The Valuation of the Basket CDS in a Primary-Subsidiary Model[1]

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Abstract This paper considers the valuation problem of basket CDSs. Based on the construction of total hazard rates, the paper develops the work of Zheng and Jiang [17] from the homogenous case to the primary-subsidiary heterogenous case in the interacting intensity framework, and obtains the corresponding joint density of the default time. Moreover, the paper derives the valuation formulae for the basket CDSs with and without counterparty risk. Numerical results robustly show that, under certain conditions, using the analytical pricing formulae derived in this paper is more efficient than the Monte Carlo method for the basket CDS valuation.

Keywords Default intensity, Contagion, Counterparty risk, Primary-subsidiary model, Basket CDS

§1 Introduction

Credit default swaps (CDSs) continue to play an important role in today's global economy (by Eraj Shirvani, ISDA Chairman), and its market keeps on growing even in recent financial crisis. The nontional amount had grown to \$26.5 trillion by July 2009 according to DTCC. In terms of basket CDS, the derivative often used to transfer the underlying credit risk of portfolios to the counterparty, its valuation problem is one of the most vital issues, principally because it can bring about far more convenience to the investors, who make efficient use of basket CDSs for the management of the corresponding credit risk of portfolios. This paper considers the pricing problem of basket CDSs, especially the kth-to-default swap's valuation.

Default correlation is one of the most important problems for pricing basket CDSs. Three methodologies, including the copula model, the conditional independence model as well as the contagion model, are widely used in literature. The copula model characterizes the correlation through a so called copula function, which links the joint distribution to the corresponding marginal distributions. See the book by Cherubini et al [4] for a detail. Li [11] is one of the first to apply the Gaussian copula to the default correlation problems. Other copulas, such as the T-copula and the Archimedean copula, were concerned

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later on, for example Demarta and Mcneil [5], and Shaw [14]. Different from the copula model, the conditional independence model characterizes correlation through a common factor, and conditioning on this common factor, all the default events are independent. One of the advantages of this model is the joint distribution of default times can be expressed semi-analytically, see Gregory [7] and Schonbucher [15]. However, this model cannot deal with the direct interaction of obligors, which resulting in underestimation of the portfolio credit risk, see Das et al [6] and the report by BCBS [2].

The contagion model throws light on the correlation of default through the construction of default intensity or hazard rate of each party. In fact, the default of one party often influences the intensities of others, which engenders the surging of the default rates of other underlying names. The general form of the intensity can be described as follows

$$\lambda_t^i = a_{i0} + \sum_{j=1, j\neq i}^{n} a_{ij} 1_{\{\tau_j \leq t\}}, \qquad i = 1, 2, \cdots, n \tag{1.1}$$

where τ_i denotes the default time of the company i, and λ_t^i denotes the corresponding default intensity. The parameters a_{i0} and a_{ij} represent the default parameter of the company i and the influence parameter by the company i respectively. We assume $a_{i0} > 0$ and a_{ij} such that λ^i is positive. (1.1) shows that the intensity of the company i will have a jump by the amount a_{ij} if the company j defaults. The sign of a_{ij} can be used to characterize the dependency of intensities between i and j.

Jarrow and Yu [9] are the avant-garde researchers who introduced such intensity model into the field of the credit risk of portfolios. However, they had to resort to the one-way credit risk to obtain the joint default probability of two companies due to the difficulty of the interactive correlation in the contagion model. Collin et al [3] employed the technique of changing measure to overcome the difficulty of the loop correlation and applied it to the bond pricing. Presuming on the result of Collin et al [3], Leung and Kwok [10] extended it to the case of three companies and obtained the value of the single-name CDS with counterparty risk. Yu [16] shed new light on the issue of the joint default probability of three companies through the construction of the total hazard rates which was introduced by Norros [12] and Shaked and Shanthikumar [13]. However, he had to reply on the Monte Carlo method when employing the construction of the total hazard rates to the basket CDS pricing. Herbertsson and Rootzen [8] employed the matrix analysis to price the basket CDS and got the corresponding valuation formula. This method, however, needs the inverse of the exponential matrix, which gives rise to computational difficulty especially with the large dimensional matrix. Zheng and Jiang [17] firstly derived the joint default probability of n companies based on the method of the construction of total hazard rates and applied it to the pricing of the basket CDS in the homogeneous case. In their model, the intensities of n underlying companies were assumed as follows

$$\lambda_t^i = a + b \sum_{j=1, j\neq i}^{n} 1_{\{\tau_j \leq t\}}, \qquad i = 1, 2, \cdots, n \tag{1.2}$$

Revealing the fact that the default time of the company $i, i = 1, \ldots, n$ can be linearly expressed through the corresponding n i.i.d. unit exponential random variables, and taking the advantage of the symmetry

of the intensities of n companies, they merely had to obtain the joint default probability in one region, and timed it by $n!$, making the result fairly succinct. Notwithstanding, if the intensities of n companies are totally different, the joint default probability has to be calculated within each region, and then the $n!$ joint default probabilities are added together to get the sum total. Although Zheng and Jiang [17] provided the permutation method to get the joint default probability of the n companies in heterogenous case, it is too hard to be employed as it involves myriads of summation elements.

Therefore, in order to provide a useful and feasible solution to pricing of portfolio credit derivatives such as kth-to-default swaps in the heterogeneous case, we consider the case of the primary-subsidiary model, which often caters to the real world. Specifically, the underlying comprises one primary company and n subsidiary companies, the intensities of which are given as follows

$$\lambda_t^{n+1} = a_1 + \sum_{j=1}^{n} b_1 1_{\{\tau_j \le t\}}, \tag{1.3}$$

$$\lambda_t^{i} = a_2 + \sum_{j=1, j\neq i}^{n} b_2 1_{\{\tau_j \le t\}} + c 1_{\{\tau_{n+1} \le t\}}, \qquad i = 1, 2, \ldots, n \tag{1.4}$$

where τ_{n+1} and τ_i denote the default times of the primary company and the subsidiary i company respectively. λ_t^{n+1} and λ_t^i denote the corresponding default intensities. We assume that $a_1 > 0, a_2 > 0$ and b_1, b_2, c are constants such that $\lambda_t^i (i = 1, \cdots, n+1)$ are all positive. We further assume that $c > |b_1|$ and $c > |b_2|$, which reflect the fact that the influence of the subsidiary's default is by no means greater than that of the primary's default. (1.3) indicates that the default intensity of the primary company is influenced by both its own default intensity and the default times of the other n subsidiaries, whereas (1.4) reveals that the default intensity of the subsidiary i company is affected by its own default intensity, the default times of the primary company and the rest $n-1$ subsidiaries.

The model introduced in this paper is obviously more useful for the investors holding the securities issued by the primary company. Moreover, it is feasible to obtain the joint default probability due to the convenience brought about by the symmetric assumption of the subsidiary companies. Based on Zheng and Jiang [17] and taking the advantage of the model introduced in this paper, we derive the valuation formulae for the basket CDS with and without counterparty risk in the heterogeneous case.

The paper is organized as follows, we firstly introduce Norros [12] and Shaked and Shanthikumar [13] methodology to construct the total hazard rates, and then obtain the joint default probability of the default times in the primary-subsidiary heterogeneous case in section 2. Furthermore, in section 3 we derive the density of the kth default time. Then we give the explicit pricing formulae for basket CDS with and without counterparty risk in section 4 and section 5 respectively. Numerical results are provided and analyzed in section 6. We conclude the paper in section 7, whereas the proofs of the formulae are supplied in the Appendix.

§2 Total hazard rate construction and the joint density of default times

In this section we will construct the total hazard rate of the default time, which was originally from Norros [12] and Shaked and Shanthikumar [13], and then the joint density of the default times is provided.

We consider an economy where uncertainty is represented by a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ models the default information over time, which is right-continuous and complete. For firm $i (i = 1, 2, \cdots, n+1)$, we model its default time as a totally accessible stopping time $\tau_i : \Omega \to (0, \infty]$ and its default as a counting process $1_{\{\tau_i \leq t\}}$. By Doob-Meyer decomposition, there exists a compensator A_t^i, which is continuous and increasing, such that

$$M_t^i = 1_{\{\tau_i \leq t\}} - A_t^i$$

is a uniformly integrable $(\mathcal{F}_t, P)$-martingale. We further assume A_t^i is absolutely continuous, and hence it processes an intensity λ_t^i satisfying $E[\int_0^t \lambda_s^i ds] < \infty$ for $t \geq 0$ such that

$$A_t^i = \int_0^{t\wedge\tau_i} \lambda_s^i ds.$$

Based on the intensity model introduced in section 1 and noting the explicit dependence of λ_t^i on other default times, we use the notation $\lambda_i(t|t_{J_k})$ to denote the intensity of the company i given the observed default times of k other companies $t_{j_1}, \cdots, t_{j_k}$ such that $0 = t_{j_0} < t_{j_1} < \cdots < t_{j_k} < t < \tau_i$, where $J_k = \{j_1, \cdots, j_k\} \subset \{1, 2, \cdots, n+1\}$. In this case, the company i's conditional intensity at the time t is

$$\lambda_i(t|t_{J_k}) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P(t < \tau_i \leq t + \Delta t | \tau_j = t_j, j \in J_k). \tag{2.1}$$

And its accumulated hazard rate during the time interval $[t_{j_k}, t]$ is

$$\Lambda_i(t - t_{j_k}|t_{J_k}) = \int_{t_{j_k}}^t \lambda_i(u|t_{J_k}) du. \tag{2.2}$$

Then define the total hazard rate accumulated by the company i by the time t given k observed defaults as follows

$$\psi_i(t|t_{J_k}) = \sum_{l=0}^{k-1} \Lambda_i(t_{j_{l+1}} - t_{j_l}|t_{J_l}) + \Lambda_i(t - t_{j_k}|t_{J_k}). \tag{2.3}$$

The result of Aalen and Hoem [1] shows that the total hazard rates accumulated by the firms until they default are independent unit exponential random variables. With the increasing process $\Lambda_i(t|t_{J_k})$, for any $x \geq 0$, we can associate the generalized inverse process $\Lambda_i^{-1}(x|t_{J_k})$, which is in the spirit of random change of time,

$$\Lambda_i^{-1}(x|t_{J_k}) = \inf\{s \geq 0 : \Lambda_i(s|t_{J_k}) \geq x\}. \tag{2.4}$$

Based on a collection of i.i.d unit exponential random variables $(E_1, \cdots, E_{n+1})$, we will construct a collection of random variables $(\hat{\tau}_1, \cdots, \hat{\tau}_{n+1})$ which are to be employed as the desired default times.

