BIFURCATIONS OF TWO-DIMENSIONAL DIFFEOMORPHISMS WITH NON-ROUGH HOMOCLINIC CONTOURS

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Two-dimensional diffeomorphisms having two saddle points for which one pair of stable and unstable manifolds intersect transversely and the other pair has a quadratic tangency are considered.

1. Introduction

Consider a $C^r$-smooth ($r \geq 3$) two-dimensional diffeomorphism $f$ having two saddle fixed points $O_1$ and $O_2$ with multipliers $\lambda_i, \gamma_i$ where $|\lambda_i| < 1, |\gamma_i| > 1, i = 1, 2$. Suppose $W^u(O_1)$ and $W^s(O_2)$ intersect each other transversely along a heteroclinic orbit $\Gamma_{12}$ and $W^u(O_2)$ and $W^s(O_1)$ exhibit a quadratic tangency along a heteroclinic orbit $\Gamma_{21}$ (Fig. 1). We will say that $f$ has a non-rough homoclinic contour $C = O_1 \cup O_2 \cup \Gamma_{12} \cup \Gamma_{21}$.

Denote by $N$ the set of orbits of $f$ which lie in a sufficiently small neighbourhood $U$ of $C$. We show that such diffeomorphisms may be divided into three classes, depending on the character of tangency of $W^u(O_2)$ and $W^s(O_1)$ along $\Gamma_{21}$ and give a description of the structure of the set $N$. For the first class the structure is trivial, for the second class it admits a complete description in terms of symbolic dynamics. For the third class...
the following results are obtained: the principal moduli of $\Omega$-conjugacy are found; the density of systems possessing non-rough periodic orbits and the density of systems having infinitely many stable or (and) completely unstable periodic orbits are proved.

2. Three Classes of Diffeomorphisms

It is shown in [1, 2] that in a small neighbourhood $U_i$ of the point $O_i$ there exist $C^{r-1}$-coordinates $(x_i, y_i)$ such that the map $f_{|U_i}$ has the form

$$
\bar{x}_i = \lambda_i x_i + f_i(x_i, y_i)x_i y_i, \quad \bar{y}_i = \gamma_i y_i + g_i(x_i, y_i)x_i y_i,
$$

where $f_i(0, y_i) \equiv 0$, $g_i(x_i, 0) \equiv 0$. Equations of the manifolds $W^s_{\text{loc}}(O_i)$ and $W^u_{\text{loc}}(O_i)$ are $y_i = 0$ and $x_i = 0$, respectively.

Choose two points of orbit $T_{12}$: the point $M^-(0, y^-) \in U_2$ and the point $M^+(x^+, 0) \in U_1$. Evidently, $f^n(M^-) = M^+$ for some integer $n$. The map $T \equiv f^n$ from some neighbourhood of $M^-$ into some neighbourhood of $M^+$ may be written in the form

$$
\bar{x}_1 - x^+ = ax_2 + b(y_2 - y^-) + ..., \quad \bar{y}_1 = cx_2 + d(y_2 - y^-)^2 + ...
$$

Note that $T$ is a diffeomorphism, therefore, the Jacobian does not vanish at $M^-$: $bc \neq 0$. Also, $d \neq 0$ because the tangency is quadratic. The character of adjoining of $W^u(O_2)$ to $W^s(O_1)$ at $M^+$ is determined by signs of $c$ and $d$ (Fig. 2). For instance, $W^u(O_2)$ touches $W^s(O_1)$ from below if $d < 0$, and from above if $d > 0$.

The diffeomorphisms with non-rough homoclinic contours are divided into three classes, depending on signs of $\lambda_1$, $\gamma_2$, $c$ and $d$. The combinations $\lambda_1 > 0$, $\gamma_2 > 0$, $c < 0$, $d < 0$ and $\lambda_1 > 0$, $\gamma_2 > 0$, $c < 0$, $d > 0$ correspond to the first and the second classes, respectively (Fig. 2 a and Fig. 2 b). The remaining cases (among them those with negative $\lambda_1$ and $\gamma_2$) correspond to the third class.

For the diffeomorphisms of the first class the set $N$ is trivial: $N = C$. For the diffeomorphisms of the second class the structure of $N$ is non-trivial. Here, all orbits of the set $N \setminus \Gamma_{21}$ are of saddle type and $N$ admits a complete description in terms of symbolic dynamics.

