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Macaulay durations for nonparallel shifts

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Abstract Macaulay duration is a well-known and widely used interest rate risk measure. It is commonly believed that it only works for parallel shifts of interest rates. We show in this paper that this limitation is largely due to the traditional parametric modelling and the derivative approach, the Macaulay duration works for non-parallel shifts as well when the non-parametric modelling and the equivalent zero coupon bond approach are used. We show that the Macaulay duration provides the best one-number sensitivity information for non-parallel interest rate changes and that a Macaulay duration matched portfolio is least vulnerable to the downside risk caused by non-parallel rate changes under some verifiable conditions.

Keywords Macaulay duration · Non-parallel shifts · Immunization · Linear programming

AMS Classification 65K10 · 90C90

Macaulay duration is a well-known interest rate risk measure for a portfolio of regular bonds (options-free and default-free). It measures the percentage change in portfolio value due to the instantaneous change in interest rates. It also defines a portfolio immunization horizon over which the portfolio value remains immunized from an instantaneous shock in interest rates. A simple strategy in immunization is to match the present value and the Macaulay duration of an asset portfolio with those of the liability. The portfolio is then immunized to instantaneous small parallel movements in yields (the whole yield curve is shifted up or down by the same increment). A dynamic strategy (continuous rebalancing) is needed to keep the portfolio immunized against small parallel shifts.

The main criticism to the Macaulay duration is that it is valid only to small parallel shifts. Much effort has been made to extend it to non-parallel shifts. Some other durations have been suggested, for example, log-stochastic process duration (Khang, 1979) when short-term rates are more volatile than long-term rates, key-rate duration (Ho, 1992) when there

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are several key rates whose changes determine changes of other rates, multidimensional duration (Rzadkowski and Zaremba, 2000) when several factors affect changes of the term structure, etc. These durations are defined for more general patterns of rate changes and one would expect they perform better in immunization than the Macaulay duration does. However, this is not the case in many empirical tests, as pointed out by Jorion and Khoury (1996, page 108): "The Macaulay duration was also compared to additive, multiplicative, and log-multiplicative process duration. Somewhat unexpectedly, they reported little difference among duration strategies and concluded that the simplest Macaulay duration provides the most cost-effective immunization method."

The common feature of those mentioned durations is that interest rate changes are specified exogenously by some parametric models. Durations are defined simply as derivatives of bond price with respect to underlying factors. The immunization based on these durations works well if rate changes do follow specified models, but may perform poorly if they do not. This intuitively explains why those more recent durations do not have consistent better performance than the simple Macaulay duration simply because it is unlikely one can accurately predict rate changes in practice.

Fong and Vasicek (1984) define a measure of immunization risk, called M-squared, and show that a Macaulay duration matched portfolio structured with minimum M-squared is less vulnerable to any interest rate movements with bounded slopes. Two questions remain unanswered in Fong and Vasicek (1984): 1. How to choose an immunization portfolio when a Macaulay duration matched portfolio does not exist, and 2. Is a Macaulay duration matched minimum M-squared portfolio least vulnerable to non-parallel interest rate changes? The reason for these questions is that in Fong and Vasicek (1984) a portfolio is assumed to be Macaulay duration matched and therefore no further effort is made on other possibilities.

Nawalkha and Chambers (1996) suggest another immunization risk measure, called M-absolute, which is valid for any portfolios without duration constraints, and show that a portfolio structured with minimum M-absolute is less vulnerable to any bounded interest rate changes. M-absolute of a bond is the weighted sum of time differences of cash flows with a specified holding period. The optimal solution of a minimum M-absolute portfolio is unstable in the sense that it can easily choose bonds of very different durations with a small change of data.

Zheng, Thomas, and Allen (2003) suggest an alternative interest rate risk measure, called approximate duration, for regular bonds in order to address the model misspecification risk. The approximate duration measures the bond price sensitivity to rate changes without assuming prior any particular patterns of rate changes. The immunization based on approximate duration does not completely remove the interest rate risk, but it minimizes the overall downside risk to any patterns of rate changes, which is in the same spirit as that of Nawalkha and Chambers (1996). Vinter and Zheng (2003) extend the approximate duration further to instantaneous forward rates with bounded measurable rate changes. The approximate duration is characterized as the median time of the discounted cash flows with nonsmooth optimization.