Step 1 Simulate i.i.d. unit exponential random variables $(E_1, \cdots, E_{n+1})$.

Step 2 Let

$$j_1 = \texttt{argmin}\{\Lambda_i^{-1}(E_i) : i = 1, \ldots, n+1\}$$

and let

$$\hat{\tau}_{j_1} = \Lambda_{j_1}^{-1}(E_{j_1}) \tag{2.5}$$

Define $J_1 = \{j_1\}$.

...

Step $k+1$ Provided that the first k steps are given, i.e. $\hat{\tau}_j, j \in J_k$ are already known, then for the $(k+1)$th step, let

$$j_{k+1} = \texttt{argmin}\{\Lambda_i^{-1}[E_i - \psi_i(\hat{\tau}_{j_k}|\hat{\tau}_{J_k})|\hat{\tau}_{J_k}] : i \notin J_k\}$$

and let

$$\hat{\tau}_{j_{k+1}} = \hat{\tau}_{j_k} + \Lambda_{j_{k+1}}^{-1}[E_{j_{k+1}} - \psi_{j_{k+1}}(\hat{\tau}_{j_k}|\hat{\tau}_{J_k})|\hat{\tau}_{J_k}] \tag{2.6}$$

Define $J_{k+1} = J_k \cup \{j_{k+1}\}$.

...

Continue this procedure until $J_{n+1} = \{1, 2, \cdots, n+1\}$.

Norros [12], Shaked and Shanthikumar [13] proved the above construction of $(\hat{\tau}_1, \cdots, \hat{\tau}_{n+1})$ are identical to $(\tau_1, \cdots, \tau_{n+1})$ in distribution. Therefore we are able to derive the joint density of $(\tau_1, \cdots, \tau_{n+1})$ through the joint density of $(\hat{\tau}_1, \cdots, \hat{\tau}_{n+1})$.

Next, based on the foregoing total hazard rate construction methodology, we will derive the joint density of the default times under the intensities modeled by (1.3) and (1.4). Thanks to the symmetry property of the subsidiary intensities, we can classify the issue into $(n+1)$ cases according to the position of the primary company's default time. Let

$$U_j = \{\omega|\hat{\tau}_1(\omega) < \cdots < \hat{\tau}_{j-1}(\omega) < \hat{\tau}_{n+1}(\omega) < \hat{\tau}_j(\omega) < \cdots < \hat{\tau}_n(\omega)\}, \quad j = 1, 2, \cdots, n+1$$

be the case where the primary company is the jth default. We will then give the joint densities of the default times in the $(n+1)$ cases respectively.

To start with, rewrite the intensity model for notation convenience

$$\lambda_i(t) = a_{i0} + \sum_{j=1, j\neq i}^{n+1} a_{ij} 1_{\{\tau_j \leq t\}}. \tag{2.7}$$

where $a_{i0} = a_2, a_{ij} = b_2, a_{ii} = 0, a_{n+1,0} = a_1, a_{n+1,j} = b_1, a_{i,n+1} = c$ for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$.

Given j, $(j = 1, 2, \ldots, n+1)$, for $\omega \in U_j$, i.e. $\hat{\tau}_1 < \cdots < \hat{\tau}_{j-1} < \hat{\tau}_{n+1} < \hat{\tau}_j < \cdots < \hat{\tau}_n$, according to the total hazard rate construction methodology, we can express default times $\hat{\tau}_1 < \cdots < \hat{\tau}_{j-1} < \hat{\tau}_{n+1} < \hat{\tau}_j < \cdots < \hat{\tau}_n$ in terms of standard exponential variables $(E_1, \ldots, E_{j-1}, E_{n+1}, E_j, \ldots, E_n)$, and vice versa

as follows

$$
\begin{aligned}
E_1 &= a_{10}\hat{\tau}_1 \\
E_2 &= a_{20}\hat{\tau}_2 + a_{21}(\hat{\tau}_2 - \hat{\tau}_1) \\
E_m &= a_{m0}\hat{\tau}_m + a_{m1}(\hat{\tau}_m - \hat{\tau}_1) + \cdots + a_{m,m-1}(\hat{\tau}_m - \hat{\tau}_{m-1}) \text{ for } m = 2, 3, \cdots, j-1 \\
E_{n+1} &= a_{n+1,0}\hat{\tau}_{n+1} + a_{n+1,1}(\hat{\tau}_{n+1} - \hat{\tau}_1) + \cdots + a_{n+1,j-1}(\hat{\tau}_{n+1} - \hat{\tau}_{j-1}) \\
E_j &= a_{j0}\hat{\tau}_j + a_{j1}(\hat{\tau}_j - \hat{\tau}_1) + \cdots + a_{j,j-1}(\hat{\tau}_j - \hat{\tau}_{j-1}) + a_{j,n+1}(\hat{\tau}_j - \hat{\tau}_{n+1}) \\
&\cdots \\
E_n &= a_{n0}\hat{\tau}_n + \cdots + a_{n,n-1}(\hat{\tau}_n - \hat{\tau}_{n-1}) + a_{n,n+1}(\hat{\tau}_n - \hat{\tau}_{n+1})
\end{aligned}
$$

The Jacobi determinant of $(E_1, \ldots, E_{j-1}, E_{n+1}, E_j, \ldots, E_n)$ with respect to $(\hat{\tau}_1, \ldots, \hat{\tau}_{j-1}, \hat{\tau}_{n+1}, \hat{\tau}_j, \ldots, \hat{\tau}_n)$ is given by

$$
\begin{aligned}
c^j &= \left|\det\left(\frac{\partial(E_1, \ldots, E_{j-1}, E_{n+1}, E_j, \ldots, E_n)}{\partial(\hat{\tau}_1, \ldots, \hat{\tau}_{j-1}, \hat{\tau}_{n+1}, \hat{\tau}_j, \ldots, \hat{\tau}_n)}\right)\right| \\
&= a_{10}(a_{20} + a_{21}) \cdots (a_{j-1,0} + \cdots + a_{j-1,j-2})(a_{n+1,0} + \cdots + a_{n+1,j-1}) \times \\
&\quad (a_{j0} + \cdots + a_{j,j-1} + a_{j,n+1}) \cdots (a_{n0} + \cdots + a_{n,n-1} + a_{n,n+1})
\end{aligned}
$$

The density of $(\hat{\tau}_1, \hat{\tau}_2, \cdots, \hat{\tau}_{n+1})$ for $\hat{\tau}_1 < \cdots < \hat{\tau}_{j-1} < \hat{\tau}_{n+1} < \hat{\tau}_j < \cdots < \hat{\tau}_n$ is therefore given by

$$
f^j(\hat{\tau}_1, \hat{\tau}_2, \cdots, \hat{\tau}_{n+1}) = c^j e^{-(E_1 + \cdots + E_{j-1} + E_{n+1} + E_j + \cdots + E_n)}.
$$

Substituting the expression of $(E_1, \cdots, E_{j-1}, E_{n+1}, E_j, \cdots, E_n)$ for f^j, we get the joint density for the case $0 < t_1 < \cdots < t_{j-1} < t_{n+1} < t_j \cdots < t_n$

$$
f^j(t_1, t_2, \cdots, t_{n+1}) = c^j e^{-\left(w_1^j t_1 + \cdots + w_{j-1}^j t_{j-1} + w_j^j t_{n+1} + \cdots + w_{n+1}^j t_n\right)}. \tag{2.8}
$$

where

$$
\begin{aligned}
w_1^j &= a_{10} - (a_{21} + \cdots + a_{n+1,1}) \\
w_2^j &= (a_{20} + a_{21}) - (a_{32} + \cdots + a_{n+1,2}) \\
w_m^j &= (a_{m0} + \cdots + a_{m,m-1}) - (a_{m+1,m} + \cdots + a_{n+1,m}) \text{ for } m = 2, 3, \cdots j-1 \\
w_j^j &= (a_{n+1,0} + \cdots + a_{n+1,j-1}) - (a_{j,n+1} + \cdots + a_{n,n+1}) \\
w_{j+1}^j &= (a_{j0} + \cdots + a_{j,j-1} + a_{j,n+1}) - (a_{j+1,j} + \cdots + a_{n,j}) \\
&\cdots \\
w_{n+1}^j &= (a_{n0} + \cdots + a_{n,n-1} + a_{n,n+1})
\end{aligned}
$$

Due to the notation in (2.7), $c^j, w_l^j, (l = 1, 2, \cdots, n+1)$ are hence as follows

$$
c^j = (a_1 + (j-1)b_1) \prod_{m=1}^{j-1}(a_2 + (m-1)b_2) \prod_{m=j+1}^{n+1}(a_2 + (m-2)b_2 + c) \tag{2.9}
$$

$$
\begin{aligned}
\omega_l^j &= [a_2 + (2l - n - 1)b_2 - b_1] + [-2b_2 + c + b_1]1_{\{l \geq j\}} \\
&+ [-a_2 + (n - 2j + 3)b_2 + (j - n - 2)c + (j-1)b_1 + a_1]1_{\{l = j\}}
\end{aligned} \tag{2.10}
$$

§3 The density of the kth default time

For the application of the joint default probability of the default times in the primary-subsidiary model to the valuation of basket CDSs, the density of the kth default time τ^k needs to be calculated beforehand, which will be considered in this section.

Let $F_k(t)$ be the distribution of τ^k, and let $F_{k,i}(t)$ be the default probability in which the company i is the kth default. $f_k(t)$ and $f_{k,i}(t)$ are the corresponding densities respectively. i.e. $F_k(t) = P(\tau^k \leq t), F_{k,i}(t) = P(\tau^k \leq t, \tau^k = \tau_i)$, and $f_k(t) = \frac{dF_k(t)}{dt}, f_{k,i}(t) = \frac{\partial F_{k,i}(t)}{\partial t}$.

