) N_u^{-1} | \mathcal{F}_t) \), and \( F_t^i(u) = P(\tau^i \leq u | \tau^i > t) \), \( i = 1, 2 \).

**Remark 1.** If there is only credit risk but no liquidity risk then \( F_t^2 = P(\tau^2 \leq u | \mathcal{F}_u) = 0 \) for all \( u \in (0,T] \). The liquidity exposure \( L_t^2 \) is zero and the only exposure of the contract is the credit exposure \( L_t^1 \).

**Remark 2.** The total yield spread \( s_t^0 \), the credit yield spread \( s_t^1 \), and the liquidity yield spread \( s_t^2 \) between a defaulatable illiquid contract and a default-free liquid contract are defined by

\[ s_t^i = -\frac{1}{T-t} \ln \left( 1 - \frac{L_t^i}{V_t^i} \right), \quad i = 0,1,2. \]

Denote \( x^1 = L_t^1/L_t^0 \) and \( x^2 = L_t^2/L_t^0 \). We have

\[ s_t^1 + s_t^2 = -\frac{1}{T-t} \ln(x^2 + x^1 e^{-s_t^0(T-t)}) - \frac{1}{T-t} \ln(x^1 + x^2 e^{-s_t^0(T-t)}) \]

\[ = -\frac{1}{T-t} \ln(e^{-s_t^0(T-t)} + x^1 x^2 (e^{-s_t^0(T-t)} - 1)^2) \leq -\frac{1}{T-t} \ln(e^{-s_t^0(T-t)}) = s_t^0. \]

The equality holds if and only if \( x^1 = 0 \) or \( x^2 = 0 \), i.e., there is no credit risk or liquidity risk. Note that the total exposure \( L_t^0 \) is the sum of the credit exposure \( L_t^1 \) and the liquidity exposure \( L_t^2 \), but the total yield spread \( s_t^0 \) dominates the sum of the credit yield spread \( s_t^1 \) and the liquid yield spread \( s_t^2 \).

**Remark 3.** If the value of the contract \( V_u \) is non-negative for all \( u \) (e.g., bonds and options) and exogenous constants \( \delta^1 = \delta^2 = \delta \) and \( \delta^3 = 0 \), then the exposure can be simplified to
Furthermore, if \( F \) is positive and negative (e.g., swaps) or the exogenous constants \( \delta^j \) are different, then the credit exposure and the liquidity exposure are different measures and can be distinguished. This is one reason why many papers only focus on credit risk when studying corporate bonds, but not liquidity risk. We know, however, that if \( V_u \) can be positive and negative (e.g., swaps) or the exogenous constants \( \delta^j \) are different, then the credit exposure and the liquidity exposure are different measures and can be identified.

**Remark 4.** When \( \mathcal{H}^1 \) and \( \mathcal{H}^2 \) are not conditionally independent, we can either specify their joint distribution directly or use a copula to specify their dependence structure. These two approaches are essentially equivalent due to the Sklar theorem: Let \( X_1, \ldots, X_N \) be random variables with marginal distributions \( F_1, \ldots, F_N \) and joint distribution \( F \), then there exists a copula \( C \) such that for all \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \),

\[
F(x) = C(F_1(x_1), \ldots, F_N(x_N)).
\]

Furthermore, if \( F_1, \ldots, F_N \) are continuous, then \( C \) is unique. The copula approach highlights the dependence structure and has become an increasingly popular modeling technique. Many different copulas (product, normal, exponential, Archimedean, etc.) have been suggested and applied in the portfolio risk measurement and management (Embrechts et al., 2002). The impact on valuation of defaultable illiquid contracts when \( \mathcal{F} \), \( \mathcal{H}^1 \) and \( \mathcal{H}^2 \) are fully dependent is to be studied in a separate paper.
3. Exposure of standard market instruments

We assume in the rest of the paper that $\mathcal{H}^1$, $\mathcal{H}^2$, $\mathcal{F}$ are independent, and $\delta^3 = 0$.