This paper discusses the measurement and immunization of a bond (or a portfolio of bonds) with respect to Lipschitz (non-parallel) changes of interest rates, i.e., the rate of change is bounded. In practice rate changes are always Lipschitz when yield curves are calibrated with linear splines (Ho, 1997) or other smooth splines from the observed market data. The objective of this paper is to answer the following two questions: 1. Is Macaulay duration the best one-number sensitivity measure for Lipschitz changes of interest rates? and 2. Is the Macaulay duration matched minimum M-squared portfolio least vulnerable to Lipschitz changes of interest rates? The main contribution of the paper is to give positive answers to both questions (under some conditions for the second one), which establishes the Macaulay duration as a good interest rate risk measure for non-parallel shifts as well and explains partly many apparent paradoxes in the empirical literature on durations.

The paper is organized as follows: Section 2 reviews the equivalent ZCB approach to studying the sensitivity of regular bonds and shows the Macaulay duration provides the best one-number sensitivity information when rate changes are Lipschitz. Section 3 discusses the immunization with the minimum downside risk and characterizes the conditions under which the Macaulay duration matched minimum M-squared portfolio is least vulnerable to non-parallel rate changes. The conclusions summarizes the main results and open questions. The appendix contains the proofs of theorems.

1 Nonparametric duration measures

Assume that the term structure of interest rates is flat with rate r (continuous compounding) and that a bond has cash flows c_i at time t_i, $i = 1, \ldots, N$, until maturity $t_N = T$. The bond price at time 0 is equal to $P = \sum_{i=1}^N c_i e^{-rt_i}$ and the Macaulay duration is defined by $D = \sum_{i=1}^N t_i c_i e^{-rt_i}/P$.[1] A standard method to derive the Macaulay duration is to set $D = -P'/P$ where P' is the derivative of P with respect to r. This approach gives clear financial interpretation to the Macaulay duration, i.e., it indicates the magnitude of percentage price changes to yield changes. An alternative method to derive the Macaulay duration is to find an equivalent zero-coupon bond with the same present value and interest rate sensitivity as the given bond, i.e., the face value F and the maturity D of the equivalent ZCB are determined from equations $P = P_0 := Fe^{-rD}$ and $P' = P_0'$. The solution D is again the Macaulay duration.

When there are several factors affecting rate changes, the two methods produce divergent results. The derivative method changes to the partial derivative method and the duration becomes a vector, which implies the change of the term structure must be specified parametrically and the model risk is inherent. The equivalent zero coupon bond method keeps the same spirit with a generalized equivalence relation and the duration is still a single number. The first approach is discussed by Ho (1997), Rzadkowski and Zaremba (2000), etc. The second approach is investigated by Zheng, Thomas, and Allen (2003), Vinter and Zheng (2003). We first review the idea of the second approach to general rate changes.

Assume the initial term structure of forward rates f is given, that is, $f(t)$ is the instantaneous forward rate (continuous compounding) at time t, seen at time 0. The present value of a bond is given by

$$P(f) = \sum_i c_i v(t_i)$$

where $v(t) = e^{-\int_0^t f(u)du}$ is the discount factor at time t. If there is an instantaneous shift of forward rates from f to $f + g$, where g is a function defined on a space S with a norm $\|\cdot\|$, then the new bond price $P(f + g)$ can be approximated (the first order Taylor expansion) by $P(f) + P'(f; g)$ where

$$P'(f; g) = \sum_i c_i v(t_i)\left(-\int_0^{t_i} g(u)du\right)$$

is the directional derivative of P at f in the direction g. Consider now a ZCB with face value F and maturity D. Its present value is

$$P_0(f) = Fv(D)$$

and its directional derivative is

$$P_0'(f;g) = P_0(f)\left(-\int_0^D g(u)du\right).$$

The ZCB is said to be equivalent to the given bond if

$$P(f) = P_0(f) \tag{1}$$

and

$$P'(f;g) = P_0'(f;g) \quad \text{for all } g. \tag{2}$$

Divide (2) with (1) and set

$$w_i = c_i v(t_i)/P(f) \tag{3}$$

to get

$$\sum_i w_i \int_0^{t_i} g(u)du = \int_0^D g(u)du \quad \text{for all } g$$

which is equivalent to equation $H(D) = 0$ where

$$H(D) = \max_{\|g\|\le 1} \left|\sum_i w_i \int_D^{t_i} g(u)du\right| \tag{4}$$

Note $H(D)$ is the normalized maximum deviation of interest rate sensitivities of the two bonds. Once D is derived F is computed from (1).