Seizing on the joint density of the default times we got in section 2, the density of τ^k is given as the following lemma.

Lemma 3.1. *For any $k, j \in \{1, 2 \cdots, n+1\}$, provided that $\sum_{l=\eta+1}^{m} w_l^j \neq 0$ for any pair $(\eta, m), 0 \leq \eta < m \leq k-1$, the density of the kth default is as follows*

$$f_k(t) = n! \sum_{j=1}^{n+1} \sum_{\eta=0}^{k-1} \alpha(j, \eta; k) e^{-\psi(j,\eta;n+1)t}, \tag{3.1}$$

where $\psi(j, \eta; n+1)$ and $\alpha(j, \eta; k)$ are given by

if $j \geq k$, $\quad \psi(j, \eta; n+1) = (n-\eta)a_2 + \eta(n-\eta)b_2 + \eta b_1 + a_1,$

$$\alpha(j, \eta; k) = \frac{G(j, \eta; k, n, a_1, b_1, a_2, b_2, 0)\{\prod_{m=0}^{k-1}[a_2 + (\eta + m - n)b_2 - b_1]\}^{-1}}{(k-1-\eta)!\eta!(n-j+1)! \prod_{m=k+1}^{j}[(n-m+1)a_2 + (n-m+1)(m-1)b_2 + (m-1)b_1 + a_1]} \tag{3.1$'$}$$

if $j \leq k-1, \eta \leq j-1$, $\quad \psi(j, \eta; n+1) = (n-\eta)a_2 + \eta(n-\eta)b_2 + \eta b_1 + a_1,$

$$\alpha(j, \eta; k) = \frac{G(j, \eta; k, n, a_1, b_1, a_2, b_2, 0) \prod_{m=j+1}^{k}[a_2 + (m-2)b_2 + c]\{(n-k+1)! \prod_{m=0}^{j-1}[a_2 + (\eta + m - n)b_2 - b_1]\}^{-1}}{(j-1-\eta)!\eta! \prod_{m=j}^{k-1}[(m-\eta-1)a_2 + (m-\eta-1)(\eta+m-n-1)b_2 + (m-n-1)c + \eta b_1 + a_1]} \tag{3.1$''$}$$

if $j \leq k-1, \eta \geq j$, $\quad \psi(j, \eta; n+1) = (n-\eta+1)[a_2 + (\eta-1)b_2 + c],$

$$\alpha(j, \eta; k) = \frac{G(j, \eta; k, n+2, a_1, 0, a_2, b_2, c) \prod_{m=j+1}^{k}[a_2 + (m-2)b_2 + c]\{(n-k+1)! \prod_{m=j}^{k-1}[a_2 + (\eta + m - n - 2)b_2 + c]\}^{-1}}{(k-1-\eta)!(\eta-j)! \prod_{m=1}^{j}[(\eta-m)a_2 + (\eta-m)(\eta+m-n-2)b_2 + (\eta-n-1)c + (m-1)b_1 + a_1]} \tag{3.1$'''$}$$

$$G(j, \eta; k, n, a_1, b_1, a_2, b_2, c) = (-1)^{k-1-\eta}[a_1 + (j-1)b_1][a_2 + (2\eta - n)b_2 - b_1 + c] \prod_{m=1}^{j-1}[a_2 + (m-1)b_2].$$

In particular, as $a_1 = a_2 = a, b_1 = b_2 = c = b$, formula (3.1) then reduces to the expression determined

by Zheng and Jiang [17] for homogeneous contagion model:

$$f_k(t) = (n+1)! \sum_{\eta=0}^{k-1} \frac{(-1)^{k-1-\eta} \prod_{m=0}^{k-1} (a+mb) e^{-(n+1-\eta)(a+\eta b)t}}{(n+1-k)!\eta!(k-1-\eta)! \prod_{m=0, m\neq\eta}^{k-1} (a+(m+\eta-n-1)b)}.$$

Proof. Since in primary-subsidiary model the intensities of the companies are identical to each other except the primary company, for any $i \in \{0, 1, \cdots, k-1\}$,

$$\begin{aligned} P(\tau^i \leq t < \tau^{i+1}) &= n! \sum_{j=1}^{i} P(\tau_1 < \cdots < \tau_{j-1} < \tau_{n+1} < \cdots < t < \tau_i < \cdots < \tau_n) \\ &+ n! \sum_{j=i+1}^{n+1} P(\tau_1 < \cdots < \tau_i < t < \cdots < \tau_{j-1} < \tau_{n+1} < \cdots < \tau_n), \end{aligned}$$

where for $j \in \{1, 2 \cdots, i\}$,

$$\begin{aligned} & P(\tau_1 < \cdots < \tau_{j-1} < \tau_{n+1} < \cdots < t < \tau_i < \cdots < \tau_n) \\ = & \int_0^t dt_1 \cdots \int_{t_{j-2}}^t dt_{j-1} \int_{t_{j-1}}^t dt_{n+1} \int_{t_{n+1}}^t dt_j \cdots \int_{t_{i-2}}^t dt_{i-1} \int_t^\infty dt_i \cdots \int_{t_{n-1}}^\infty dt_n \times \\ & \qquad c^j e^{-(\omega_1^j t_1 + \cdots + \omega_{j-1}^j t_{j-1} + \omega_j^j t_{n+1} \cdots + \omega_{n+1}^j t_n)} \\ = & \int_0^t d\tilde{t}_1 \int_{\tilde{t}_1}^t d\tilde{t}_2 \cdots \int_{\tilde{t}_{i-1}}^t d\tilde{t}_i \int_t^\infty d\tilde{t}_{i+1} \cdots \int_{\tilde{t}_n}^\infty d\tilde{t}_{n+1} \; c^j e^{-(\omega_1^j \tilde{t}_1 + \omega_2^j \tilde{t}_2 + \cdots + \omega_{n+1}^j \tilde{t}_{n+1})} \\ = & A_i(t, \omega_1^j, \omega_2^j, \ldots, \omega_i^j) \left(\frac{c^j}{\prod_{m=i+1}^{n+1} (\sum_{l=m}^{n+1} \omega_l^j)} e^{-(\sum_{l=i+1}^{n+1} \omega_l^j)t} \right), \end{aligned}$$

and for $j \in \{i+1 \cdots, n+1\}$,

$$\begin{aligned} & P(\tau_1 < \cdots < \tau_i < t < \cdots < \tau_{j-1} < \tau_{n+1} < \cdots < \tau_n) \\ = & \int_0^t dt_1 \cdots \int_{t_{i-1}}^t dt_i \int_t^\infty dt_{i+1} \cdots \int_{t_{j-1}}^\infty dt_{n+1} \int_{t_{n+1}}^\infty dt_j \cdots \int_{t_{n-1}}^\infty dt_n \times \\ & \qquad c^j e^{-(\omega_1^j t_1 + \cdots + \omega_{j-1}^j t_{j-1} + \omega_j^j t_{n+1} \cdots + \omega_{n+1}^j t_n)} \\ = & \int_0^t d\tilde{t}_1 \int_{\tilde{t}_1}^t d\tilde{t}_2 \cdots \int_{\tilde{t}_{i-1}}^t d\tilde{t}_i \int_t^\infty d\tilde{t}_{i+1} \cdots \int_{\tilde{t}_n}^\infty d\tilde{t}_{n+1} \; c^j e^{-(\omega_1^j \tilde{t}_1 + \omega_2^j \tilde{t}_2 + \cdots + \omega_{n+1}^j \tilde{t}_{n+1})} \\ = & A_i(t, \omega_1^j, \omega_2^j, \ldots, \omega_i^j) \left(\frac{c^j}{\prod_{m=i+1}^{n+1} (\sum_{l=m}^{n+1} \omega_l^j)} e^{-(\sum_{l=i+1}^{n+1} \omega_l^j)t} \right). \end{aligned}$$

So

$$P(\tau^i \leq t < \tau^{i+1}) = n! \sum_{j=1}^{n+1} A_i(t, \omega_1^j, \omega_2^j, \ldots, \omega_i^j) \left(\frac{c^j}{\prod_{m=i+1}^{n+1} (\sum_{l=m}^{n+1} \omega_l^j)} e^{-(\sum_{l=i+1}^{n+1} \omega_l^j)t} \right),$$

where

$$A_i(t,\omega_1^j,\ldots,\omega_i^j)=\int_0^t\int_{t_1}^t\cdots\int_{t_{i-1}}^t e^{-(\omega_1^j t_1+\cdots+\omega_i^j t_i)}dt_1\ldots dt_i. \tag{3.2}$$

Using the induction method, (3.2) is showned to be

$$A_i(t,\omega_1^j,\ldots,\omega_i^j)=\sum_{\eta=0}^{i}\bar{\gamma}_{i,\eta}(\omega_1^j,\ldots,\omega_i^j)e^{-(\sum\limits_{l=\eta+1}^{i}\omega_l^j)t}, \tag{3.3}$$

where

$$\bar{\gamma}_{i,\eta}(\omega_1^j,\ldots,\omega_i^j)=\frac{(-1)^{i-\eta}}{\left\{\prod_{m=\eta+1}^{i}\left(\sum_{l=\eta+1}^{m}\omega_l^j\right)\right\}\cdot\left\{\prod_{m=1}^{\eta}\left(\sum_{l=m}^{\eta}\omega_l^j\right)\right\}},$$

For the product terms and the sum terms above, define $\prod_{l=\eta}^{i}a_l=1$ and $\sum_{l=\eta}^{i}a_l=0$ if $i<\eta$. In fact integrating with respect to (3.2), we get the recursive formula as follows

$$A_i(t,\omega_1^j,\ldots,\omega_i^j)=-\frac{1}{\omega_i^j}A_{i-1}(t,\omega_1^j,\ldots,\omega_{i-1}^j)+\frac{1}{\omega_i^j}A_{i-1}(t,\omega_1^j,\ldots,\omega_{i-2}^j,\omega_{i-1}^j+\omega_i^j),$$

(3.3) can be verified then by the induction method.