3.1. Zero coupon bonds

Assume the contract is a ZCB with face value 1 and maturity $T$. If there is no default risk nor liquidity risk, then the value of ZCB at time $u \leq T$ is $V_u = D_{uT}$, the discount factor. If we choose the numeraire $N_u = D_{uT}$, then $V^1_u = \delta^1 D_{uT}$ and $V^2_u = \delta^3 D_{uT}$. The total exposure of counterparty $A$ at time $t < T$ is given by (3).

The credit and liquidity yield spreads are given by

$$s^1_t = -\frac{1}{T-t} \ln \left( 1 - \delta^1 \int_{(t,T)} (1 - F^2_t(u)) dF^1_t(u) \right),$$

$$s^2_t = -\frac{1}{T-t} \ln \left( 1 - \delta^2 \int_{(t,T)} (1 - F^1_t(u)) dF^2_t(u) \right),$$

respectively.

3.2. Forward rate agreements

Assume that the contract is a payers FRA that pays the fixed rate $K$ and receives the floating rate $L_{T_t}$ at time $T$ where $L_{T_0}$ is the LIBOR rate for the period $[T_0, T]$. The payoff of the contract at time $T$ is equal to $\alpha(L_{T_0} - K)$ where $\alpha = T - T_0$ is the tenor. Assume $t < T_0$, i.e., the FRA has not started at time $t$. (The case $t \geq T_0$ is trivial as the LIBOR rate $L_{T_0}$ is known at time $t$.) The value of FRA at time $u \in (t, T]$ is equal to $V_u = \alpha(L_{T_0} - K)D_{uT}$. If we use the numeraire $N_u = D_{uT}$ then $V^1_t(u) = \delta^1 \alpha D_{uT}E((L_{T_0} - K)\mid \mathcal{F}_t)$ and $V^2_t(u) = \delta^2 \alpha D_{uT}E(|L_{T_0} - K|\mid \mathcal{F}_t)$. The forward LIBOR rate $L_t$ for the period $[T_0, T]$ is defined by

$$L_t = \frac{D_{uT} - D_{uT}}{\alpha D_{uT}}$$

and $L_t$ is assumed to follow a log-normal martingale process

$$dL_t = \sigma L_t dW_t,$$

where $W_t$ is a standard Brownian motion and $\sigma$ is a constant volatility (Hunt and Kennedy, 2000). $V^1_t(u)$ is then equal to the value of a European call option on forward LIBOR rate $L_t$ with exercise price $K$ and exercise time $T_0$, therefore,

$$V^1_t(u) = \delta^1 \alpha D_{uT} \text{BSC}(L_t, K, t, T_0, \sigma),$$

where BSC$(S, K, t, T, \sigma)$ is the Black–Scholes European call value, i.e.,

$$\text{BSC}(S, K, t, T, \sigma) = S \Phi(d_1) - K \Phi(d_2)$$
and \( \Phi \) is the standard normal distribution function and

\[
d_1 = \frac{\ln(S/K)}{\sigma \sqrt{T-t}} + \frac{1}{2} \sigma \sqrt{T-t} \quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{T-t}.
\]

Since \( |LT_0 - K| = (LT_0 - K)^+ + (K - LT_0)^+ \) we have

\[
V^2_t(u) = \delta^2 zD_t \text{BSCP}(L_t, K, t, T_0, \sigma),
\]

where BSCP\((S, K, t, T, \sigma)\) is the sum of the Black–Scholes European call and put values, i.e.,

\[
\text{BSCP}(S, K, t, T, \sigma) = S \Phi(d_1) - K \Phi(d_2) + K \Phi(-d_2) - S \Phi(-d_1).
\]

The total exposure at time \( t < T_0 \) is given by (3).