If rate changes are parallel, i.e., g is a constant function, then $H(D) = |\sum_i w_i t_i - D|$ and the solution to $H(D) = 0$ is the Macaulay duration $D = \sum_i w_i t_i$. However, if rate changes are not parallel then there is no solution to $H(D) = 0$ unless the coupon bond itself has only one cash flow. We need to generalize the equivalence from the relation $H(D) = 0$ to something else. A natural candidate is to make $H(D)$ as close to zero as possible. We therefore call a ZCB with face value F and maturity D equivalent to the given bond if D is the optimal solution to the problem:

$$\text{minimize} \quad H(D) \qquad \text{subject to} \quad D \ge 0 \tag{5}$$

and F satisfies (1).

Vinter and Zheng (2003) discuss, among some other finance problems, the equivalence relation in the space $S = L^\infty[0,\infty)$ with the norm $\|g\| = \sup_x |g(x)|$. The optimal solution D_a to (5) is the median time of the discounted cash flows, i.e., $D_a = t_{i_0}$ if $\sum_{i<i_0} w_i \le \sum_{i\ge i_0} w_i$

and $\sum_{i\leq i_0} w_i \geq \sum_{i>i_0} w_i$ for some integer i_0 and D_a is any number in the interval $[t_{i_0}, t_{i_0+1}]$ if $\sum_{i\leq i_0} w_i = \sum_{i\geq i_0+1} w_i$. The result implies that if we know nothing about rate changes (always measurably bounded) then the ZCB that best approximates a coupon bond should have the maturity equal to the median time of the discounted cash flows of the given bond. However, the resulting ZCB is unstable in the sense that its maturity can vary greatly even there are only small changes of cash flows. For example, if a bond has two cash flows of equal present values, one in one year and the other in ten years, then the equivalent ZCB has duration of either one year or ten years with any slight tipping of balance of cash flows. Note that D_a is intrinsically equal to one of cash flow dates t_{i_0}. It is undecided only when $\sum_{i\leq i_0} w_i = \sum_{i\geq i_0+1} w_i$, but that relation is transient since the weights w_i change continuously as time t passes by. The optimal solution D_a is the same as the approximate duration, an interest rate risk measure discussed in Zheng, Thomas, and Allen (2003) in comparing duration-based immunization strategies.

In this paper we study the equivalent ZCB when S is a Lipschitz functional space, i.e., if $g \in S$ then there exists a $K > 0$, called Lipschitz constant, such that $|g(x) - g(y)| \leq K|x - y|$ for all $x, y \geq 0$. If g is continuously differentiable with bounded derivatives, then g is Lipschitz with $K = \|g'\|_\infty$. The converse is not true, for example, linear splines are Lipschitz, but are not differentiable. The next result shows that the Macaulay duration is a stable interest rate sensitivity measure valid not only for parallel shifts but also for non-parallel shifts.

Theorem 1. *Let the change of the term structure g be a Lipschitz function with a D-dependent norm* $\|g\|_D$ *defined by* $\|g\|_D = \max\{|g(D)|, K\}$.[2] *Let all cash flows of the underlying bond be nonnegative. Then the maximum deviation* $H(D)$ *is characterized by*

$$H(D) = \frac{1}{2}\sum_i w_i(t_i - D)^2 + \left|\sum_i w_i t_i - D\right| \tag{6}$$

and the optimal solution to (5) *is the mean time of the discounted cash flows, i.e., the Macaulay duration* $D_m = \sum_i w_i t_i$.

Theorem 1 shows that the Macaulay duration D_m not only gives exact sensitivity information when rate changes are parallel but also provides the best approximation of that information when rate changes are not parallel as long as the slope of rate changes is not too steep. The Macaulay duration D_m is also stable, i.e., small changes of cash flows result in small changes of the Macaulay duration, this is because D_m is the mean time, not the median time, of the discounted cash flows.

In practice we do not observe the instantaneous forward rate curve f directly, but we can easily construct it based on the observed market data to any degree of accuracy (depending on the availability of the data). Let $R(0)$ be the short rate at time $t_0 = 0$ and $R(t_i)$ be the zero rate at time t_i, $i = 1, \ldots, N$.[3] The relation between $R(t)$ and $f(u)$, $0 \leq u \leq t$, is $R(t) = (1/t)\int_0^t f(u)du$ with $R(0) = \lim_{t\downarrow 0} R(t) = f(0)$. Assume a linear spline (the same technique applies to other splines) is used to construct the instantaneous forward rate curve f from rates $f(t_i)$ at time t_i, $i = 0, 1, \ldots, N$. We can set $f(0) = R(0)$ and compute $f(t_i)$ from the recursive formula $f(t_i) = 2(t_i R(t_i) - t_{i-1}R(t_{i-1}))/(t_i - t_{i-1}) - f_{i-1}$ for $i = 1, \ldots, N$. The curve f and the change g constructed this way are Lipschitz.