Next in terms of the density of the kth default, Since

$$f_k(t)=-\frac{\mathrm{d}P(\tau^k>t)}{\mathrm{d}t}=-\sum_{i=0}^{k-1}\frac{\mathrm{d}P(\tau^i\le t<\tau^{i+1})}{\mathrm{d}t},$$

and $\frac{\mathrm{d}}{\mathrm{d}t}A_i(t,\omega_1^j,\ldots,\omega_i^j)=e^{-\omega_i^j t}A_{i-1}(t,\omega_1^j,\ldots,\omega_{i-1}^j)$, we give $f_k(t)$ as follows

$$\begin{aligned}f_k(t)&=n!\sum_{j=1}^{n+1}A_{k-1}(t,\omega_1^j,\ldots,\omega_{k-1}^j)\left(\frac{c^j}{\prod\limits_{m=k+1}^{n+1}(\sum\limits_{l=m}^{n+1}\omega_l^j)}e^{-(\sum\limits_{l=k}^{n+1}\omega_l^j)t}\right)\\&=n!\sum_{j=1}^{n+1}\sum_{\eta=0}^{k-1}\frac{(-1)^{k-1-\eta}c^j e^{-(\sum\limits_{l=\eta+1}^{n+1}\omega_l^j)t}}{\left\{\prod\limits_{m=\eta+1}^{k-1}\left(\sum\limits_{l=\eta+1}^{m}\omega_l^j\right)\right\}\cdot\left\{\prod\limits_{m=1}^{\eta}\left(\sum\limits_{l=m}^{\eta}\omega_l^j\right)\right\}\left\{\prod\limits_{m=k+1}^{n+1}\left(\sum\limits_{l=m}^{n+1}\omega_l^j\right)\right\}}.\end{aligned}$$

For notation convenience, define $\alpha(j,\eta;k)$ and $\psi(j,\eta;n+1)$ as follows

$$\alpha(j,\eta;k)=\frac{(-1)^{k-1-\eta}c^j}{\left\{\prod\limits_{m=\eta+1}^{k-1}\left(\sum\limits_{l=\eta+1}^{m}\omega_l^j\right)\right\}\cdot\left\{\prod\limits_{m=1}^{\eta}\left(\sum\limits_{l=m}^{\eta}\omega_l^j\right)\right\}\left\{\prod\limits_{m=k+1}^{n+1}\left(\sum\limits_{l=m}^{n+1}\omega_l^j\right)\right\}}, \tag{3.4}$$

$$\psi(j,\eta;n+1)=\sum_{l=\eta+1}^{n+1}\omega_l^j. \tag{3.5}$$

Finally, the foregoing expression of $f_k(t)$ is well defined because of the assumption of the lemma. □

The foregoing formula for $f_k(t)$ is valid provided that the assumption of lemma 3.1 is satisfied. Specifically, none of the product terms in the denominator of $\alpha(j,\eta;k)$ can be nought. However, if there exists some η_0 such that some product terms in the denominator of $\alpha(j,\eta_0;k)$ are nought, $\alpha(j,\eta_0;k)$ is undefined in this situation. In order to circumvent this issue, we employ L'Hospital rule to redefine $\alpha(j,\eta;k)$ as follows[2]:

Lemma 3.2. *For any $k,j \in \{1,2\cdots,n+1\}$, there exist at most two product terms being zero in the denominator of $\alpha(j,\eta;k)$. Moreover, $f_k(t)$ is given as follows*

$$f_k(t) = n!\sum_{j=1}^{n+1}\sum_{\eta=0}^{k-1}\hat{\alpha}(j,\eta;k)e^{-\psi(j,\eta;n+1)t}, \tag{3.6}$$

where $\hat{\alpha}(j,\eta;k)$ is given by

(1) if there exists some $\eta_0 \in \{0,1,\cdots,k-1\}$ such that only one product term is zero, i.e. there exists some pair $(\eta_0,m_0), 0 \le \eta_0 < m_0 \le k-1$ such that $\sum\limits_{l=\eta_0+1}^{m_0} w_l^j = 0$, then

$$\hat{\alpha}(j,\eta_0;k) = \hat{\alpha}(j,m_0;k) = \frac{1}{2}[A_k^*(\eta_0,m_0,j) * (B_k^*(\eta_0,m_0,j)+t)] * \beta(j;k,n+1);$$

(2) if there exists some $\eta_1 \in \{0,1,\cdots,k-1\}$ such that two product terms are identical to nought, i.e. there exists some pair $(\eta_1,m_1,\bar{m}), 0 \le \eta_1 < m_1 < \bar{m} \le k-1$ such that $\sum\limits_{l=\eta_1+1}^{m_1} w_l^j = 0$, $\sum\limits_{l=m_1+1}^{\bar{m}} w_l^j = 0$, then

$$\begin{aligned} & \hat{\alpha}(j,\eta_1;k) = \hat{\alpha}(j,m_1;k) = \hat{\alpha}(j,\bar{m};k) \\ = \; & \frac{1}{3}[\bar{A}_k^*(\eta_1,m_1,\bar{m},j) * [\bar{B}_k^*(\eta_1,m_1,\bar{m},j) - t(\frac{t}{2} - \bar{C}_k^*(\eta_1,m_1,\bar{m},j))]] * \beta(j;k,n+1); \end{aligned}$$

(3) if none of the product terms is zero, then

$$\hat{\alpha}(j,\eta;k) = \alpha(j,\eta;k);$$

where

$$A_k^*(\eta_0,m_0,j) = (-1)^{k-\eta_0+1}\left[\widetilde{\phi}_{j,\eta_0,m_0}^k\right]^{-1},$$

$$B_k^*(\eta_0,m_0,j) = \left[\sum_{i=1}^{\eta_0}\frac{1}{\varphi_{i,\eta_0}^j} - \sum_{i=\eta_0+1}^{m_0-1}\frac{1}{\varphi_{\eta_0+1,i}^j} - \sum_{i=m_0+1}^{k-1}\frac{1}{\varphi_{m_0+1,i}^j}\right],$$

$$\bar{A}_k^*(\eta_1,m_1,\bar{m}j) = (-1)^{k-\eta_1+1}\left[\bar{\phi}_{j,\eta_1,m_1,\bar{m}}^k\right]^{-1},$$

[2]Due to the limited space, we omit the proof of lemma 3.2. The readers interested in the proof can consult the authors about the detail.

$$
\begin{aligned}
&\bar{B}_k^*(\eta_1, m_1, \bar{m}j) = \\
&\left\{ \left(\sum_{i=\eta_1+1}^{m_1-1} \frac{1}{\varphi^j_{\eta_1,i}} \right)^2 - \left(\sum_{i=\eta_1+1}^{m_1-1} \frac{1}{\varphi^j_{\eta_1,i}} \right) \left(\sum_{\delta=1}^{\eta_1} \frac{1}{\varphi^j_{\delta,\eta_1}} \right) + \left(\sum_{i=1}^{\eta_1} \frac{1}{\varphi^j_{i,\eta_1}} \right)^2 \right\} \\
&-\frac{1}{2} \left\{ \sum_{i=m_1+1}^{\bar{m}-1} \sum_{\substack{\delta=m_1+1 \\ \delta\neq i}}^{\bar{m}-1} \left(\frac{1}{\varphi^j_{m_1+1,i}} \frac{1}{\varphi^j_{m_1+1,\delta}} \right) + \sum_{i=\eta_1+1}^{m_1-1} \sum_{\substack{\delta=\eta_1+1 \\ \delta\neq i}}^{m_1-1} \left(\frac{1}{\varphi^j_{\eta_1+1,i}} \frac{1}{\varphi^j_{\eta_1+1,\delta}} \right) \right. \\
&\left. + \sum_{i=1}^{\eta_1} \sum_{\substack{\delta=1 \\ \delta\neq i}}^{\eta_1} \left(\frac{1}{\varphi^j_{i,\eta_1}} \frac{1}{\varphi^j_{\delta,\eta_1}} \right) + \sum_{i=\bar{m}+1}^{k-1} \sum_{\substack{\delta=\bar{m}+1 \\ \delta\neq i}}^{k-1} \left(\frac{1}{\varphi^j_{\bar{m}+1,i}} \frac{1}{\varphi^j_{\bar{m}+1,\delta}} \right) \right\} \\
&+ \left\{ \left(\sum_{i=\bar{m}+1}^{k-1} \frac{1}{\varphi^j_{\bar{m}+1,i}} \right) \left(\sum_{\delta=\eta_1+1}^{m_1-1} \frac{1}{\varphi^j_{\eta_1+1,\delta}} \right) + \left(\sum_{i=m_1+1}^{\bar{m}-1} \frac{1}{\varphi^j_{m_1,i}} \right) \left(\sum_{\delta=\eta_1+1}^{m_1-1} \frac{1}{\varphi^j_{\eta_1+1,\delta}} \right) \right. \\
&\left. + \left(\sum_{i=m_1+1}^{\bar{m}-1} \frac{1}{\varphi^j_{m_1+1,i}} \right)^2 - \left(\sum_{i=m_1+1}^{\bar{m}-1} \frac{1}{\varphi^j_{m_1+1,i}} \right) \left(\sum_{\delta=1}^{\eta_1} \frac{1}{\varphi^j_{\delta,\eta_1}} \right) - \left(\sum_{i=\bar{m}+1}^{k-1} \frac{1}{\varphi^j_{\bar{m}+1,i}} \right) \left(\sum_{\delta=1}^{\eta_1} \frac{1}{\varphi^j_{\delta,\eta_1}} \right) \right\} \\
&+ \left\{ \left(\sum_{i=\bar{m}+1}^{k-1} \frac{1}{\varphi^j_{\bar{m}+1,i}} \right)^2 + \left(\sum_{i=m_1+1}^{\bar{m}-1} \frac{1}{\varphi^j_{m_1+1,i}} \right) \left(\sum_{\delta=\bar{m}+1}^{k-1} \frac{1}{\varphi^j_{\bar{m}+1,\delta}} \right) \right\},
\end{aligned}
$$

$$
\bar{C}_k^*(\eta_1, m_1, \bar{m}, j) = \left\{ \left(\sum_{i=1}^{\eta_1} \frac{1}{\varphi^j_{i,\eta_1}} \right) - \left(\sum_{i=\eta_1+1}^{m_1-1} \frac{1}{\varphi^j_{\eta_1+1,i}} \right) - \left(\sum_{i=m_1+1}^{\bar{m}-1} \frac{1}{\varphi^j_{m_1+1,i}} \right) - \left(\sum_{i=\bar{m}+1}^{k-1} \frac{1}{\varphi^j_{\bar{m}+1,i}} \right) \right\},
$$

$$
\beta(j; k, n+1) = \frac{[a_1 + (j-1)b_1] \prod_{m=1}^{j-1} [a_2 + (m-1)b_2] \prod_{m=j+1}^{k} [a_2 + (m-2)b_2 + c]}{(n-k+1)!}.
$$

$\alpha(j, \eta; k), \psi(j, \eta; n+1)$ are the same with the ones in lemma 3.1, and $\widetilde{\phi}^k_{j,\eta_0,m_0}, \bar{\phi}^k_{j,\eta_1,m_1,\bar{m}}, \varphi^j_{\eta_1,\eta_2}$ are provided in appendix A.

