# M3S4/M4S4: Applied probability: 2007-8
# Problems 6: Continuous time Markov processes

1. Show that $\Pi_i(s,t)$ – the pgf for the population size at time $t$ given there are $i$ individuals at time 0, for a pure death process with rate $\nu_n = n\nu$, satisfies the partial differential equation
$$\frac{\partial}{\partial t}\Pi_i(s,t) = \nu(1-s)\frac{\partial}{\partial s}\Pi_i(s,t).$$

2. Write down the forward differential difference equations for

   (a) a Poisson process ($\beta_n = \lambda,\ \nu_n = 0$).

   (b) a linear birth process ($\beta_n = \beta n,\ \nu_n = 0$).

3. A machine can be in one of two states: working or being repaired. When it is in the 'working' state it functions for a time that is exponentially distributed, $Exponential(\lambda)$, before switching to the 'being repaired' state. When it is in the 'being repaired' state it functions for a time that is exponentially distributed, $Exponential(\nu)$, before switching to the 'working' state. Given that the machine starts in the 'working' state, what is the mean time until

   (a) it breaks down for the first time?

   (b) it breaks for the third time?

   What is the variance of the time until

   (c) it breaks down for the first time?

   (d) it breaks down for the third time?

4. A colony of $N(>1)$ creatures inhabit a planet which has continual daylight, and the pattern of waking and sleeping follows a Markov process. The probability that a particular sleeping individual awakes during a time interval of length $\delta t$ is $\beta\delta t + o(\delta t)$, and the probability that a particular awake individual falls asleep during a time interval of length $\delta t$ is $\nu\delta t + o(\delta t)$. Assume that individuals behave independently of each other. We are interested in the number of individuals awake at time $t$.

   (a) Find the $Q$-matrix for this process.

   (b) What is the stationary distribution?

   Consider the 2-state Markov chain (with states $s$:sleep and $w$:wake) for one individual with transition matrix

$$P(t) = \begin{array}{c} s \\ w \end{array} \begin{pmatrix} 1 - p_{sw}(t) & p_{sw}(t) \\ 1 - p_{ww}(t) & p_{ww}(t) \end{pmatrix}$$

   (c) Write down the $Q$-matrix for this 2-state process.

   (d) Calculate $p_{ww}(t)$ and $p_{sw}(t)$ using the forward differential equations.

   (e) If $X_m(t)$ denotes the number awake at time $t$ given there are $m(< N)$, what is $\mathrm{E}(X_m(t))$?
   Hint: use the fact that $X_m(t) \sim Binomial(m, p_{ww}(t)) + Binomial(N-m, p_{sw}(t))$ (easier than evaluating $\Pi'_m(1,t)$ for the entire population - though you might like to try!)

5. Consider the linear birth and death process (with $\nu_n = n\nu$ and $\beta_n = n\beta$).

   (a) Use the backward differential equations to show

$$\frac{\partial}{\partial t}\Pi_i(s,t) = i\nu\Pi_{i-1}(s,t) - i(\nu+\beta)\Pi_i(s,t) + i\beta\Pi_{i+1}(s,t) \quad i \geq 1$$

   (b) Show that $\Pi_i(s,t) = [\Pi_1(s,t)]^i$, and use these result to:

   (c) find $\Pi_1(s,t)$ for both $\beta = \nu$ and $\beta \neq \nu$.
   Check that this agrees with the results using the forward equations derived in lectures.