Heath et al. (1992) discuss the risk-neutral forward rate process modelling and its applications in pricing interest rate derivatives. They specify the whole term structure of $f(s, t)$, the instantaneous forward rates at time t seen at time $s \leq t$, and assume $f(\cdot, t)$ is driven by some diffusion processes for fixed t. Therefore with probability one the realized path $f(\cdot, t)$ is nowhere differentiable, certainly not Lipschitz. However, the HJM forward rate model can not be directly applied in immunization context because immunization is concerned with the impact of an unpredictable regime change from f to $f+g$ at a fixed time s, not the dynamic process f itself over the whole period.

Note that the traditional Macaulay duration is well-defined for bonds with positive or negative cash flows but the Macaulay duration in Theorem 1 is only defined for bonds with nonnegative cash flows. This disparity is due to the different derivation techniques: one with derivative (parallel shift, valid in two directions) and the other with minimum deviation (any pattern, valid only in one direction). This observation implies that the short-selling (negative cash flows) is not allowed when the Macaulay duration is used for Lipschitz (non-parallel) rate changes.

2 Downside risk minimizing immunization

Suppose a bond (or a portfolio of bonds) is to be held for a period D. The value of the bond at time D is

$$FV(f) = \sum_i c_i e^{\int_{t_i}^D f(u)du} = P(f) e^{\int_0^D f(u)du}.$$

In practice $FV(f)$ may represent the value of the liability at time D to be paid. If forward rates change from f to $f+g$, then the value of the bond changes to $FV(f+g)$. The immunization is to choose a bond (or a portfolio of bonds) such that $FV(f+g) \geq FV(f)$ for any rate changes g. The next result provides a lower bound for $FV(f+g)$ and shows that the Macaulay duration maximizes the lower bound.

Theorem 2. *Let the change of forward rates g be a Lipschitz function* (*non-parallel shifts*). *Then for any holding period $D > 0$*

$$FV(f+g) \geq FV(f) - \|g\|_D H(D) FV(f) \quad \textit{for all } g \tag{7}$$

where $H(D)$ is given by (6). *Furthermore, if the change of forward rates g is a constant function* (*parallel shifts*) *and D is the Macaulay duration, then*

$$FV(f+g) \geq FV(f) \quad \textit{for all } g.$$

Theorem 2 shows that a Macaulay duration matched bond immunizes the interest rate risk only when rate changes are parallel. In general it is impossible to have a bond (or a portfolio of bonds) such that $FV(f+g) \geq FV(f)$ for *all* rate changes g. The best one can hope for is to make the loss (the downside risk), when it occurs, as small as possible. (7) shows that the percentage loss is bounded by $\|g\|_D H(D)$, the product of the magnitude of interest rate changes and the maximum deviation of a bond with a holding period D. To minimize the downside risk one should choose a bond (or a portfolio of bonds) with minimum $H(D)$ as $\|g\|_D$ is uncontrollable. If the holding period D is a decision variable, then the Macaulay duration D_m minimizes $H(D)$. If the holding period D is fixed as in immunization, then a bond with the smallest $H(D)$ in comparison with other bonds is likely to be least vulnerable to the loss caused by non-parallel rate changes. This raises a natural question as whether a

Macaulay duration matched bond (or a portfolio of bonds) is still a good choice in controlling the downside risk.

Fong and Vasicek (1984) define $M^2 = \sum_i w_i(t_i - D)^2$ as the time variance of a bond with a holding period D. They claim a Macaulay duration matched portfolio has the minimum exposure to interest rate changes when M^2 is minimized. We can recover the same conclusion from Theorem 2 under the weak assumption (rate changes are Lipschitz, not continuously differentiable) and provide a better objective function $H(D)$ than M^2 in choosing a bond portfolio with the minimum downside risk.