Lemma 3.3. *For any $k, i \in \{1, 2 \cdots, n+1\}$, the density $f_{k,i}(t)$ is given as follows*

$$
f_{k,n+1}(t) = n! \sum_{\eta=0}^{k-1} \hat{\alpha}(k, \eta; k) e^{-\psi(k,\eta;n+1)t}, \tag{3.7}
$$

$$
f_{k,i}(t) = (n-1)! \sum_{\substack{j=1 \\ j\neq k}}^{n+1} \sum_{\eta=0}^{k-1} \hat{\alpha}(j, \eta; k) e^{-\psi(j,\eta;n+1)t}, \quad i = 1, 2, \cdots n, \tag{3.8}
$$

where $\hat{\alpha}(j, \eta; k), \psi(j, \eta; n+1)$ are the same with the ones in lemma 3.2.

The proofs of lemma 3.3 is provided in appendix B.

§4 Basket CDS pricing without counterparty risk

In this section, we will price the kth-to-default basket CDS without counterparty risk in the primary-subsidiary model.

Consider the contract consisting of $(n+1)$ underlying names, one of which is the primary company $(n+1)$, and the others of which are subsidiaries $1,\ldots,n$. Their default times are denoted to be $\tau_1,\ldots,\tau_{n+1}$. Let τ^k be the kth default time, and let X_k be the premium rate of the basket CDS contract, which is paid at the time $t_m,(m=1,\ldots,N)$, where $0=t_0<t_1<\cdots<t_N=T$ and T is the maturity of the contract. Denote $\Delta_i=t_m-t_{m-1}$ and $R_i,(i=1,\ldots,n+1)$ to be the time interval of the premium payment and the recovery rate of the company i respectively. r is the risk-free interest rate, and δ is the settlement period.

On one hand, if the kth default time is prior to the maturity of the contract, the seller will undertake the default loss of this kth underlying name. So the present value of the contingent leg is equal to

$$C_k=\sum_{i=1}^{n+1}(1-R_i)[1_{\{\tau^k\le T\}}1_{\{\tau^k=\tau_i\}}e^{-r(\delta+\tau^k)}], \tag{4.1}$$

On the other hand, the buyer of the contract will have to pay the seller for the premium until the kth default happens. So the present value of the fee leg is equal to

$$F_k=\sum_{m=1}^{N}\left[X_k\Delta_m e^{-rt_m}1_{\{\tau^k>t_m\}}+X_k\left(\tau^k-t_{m-1}\right)e^{-r\tau^k}1_{\{t_{m-1}<\tau^k\le t_m\}}\right]. \tag{4.2}$$

The value of the premium rate X_k is determined by $E(C_k)=E(F_k)$, so X_k can be expressed as follows：

$$X_k=\frac{\sum\limits_{i=1}^{n+1}(1-R_i)E[1_{\{\tau^k\le T\}}1_{\{\tau^k=\tau_i\}}e^{-r(\delta+\tau^k)}]}{\sum\limits_{m=1}^{N}\left[\Delta_m e^{-rt_m}E[1_{\{\tau^k>t_m\}}]+E[(\tau^k-t_{m-1})e^{-r\tau^k}1_{\{t_{m-1}<\tau^k\le t_m\}}]\right]}. \tag{4.3}$$

Since $F_k(t)=P(\tau^k\le t)$ and $F_{k,i}(t)=P(\tau^k\le t,\tau^k=\tau_i)$, X_k in fact is：

$$X_k=\frac{\sum\limits_{i=1}^{n+1}(1-R_i)\int\limits_0^T e^{-r(\delta+s)}dF_{k,i}(s)}{\sum\limits_{m=1}^{N}\left[\Delta_m e^{-rt_m}\int_{t_m}^{+\infty}dF_k(s)+\int\limits_{t_{m-1}}^{t_m}(s-t_{m-1})e^{-rs}dF_k(s)]\right]}, \tag{4.4}$$

It is showned in (4.4) that the essential step to get the premium is to compute $F_k(t)$ and $F_{k,i}(t)$, both of which can be derived through integrating with respect to (3.6) (3.7) and (3.8) respectively.

Theorem 4.1. *Let the hazard rate processes $\lambda_t^i,(i=1,2,\cdots,n+1)$, be given by (1.3) and (1.4). Then the present value of the contingent leg and the fee leg can be expressed respectively as follows*

$$\begin{aligned}E(C_k)\quad&=n!(1-R_{n+1})e^{-r\delta}\sum_{\eta=0}^{k-1}\hat{\alpha}(k,\eta;k)\left\{\frac{1}{r+\psi(k,\eta;n+1)}\left[1-e^{-[r+\psi(k,\eta;n+1)]T}\right]\right\}\\&+n!(1-\frac{\sum\limits_{i=1}^{n}R_i}{n})e^{-r\delta}\sum_{\substack{j=1\\j\neq k}}^{n+1}\sum_{\eta=0}^{k-1}\hat{\alpha}(j,\eta;k)\left\{\frac{1}{r+\psi(j,\eta;n+1)}\left[1-e^{-[r+\psi(j,\eta;n+1)]T}\right]\right\},\end{aligned} \tag{4.5}$$

$$
\begin{aligned}
E(F_k) &= X_k n! \sum_{m=1}^{N} \sum_{j=1}^{n+1} \sum_{\eta=0}^{k-1} \hat{\alpha}(j,\eta;k) \left[r+\psi(j,\eta;n+1)\right]^{-2} \times \\
&\left\{ e^{-[r+\psi(j,\eta;n+1)]t_{m-1}} + \left[\frac{r[r+\psi(j,\eta;n+1)]\Delta_m}{\psi(j,\eta;n+1)} - 1 \right] e^{-[r+\psi(j,\eta;n+1)]t_m} \right\},
\end{aligned} \tag{4.6}
$$

where $\hat{\alpha}(j,\eta;k)$ and $\psi(j,\eta;n+1)$ are the same with the ones in lemma 3.2.

§5 Basket CDS valuation with counterparty risk

Counterparty risk is one of the underlying causes of recent financial crisis, and a direct focus on the quality and transparency of balance sheets of sellser such as banks would be appropriate, either by requiring more transparency, dealing directly with the increasing number of CDS defaults, or looking for ways to bring more capital into the banks and other financial institutions. In this section. We will consider the counterparty risk, meaning that the seller also has the default intensity, which will surge whenever the kth default of the underlying name occurs.

Consider the same contract as in section 4. For the contract seller, denote τ^B to be its default time. The seller's default intensity is characterized as follows

$$
\lambda_t^B = \lambda_0 + \sum_{i=1}^{n+1} \bar{\lambda}_i 1_{\{\tau^k \le t, \tau^k = \tau_i\}}, \tag{5.1}
$$

where λ_0 is seller's own default intensity, whereas $\bar{\lambda}_i, i = 1, \ldots, n+1$ is the influence of the company i provided that it is the kth default.

On one hand, if the kth default time is prior to the maturity T, and the seller does not default before the settlement period, the seller will undertake the default loss of this kth underlying name. So the present value of the contingent leg is equal to

$$
C_k = \sum_{i=1}^{n+1} (1-R_i)[1_{\{\tau^k \le T, \tau^B > \tau^k + \delta\}} 1_{\{\tau^k = \tau_i\}} e^{-r(\delta+\tau^k)}], \tag{5.2}
$$

On the other hand, the buyer of the contract will have to pay the seller for the premium until one of the following credit events is triggered: the default of the seller or the default of the kth company in the underlying pool. So the present value of the fee leg is equal to

$$
F_k = \sum_{m=1}^{N} \left[X_k \Delta_m e^{-rt_m} 1_{\{\tau^k > t_m, \tau^B > t_m\}} + X_k \left(\tau^k - t_{m-1}\right) e^{-r\tau^k} 1_{\{t_{m-1} < \tau^k \le t_m, \tau^B > \tau^k\}} \right]. \tag{5.3}
$$

Likewise, $E(C_k) = E(F_k)$ determines the premium rate X_k. From (5.2) and (5.3), we should compute the following three terms

$E[1_{\{\tau^k \le T\}} 1_{\{\tau^k = \tau_i\}} 1_{\{\tau^B > \tau^k + \delta\}} e^{-r(\delta+\tau^k)}], E[1_{\{\tau^k > t_m, \tau^B > t_m\}}]$ and $E\left[\left(\tau^k - t_{m-1}\right) e^{-r\tau^k} 1_{\{t_{m-1} < \tau^k \le t_m, \tau^B > \tau^k\}}\right]$.