### 3.3. Interest rate swaps

Assume the contract is a payers swap that pays the fixed rate \( K \) and receives the floating rate \( L_{t_{i-1}} \) at time \( t_i, i = 1, \ldots, n \), where \( L_{t_{i-1}} \) is the LIBOR rate for the period \([t_{i-1}, t_i] \). The payoff at time \( t_i \) is \( \alpha_i(L_{t_{i-1}} - K) \) where \( \alpha_i = t_i - t_{i-1} \) is the tenor. Since an interest rate swap is a portfolio of FRAs, its exposure can be defined as the sum of those of FRAs. However, such a definition overestimates the risk due to the following inequalities: \( \left( \sum x_i \right)^+ \leq \sum x_i^+ \) and \( |\sum x_i| \leq \sum |x_i| \). The FRAs that form the swap do tend to have both positive and negative values and therefore the total risk of a swap is less than the sum of individual risks of FRAs.

Assume \( t < t_0 \), i.e., the swap has not started at time \( t \). The swap at time \( u \in (t_{j-1}, t_j], j = 1, \ldots, n-1 \), is composed of two parts: a known payment \( \alpha_j(L_{t_{j-1}} - K) \) at time \( t_j \) and a new swap that starts at time \( t_j \) with payment dates \( t_{j+1}, \ldots, t_n \). The value of the known payment at time \( u \) is \( \alpha_j(L_{t_{j-1}} - K)D_{ut_j} \) and that of the new swap is \( D_{ut_j} - D_{ut_n} - K \sum_{k=j+1}^n x_k D_{ut_k} \). The value of the swap at time \( u \) is given by

\[
V_u = \alpha_j(L_{t_{j-1}} - K)D_{ut_j} + (y_u^f - K)P_u(t_j, t_n),
\]

where \( P_u(t_j, t_n) = \sum_{k=j+1}^n \alpha_k D_{ut_k} \) is the present value of a basis point and \( y_u^f \) is the forward swap rate for the period \([t_j, t_n] \) defined by

\[
y_u^f = \frac{D_{ut_j} - D_{ut_n}}{P_u(t_j, t_n)}.
\]

\( V^1_t(u) \) is defined to be

\[
V^1_t(u) = N_f E(\alpha_j(L_{t_{j-1}} - K)^+D_{ut_j}N_u^{-1}|\mathcal{F}_t) + N_f E((y_u^f - K)^+P_u(t_j, t_n)N_u^{-1}|\mathcal{F}_t).
\]

If we use the numeraire \( N_u = D_{ut_j} \) in the first term and the numeraire \( N_u = P_u(t_j, t_n) \) in the second term, and also assume that the forward LIBOR rate \( L_t^f \) and the forward swap rate \( y_u^f \) for the period \([t_j, t_n] \) follow some log-normal martingale processes with volatilities \( \sigma^L_t \) and \( \sigma^f_t \), respectively, we then get

\[
V^1_t(u) = \alpha_j D_{ut_j} E((L_{t_{j-1}} - K)^+|\mathcal{F}_t) + P_t(t_j, t_n) E((y_u^f - K)^+|\mathcal{F}_t) = \alpha_j D_{ut_j} \text{BSCP}(L^f_t, K, t, t_{j-1}, \sigma^L_t) + P_t(t_j, t_n) \text{BSCP}(y_u^f, K, t, u, \sigma^f_u).
\]
If \( u \in (t, t_0] \) then there is no payment at time \( t_0 \) and
\[
V^1_1(u) = P_t(t_0, t_u) \text{BSC}(v^0_t, K, t, u, \sigma^0_t).
\]
If \( u \in (t_{n-1}, t_n] \) then there is no swap that starts at \( t_n \) and
\[
V^1_1(u) = \alpha_n D_{t_n} \text{BSC}(L^n_k, K, t, t_{n-1}, \sigma^n_t).
\]

\( V^2_1(u) \) can be computed similarly and has the same expression as \( V^1_1(u) \) except that the Black–Scholes call value \( \text{BSC}(\cdot) \) is replaced by the sum of the Black–Scholes call and put values \( \text{BSCP}(\cdot) \). The total exposure \( L^0_t \) at time \( t < t_0 \) is given by (3).