Diffeomorphisms of the third class also have nontrivial hyperbolic subsets. These subsets, however, may not exhaust all the set $N \setminus \Gamma_{21}$. Moreover, the structure of $N$ changes when the value of the invariant $\theta = -((\ln |\lambda_2|)/(\ln |\gamma_1|))$ changes. Namely, let $H_3$ be a codimension one bifurcation surface in the space of dynamical systems, composed by diffeomorphisms of the third class.

**Theorem 1.** If $f$, $f' \in H_3$ and $f$ is $\Omega$-conjugate to $f'$, then $\theta = \theta'$.

Theorem 1 means that $\theta$ is a modulus of $\Omega$-conjugacy [1, 2] for the third class. Moreover, similarly to the case of homoclinic tangency of invariant manifolds of a single saddle point [3, 4], the following result is proved:

**Theorem 2.** Systems with a countable set of moduli of $\Omega$-conjugacy are dense in $H_3$.

As it is argued in [3, 4], if a system has $\Omega$-moduli, then changing the values of the moduli causes bifurcations of nonwandering orbits (in particular, periodic and homoclinic orbits). Moreover, when a countable set of $\Omega$-moduli exists, we deduce that those bifurcations may be very complex.
THEOREM 3. Systems with non-rough periodic orbits of any order of degeneracy and systems with a homoclinic tangencies of any order are dense in $\mathbb{H}_3$.

3. Sinks and Sources of Diffeomorphisms of the Third Class

Note, that saddle-node periodic orbits having one unit multiplier and non-zero first Lyapunov value are the simplest form of non-rough periodic orbits mentioned in the Theorem 3. Stable (sinks) or completely unstable (sources) periodic orbits may appear when the saddle-nodes bifurcate. The type of stability of these orbits depends, first of all, on the saddle values $\sigma_i = |\lambda_s| |\gamma_i|$ of the points $O_i$.

THEOREM 4.
1. Systems with a countable set of sinks (resp., sources) are dense in $\mathbb{H}_3$ in the case $\sigma_1 < 1$, $\sigma_2 < 1$ (resp., in the case $\sigma_1 > 1$, $\sigma_2 > 1$).

2. If $\sigma_1 < 1$, $\sigma_2 < 1$, then neither $f$ nor any nearby system has sources in $U$. If $\sigma_1 > 1$, $\sigma_2 > 1$, then neither $f$ nor any nearby system has sinks in $U$.

In the case where one of the saddle values is less than 1 and the other is greater than 1, systems in $\mathbb{H}_3$ may have sinks and sources simultaneously. Here, an important quantity is also $\alpha = \sigma_1^\theta \sigma_2$. Divide the surface $\mathbb{H}_3$ into two parts. Denote as $H_s$ the part of $\mathbb{H}_3$ where $\alpha < 1$, and as $H_u$ the part of $\mathbb{H}_3$ where $\alpha > 1$. 
THEOREM 5. Systems with a countable set of sinks (sources) are dense in $H_s$ (in $H_u$).

THEOREM 6. In the cases where $\lambda_1$ or $\gamma_2$ are negative, systems having a countable set of sinks and sources simultaneously are dense in $H_s \cup H_u$.

In the case where $\lambda_1$ and $\gamma_2$ are positive, let $H_{s+}$ be the subset of $H_s$ for which $d > 0, \sigma_1 > 1, \sigma_2 < 1$ or $d < 0, \sigma_1 < 1, \sigma_2 > 1$, let $H_{u+}$ be the subset of $H_u$ for which $d > 0, \sigma_1 < 1, \sigma_2 > 1$ or $d < 0, \sigma_1 > 1, \sigma_2 < 1$.

THEOREM 7.
1. Systems in $H_{s+}$ do not have sources and systems in $H_{u+}$ do not have sinks.
2. Systems having a countable set of sinks and sources simultaneously are dense in $(H_s \setminus H_{s+}) \cup (H_u \setminus H_{u+})$.

4. Newhouse Regions

The coexistence of periodic sinks and sources is not only the property of diffeomorphisms on $H_3$; in fact, it is a general situation in the following sense

THEOREM 8. Let $f$ be a diffeomorphism with non-rough homoclinic contour ($f$ is not assumed here to belong to the third class) and let one of the saddle values be less than 1 and the other be greater than 1. Then, in any neighbourhood of $f$ in the space of dynamical systems there exist open domains (Newhouse regions) where systems having a countable set of sinks and sources simultaneously are dense.

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References