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ABSTRACT

We study the Schrödinger equation $i\partial_t\psi = -\Delta\psi + V\psi$ on $L^2((0,1),\mathbb{C})$ where V is a very high and localized potential wall. We consider the process where the position and the height of the wall change as follows: First, the potential increases from zero to a very large value, and so a narrow potential wall is formed and almost splits the interval into two parts; then, the wall moves to a different position, after which the height of the wall decreases to zero again. We show that even though the rate of variation of the potential's parameters can be arbitrarily slow, this process alternates adiabatic and non-adiabatic dynamics, leading to a non-trivial permutation of the instantaneous energy eigenstates. Furthermore, we consider potentials with several narrow walls and show how an arbitrarily slow motion of the walls can lead the system from any given state to an arbitrarily small neighborhood of any other state, thus proving the approximate controllability of the above Schrödinger equation by means of a soft, quasi-adiabatic variation of the potential.

I. INTRODUCTION

In this paper, we control the eigenstates of the Schrödinger equation in a bounded interval by moving very slowly a sharp and narrow potential wall. Our aim is twofold. First, we show how to *permute eigenstates through a slow cyclic process*. Second, we provide a *method for approximate control* of the Schrödinger equation.

One of the basic principles of quantum mechanics is that quantum numbers are preserved when the parameters of the system change sufficiently slowly.[1–3] That is, a slow variation of the Hamiltonian operator makes the system evolve adiabatically: if one prepares the initial state with a definite energy and, then, changes the Hamiltonian slowly, the system will stay close to the state defined by the same set of quantum numbers for a very long time. In the absence of symmetries, this means that if we order the eigenstates of the Hamiltonian by the increase in energy, then the adiabatic evolution along a generic path from a Hamiltonian H_1 to a Hamiltonian H_2 must lead the system from the kth eigenstate of H_1 to the kth eigenstate of H_2 with the same k. In particular, a slow cyclic (time-periodic) variation of the Hamiltonian is expected to return the system to the initial eigenstate (up to a phase change) after each period, for many periods.

A class of (quasi)adiabatic processes we consider in this paper violates this principle in the following sense. Consider a particle in a bounded region subject to an external potential. We show how an arbitrarily slow cyclic change in the potential can lead the system close to a different eigenstate after each cycle, provided that the potential wall almost separates the region into disjoint pieces during a part of the cycle. Moreover, the evolution over the cycle of our process may act as any finite permutation of the energy eigenstates. By building further on this, we are able to perform a global approximate control of the Schrödinger equation using adiabatic arguments: by a slow change in potential, we can lead the system from any given state to any other one, up to an arbitrarily small error.

The effect is due to the fact that when the region is separated into parts, an additional quantum number emerges (the number of the part where the particle is localized). It is an implementation of a general idea from Ref. 4. Let the Hamiltonian vary slowly and periodically such that, for a part of the period, the system acquires an additional quantum number that gets destroyed for the rest of the period. The system will evolve adiabatically, but it can find itself close to a different eigenstate at the end of each period. Thus, the periodic creation and destruction of an additional quantum integral gives rise to much more general and diverse types of adiabatic processes than it was previously thought.

We consider the one-dimensional Schrödinger equation in $L^2((0,1),\mathbb{C})$ with a locally supported potential with time-dependent position and height,

$$\begin{cases} i\partial_t u(t,x) = -\partial_{xx}^2 u(t,x) + V(t,x)u(t,x), & x \in (0,1),\ t > 0, \\ u(t,0) = u(t,1) = 0, & t \geq 0, \\ u(t=0,\cdot) = u_0(\cdot) \in L^2((0,1),\mathbb{C}). \end{cases} \tag{1.1}$$

