ON THE CLASSIFICATION OF SELFLOCALIZED STATES OF ELECTROMAGNETIC FIELD WITHIN NONLINEAR MEDIUM

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The problem of identifying and classifying soliton-type states for nonintegrable equations is one of essential problems in nonlinear physics. The difficulty of its solution is explained by the fact that from the mathematical point of view this problem is reduced to the analysis of nonintegrable conservative systems (in particular, Hamiltonian ones) with two or more degrees of freedom: the selflocalized states are related to the homoclinic loops of saddle equilibrium states of dynamical systems.

The electrodynamics' equations for nonlinear nondissipative medium in the approximation of a high-frequency potential (i.e., we take into account the self-action but neglect the excitation by multiple frequencies) for the flat geometry of the field lead to the following dynamical system with two degrees of freedom:

\[
\begin{align*}
\frac{dE}{dx} &= (k_z^2 - k^2) \mathcal{E} (\omega, E^2) E_y, \\
\frac{dE}{dy} &= P, \\
\frac{dE}{dz} &= (k_z^2 - k^2) \mathcal{E} (\omega, E^2) E_x, \\
\frac{dE}{dx} &= \mathcal{Q} (E, P) = \frac{\mathcal{E} + 2E^2 \frac{d\mathcal{E}}{dE}}{dE}.
\end{align*}
\]

Here \( \mathcal{Q} \) is the dielectric permittivity; \( k_z \) is a wave vector; \( k = \omega / c; \omega \) is the frequency of the electromagnetic wave of the type \( E_j(x) \exp(ik_z z), j=x,y,z \). The system (1) has a first integral

\[
\mathcal{H} = P^2 - k_z^2 (E_x^2 + E_y^2) + k_z^2 \int_0^2 \mathcal{E} (\omega, s) ds + (k_z^2 - k^2) \mathcal{E} )^2 E_x^2 / k_z.
\]

For \( k_z^2 - k^2 \mathcal{E} (\omega, 0) > 0 \), homoclinic loops of the singular point \( 0 (E_x = E_y = E_z = P = 0) \) in the phase space \( R^4 \) on the level of the first integral \( \mathcal{H} = 0 \) correspond to selflocalized waveguide states of the field, for which \( \lim_{x \to +\infty} E = 0, \lim_{x \to -\infty} P = 0 \). Note that 0 is a...
saddle with multiple characteristic exponents \( \lambda, \lambda, -\lambda, -\lambda \) where 
\[ \lambda = \left( k_x^2 - k_y^2 \right)^{1/2}. \]
For the TE- and TM-type fields described by electrical vectors \((0, E_y, 0)\) and \((E_x, 0, E_z)\), the image of known self-localized states is given by the four homoclinic loops \( S_1, S_2, S_3, S_4 \) lying in invariant planes: \( S_1 \) and \( S_2 \) in \((E_y, P)\), and \( S_3 \) and \( S_4 \) in \((E_x, E_z)\).

A numerical study has been performed for 
\[ E = E_0 + E_2 E^2, \]
where 
\[ E_0(\omega) > 0, \quad E_2(\omega) > 0. \]
Simple scaling transformations of the electrical vector and the independent variable show that the system contains the only structural parameter 
\[ \gamma^2 = k_x^2/(k_y^2 E_0(\omega)). \]
The computations have been done for \( \gamma^2 = 2. \)

Below the results of qualitative and numerical analysis are outlined.

1) Numerically detected have been a saddle periodic motion \( L \) and a heteroclinic trajectory \( g_1 \) connecting \( L \) with 0. The presence of discrete symmetries in the system (1) show that the system contains a second heteroclinic curve \( g_2 \) connecting 0 with \( L \). In addition, a curve \( g_L \) homoclinic to \( L \) has been found. We have checked that \( W_L^U \) with \( W_L^S \), \( W_0^U \) with \( W_L^S \) and \( W_L^U \) with \( W_L^S \) intersect along, respectively, \( g_1, g_2 \) and \( g_L \) in a stable way. Here we assert the following [4,5]: within each neighbourhood of the homoclinic contour \( L U g_1 U g_2 U g_L U 0 \) there exists a countable set of homoclinic loops to 0 being in one-to-one correspondence with the set of all finite words composed of symbols \( L \) and \( g_L \) (in accordance with the principles of symbolic dynamics, the symbol \( L \) stands for one rotation near the trajectory \( L \), and the symbol \( g_L \) stands for a passage of the loop in a neighbourhood of the trajectory \( g_L \)).