Assume that the liability portfolio is a ZCB with the present value V and the duration D, and that the asset portfolio is made of N bonds with a holding period D. Assume that there are x_j units of bond j which has the present value P_j, the Macaulay duration D_j, and the time variance M_j^2, $j = 1, \dots, N$. Then the present value $V(x)$, the Macaulay duration $D(x)$, and the maximum deviation $H(x)$ of the asset portfolio are given by

$$\begin{aligned} V(x) &= \sum_j P_j x_j \\ D(x) &= \sum_j (P_j x_j / V(x)) D_j \\ H(x) &= (1/2) \sum_j (P_j x_j / V(x)) M_j^2 + \left| \sum_j (P_j x_j / V(x)) D_j - D \right|. \end{aligned} \tag{8}$$

To match the present values of the asset and liability portfolios decision variables x_j must satisfy the relation $V(x) = V$. Denote $y_j = (x_j P_j)/V$ the proportion of bond j in the whole portfolio for $j = 1, \dots, N$. From (8) we have $\sum_j y_j = 1$ and $y_j \geq 0$ for all j. The Macaulay duration and the maximum deviation of the portfolio are given by $D(y) = \sum_j y_j D_j$ and $H(y) = (1/2)\sum_j y_j M_j^2 + |\sum_j y_j D_j - D|$, respectively. We can set up an optimal portfolio (in the sense of minimum downside risk) by solving the following optimization problem

$$\begin{aligned} &\text{minimize} \quad (1/2) \sum_j y_j M_j^2 + \left| \sum_j y_j D_j - D \right| \\ &\text{subject to} \quad \sum_j y_j = 1 \ \text{ and } \ y_j \geq 0, \ \forall j. \end{aligned} \tag{9}$$

(9) can be easily formulated by an equivalent linear programming problem.

Fong and Vasicek (1984) choose an optimal portfolio by minimizing the M^2 of the portfolio, i.e.,

$$\begin{aligned} &\text{minimize} \quad (1/2) \sum_j y_j M_j^2 \\ &\text{subject to} \quad \sum_j y_j D_j = D \\ &\qquad\qquad\quad \sum_j y_j = 1 \ \text{ and } \ y_j \geq 0, \ \forall j. \end{aligned} \tag{10}$$

(9) is a better formulated optimization problem than (10) in two aspects: 1. The minimum value of (9) is less than or equal to that of (10) because any feasible solution to (10) is a

feasible solution to (9) with the same objective function value, which implies that a portfolio chosen with (9) has smaller bound for the downside risk than that chosen with (10), and the downside risk itself is likely to be smaller as a result. 2. The existence of the optimal solution to (9) is guaranteed because the feasible region is a nonempty compact convex set in R^N, however, the feasible region to (10) can be an empty set, which implies that a portfolio can always be chosen with (9), but not necessarily with (10). The example after Theorem 3 illustrates these points.

Next we specify the conditions under which optimization problems (9) and (10) are equivalent, i.e., a Macaulay duration matched minimum M^2 portfolio is the same as a minimum maximum deviation portfolio which is least vulnerable to non-parallel rate changes.

Theorem 3. *The optimization problems* (9) *and* (10) *have the same optimal solution if and only if the following two conditions are satisfied*:

(i) *if* $(1/2)M_k^2 - D_k = \min\{(1/2)M_j^2 - D_j : j = 1, \dots, N\}$ *then* $D_k \geq D$;
(ii) *if* $(1/2)M_k^2 + D_k = \min\{(1/2)M_j^2 + D_j : j = 1, \dots, N\}$ *then* $D_k \leq D$.

Furthermore, if either $(1/2)M_k^2 - D_k \leq (1/2)M_j^2 - D_j$ *for all* j *and* $D_k < D$ *or* $(1/2)M_k^2 + D_k \leq (1/2)M_j^2 + D_j$ *for all* j *and* $D_k > D$, *then the optimal solution to* (9) *is* $y_k = 1$ *and* $y_j = 0$ *for* $j \neq k$, *and the minimum value of* (9) *is strictly less than that of* (10).

Example. Consider an asset portfolio of 3 bonds: bond 1 has 50% cash flow at time 4 and the rest at time 8 (in terms of present values), bond 2 has 50% cash flow at time 10 and the rest at time 14, bond 3 has only one cash flow at time 8.

(a) Assume the holding period is $D = 10$. The Macaulay durations and the time variances of these bonds are given by $D_1 = 6$, $M_1^2 = 20$, $D_2 = 12$, $M_2^2 = 8$, and $D_3 = 8$, $M_3^2 = 4$. We can check that $\min\{(1/2)M_j^2 - D_j\} = (1/2)M_2^2 - D_2$ and $D_2 > D$, and that $\min\{(1/2)M_j^2 + D_j\} = (1/2)M_3^2 + D_3$ and $D_3 < D$, therefore conditions (i) and (ii) in Theorem 3 are satisfied. Problems (9) and (10) have the same optimal solution $y_1 = 0$, $y_2 = 0.5$, $y_3 = 0.5$ and the optimal value 3. In this case a Macaulay duration matched minimum M^2 portfolio is the same as a minimum maximum deviation portfolio.