4. Modelling of default and liquidity events

The previous sections discuss how to derive risk-neutral exposures of defaultable illiquid contracts provided distributions of default event and liquidity event are specified. We assume throughout this section that the liquidity time \( \tau^2 \) of counterparty \( A \) is an exponential random variable with parameter \( \lambda \), i.e., the conditional distribution function of \( \tau^2 \) is given by
\[
F^2_t(u) = P(\tau^2 \leq u | \tau^2 > t) = 1 - \exp(-\lambda(u - t)).
\]
(5)

This assumption is consistent with the findings of Ericsson and Renault (2001) and others.

4.1. First passage time default model

This model is suitable to study an individual firm where the asset process \( A_t \) of counterparty \( B \) follows a log-normal process:
\[
dA_t = rA_t dt + \sigma A_t dW_t,
\]
(6)
where \( r \) is the risk free interest rate, \( \sigma \) is the asset volatility, \( r \) and \( \sigma \) are constants, and \( W_t \) is a standard Brownian motion under \( \mathbb{P} \) with respect to \( \mathcal{F} \). The solution \( A_u \) to the Eq. (6) is given by
\[
A_u = A_0 e^{mu + \sigma W_u} = A_0 e^{m(u-t) + \sigma (W_u - W_t)}, \quad \forall u > t,
\]
where \( m = r - \frac{1}{2} \sigma^2 \) and \( A_0 \) is the asset value at time 0. The first passage time model assumes that counterparty \( B \) defaults if the asset value \( A_t \) is below an exogenously given constant boundary \( L \). The default time \( \tau^1 \) is the first hitting time of \( A_u \) to the barrier \( L \), i.e.,
\[
\tau^1 = \inf \{ u : A_u < L \}.
\]
For every \( u > t \), on the set \( \{ \tau^1 > t \} \), the conditional distribution function of \( \tau^1 \) is characterized by
\[ F_1^t(u) = P(\tau^1 \leq u | \tau_1 > t) = P(\min_{1 < i \leq u} A_i \leq L | \tau_1 > t) \]
\[ = P(\min_{1 < i \leq u} (s - t) + \sigma(W_s - W_t)) \leq \ln(L/A_t)) \]
\[ = \Phi(d_1^i(u)) + K_t \Phi(d_2^i(u)), \]  

(7)

where \( d_1^i(u) \), \( d_2^i(u) \), and \( K_t \) are defined by
\[ d_1^i(u) = \frac{\ln(L/A_t) - m(u - t)}{\sigma \sqrt{u - t}}, \]
\[ d_2^i(u) = \frac{\ln(L/A_t) + m(u - t)}{\sigma \sqrt{u - t}}, \]
\[ K_t = (L/A_t)^{2\varpi}. \]

Since \( \tau^1 > t \) we know the asset value of the firm \( A_t \) at time \( t \) is greater than \( L \), i.e., \( \ln(L/A_t) < 0 \), which implies \( d_1^i(u) \) and \( d_2^i(u) \) tend to \(-\infty\) and \( F_1^t(u) \) tends to 0 as \( u \) tends to \( t \). Therefore the default probability is negligible if the remaining life of the contract is very short.

Assume the contract is a ZCB with maturity \( T \). The credit yield spread \( s_1^T \) and the liquidity yield spread \( s_2^T \) are given by (4) where \( F_1^t(u) \) and \( F_2^t(u) \) are defined in (7) and (5), respectively. A simple calculation shows that \( \lim_{T \to \infty} s_1^T = 0 \) and \( \lim_{T \to \infty} s_2^T = \delta^2 \lambda \), which implies that the short-term yield spread is mainly due to the liquidity risk, not the credit risk. This intuitively makes sense. We do not in general expect a BB company to go bust in a day, but we may have difficulty to sell a BB company ZCBs in a day without incurring significant liquidity premium (wide bid–ask spreads).