The potential V is defined as follows: Take a function $\rho \in \mathcal{C}^2(\mathbb{R},\mathbb{R}^+)$ supported in $[-1,1]$ such that $\int_{\mathbb{R}} \rho(s)\,\mathrm{d}s = 1$. Take $\eta \in \mathcal{C}^2(\mathbb{R},\mathbb{R}^+)$, $I \in \mathcal{C}^2(\mathbb{R},\mathbb{R}^+)$, and $a \in \mathcal{C}^2(\mathbb{R},(0,1))$ and set

$$V(t,x) \;=\; I(t)\,\rho^{\eta(t)}(x - a(t)), \qquad \text{where} \quad \rho^{\eta}(x) = \eta\,\rho(\eta x). \tag{1.2}$$

When I and η are very large, this describes a very high and thin wall (where I and η determine the wall's height and sharpness, respectively) that splits the interval $(0,1)$ into two parts $(0,a)$ and $(a,1)$. We are interested in controlling the eigenmodes of the Schrödinger equation by suitably changing η, I, and a.

Note that (1.1) is a free Schrödinger equation when $I \equiv 0$ and that ρ^{η} is close to the Dirac delta-function for large η. Thus, in the limit $\eta = +\infty$, the system can be written as

$$\begin{cases} i\partial_t u(t) = -\partial_{xx}^2 u(t), & x \in (0,a) \cup (a,1), \\ u(t,0) = u(t,1) = 0, \qquad u(t,a-0) = u(t,a+0), & t \geq 0, \\ \alpha u(t,a) + (1-\alpha)(\partial_x u(t,a+0) - \partial_x u(t,a-0)) = 0, & t \geq 0, \\ u(t=0,\cdot) = u_0(\cdot) \in L^2((0,a)\cup(a,1),\mathbb{C}), \end{cases} \tag{1.3}$$

where $\alpha = \frac{I}{I+1}$. Let $\alpha(t)$ change slowly in such a way that $\alpha \equiv 0$ for t close to 0 and T and $\alpha \equiv 1$ for a closed subinterval of $(0,T)$. Then, system (1.3) formally describes adiabatic transitions from a free particle confined in the interval $(0,1)$ (when $\alpha = 0$) to a free particle confined in the pair of disjoint intervals $(0,a)$ and $(a,1)$ (when $\alpha = 1$) and back to a free particle in $(0,1)$. In Ref. 4, it was proposed that the adiabatic evolution starting at $t = 0$ with an eigenstate of the Laplacian leads this system, typically, to the vicinity of a different eigenstate at $t = T$ and that repeating the same process for many time periods T makes the energy grow exponentially in time.

The analysis of Ref. 4 is non-rigorous because of the singularity of the Dirac potential. However, we show that the evolution defined by the well-posed system (1.1) can mimic the permutations of eigenmodes described in Ref. 4, even though it lets the tunneling effect appear since the potential in (1.1) is not a perfect infinite wall. Moreover, the tunneling effect can be used to obtain the global controllability of the system by tuning the speed of the slow motion of the potential wall.

A. Main results: Permutation of the eigenstates

We start by formally introducing the permutation of eigenmodes defined by Eq. (1.3). Let $a \in (0,1)$, and let $\mathbb{N}^*$ be the set of strictly positive integers. Set

$$\mu_p^l(a) = \frac{p^2\pi^2}{a^2} \quad \text{and} \quad \mu_q^r(a) = \frac{q^2\pi^2}{(1-a)^2} \quad \text{with} \quad p,q \in \mathbb{N}^*. \tag{1.4}$$