(b) Bonds 1 and 2 are the same as those in (a) but bond 3 is replaced by a cash flow at time 11. Assume the holding period is still $D = 10$. We have $D_3 = 11$ and $M_3^2 = 1$. We can check that condition (ii) is not satisfied because $\min\{(1/2)M_j^2 + D_j\} = (1/2)M_3^2 + D_3$ and $D_3 > D$. Problem (9) has the optimal solution $y_1 = 0$, $y_2 = 0$, $y_3 = 1$ and the optimal value 1.5. Problem (10) has the optimal solution $y_1 = 0.2$, $y_2 = 0$, $y_3 = 0.8$ and the optimal value 2.4. In this case a Macaulay duration matched minimum M^2 portfolio is not as good as a minimum maximum deviation portfolio.

(c) All three bonds are the same as those in (b) but the holding period is replaced by $D = 14$. The time variances of these bonds are changed to $M_1^2 = 68$, $M_2^2 = 8$, and $M_3^2 = 9$. We can check that condition (i) is not satisfied because $\min\{(1/2)M_j^2 - D_j\} = (1/2)M_2^2 - D_2$ and $D_2 < D$. Problem (9) has the optimal solution $y_1 = 0$, $y_2 = 1$, $y_3 = 0$ and the optimal value 6. Problem (10) has no feasible solution due to $D_j < D$ for all j. In this case a Macaulay duration matched minimum M^2 portfolio simply does not exist whereas a minimum maximum deviation portfolio is still well defined.

An immunized portfolio requires continuous rebalancing to keep it immunized. Such a strategy is untenable when there are transaction costs in trading bonds. One has to strike a balance between two conflicting objectives of minimizing the transaction cost and of

Table 1 Profits/losses (Volumes) of immunization strategies

	1994	1995	1996	1997	1998	1999	2000	2001
Macaulay	0.00	3.39	−5.19	3.20	0.10	−0.14	−0.29	0.00
	(6.60)	(4.67)	(8.30)	(11.27)	(6.97)	(10.15)	(13.26)	(0.00)
M-Absolute	0.00	4.26	−2.34	4.85	−3.30	−1.48	1.32	0.00
	(6.55)	(3.73)	(12.99)	(0.55)	(0.76)	(0.80)	(0.84)	(0.00)

minimizing the maximum deviation. We can achieve this by solving the following LP:

$$\begin{aligned} &\text{minimize} \quad (1-\lambda)\sum_j a_j y_j + \lambda\left((1/2)\sum_j y_j M_j^2 + \left|\sum_j y_j D_j - D\right|\right) \\ &\text{subject to} \quad \sum_j y_j = 1 \ \text{ and } \ y_j \geq 0, \ \forall j \end{aligned}$$

where a_j is the transaction cost associated with bond j and $0 \leq \lambda \leq 1$ is a preference parameter. If $\lambda = 0$ then the objective is to minimize the transaction cost. If $\lambda = 1$ then the objective is to minimize the maximum deviation. A family of optimal solutions can be constructed by varying parameter λ, which is similar to the Markowitz's mean-variance efficient frontier.

We perform a simple empirical test to compare the Macaulay duration strategy and the M-absolute strategy with the objective of minimizing the downside risk. The data used are the observed US Treasury bonds and STRIPS rates. The data source is the Wall Street Journal (NY edition) around February 15 from 1994 to 2001. Each year six new Treasury bonds (maturity in one, two, three, five, ten, and twenty-five years) are added to the selection universe. All bonds are options free with face value 100. Coupons are assumed to be paid annually for ease of calculation. STRIPS rates are used as zero rates. Assume the holding period is seven years from February 1994 (maturity in February 2001) and the target value is one million dollars. The portfolio is rebalanced in every February.

Table 1 displays the profits/losses (000's) and the number of bonds traded (000's, in parentheses) of the portfolio each year. The Macaulay and M-absolute strategies are stable and have similar performances. Note that the M-absolute strategy has much fewer transactions than that by the Macaulay duration strategy from 1997. The reason is that in 1996 the liability has five years to maturity and there are five year bonds available in the asset portfolio, the M-absolute strategy switches its bond holding from bonds of other maturities to those of five years. After that only small adjustment is needed at rebalancing time. The portfolio has a bullet structure and can significantly save the cost if transaction costs are not negligible or if the portfolio is frequently rebalanced. On the other hand, such a concentration may have adverse effect if there is default risk of underlying bonds (a topic not discussed in this paper).

