2 Complex Numbers

The complex number \( i \) is that object we define which satisfies the equation

\[ i^2 + 1 = 0. \]

\( i \) is the **imaginary number**. \( i = \sqrt{-1}. \)

From here, using the usual rules of arithmetic we uncover some properties that turn out to be very important for applications in wider mathematics and engineering.

2.1 The Basics

A complex number has a so-called Cartesian representation in the form

\[ z = x + iy \in \mathbb{C}, \]

where \( x \) and \( y \) are both **real numbers**. We say that \( x \) is the real part and \( y \) the imaginary part of \( z \),

\[ x = \text{Re}(z), \; y = \text{Im}(z). \]

This means that a complex number can be thought of as a two-dimensional number, with the real part \( x \) represented along the horizontal axis and the imaginary part \( y \) along the vertical axis.

The modulus of the complex number \(|z|\) is defined by

\[ |z| = \sqrt{x^2 + y^2}, \]

and the conjugate of \( z \) is

\[ \overline{z} = x - iy. \]

Using

\[ z\overline{z} = (x + iy)(x - iy) = x^2 + ixy - ixy - (i^2)y^2 = x^2 + y^2, \]

we find

\[ z\overline{z} = |z|^2. \]

Hence

\[ \frac{1}{z} = \frac{\overline{z}}{|z|^2}. \]
From this we can deduce how to divide using complex numbers. If \( z \) and \( w \) are both complex numbers then

\[
\frac{w}{z} = \frac{w\overline{z}}{|z|^2}.
\]

Let us look at some examples of how this works in practice. Remember, you use the usual rules of arithmetic with the proviso that \( i \) satisfies \( i^2 + 1 = 0 \).

**Example 1** Let \( z = 2 + i \) and let \( w = 1 - i \), then

\[
\frac{1}{z} = \frac{1}{2 + i} = \frac{1}{2 + i} \times \frac{2 - i}{2 - i} = \frac{2 - i}{4 - 2i + 2i - (-1)} = \frac{2 - i}{5},
\]

also

\[
|z| = \sqrt{4 + 1} = \sqrt{5}, \quad |w| = \sqrt{1 + 1} = \sqrt{2}.
\]

And

\[
zw = (2 + i)(1 - i) = 2 + i - 2i - i^2 = 2 - i - (-1) = 3 - i,
\]

also

\[
z + w = 2 + i + 1 - i = 3.
\]

### 2.2 The Argand Diagram

Given that a complex number \( z \) has a real and imaginary part, \( x \) and \( y \) say, so that

\[
z = x + iy,
\]

then we can represent \( z \) on a two-dimensional diagram by using the \( x \)-axis horizontally and the \( y \)-axis vertically. This representation is often called the Argand Plane or Diagram. Of course, when you draw such a diagram, you only need to draw the axes and the location of the complex number itself, the remaining arrows are just for illustration.

### 2.3 Polar Form

Consider the following power series which you have studied during the first part of this course last term,

\[
\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad \sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad (z \text{ in radians})
\]
and
\[
\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.
\]

Using the fact that powers of \(i\) cycle between the four values

\[1, i, -1, -i,\]

we find after a little algebra that

\[\cos(z) + i\sin(z) = \exp(iz),\]

holds for all complex \(z\). In particular,

\[\cos(\theta) + i\sin(\theta) = \exp(i\theta),\]

holds for all \(\theta \in \mathbb{R}\). So, this means that if there are real numbers \(r \geq 0\) and \(\theta\) such that

\[z = r \exp(i\theta),\]

then

\[z = r(\cos(\theta) + i\sin(\theta)).\]

**Example 2** The complex number \(-1\) can be written in polar form

\[e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1,\]
and the complex number 1 can be written in the form
\[ e^{i0} = \cos(0) + i\sin(0) = 1 \]

Less trivially, we see that
\[ 1 + i = \sqrt{2}e^{i\pi/4}, \]
because \( \sqrt{2}e^{i\pi/4} = \sqrt{2}(\cos \pi/4 + i\sin \pi/4) \) and \( \cos \pi/4 = 1/\sqrt{2} \).
Moreover, note that
\[ e^{2k\pi i} = 1, \]
whenever \( k \) is an integer.

The point of this is that we can manipulate complex expressions using the various properties of the exponential function, this can provide simpler methods of solving certain problems when contrasted with the Cartesian representation of complex numbers.