One main difficulty in using the structural model is that we need to know the asset value \( A_t \) and asset volatility \( \sigma \) to estimate the exposure of the credit risk, however, both are unobservable. One approach is to use the Black–Scholes formula to derive the value and the volatility of the asset from those of the equity, which are observable market variables, see Hull (2002). The other approach is to use the distance to default measure to approximate the asset value and the asset volatility, see Finger (2002).

4.2. Discrete time default model

This model is suitable to study term structures of credit and liquidity yield spreads. Assume that the default event can only happen at times \( t_k \) with probabilities \( p_k, k = 1, \ldots, N + 1 \), where \( t_{N+1} = \infty \) denotes the event that default does not occur. At time \( t \in (t_{k-1}, t_k] \) (denote \( t_{-1} = 0 \)) the distribution function \( F^t_i \) is given by \( u > t \)
\[ F^t_i(u) = \sum_{k: t < t_k \leq u} q_k, \]

where \( q_k = p_k/(\sum_{i: t_k > p_i} \) is the conditional probability of the event \( \tau^1 = t_k \) given \( \tau^1 > t \) and, by convention, \( F^t_i(u) = 0 \) if there is no such \( k \) satisfying \( t < t_k \leq u \). The credit and liquidity yield spreads are given by (4) where
\[
\int_{(t,T)} (1 - F^2_t(u)) \, dF^1_t(u) = \sum_{k: \, t_k < T} \exp(-\delta_t(t_k - t)) q_k
\]

and

\[
\int_{(t,T)} (1 - F^1_t(u)) \, dF^2_t(u) = \int_t^T \sum_{k: \, t_k > t} q_k \, \lambda \exp(-\lambda(u - t)) \, du \\
= \sum_{k: \, t_k > t} \int_t^{t_k \land T} q_k \, \lambda \exp(-\lambda(u - t)) \, du \\
= \sum_{k: \, t_k > t} q_k \, (1 - \exp(-\lambda(t_k \land T - t))) \\
= 1 - \sum_{k: \, t_k > t} q_k \exp(-\lambda(t_k \land T - t)).
\]

If there is no credit risk, then we can set \( p_1 = \cdots = p_N = 0 \) and \( p_{N+1} = 1 \) (the probability corresponding to time \( t_{N+1} = \infty \)), which implies that \( q_k = 0 \) for all \( k \) such that \( t < t_k \leq T \) and \( q_{N+1} = 1 \). The forward yield spreads are given by \( s^1_t = 0 \) and \( s^2_t = -\frac{1}{T-t} \ln(1 - \delta^2 + \delta^2 \exp(-\lambda(T - t))) \). If there is no liquidity risk, then we can set \( \lambda = 0 \). The forward yield spreads are given by \( s^1_t = -\frac{1}{T-t} \ln(1 - \delta^1 \sum_{k: \, t < t_k \leq T} q_k) \) and \( s^2_t = 0 \).

Fig. 1 illustrates the contribution of the credit yield spread and the liquidity yield spread to the total yield spread when \( \delta^1 = 0.5 \), \( \delta^2 = 0.3 \), \( \lambda = 0.5 \), \( p_1 = 0.02 \), and \( p_k = 0.03 \) for \( k \geq 2 \). Short-term yield spreads are mainly due to the liquidity risk, but long-term yield spreads are dominated by the credit risk. By choosing suitable

![Fig. 1. Credit and liquidity yield spreads.](image-url)
default probabilities $p_1, \ldots, p_{N+1}$ and liquidity shock intensity $\lambda$, we can reproduce commonly seen term structures of yield spreads, including upward-sloping, downward-sloping, and humped ones.