These numbers are the eigenvalues of the Dirichlet–Laplacian operator in the left and right parts of the split interval $(0,a)\cup(a,1)$. For any initial and final positions a_{i} and a_{f} in $(0,1)\backslash\mathbb{Q}$, we define the *quasi-adiabatic permutation* $\sigma_{a_{\mathrm{i}}}^{a_{\mathrm{f}}} : \mathbb{N}^* \to \mathbb{N}^*$ as follows: Since a_{i} is irrational, it follows that $\mu_p^l \neq \mu_q^r$ for all $p,q \in \mathbb{N}^*$, and so we can order the set $\{\mu_p^l(a_{\mathrm{i}})\}_{p\in\mathbb{N}^*} \cup \{\mu_q^r(a_{\mathrm{i}})\}_{q\in\mathbb{N}^*}$ as a strictly increasing sequence. Let $k \in \mathbb{N}^*$. The kth element of this sequence is either a number $\mu_{p_0}^l(a_{\mathrm{i}})$ or a number $\mu_{q_0}^r(a_{\mathrm{i}})$ for some p_0 or q_0. When a changes from a_{i} to a_{f}, the corresponding eigenvalue $\mu_{p_0}^l(a_{\mathrm{i}})$ or $\mu_{q_0}^r(a_{\mathrm{i}})$ smoothly goes to $\mu_{p_0}^l(a_{\mathrm{f}})$ (or, respectively, $\mu_{q_0}^r(a_{\mathrm{f}})$) with the same p_0 (or q_0). As $a_{\mathrm{f}} \notin \mathbb{Q}$, the union $\{\mu_p^l(a_{\mathrm{f}})\}_{p\in\mathbb{N}^*} \cup \{\mu_q^r(a_{\mathrm{f}})\}_{q\in\mathbb{N}^*}$ can, once again, be ordered as a strictly increasing sequence in a unique way. Thus, we define $\sigma_{a_{\mathrm{i}}}^{a_{\mathrm{f}}}(k)$ as the number such that $\mu_{p_0}^l(a_{\mathrm{f}})$ (or $\mu_{q_0}^r(a_{\mathrm{f}})$) is the $\sigma_{a_{\mathrm{i}}}^{a_{\mathrm{f}}}(k)$−th element of the sequence obtained by the ordering of $\{\mu_p^l(a_{\mathrm{f}})\}_{p\in\mathbb{N}^*} \cup \{\mu_q^r(a_{\mathrm{f}})\}_{q\in\mathbb{N}^*}$. Figure 1 illustrates an example of this permutation.

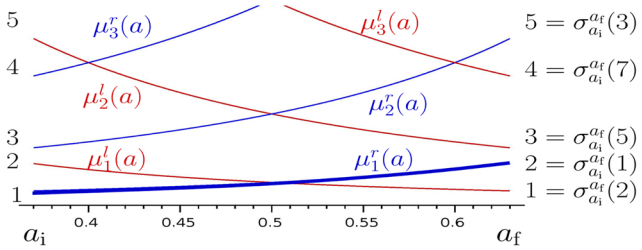

FIG. 1. An example of the quasi-adiabatic permutation $\sigma_{a_i}^{a_f}$. The eigenvalues of the Dirichlet–Laplacian on $(0, a) \cup (a, 1)$ with $a \in [a_i, a_f]$ are decomposed into two sets given by (1.4). When a goes from a_i to a_f, the eigenvalues from the different sets can cross, which yields the permutation $\sigma_{a_i}^{a_f}$. The line of eigenvalues $\mu_1^r(a)$ corresponds to the evolution of the instantaneous energy value when the potential wall moves from $a = a_i$ to $a = a_f$, transforming the first eigenmode into the second one (see Fig. 2).

Theorem 1. *Let $a_i, a_f \in (0,1)\backslash\mathbb{Q}$, $N \in \mathbb{N}^*$, $\varepsilon > 0$, and $\kappa > 0$. There exist $T > 0$ and*