$$
\begin{aligned}
(1) \quad & E[1_{\{\tau^k \le T\}} 1_{\{\tau^k = \tau_i\}} 1_{\{\tau^B > \tau^k + \delta\}} e^{-r(\delta+\tau^k)}] \\
&= E_{\tau^k, \tau^k = \tau_i}[1_{\{\tau^k \le T\}} 1_{\{\tau^k = \tau_i\}} e^{-r(\delta+\tau^k)} E[1_{\{\tau^B > \tau^k + \delta\}} | \tau^k, \tau^k = \tau_i]] \\
&= E_{\tau^k, \tau^k = \tau_i}[1_{\{\tau^k \le T\}} 1_{\{\tau^k = \tau_i\}} e^{-(r+\lambda_0)(\delta+\tau^k)}] e^{-\bar{\lambda}_i \delta},
\end{aligned}
$$

where

$$E[1_{\{\tau^B>\tau^k+\delta\}}|\tau^k,\tau^k=\tau_i]=\int_{\tau^k+\delta}^{\infty}\lambda_B(t)e^{-\int_0^t\lambda_B(s)ds}dt=e^{-\int_0^{\tau^k+\delta}\lambda_B(s)ds}=e^{-(\lambda_0+\bar{\lambda}_i)\delta-\lambda_0\tau^k}.$$

$$(2)\quad E[1_{\{\tau^k>t_m,\tau^B>t_m\}}]=E_{\tau^k}[1_{\{\tau^k>t_m\}}E[1_{\{\tau^B>t_m\}}|\tau^k]]=e^{-\lambda_0 t_m}E_{\tau^k}[1_{\{\tau^k>t_m\}}],$$

where

$$E[1_{\{\tau^B>t_m\}}|\tau^k]=\int_{t_m}^{\infty}\lambda_B(t)e^{-\int_0^t\lambda_B(s)ds}dt=e^{-\int_0^{t_m}\lambda_B(s)ds}=e^{-\lambda_0 t_m}.$$

$$\begin{aligned}(3)\quad &E\left[\left(\tau^k-t_{m-1}\right)e^{-r\tau^k}1_{\{t_{m-1}<\tau^k\le t_m,\tau^B>\tau^k\}}\right]\\ &=E_{\tau^k}\left[\left(\tau^k-t_{m-1}\right)e^{-r\tau^k}1_{\{t_{m-1}<\tau^k\le t_m\}}E[1_{\{\tau^B>\tau^k\}}|\tau^k]\right]\\ &=E_{\tau^k}\left[\left(\tau^k-t_{m-1}\right)e^{-(r+\lambda_0)\tau^k}1_{\{t_{m-1}<\tau^k\le t_m\}}\right]\end{aligned}$$

where

$$E[1_{\{\tau^B>\tau^k\}}|\tau^k]=e^{-\int_0^{\tau^k}\lambda_B(s)ds}=e^{-\lambda_0\tau^k}.$$

Employing (1) (2) and (3), integrate with respect to (3.6) (3.7) and (3.8) respectively, we get the following theorem.

Theorem 5.1. *Let the hazard rate processes $\lambda_t^i,(i=1,2,\cdots,n+1)$, be given by (1.3) and (1.4), and let $\lambda_B(t)$ satisfy (5.1). Then the present value of the contingent leg and the fee leg can be expressed respectively as follows*

$$\begin{aligned}E(C_k)\quad &=n!(1-R_{n+1})e^{-(r+\lambda_0+\bar{\lambda}_{n+1})\delta}\sum_{\eta=0}^{k-1}\hat{\alpha}(k,\eta;k)\left\{\frac{1}{r+\lambda_0+\psi(k,\eta;n+1)}\left[1-e^{-[r+\lambda_0+\psi(k,\eta;n+1)]T}\right]\right\}\\ &+(n-1)!\sum_{i=1}^{n}(1-R_i)e^{-(r+\lambda_0+\bar{\lambda}_i)\delta}\sum_{\substack{j=1\\ j\neq k}}^{n+1}\sum_{\eta=0}^{k-1}\hat{\alpha}(j,\eta;k)\left\{\frac{1}{r+\lambda_0+\psi(j,\eta;n+1)}\left[1-e^{-[r+\lambda_0+\psi(j,\eta;n+1)]T}\right]\right\},\end{aligned}\tag{5.4}$$

$$\begin{aligned}E(F_k)\quad &=X_k n!\sum_{m=1}^{N}\sum_{j=1}^{n+1}\sum_{\eta=0}^{k-1}\hat{\alpha}(j,\eta;k)\left[r+\lambda_0+\psi(j,\eta;n+1)\right]^{-2}\times\\ &\left\{e^{-[r+\lambda_0+\psi(j,\eta;n+1)]t_{m-1}}+\left[\frac{(r+\lambda_0)[r+\lambda_0+\psi(j,\eta;n+1)]\Delta_m}{\psi(j,\eta;n+1)}-1\right]e^{-[r+\lambda_0+\psi(j,\eta;n+1)]t_m}\right\},\end{aligned}\tag{5.5}$$

where $\hat{\alpha}(j,\eta;k),\psi(j,\eta;n+1)$ are the same with the ones in lemma 3.2.

From the expression (5.4) and (5.5) we find that the parameter $\bar{\lambda}_i$ has nothing to do with the premium rate if the settlement period $\delta=0$. It is because of the model we give in (5.1), which implies that $\bar{\lambda}_i$ will take effect only after the kth default. However, due to the condition that $\delta=0$, the seller has to undertake the loss whenever the $k$the default occurs. Therefore X_k is not influenced by $\bar{\lambda}_i$ in this situation.

§6 Numerical results

In this section, using the valuation formulae for the basket CDS (4.5) (4.6) and (5.4) (5.5) we have obtained in the previous sections, we study the impacts of the parameters on the premium rate X_k.

Due to the symmetry property of the subsidiaries, we assume that their recovery rates are identical $R_i = R_1, (i = 1, 2, \cdots, n)$; the primary company's recovery rate is $R_{n+1} = R_2$. Let $\bar{\lambda}_i = \lambda_{11}, (i = 1, 2 \cdots, n)$; $\bar{\lambda}_{n+1} = \lambda_{22}$. The parameter are chosen as follows: $a_2 = 0.3; b_1 = 0.005; b_2 = 0.015; c = 0.1; a_1 = 0.1; k = 1; n = 15; r = 0.05$; $R_1 = 0.3; R_2 = 0.7$; $T = 5; \Delta_m = 0.25; \delta = 0.25; \lambda_0 = 0.1; \lambda_{11} = 0.1$ and $\lambda_{22} = 0.2$. Unless otherwise noted, the parameters are the same as the ones given above.

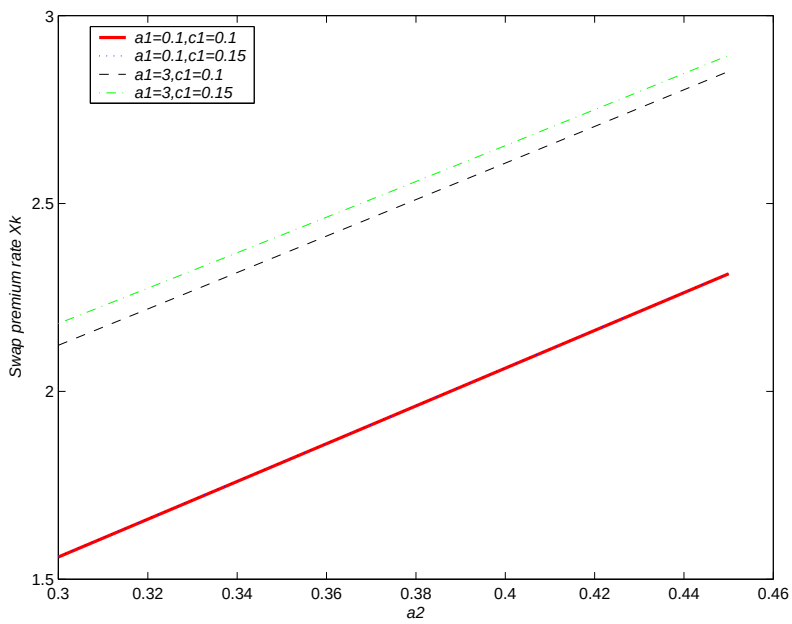

Fig 1: k=2

Figure 1 displays the premium rate of the basket CDS without counterparty risk against the values of default intensity parameters in the primary-subsidiary model. The premium rate X_k will increase when a_2 is largerer, which means the protection buyer is willing to pay a higher premium to the contract seller with more risky underlying assets. Similiar to a_2, the premium rate X_k will also surge as a_1 increases. In addition, it is noted that the premium rate X_k is not sensitive to the influence factor c of the primary company to subsidiary companies when a_1 is small ($a_1 = 0.1$). But it is sensitive when the value of a_1 becomes large ($a_1 = 3$). This reflects the fact that a_1 puts more weight on default than c does.

Table 1

	$X_k, k=1$	$X_k, k=2$	$X_k, k=3$	$X_k, k=4$
$b_2 = 0.005$	3.1565	1.5342	0.9917	0.7194
0.0075	3.1565	1.5405	0.9999	0.7283
0.01	3.1565	1.5468	1.0080	0.7371
0.0125	3.1565	1.5530	1.0160	0.7457
0.0150	3.1565	1.5592	1.0240	0.7543
0.0175	3.1565	1.5654	1.0318	0.7628
0.02	3.1565	1.5715	1.0396	0.7711

Table 2

	$X_k, k=1$	$X_k, k=2$	$X_k, k=3$	$X_k, k=4$
$c = 0.05$	3.1565	1.5574	1.0215	0.7515
0.075	3.1565	1.5584	1.0228	0.7530
0.1	3.1565	1.5592	1.0240	0.7543
0.125	3.1565	1.5600	1.0250	0.7554
0.15	3.1565	1.5607	1.0259	0.7565
0.175	3.1565	1.5613	1.0267	0.7574
0.2	3.1565	1.5619	1.0275	0.7582

Table 1 and Table 2 reveal the relationship amongst the contagion parameters b_2 c, and the premium rate X_k. They illustrate that an increasing value of b_2 or c gives rise to a higher premium rate X_k. On the other hand, the premium rate X_k will decrease as k increases. Moreover, the decreasing rate of X_k with respect to k becomes smaller as b_2 or c increases, which indicates that with a higher value of b_2 or c a default is more likely to trigger "infectious defaults" of others. In addition, both of them also show that when the contract is the first-to-default, the contagion parameters will not influence X_k. In fact, the first-to-default contract will expire when the first default occurs, so the correlation effect will never be evolved in this situation.