2) Numerically found has been a series of homoclinic loops performing a lot of rotations in the vicinity of TE- and TM-loops; namely, the following loops have been found: 
\[ S_3 g_2 S_2 g_21 S_2, S_3 g_1 S_1 g_2 S_2, S_1 g_1 S_1 g_2 S_2, S_3 g_2 S_2 g_21 S_2, S_1 g_1 S_1 g_2 S_2. \]
Here the symbols \( S_1, S_2 \) code the parts of the trajectory that are close to the TE-loops, and the symbols \( S_3, S_4 \) code the parts close to the TM-loops. The symbols \( g_{ij} \) denote the trajectory parts that, in projection to the plane \((E_x, E_z)\), are close to \( S_4 \) (symbols \( g_{11} \)) or to \( S_3 \) (symbols \( g_{12} \)).
the coordinate $E_y$ is somewhat biased from zero: for $\varepsilon_{2j}$ to the positive side, and for $\varepsilon_{1j}$ to the negative side.

PROPOSITION 5. Let a dynamical system $X$ in $\mathbb{R}^4$ be symmetric with respect to transformations

$$(u_1, u_2, v_1, v_2) \mapsto (-u_1, u_2, -v_1, v_2), \quad (u_1, u_2, v_1, v_2) \mapsto (u_1, -u_2, v_1, -v_2)$$

and have a smooth first integral $H = u_1 v_1 - u_2 v_2 + \cdots$ (Here $(u_1, u_2, v_1, v_2)$ are coordinates in $\mathbb{R}^4$). Suppose a saddle equilibrium state $0$ with multiple characteristic exponents lies in the level $H=0$. Let $0$ have in invariant planes $(u_1=v_1=0)$ and $(u_2=v_2=0)$, two homoclinic eights $(S_1, S_2)$ and, resp., $(S_3, S_4)$, along which $W^u_0$ and $W^s_0$ intersect transversely. Suppose also that the coefficients $\beta_1$ and $\beta_2$ of the terms $u^2 v_1$ and $u_1^2 v_2$ in the normal form of the system in the saddle are different from zero. Then the bunch $B = S_1 \cup S_2 \cup S_3 \cup S_4$ possesses two-dimensional stable and unstable manifolds $W^s_B$ and $W^u_B$.

As $\mathcal{X} \to \infty$ (i.e. $\infty \to \mathcal{X}$) all trajectories from $W^s_B$ (resp. $W^u_B$) tend to $B$. For $\delta^2 = 2$ in system (1), the manifolds $W^u_B \setminus B$ and $W^s_B \setminus B$ consist of two connected components $W^u_1, W^u_2$ and $W^s_1, W^s_2$; the motion near $W^u_1(s)$ and $W^u_2(s)$ is a repetitive passage along the loop groups $S_1 S_3 S_1 S_4$ and $S_2 S_3 S_2 S_4$, respectively.

As $\mathcal{X} \to \infty$ all the trajectories from a small neighbourhood $V$ of $B$ that do not lie in $W^u_B$ leave $V$ along $W^u_B$. The neighbourhood $V$ evidently has no other homoclinic loops but $S_1$. As system (1) has homoclinic loops close in their initial and final parts, to the loops of $B$ this makes it possible to suppose the existence of homoclinic to $B$ trajectories along which the manifolds $W^u_B$ and $W^s_B$ intersect in a stable way: $\{\varepsilon_{11}, \varepsilon_{12}\} \subseteq W^u_1 \cap W^s_1$.

Note that a homoclinic trajectory to a bunch of homoclinic loops is a new object for the theory of dynamical systems.

PROPOSITION 5. In any neighbourhood of the contour $B \cup \bigcup_{i,j} \varepsilon_{ij} \cup 0$ there exists a countable set of homoclinic loops being in one-to-one correspondence to the set of the words.
of the type $U_1^{p_1}g_1^{i_1}j_1 S_1^{q_1} U_2^{p_2} g_2^{i_2} j_2 \ldots g_n^{i_n} S_n^{q_n}$. Here the symbol $g_{ij}$ denotes a passage of the loop near the trajectory $g_{ij}$; $i_j \in \{1,2\}$; $S_i^q$ and $U_i^p$ are groups of symbols $S_1, S_2, S_3, S_4$ denoting respectively $q$ and $p$ rotations near $w_i^q$ and $w_i^p$. $S_1^q$ start with $S_1$ or $S_2$, and $U_i^p$ terminate with $S_3$ or $S_4$; $p_j$ and $q_j$ are sufficiently large integers, all even except, perhaps, for $p_1$ and $q_1$; $q_j$ can be taken arbitrary, while $p_{j+1}$ is uniquely determined by $q_j$; for each given $i_j$, the freedom in the choice of the $i_{j+1}$ is restricted by the only condition that the group $S_{i_j} U_{i_j}^{p_{j+1}}$ cannot contain the following quadruplets of symbols: $S_1 S_2 S_3 S_4$, $S_2 S_3 S_4 S_1$, and $S_3 S_4 S_1 S_2$.

Similar problems of classifying the soliton states and identifying the basic series of solitons arise for the system of two coupled Shrödinger equations. Here, in particular, bifurcations can occur that produce vector solitons from the polarized solitons.

REFERENCES

1. Eleonsky V.M., Oganesjantz L.G. and Silin V.P., Radiophizika, 17, 1812-1816, 1974