- $\eta \in \mathcal{C}^\infty([0,T], \mathbb{R}^+)$ *with* $\|\eta'\|_{L^\infty([0,T],\mathbb{R})} \leq \kappa$,
- $I \in \mathcal{C}^\infty([0,T], \mathbb{R}^+)$ *with* $\|I'\|_{L^\infty([0,T],\mathbb{R})} \leq \kappa$ *and* $I(0) = I(T) = 0$, *and*
- $a \in \mathcal{C}^\infty([0,T], (0,1))$ *with* $\|a'\|_{L^\infty([0,T],\mathbb{R})} \leq \kappa$, $a(0) = a_i$, *and* $a(T) = a_f$

such that the evolution defined by the linear Schrödinger equation (1.1) with the potential V given by (1.2) realizes the quasi-adiabatic permutation $\sigma_{a_i}^{a_f}$. Specifically, let Γ_s^t be the unitary propagator generated in the time interval $[s,t] \subset [0,T]$ by Eq. (1.1) with the potential (1.2). Then, for all $k \leq N$, there exist $\alpha_k \in \mathbb{C}$ with $|\alpha_k| = 1$ such that

$$\left\| \Gamma_0^T \sin(k\pi x) - \alpha_k \sin(\sigma_{a_i}^{a_f}(k)\pi x) \right\|_{L^2} \leq \varepsilon.$$

The trajectory of the potential wall V in Theorem 1 is shown in Fig. 2. First, we slowly grow a thin potential wall at $a = a_i$ until its height I reaches a sufficiently large value, and so the interval gets almost split into two parts. During this stage, the system evolves adiabatically: starting with the kth eigenstate $\sin(k\pi x)$ of the Dirichlet–Laplacian on $(0,1)$, it arrives close to the kth eigenstate of the Dirichlet–Laplacian on $(0, a_i) \cup (a_i, 1)$. The latter eigenstates are localized either in $(0, a_i)$ or in $(a_i, 1)$ (see Fig. 2), and so the particle gets almost completely localized in one of these intervals. After that, we slowly change the wall's position a. When the kth eigenvalue of the Dirichlet–Laplacian on $(0, a) \cup (a, 1)$ is a double eigenvalue, a fully adiabatic evolution would lead to a strong tunneling, and so we keep the slow wall speed fast enough so that the particle remains trapped in $(0, a)$ or $(a, 1)$. Thus, the system stays close to the corresponding "left" eigenstate [with the eigenvalue $\mu_{p_0}^l(a)$] or "right" eigenstate [with the eigenvalue $\mu_{q_0}^r(a)$], and so at the end of this stage, the system arrives close to the $\sigma_{a_i}^{a_f}(k)$−th eigenstate of the Dirichlet–Laplacian on $(0, a_f) \cup (a_f, 1)$. Finally, we adiabatically decrease the potential V to zero—the system finds itself close to the eigenstate $\sin(\sigma_{a_i}^{a_f}(k)\pi x)$.

We stress that the time-dependence in $V(t,x)$ can be as slow as desired (the constant κ that bounds the rate of the parameters' change can be as small as we need). This ensures the softness of the control we apply to permute the eigenstates. However, it would be wrong to think of this process as fully adiabatic (we call it "quasi-adiabatic" instead). Fully adiabatic dynamics would preserve the ordering of the eigenmodes since the spectrum of the Hamiltonian $-\partial_{xx}^2 + V(t,\cdot)$ is simple for every $t \in [0,T]$, as shown in Fig. 3 (see Proposition 2). Would the speed of the parameters' change be too slow, the evolution would trace the eigenstates of the instantaneous Hamiltonian too tightly, making no permutation of the eigenstates possible. The permutation of eigenstates is achieved by choosing the slow speed of the potential wall to be