5. Model calibration

To price defaultable illiquid bonds it is essential to have the information on credit and liquidity yield spreads. If we can estimate $\int_{(t,T]}(1-F^2_i(u))\,dF^1_i(u)$ and $\int_{(t,T]}(1-F^1_i(u))\,dF^2_i(u)$ then we can get $s^1_i$ and $s^2_i$ from (4) and derive the defaultable illiquid ZCB price as $P(t,T)(\exp(-s^1_i(T-t)) + \exp(-s^2_i(T-t)) - 1)$. This section focuses on how to estimate these conditional probabilities from the market data. To this end we need first to construct a benchmark yield curve. This can be achieved by choosing a set of default-free liquid bonds (e.g., Treasury securities) and applying the boot-strapping or linear programming (Allen et al., 2000) to these bonds to construct a zero yield curve. (If bonds do not cover all maturities then some spline functions may be needed to fill the gap.)

If there is a set of credit risky but liquid bonds on the market, we can use them to construct the corresponding zero yield curve, which can then be compared with the benchmark zero yield curve to get credit yield spreads. This method can be used to construct a family of credit yield spread curves of different credit qualities. The same approach can be used to find liquidity yield spreads if there is a set of default-free but illiquid bonds of the same level of illiquidity (e.g., similar percentage of bid–ask spreads) on the market.

In practice we may not be able to find sufficiently large number of bonds which contain only credit risk or liquidity risk. To derive yield spreads we have to use bonds with both risks. Assume the current time $t = 0$, the default can only happen at times $t_k$ with probabilities $p_k$, $k = 1, \ldots, N$, and the liquidity time follows an exponential distribution with parameter $\lambda$. Consider a defaultable illiquid ZCB with maturity $T = t_i$. Applying the result of Section 4, we know that $q_k = p_k$ and

$$
\int_{(0,t_i]} (1-F^2_i(u))\,dF^1_i(u) = \sum_{k=1}^j \exp(-\lambda t_k)p_k
$$

and

$$
\int_{(0,t_i]} (1-F^1_i(u))\,dF^2_i(u) = 1 - \sum_{k \geq 1} p_k \exp(-\lambda (t_k \wedge t_i))
= 1 - \sum_{k=1} p_k \exp(-\lambda t_k) - \sum_{k \geq t_i+1} p_k \exp(-\lambda t_i)
= 1 - \exp(-\lambda t_i) - \sum_{k=1}^i p_k (\exp(-\lambda t_k) - \exp(-\lambda t_i)).
$$

The price of a defaultable illiquid ZCB is given by $D_{0,t_i} - L_{t_i}$, i.e.,
\[
D_{0i}\left( g_{0i}(\lambda) + \sum_{k=1}^{i} p_k g_{ki}(\lambda) \right),
\]
where
\[
g_{0i}(\lambda) = 1 - \delta^2 + \delta^2 \exp(-\lambda t_i),
\]
\[
g_{ki}(\lambda) = -\delta^2 \exp(-\lambda t_k) + \delta^2 \exp(-\lambda t_k) - \delta^2 \exp(-\lambda t_i).
\]

Assume that we have a set \( J \) of defaultable illiquid bonds with the same level of the credit risk and the liquidity risk (e.g., the same credit rating and the same bid–ask spread), each bond \( j \in J \) has non-negative cash flow \( c_i^j \) at time \( t_i \) for \( i = 1, \ldots, N \). (Note that \( c_i^j \) may be zero for some \( i \) since \( t_i, i = 1, \ldots, N, \) are cash flow times for all bonds in set \( J \).) The price of bond \( j \) is given by
\[
V^j = \sum_{i=1}^{N} c_i^j D_{0i}\left( g_{0i}(\lambda) + \sum_{k=1}^{i} p_k g_{ki}(\lambda) \right)
= \sum_{i=1}^{N} c_i^j D_{0i} g_{0i}(\lambda) + \sum_{k=1}^{N} \sum_{i=k}^{N} c_i^j D_{0i} p_k g_{ki}(\lambda)
= H^j_0(\lambda) + \sum_{k=1}^{N} p_k H^j_k(\lambda),
\]
where
\[
H^j_0(\lambda) = \sum_{i=1}^{N} c_i^j D_{0i} g_{0i}(\lambda),
\]
\[
H^j_k(\lambda) = \sum_{i=k}^{N} c_i^j D_{0i} g_{ki}(\lambda).
\]