3 Conclusions

In this paper we show that the Macaulay duration works well for non-parallel shifts of interest rates. It provides the best one-number interest rate sensitivity information for Lipschitz rate changes. A Macaulay duration matched minimum M-squared portfolio has the minimum downside risk under some easily-verified conditions. These results are valid for regular bonds, i.e., there is no uncertainty in timing and amount of cash flows, which implies that the Macaulay duration may work well for regular Treasury bonds (options-free and default-free). However, if a bond has an embedded option (callable bond, etc) or has credit risk (corporate bond, etc.) then one should be cautious in applying the results discussed in this paper, especially in immunization. This is because the optimal portfolio set up with the help of (9) contains only one or two bonds and is subject to severe credit/option risk. More research is needed on the role of the Macaulay duration (or effective duration) for option-embedded credit risky bonds.

Appendix

Proof of Theorem 1. Notice first that

$$\left|\int_D^{t_i}(g(u)-g(D))du\right| \le \int_D^{t_i} K(u-D)du \le \frac{K}{2}(t_i-D)^2. \tag{11}$$

The above calculation is valid for both cases $D \le t_i$, or $D > t_i$.

We can now show "$\le$" inequality of (6) as follows.

$$\begin{aligned}\left|\sum_i w_i \int_D^{t_i} g(u)du\right| &= \left|\sum_i w_i\left(\int_D^{t_i}(g(u)-g(D)\right)du + \sum_i w_i g(D)(t_i-D)\right|\\ &\le \sum_i w_i\left|\int_D^{t_i}(g(u)-g(D))du\right| + \left|\sum_i w_i g(D)(t_i-D)\right|\\ &\le \frac{K}{2}\sum_i w_i(t_i-D)^2 + \left|\sum_i w_i t_i - D\right||g(D)|\\ &\le \frac{1}{2}\sum_i w_i(t_i-D)^2 + \left|\sum_i w_i t_i - D\right|. \end{aligned}\tag{12}$$

The last inequality is due to $\|g\|_D \le 1$.

To show "$\ge$" inequality of (6) we define $g(D) = \mathrm{sgn}(\sum_i w_i t_i - D)$, where $\mathrm{sgn}(x) = 1$ if $x > 0$ and -1 if $x < 0$, and $g(u) = (u-D) + g(D)$. Then g is Lipschitz with $\|g\|_D = 1$. Now compute $\int_D^{t_i} g(u)du$ to get

$$\int_D^{t_i} g(u)du = \frac{1}{2}(t_i-D)^2 + g(D)(t_i-D).$$

Therefore

$$\begin{aligned}H(D) &\ge \frac{1}{2}\sum_i w_i(t_i-D)^2 + \sum_i w_i(t_i-D)g(D)\\ &= \frac{1}{2}\sum_i w_i(t_i-D)^2 + \left|\sum_i w_i t_i - D\right|. \end{aligned}\tag{13}$$

(12) and (13) imply (6). Note that $H(D)$ in (6) is a nonsmooth convex function and, in general, nonsmooth optimization is needed to find the minimum solution. However, it is easy to solve this particular problem. The first term of (6) reaches the minimum when $D = \sum_i w_i t_i$ and the second term is zero with this choice of D. Therefore the Macaulay duration $D_m = \sum_i w_i t_i$ minimizes $H(D)$.

Proof of Theorem 2. A simple calculation shows that

$$\frac{FV(f+g)}{FV(f)} = \sum_i w_i e^{\int_{t_i}^D g(u)du} \geq 1 + \sum_i w_i \int_{t_i}^D g(u)du. \tag{14}$$

The last inequality is due to $e^x \geq 1 + x$ for any real number x and $w_i \geq 0$ and $\sum_i w_i = 1$ (see (3)). If the change of forward rates g is a constant function, i.e., $g(u) = c$ for some constant c, then (14) implies

$$\frac{FV(f+g)}{FV(f)} \geq 1 + c\left(D - \sum_i w_i t_i\right).$$

If the holding period D is chosen to be the Macaulay duration $D = \sum_i w_i t_i$ then we have $FV(f+g) \geq FV(f)$ for all constant functions g.

If the change of forward rates g is a Lipschitz function, then (14) and (11) imply that

$$\begin{aligned}\frac{FV(f+g)}{FV(f)} &\geq 1 + \sum_i w_i(D - t_i)g(D) - \frac{K}{2}\sum_i w_i(t_i - D)^2 \\ &\geq 1 - \left|\sum_i w_i t_i - D\right|\left|g(D)\right| - \frac{K}{2}\sum_i w_i(t_i - D)^2 \\ &\geq 1 - \|g\|_D H(D)\end{aligned}$$

where $H(D)$ is given by (6).