Returning to the exponential/trigonometric properties of complex numbers, we note that \( |z|^2 = r^2 \cos(\theta)^2 + r^2 \sin(\theta)^2 = r^2 \) and \( \tan(\theta) = \text{Im}(z)/\text{Re}(z) \). As a result \( r = |z| \) and if we restrict \( \theta \) to lie in the interval \((-\pi, \pi]\) then \( \theta \) is uniquely defined. On the other hand, if we allow any real value for \( \text{Arg}(z) \), then \( \theta \) is only defined up to an integer multiple of \( 2\pi \). In either case we write
\[ \theta = \text{Arg}(z). \]
This is called the argument of \( z \). From now on we shall also use the notation
\[ e^z = \exp(z). \]

Using the properties of the exponential function we can derive many useful results. For instance, if
\[ z_1 = r_1e^{i\theta_1} \quad \text{and} \quad z_2 = r_2e^{i\theta_2}, \]
then (provided that \( \text{Arg} \) is used in the sense of taking any real value)
\[ \text{Arg}(z_1z_2) = \text{Arg}(r_1e^{i\theta_1}r_2e^{i\theta_2}) = \text{Arg}(r_1r_2e^{i(\theta_1+\theta_2)}) = \theta_1+\theta_2 = \text{Arg}(z_1)+\text{Arg}(z_2). \]

We can use the same idea to obtain well-known trigonometric formulae as follows. Using
\[ \exp(z + w) = \exp(z) \exp(w), \]
we find
\[ e^{i(\theta_1 + \theta_2)} = \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2), \]
which equals
\[ e^{i\theta_1} e^{i\theta_2} = [\cos(\theta_1) + i \sin(\theta_1)][\cos(\theta_2) + i \sin(\theta_2)], \]
which is
\[ \cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_1) + i[\sin(\theta_1) \cos(\theta_2) + \cos(\theta_1) \sin(\theta_2)]. \]

Hence
\[ \cos(\theta_1 + \theta_2) = \cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_1), \]
and
\[ \sin(\theta_1 + \theta_2) = \sin(\theta_1) \cos(\theta_2) + \cos(\theta_1) \sin(\theta_2). \]

Similarly, we can deduce a formula known as DeMoivre’s Theorem that
\[ \cos(n\theta) + i \sin(n\theta) = (\cos(\theta) + i \sin(\theta))^n, \]
which is a re-statement of the fact
\[ e^{in\theta} = (e^{i\theta})^n. \]

**Example 3** Let us simplify the expression
\[ (e^{i\theta})^3 = e^{3i\theta}. \]
The right-hand side is
\[ \cos(3\theta) + i \sin(3\theta), \]
and the left-hand side is, using the binomial theorem,
\[ (\cos(\theta) + i \sin(\theta))^3 = \cos^3 \theta + 3i \cos^2 \theta \sin \theta + 3(i^2) \cos \theta \sin^2 \theta + (i^3) \sin^3 \theta. \]

But \( i^2 = -1 \) and therefore \( i^3 = -i \), so that taking real and imaginary parts in the two equal expressions that we have, we find that
\[ \cos(3\theta) = \cos^3 \theta - 3 \cos \theta \sin^2 \theta = \cos^3 \theta - 3 \cos \theta (1 - \cos^2 \theta), \]
and
\[ \sin(3\theta) = 3 \cos^2 \theta \sin \theta - 3 \sin^3 \theta = 3(1 - \sin^2 \theta) \sin \theta - \sin^3 \theta. \]

Hence
\[ \cos(3\theta) = 4 \cos^3 \theta - 3 \cos \theta, \]
and
\[ \sin(3\theta) = 3 \sin \theta - 4 \sin^3 \theta. \]
We can use the above ideas to solve various types of real and complex equations.

**Example 4** Consider the problem of finding all complex numbers \( z \) that solve 
\[
z^3 = 2.
\]

By setting \( z = x + iy \) we could try and find those \( x \) and \( y \)'s that make the equation 
\[
(x + iy)^3 = 2
\]
hold. This is not a good idea, although this approach would work relatively easily for an equation like \( z^2 = -2 \! \).

If we instead put \( z = re^{i\theta} \), then 
\[
z^3 = r^3(e^{i\theta})^3 = r^3e^{3i\theta} = 2 = 2 \times 1 = 2e^{2k\pi i},
\]
where \( k \) is any integer. Here we have to use the fact that \( e^{2k\pi i} = 1 \) in order to find all the possible complex solutions available. Now, we can equate the moduli and the arguments in our equation to find that 
\[
r^3 = 2, \quad 3\theta = 2k\pi i.
\]

This means that \( r = 2^{1/3} \) and \( \theta = \frac{2}{3}k\pi \) where \( k \) is any integer. Taking **three** consecutive values of \( k \) will give all the required complex solutions, for instance \( k = -1, 0, 1 \) will do.

This last comment follows from the fact that if 
\[
p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n
\]
is a complex polynomial (in the sense that \( a_i \) are **real** and \( z \) can be **complex**), then the following is essentially the Fundamental Theorem of Algebra.

**Theorem 2.1** Counting according to multiplicity, \( p \) has exactly \( n \) complex roots. Moreover, if \( z \) is a root, so that \( p(z) = 0 \), then so is the conjugate \( \bar{z} \), so that \( p(\bar{z}) = 0 \).

The phrase ‘counting according to multiplicity’ means that if \( p \) has a root \( w \), so that 
\[
p(z) = (z - w)^2 q(z),
\]
where \( q(w) \neq 0 \), then \( w \) is called a double root. If
\[
p(z) = (z - w)^m q(z),
\]
then \( w \) is called a root of \( p \) of multiplicity \( m \) provided \( q(w) \neq 0 \).