If \( \lambda = 0 \) then a simple calculation shows \( g_{0i}(0) = 1, \ g_{ki}(0) = -\delta^1 \), \( H^j_0(0) = \sum_{i=1}^{N} c_i^j D_{0i}, \) \( H^j_k(0) = -\delta^1 \sum_{i=1}^{N} c_i^j D_{0i}, \) and bond price \( V^j = \sum_{i=1}^{N} c_i^j D_{0i} (1 - \delta^1 \sum_{i=1}^{N} p_k). \)

If \( \lambda = \infty \) then \( g_{0i}(\infty) = 1 - \delta^2, \ g_{ki}(\infty) = 0, \) \( H^j_0(\infty) = (1 - \delta^1) \sum_{i=1}^{N} c_i^j D_{0i}, \)
\( H^j_k(\infty) = 0, \) and bond price \( V^j = (1 - \delta^2) \sum_{i=1}^{N} c_i^j D_{0i}. \) Therefore \( \lambda = 0 \) corresponds to pure credit risk model while \( \lambda = \infty \) to pure liquidity risk model.

In general, the observed market prices \( \hat{V}^j \) are different from those given by the formula above. We can formulate a nonlinear programming problem to find the default probabilities \( p_1, \ldots, p_N \) and the liquidity parameter \( \lambda \) that give the best prediction of bond prices to those of observed ones.

\[
\text{minimize} \quad \sum_{j \in J} (\alpha_j + \beta_j)
\]

such that \( \hat{V}^j + \alpha_j = H^j_0(\lambda) + \sum_{k=1}^{N} p_k H^j_k(\lambda) + \beta_j \) for all \( j \in J \)
\[\sum_{k=1}^{N} p_k \leq 1\]
\[ p_k \geq 0 \quad \text{for all } k = 1, \ldots, N \]
\[ x_j, \beta_j \geq 0 \quad \text{for all } j \in J \]
\[ \lambda \geq 0. \]  

Note that if \( \lambda \) is fixed then (8) is a linear programming problem with variables \( x_j, \beta_j, \) and \( p_k \). We can solve (8) efficiently as a two-stage problem with the first stage involving only one variable \( \lambda \) and the second stage a linear programming problem. In practice we may only need to test several \( \lambda \)'s to get the approximate optimal solution.

6. Conclusion

In this paper we discuss the interaction of default risk and liquidity risk on pricing financial contracts. Under the conditional independence assumption of the default time and the liquidation time we prove that the risk-neutral expected exposure of a defaultable illiquid contract can be decomposed as the sum of the credit exposure and the liquidity exposure. We explain that two risks are almost indistinguishable for bonds and options when the write-down rate is the same as the percentage transaction cost. We discuss a first passage time default model and a discrete time default model with exponentially distributed liquidity shocks. We show that the short-term yield spread is dominated by the liquidity risk rather than the credit risk. We suggest a calibration procedure with a nonlinear optimization problem which can be solved effectively with a scalar optimization problem and a linear programming problem.