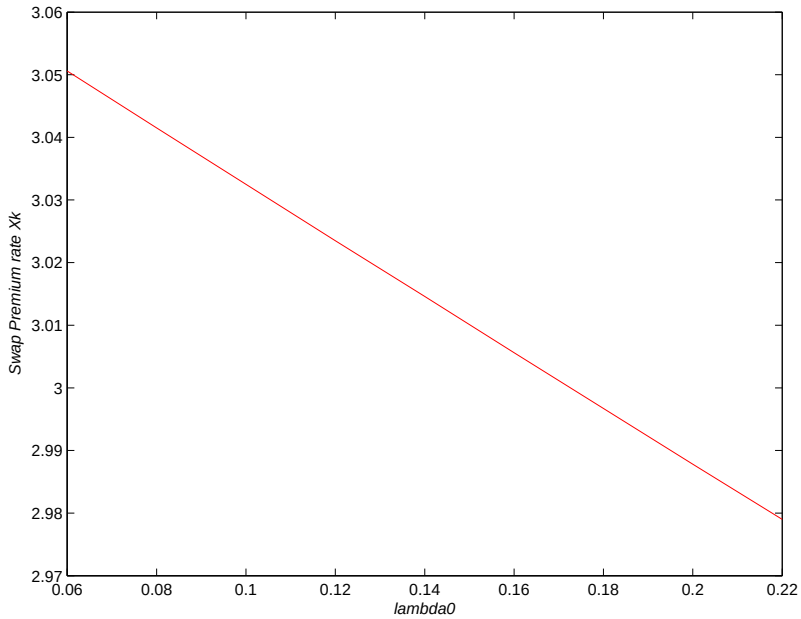

Fig 2

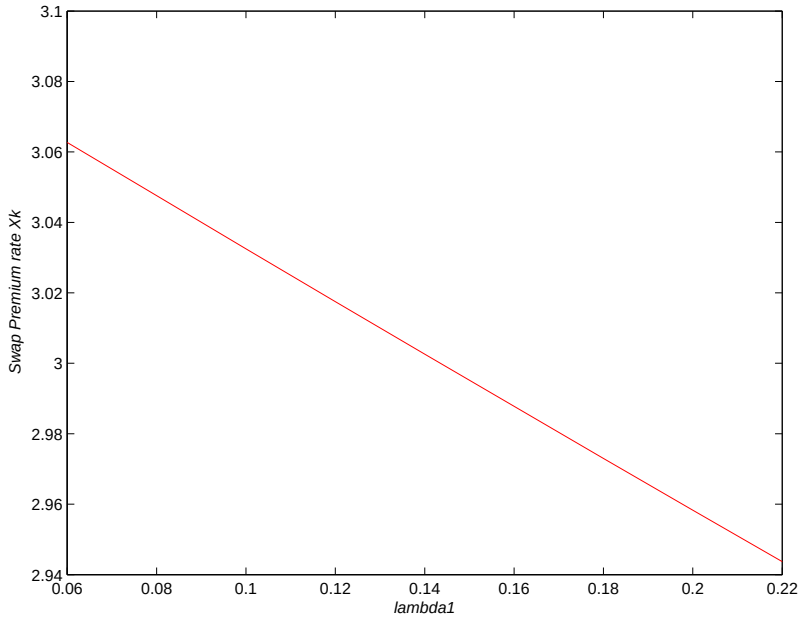

Fig 3

Figure 2 and Figure 3 show the premium rate X_k with counterparty risk. Figure 2 reveals that whilst the seller's own default intensity λ_0 rises, the premium rate will subside. It is also displayed in

Figure 3 that X_k will decrease when the contagion parameter increase. In fact, this is consistent with the fact that the protection buyer is willing to pay a lower premium when dealing with a more risky contract seller.

Table 3(the underlying subsidiary names $n = 9$)

k	1	2	5	8	10
MC	1.9083	0.9159	0.3102	0.1111	0.0182
AP	1.9067	0.9176	0.3098	0.1129	0.0186
Time(MC)	350.1720	295.6410	340.8430	412.3600	464.3120
Time(AP)	0.1400	0.2500	0.4060	0.4840	0.6720

Table 4(the underlying subsidiary names $n = 19$)

k	1	2	5	8	9	10	11	12
MC	3.9925	1.9872	0.7749	0.4619	0.4024	0.3539	0.3132	0.2782
AP	3.9891	1.9865	0.7750	0.4632	0.4035	50.8906	7.6359	-3.0585
Time(MC)	360.2660	344.5160	430.2500	528.7190	565.5320	647.6250	687.5620	734.0620
Time(AP)	0.2660	0.3750	0.6250	0.9060	0.9220	1.0310	1.0160	1.2810

Finally, in Table 3 and Table 4 we compare the computation accuracy and efficiency using our analytical pricing(AP) formulae and Monte Carlo(MC) methods. Time is displayed in second, and the MC method is tested 0.1 millon times. We consider the relatively small portfolio($n = 9$) in Table 3, in which it shows that the results computed through our AP and MC are almost the same. However, the obvious advantage of our AP is that it cost much less time compared with MC. The relatively large portfolio($n = 19$) is considered in Table 4. Our AP still gains an edge over MC when k is relatively small($k < 9$) in the contract. However, whenever k becomes larger, the accuracy is not obtained in our AP. The reason behind this is in fact from the computer limitations rather than our AP. In our AP formulae, there are quite a few product terms and factorials terms. When n and k are both large, the variables in the computer will be overflowed because of their value limitations. In this situation, one has to resort to MC instead.

§7 Conclusion

The inextricable link amongst the buyer, the seller and the underlying names arouses the difficulty of the valuation of the basket CDS and, therefore, the deus ex machina is to derive the joint default

probability of them per se. Also the heterogenous characteristics of them prohibit the possibility of the derivation of the closed formulae. Although prodigious efforts have been taken by researchers, the results are still anything but satisfactory to some extent. Based on the construction of the total hazard rates, we develop the work of Zheng and Jiang [17] from the homogenous case to the primary-subsidiary heterogenous model in the interacting intensities framework, and obtain the corresponding joint density of the default times. Moreover, we obtain the valuation formulae for the basket CDS both with and without counterparty risk. Numerical results robustly show that, under certain conditions, using the analytical pricing formulae derived in this paper is more efficient than the Monte carlo method for the basket CDS valuation.

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Appendix

Appendix A

Function tool box:

$$\begin{aligned}
\varphi^j_{n_1,n_2} = \sum_{l=n_1}^{n_2} \omega^j_l &= (n_2-n_1+1)[a_2+(n_1+n_2-n-1)b_2-b_1] \\
&+ [n_2-\max(n_1,j)+1][-2b_2+c+b_1]1_{\{n_2\geq j\}} \\
&+ [-a_2+(n-2j+3)b_2+(j-n-2)c+(j-1)b_1+a_1]1_{\{n_1\leq j, n_2\geq j\}},
\end{aligned} \tag{8.1}$$

$$\begin{aligned}
\varphi^j_{\eta+1,m} &= (m-\eta)[a_2+(\eta+m-n)b_2-b_1] \\
&+ \{(m-\eta)(-2b_2+c+b_1)1_{\{\eta+1>j\}} \\
&+ [-a_2+(n-2m+1)b_2+(m-n-1)c+mb_1+a_1]1_{\{\eta+1\leq j\}}\}1_{\{m\geq j\}},
\end{aligned}$$

$$\begin{aligned}
\varphi^j_{m,\eta} &= (\eta-m+1)[a_2+(\eta+m-n-1)b_2-b_1] \\
&+ \{(\eta-m+1)(-2b_2+c+b_1)1_{\{m>j\}} \\
&+ [-a_2+(n-2\eta+1)b_2+(\eta-n-1)c+\eta b_1+a_1]1_{\{m\leq j\}}\}1_{\{\eta\geq j\}},
\end{aligned}$$

$$\begin{aligned}
\varphi^j_{m,n+1} &= (n-m+2)[a_2+(m-2)b_2+c]1_{\{m>j\}} \\
&+ [(n-m+1)a_2+(n-m+1)(m-1)b_2+(m-1)b_1+a_1]1_{\{m\leq j\}}.
\end{aligned}$$

□

$$\text{Let} \quad \widetilde{\phi}^k_{j,\eta_0,m_0} = \{\prod_{m=\eta_0+1}^{m_0-1} \varphi^j_{\eta_0+1,m}\} \cdot \{\prod_{m=m_0+1}^{k-1} \varphi^j_{m_0+1,m}\} \cdot \{\prod_{m=1}^{\eta_0} \varphi^j_{m,\eta_0}\}, \tag{8.2}$$

$j \geq k,$

$$\widetilde{\phi}^k_{j,\eta_0,m_0} =$$
$$\{(m_0-\eta_0-1)! \prod_{m=\eta_0+1}^{m_0-1}[a_2+(\eta_0+m-n)b_2-b_1]\}\{(k-1-m_0)! \prod_{m=m_0+1}^{k}[a_2+(m_0+m-n)b_2-b_1]\}$$
$$\cdot\{\eta_0! \prod_{m=1}^{\eta_0}[a_2+(\eta_0+m-n-1)b_2-b_1]\},$$

$j \leq k-1, \eta_0 < m_0 \leq j-1,$

$$\widetilde{\phi}^k_{j,\eta_0,m_0} =$$
$$\{(m_0-\eta_0-1)! \prod_{m=\eta_0+1}^{m_0-1}[a_2+(\eta_0+m-n)b_2-b_1]\}\{(j-m_0-1)! \prod_{m=m_0+1}^{j-1}[a_2+(m_0+m-n)b_2-b_1]\}$$
$$\cdot\{\prod_{m=j}^{k}[(m-m_0-1)a_2+(m-m_0-1)(m_0+m-n-1)b_2+(m-n-1)c+m_0b_1+a_1]\}$$
$$\cdot\{\eta_0! \prod_{m=1}^{\eta_0}[a_2+(\eta_0+m-n-1)b_2-b_1]\},$$