This means that an equation like
\[
z^{100} = -3,
\]
will have 100 complex solutions which happen to lie on a regular 100-gon in the complex plane.

**Example 5**  
Find all complex solutions of
\[
z^4 - z^2 + 1 = 0.
\]

We know immediately from the fundamental theorem of algebra that there are four of them and these occur in complex conjugate pairs. If \( w = z^2 \) then
\[
w^2 - w + 1 = 0,
\]
so that
\[
w = \frac{1}{2}(1 \pm \sqrt{-3}) = \frac{1}{2} \pm i \frac{\sqrt{3}}{2}.
\]

By solving the pair of equations
\[
z^2 = \frac{1}{2} \pm i \frac{\sqrt{3}}{2},
\]
we shall be done. Now we can write
\[
\frac{1}{2} \pm i \frac{\sqrt{3}}{2} = e^{\pm i\pi/3},
\]
hence by taking square roots, we find
\[
z = e^{\pm i\pi/6} \text{ or } z = -e^{\pm i\pi/6}.
\]

This is because an equation of the form \( z^2 = w \) has a pair of solutions that we may write \( z = +w^{1/2} \) and \( z = -w^{1/2} \) and we apply the usual laws of indices when \( w \) is written in polar form.
3 Hyperbolic Functions

The hyperbolic functions are defined as follows:
\[
\cosh(z) = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh(z) = \frac{e^z - e^{-z}}{2}.
\]

Using \(\cos(-z) = \cos(z)\) and \(\sin(-z) = -\sin(z)\), we find
\[
\cos(z) + i\sin(z) = \exp(iz) \quad \text{and} \quad \cos(z) - i\sin(z) = \exp(-iz).
\]

Adding these two equations we find
\[
\cos(z) = \frac{e^{iz} + e^{-iz}}{2},
\]
and subtracting these two equations we find
\[
\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.
\]

It follows that
\[
\cosh(iz) = \cos(z) \quad \text{and} \quad -i\sinh(iz) = \sin(z).
\]

Hence the formula
\[
1 = \cos^2(-iz) + \sin^2(-iz) = \cosh^2(z) + (-\sinh(z))^2 = \cosh^2(z) - \sinh(z)^2
\]
follows.

4 Complex Logarithms

If \(e^z = w\) then the definition of the logarithm as the inverse of the exponential implies

\[
z = \ln(w).
\]

Now, if \(z = x + iy\) then
\[
e^{x+iy+2k\pi i} = w,
\]
where \(k\) is any integer. Taking the modulus of \(w\) and \(e^z\) yields
\[
|w| = |e^{x+iy+2k\pi i}| = |e^x e^{i(y+2k\pi)}| = |e^x||e^{i(y+2k\pi)}| = e^x,
\]
therefore
\[ x = \ln |w|, \]
where \( \ln \) is the real-valued logarithm. Taking the argument of \( w \) we find
\[
\text{Arg}(w) = \text{Arg}(e^{x+iy+2k\pi}) = \text{Arg}(e^x) + \text{Arg}(e^{iy+2k\pi}) = 0 + y + 2k\pi = y + 2k\pi.
\]
As a result we have the complex logarithm of \( w \):  
\[
\ln(w) = z = x + iy = \ln |w| + i\text{Arg}(w) + 2k\pi i,
\]
where \( k \) is any integer.

**Example 6** This allows us to find the logarithm of negative numbers, the result is a complex number. For instance,
\[
\ln(-1) = \ln |-1| + i\text{Arg}(-1) + 2k\pi i = \ln 1 + i\pi + 2k\pi i = (2k + 1)\pi i.
\]

5 Solution Loci

A solution locus of a complex problem just refers to the set of complex numbers that solves the particular problem at hand. For instance, consider the following problem

**Example 7** Find the solution locus in the complex plane of the problem
\[
|z + i| = |z - i|.
\]

To solve this problem, write \( z = x + iy \) and note that
\[
|z + i| = |z - i| \Rightarrow |z + i|^2 = |z - i|^2.
\]

Since
\[
|z + i|^2 = |x + iy + i|^2 = x^2 + (y + 1)^2,
\]
and
\[
|z - i|^2 = |x + iy - i|^2 = x^2 + (y - 1)^2,
\]
we find
\[
x^2 + (y + 1)^2 = x^2 + (y - 1)^2,
\]
or

\[(y + 1)^2 = (y - 1)^2.\]

Taking ± square roots, as it were, we see that

\[y + 1 = \pm(y - 1).\]

Taking the + case, \(y + 1 = y - 1\), so that no \(y\) satisfies this relationship. On the other hand, the − case gives \(y + 1 = 1 - y\), or \(y = 0\).

This means that the solution to our problem can be written in the form \(z = x + iy = x\) because \(y = 0\), but the \(x\) coordinate remains arbitrary. This means that the entire **real axis** is the solution locus of our problem.