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Appendix A

Proof of Theorem 1. If case 1 happens, then the credit exposure at time \( \tau \) is given by \( \delta^1 V_\tau^+ \). The risk-neutral expected credit exposure at time \( t \) is therefore equal to

\[ N_t E(1_{\{t < \tau \leq T, \tau = \tau \}} \delta^1 V_\tau^+ N_{\tau}^{-1} | \mathcal{F}_t). \]

Denote \( \mathcal{F}_t \lor \mathcal{H}_t^1 \) the \( \sigma \)-field generated by \( \mathcal{F}_t \) and \( \mathcal{H}_t^1 \) and \( h(u) = 1_{\{t, T\}}(u) \delta^1 V_u^+ N_u^{-1} \). Apply Lemma 5.1.2 of Bielecki and Rutkowski (2002) to get
\[ N_i E(1_{\{t \leq \tau_1 \leq \tau_2\}} \delta^1 V_{t_1}^+ N_{t_1}^{-1} | \mathcal{F}_i) = N_i E(1_{\{t < \tau_1 < \tau_2\}} h(\tau_1) | \mathcal{F}_i) \]
\[ = \frac{1}{P(\tau_2 > t | \mathcal{F}_i \cup \mathcal{H}_i^1)} N_i E(1_{\{t < \tau_1 < \tau_2\}} h(\tau_1) | \mathcal{F}_i \cup \mathcal{H}_i^1) \]
\[ = \frac{1}{P(\tau_2 > t | \mathcal{F}_i)} N_i E(1_{\{t < \tau_1 < \tau_2\}} h(\tau_1) | \mathcal{F}_i \cup \mathcal{H}_i^1) \].

The last equality is due to the conditional independence of \( \tau_1 \) and \( \tau_2 \) with respect to the filtration \( \mathcal{F} \). Applying the smoothing property of conditional expectation, the conditional independence of \( \tau_1 \) and \( \tau_2 \), Lemma 5.1.2 and Proposition 5.1.1 of Bielecki and Rutkowski (2002), we get
\[ N_i E(1_{\{t < \tau_1 < \tau_2\}} h(\tau_1) | \mathcal{F}_i \cup \mathcal{H}_i^1) = N_i E(E(1_{\{t < \tau_1 < \tau_2\}} h(\tau_1) | \mathcal{F}_i, \mathcal{H}_i^1) | \mathcal{F}_i \cup \mathcal{H}_i^1) \]
\[ = N_i E(h(\tau_1) P(\tau_2 > \tau_1 | \mathcal{F}_i, \mathcal{H}_i^1) | \mathcal{F}_i \cup \mathcal{H}_i^1) \]
\[ = N_i E(h(\tau_1) P(\tau_2 > \tau_1 | \mathcal{F}_i) | \mathcal{F}_i \cup \mathcal{H}_i^1) \]
\[ = \frac{1}{P(\tau_1 > t | \mathcal{F}_i) \mathcal{H}_i^1)} N_i E(h(\tau_1)(1 - F_{\tau_1}^2) | \mathcal{F}_i) \]
\[ = \frac{1}{P(\tau_1 > t | \mathcal{F}_i) \mathcal{H}_i^1)} N_i E(1_{\{t < \tau_1 < \tau_2\}} \delta^1 V_{t_1}^+ N_{t_1}^{-1}(1 - F_{\tau_1}^2) | \mathcal{F}_i) \]
\[ = \frac{1}{P(\tau_1 > t | \mathcal{F}_i) \mathcal{H}_i^1)} N_i E \left( \int_{\{t \leq \tau\}} \delta^1 V_{u}^+ N_{u}^{-1}(1 - F_{u}^2) dF_{u}^1 | \mathcal{F}_i \right). \]

The expected credit exposure is therefore given by
\[ \frac{1}{(1 - F_{\tau_1}^2)(1 - F_{\tau_2}^2)} N_i E \left( \int_{\{t \leq \tau\}} \delta^1 V_{u}^+ N_{u}^{-1}(1 - F_{u}^2) dF_{u}^1 | \mathcal{F}_i \right). \]

Using the same method we can get the expected liquidity exposure. \( \Box \)

If \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{F} \) are independent, then \( F_{\tau_1}^u = P(\tau_1 \leq u | \mathcal{F}_u) = P(\tau_1 \leq u) \). We can apply the Fubini theorem to interchange the integration and expectation operators to get the conclusion.

**References**