$j \leq k-1, \eta_0 < j \leq m_0,$

$$\widetilde{\phi}^k_{j,\eta_0,m_0} =$$
$$\{(j-1-\eta_0)! \prod_{m=\eta_0+1}^{j-1}[a_2+(\eta_0+m-n)b_2-b_1]\}$$
$$\cdot\{\prod_{m=j}^{m_0-1}[(m-\eta_0-1)a_2+(m-\eta_0-1)(\eta_0+m-n-1)b_2+(m-n-1)c+\eta_0b_1+a_1]\}$$
$$\cdot\{(k-1-m_0)! \prod_{m=m_0+1}^{k-1}[a_2+(m_0+m-n-2)b_2+c]\}$$
$$\cdot\{\eta_0! \prod_{m=1}^{\eta_0}[a_2+(\eta_0+m-n-1)b_2-b_1]\},$$

$j \leq k-1, j \leq \eta_0 < m_0,$

$$\widetilde{\phi}^k_{j,\eta_0,m_0} =$$
$$\{(m_0-\eta_0-1)! \prod_{m=\eta_0+1}^{m_0-1}[a_2+(\eta_0+m-n-2)b_2+c]\}$$
$$\cdot\{(k-m_0-1)! \prod_{m=m_0+1}^{k-1}[a_2+(m_0+m-n-2)b_2+c]\}$$
$$\cdot\{\prod_{m=1}^{j}[(\eta_0-m)a_2+(\eta_0-m)(\eta_0+m-n-2)b_2+(\eta_0-n-1)c+(m-1)b_1+a_1]\}$$
$$\cdot\{(\eta_0-j)! \prod_{m=j+1}^{\eta_0}[a_2+(\eta_0+m-n-3)b_2+c]\}.$$

□

Let $$\bar{\phi}^k_{j,\eta_1,m_1,\bar{m}} = \{\prod_{m=\eta_1+1}^{m_1-1}\varphi^j_{\eta_1+1,m}\}\cdot\{\prod_{m=m_1+1}^{\bar{m}-1}\varphi^j_{m_1+1,m}\}\cdot\{\prod_{m=\bar{m}+1}^{k-1}\varphi^j_{\bar{m}+1,m}\}\cdot\{\prod_{m=1}^{\eta_1}\varphi^j_{m,\eta_1}\}, \tag{8.3}$$

$j \le k-1, \eta_1 < m_1 < j < \bar{m}$,

$$\bar{\phi}^k_{j,\eta_1,m_1,\bar{m}} =$$
$$\{(m_1-\eta_1-1)! \prod_{m=\eta_1+1}^{m_1-1} [a_2+(\eta_1+m-n)b_2-b_1]\}\{(j-m_1-1)! \prod_{m=m_1+1}^{j-1} [a_2+(m_1+m-n)b_2-b_1]\}$$
$$\cdot\{\prod_{m=j}^{\bar{m}-1} [(m-m_1-1)a_2+(m-m_1-1)(m_1+m-n-1)b_2+(m-n-1)c+m_1b_1+a_1]\}$$
$$\cdot\{\prod_{m=\bar{m}+1}^{k-1} [(m-\bar{m})a_2+(m-\bar{m})(\bar{m}+m-n-2)b_2+(m-\bar{m})c]\} \cdot \{\eta_1! \prod_{m=1}^{\eta_1} [a_2+(\eta_1+m-n-1)b_2-b_1]\}.$$

□

Appendix B

Proof of lemma 3.3. For any $k, j \in \{1, 2 \cdots, n+1\}$, firstly let us consider $\sum_{l=\eta+1}^{m} w_l^j \neq 0$ for any pair $(\eta, m), 0 \le \eta < m \le k-1$,

In terms of $P(\tau^k < t, \tau^k = \tau_{n+1})$,

$$P(\tau^k < t, \tau^k = \tau_{n+1}) = E\left[1_{\{\tau^k<t\}} 1_{\{\tau^k=\tau_{n+1}\}}\right]$$
$$= n! E\left[1_{\{\tau_1<\tau_2<\cdots<\tau_{k-1}<\tau_{n+1}<t\}} 1_{\{\tau_{n+1}<\tau_k<\cdots<\tau_n\}}\right]$$
$$= n! \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{k-1}}^t dt_{n+1} \int_{t_{n+1}}^\infty dt_k \cdots \int_{t_{n-1}}^\infty dt_n c^k e^{-(\omega_1^k t_1+\omega_2^k t_2+\cdots+\omega_k^k t_{n+1}+\omega_{k+1}^k t_k+\cdots+\omega_{n+1}^k t_n)}$$
$$= n! \int_0^t d\tilde{t}_1 \int_{\tilde{t}_1}^t d\tilde{t}_2 \cdots \int_{\tilde{t}_{k-1}}^t d\tilde{t}_k \int_{\tilde{t}_k}^\infty d\tilde{t}_{k+1} \cdots \int_{\tilde{t}_n}^\infty d\tilde{t}_{n+1} c^k e^{-(\omega_1^k \tilde{t}_1+\omega_2^k \tilde{t}_2+\cdots+\omega_k^k \tilde{t}_k+\omega_{k+1}^k \tilde{t}_{k+1}+\cdots+\omega_{n+1}^k \tilde{t}_{n+1})}$$
$$= n! A_k(t, \omega_1^k, \cdots, \omega_{k-1}^k, \sum_{l=k}^{n+1} \omega_l^k) \frac{c^k}{\prod_{m=k+1}^{n+1} (\sum_{l=m}^{n+1} \omega_l^k)}.$$

Let

$$\bar{\omega}_k^j = \sum_{l=k}^{n+1} \omega_l^j, \quad \bar{\omega}_i^j = \omega_i^j, \quad i = 1, 2, \cdots, k-1,$$

So

$$
\begin{aligned}
f_{\tau^k,\tau^k=\tau_{n+1}} &= \frac{dP(\tau^k < t, \tau^k = \tau_{n+1})}{dt} \\
&= n! \sum_{\eta=0}^{k-1} \bar{\gamma}_{k,\eta}(\bar{\omega}_1^k, \ldots, \bar{\omega}_k^k) e^{-\sum\limits_{l=\eta+1}^{n+1} \omega_l^k t} \left(-\sum_{l=\eta+1}^{n+1} \omega_l^k\right) \frac{c^k}{\prod_{m=k+1}^{n+1} (\sum_{l=m}^{n+1} \omega_l^k)} \\
&= n! \sum_{\eta=0}^{k-1} \frac{(-1)^{k-\eta-1} c^k e^{-\sum\limits_{l=\eta+1}^{n+1} \omega_l^k t} \left(\sum_{l=\eta+1}^{n+1} \omega_l^k\right)}{\left\{\prod_{m=\eta+1}^{k-1} \left(\sum_{l=\eta+1}^{m} \omega_l^k\right)\right\} \cdot \left\{\sum_{l=\eta+1}^{n+1} \omega_l^k\right\} \left\{\prod_{m=1}^{\eta} \left(\sum_{l=m}^{\eta} \omega_l^k\right)\right\} \left\{\prod_{m=k+1}^{n+1} \left(\sum_{l=m}^{n+1} \omega_l^k\right)\right\}} \\
&= n! \sum_{\eta=0}^{k-1} \frac{(-1)^{k-\eta-1} c^k e^{-\sum\limits_{l=\eta+1}^{n+1} \omega_l^k t}}{\left\{\prod_{m=\eta+1}^{k-1} \left(\sum_{l=\eta+1}^{m} \omega_l^k\right)\right\} \cdot \left\{\prod_{m=1}^{\eta} \left(\sum_{l=m}^{\eta} \omega_l^k\right)\right\} \left\{\prod_{m=k+1}^{n+1} \left(\sum_{l=m}^{n+1} \omega_l^k\right)\right\}} \\
&= n! \sum_{\eta=0}^{k-1} \alpha(k, \eta; k) e^{-\psi(k,\eta;n+1)t},
\end{aligned}
$$

where the last equality is according to the definitions of $\alpha(j, \eta; k)$ and $\psi(j, \eta; n+1)$ in (3.4) and (3.5).

In terms of $P(\tau^k < t, \tau^k = \tau_i), (i = 1, 2, \cdots n)$, due to the fact that the intensities of n subsidiaries are symmetric, $P(\tau^k < t, \tau^k = \tau_1) = P(\tau^k < t, \tau^k = \tau_2) = \cdots = P(\tau^k < t, \tau^k = \tau_n)$. we therefore only have to consider $P(\tau^k < t, \tau^k = \tau_1)$.

$$
\begin{aligned}
P(\tau^k < t, \tau^k = \tau_1) &= E\left[1_{\{\tau^k < t\}} 1_{\{\tau^k = \tau_1\}}\right] \\
&= (n-1)! \sum_{\substack{j=1 \\ j \neq k}}^{n+1} E\left[1_{\{\tau_2 < \cdots < \tau_{j-1} < \tau_{n+1} < \cdots < \tau_1 < t\}} 1_{\{\tau_1 < \tau_k < \cdots < \tau_n\}}\right].
\end{aligned}
$$

Similar to the proof of $P(\tau^k < t, \tau^k = \tau_{n+1})$,

$$
P(\tau^k < t, \tau^k = \tau_1) = (n-1)! \sum_{\substack{j=1 \\ j \neq k}}^{n+1} A_k(t, \omega_1^j, \cdots, \omega_{k-1}^j, \sum_{l=k}^{n+1} \omega_l^j) \frac{c^j}{\prod_{m=k+1}^{n+1} (\sum_{l=m}^{n+1} \omega_l^j)}.
$$

So

$$
f_{\tau^k,\tau^k=\tau_1} = \frac{dP(\tau^k < t, \tau^k = \tau_1)}{dt} = (n-1)! \sum_{\substack{j=1 \\ j \neq k}}^{n+1} \sum_{\eta=0}^{k-1} \alpha(j, \eta; k) e^{-\psi(j,\eta;n+1)t}.
$$

The above proof must be presumed that $\alpha(j, \eta; k)$ is well defined, i.e. none of the denominators in $\alpha(j, \eta; k)$ is zero. Otherwise we can employ L'Hospital to redefine $\alpha(j, \eta; k)$, which has already been discussed in lemma 3.2. In other words, we substitute $\hat{\alpha}(j, \eta; k)$ for $\alpha(j, \eta; k)$ in this situation. Then the proof can be completed. □

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