Proof of Theorem 3. Denote $C_j = (1/2)M_j^2$, $j = 1, \ldots, N$. Note that (9) is equivalent to the following LP:

$$\begin{aligned}\text{minimize} \quad & \sum_j y_j C_j + z^+ + z^- \\ \text{subject to} \quad & \sum_j y_j D_j - z^+ + z^- = D \\ & \sum_j y_j = 1 \ \text{ and } \ z^+, z^-, y_j \geq 0, \ \forall j.\end{aligned} \tag{15}$$

Note also that one of z^+ and z^- must be zero for the optimal solution of (15). If both z^+ and z^- are zero then problems (10) and (15) are equivalent.

Assume conditions (i) and (ii) are satisfied but problems (9) and (10) are not equivalent. Then either z^+ or z^- is positive. If $z^+ > 0$ then z^+ is a basic variable of LP (15) and $z^- = 0$. Since there are two basic variables for two equality constraints we conclude that the other basic variable must be one of y_j, $j = 1, \ldots, N$, say y_k. The optimal basic feasible solution is given by

$$y_k = 1, \quad y_j = 0, \quad j \neq k, \quad \text{and} \quad z^+ = D_k - D, \quad z^- = 0.$$

The objective function can be written in terms of non-basic variables as

$$\begin{aligned}\sum_j C_j y_j + z^+ + z^- &= \sum_j (C_j + D_j) y_j + 2z^- - D \\ &= \sum_{j \neq k} (C_j + D_j - C_k - D_k) y_j + 2z^- + (C_k + D_k - D)\end{aligned}$$

The optimality implies that all coefficients of non-basic variables are non-negative, i.e., $C_j + D_j - C_k - D_k \geq 0$ for all $j \neq k$. On the other hand, $z^+ > 0$ implies $D_k > D$. We have arrived at a contradiction to condition (ii). Therefore $z^+ > 0$ is impossible for the optimal solution of (15) under condition (ii). Similarly, we can show $z^- > 0$ is impossible under condition (i). We have proved conditions (i) and (ii) imply the equivalence of (9) and (10).

Assume problems (9) and (10) are equivalent but conditions (i) and (ii) are not both satisfied. If condition (i) is not satisfied then $C_k - D_k \leq C_j - D_j$ for all j and $D_k < D$. We can now estimate the objective function of (15) with the help of its constraints:

$$\begin{aligned}\sum_j C_j y_j + z^+ + z^- &= \sum_j (C_j - D_j) y_j + 2z^+ + D \\ &\geq \sum_j (C_k - D_k) y_j + 2z^+ + D \\ &\geq C_k - D_k + D\end{aligned}$$

The lower bound $C_k - D_k + D$ is achieved when $y_k = 1$, $y_j = 0$ for $j \neq k$, and $z^+ = 0$, $z^- = D - D_k$, which is a basic feasible solution as $D_k < D$. However, $z^- > 0$ contradicts the equivalence of (10) and (15). Condition (i) is therefore satisfied. Note we have explicitly constructed the optimal solution to problem (15) when condition (i) is not satisfied. Condition (ii) can be shown satisfied in the same way.

Notes

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1. If bond price P is expressed in terms of its yield y by $P = \sum_i c_i (1 + y)^{-t_i}$, then its Macaulay duration is defined by $D_{mac} = \sum_i t_i c_i (1 + y)^{-t_i} / P$, and its modified duration is defined by $D_{mod} = -(1/P) dP/dy = D_{mac}/(1 + y)$. However, in continuous compounding case two durations are the same $D_{mac} = D_{mod}$.
2. The norm $\|g\|_D$ is equivalent to the standard norm $\|g\| = \max(|g(0)|, K)$ due to the Lipschitz property of g. The benefit of using $\|g\|_D$ instead of $\|g\|$ is that an equality relation (6) is established. If $\|g\|$ is used in the definition of $H(D)$ instead of $\|g\|_D$, then $H(D)$ is bounded above by $\frac{1}{2} \sum_i w_i (t_i - D)^2 + |\sum_i w_i t_i - D|(1 + D)$ and the Macaulay duration D_m provides the least upper bound.

3. Zero rates can be extracted from the observed coupon bonds with the standard bootstrapping technique (Hull (2002)) or the LP method (Allen, Thomas, and Zheng, 2000).

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