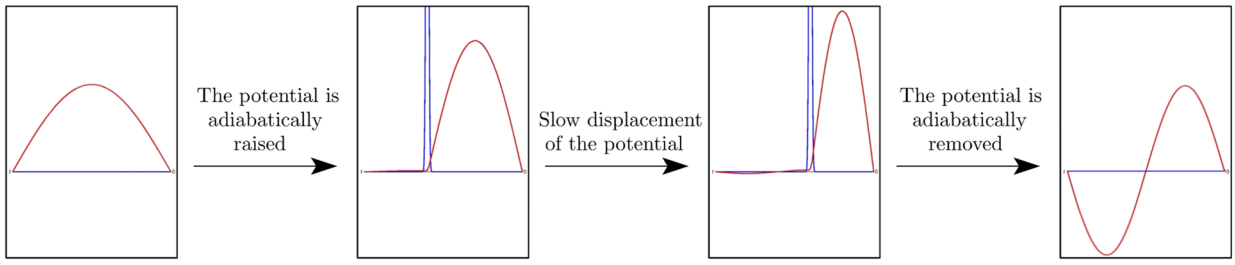

FIG. 2. A control path transforming the first mode to the second one, according to Theorem 1. The change in energy when the potential wall moves to the right follows the line of eigenvalues $\mu_1^r(a)$ shown in Fig. 1.

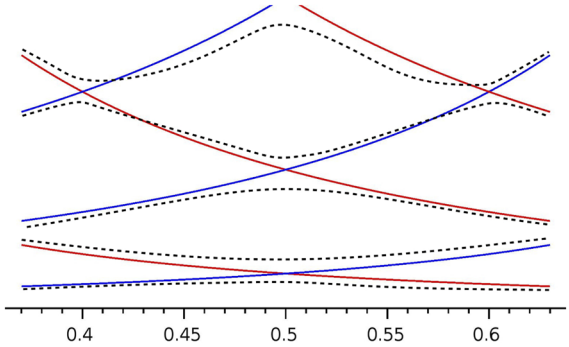

FIG. 3. The dashed lines represent the eigenvalues of the Hamiltonian $-\partial_{xx}^2 + V(t,\cdot)$ as functions of $t \in [0,T]$ with the potential given by (1.2) with sufficiently large I and η. These curves are close to the set of lines of eigenvalues of the singular problem (1.3) shown in Fig. 1, but they never intersect. As a consequence, a fully adiabatic motion would preserve the ordering of the eigenmodes in this framework.

much faster than the rate of tunneling that occurs for the values of a where different eigenvalues become close [for the singular problem (1.3), this corresponds to the crossing of left and right eigenvalues $\mu_p^l(a)$ and $\mu_q^r(a)$, see Fig. 3]. Thus, the quasi-adiabatic evolution at these moments is controlled by the rate of the parameters' variation. This key fact is used in Theorem 2.

Note that the class of controls in Theorem 1 is quite wide. The robustness of the proposed control is an important aspect of our results: the shape of ρ is not precisely constrained and the paths followed by the adiabatic and non-adiabatic parts of the control are very flexible, once they are slow or fast enough. We can also make $I(t)$ identically zero around the end points $t = 0$ and $t = T$ of the control interval. This means that the process can be repeated periodically, e.g., realizing the exponential heating process described in Ref. 4. We also remark that even though the phase shifts appearing in Theorem 1 are not relevant from a physical point of view, they can be easily removed in order to obtain $\alpha_k = 1$ for all k (see Ref. 5).

B. Main results: Control of the Schrödinger equation

Let us consider a more general class of potentials $V(t,x)$ with several slowly moving walls. Specifically, let

$$V(t,x) \;=\; \sum_{j=1}^{J} I_j(t)\rho^{\eta_j(t)}(x - a_j(t)) \tag{1.5}$$

with $\{\eta_j\}_{j\le J}, \{I_j\}_{j\le J} \subset \mathcal{C}^\infty([0,T],\mathbb{R}^+)$ and $\{a_j\}_{j\le J} \subset \mathcal{C}^\infty([0,T],(0,1))$. Increasing the number J of the walls and tuning the rate of the parameters' variation allows us to obtain the following *approximate controllability* result:

Theorem 2. *Take $\varepsilon > 0$, $\kappa > 0$ and let u_i and u_f be functions from $L^2((0,1),\mathbb{C})$ with $\|u_\mathrm{i}\|_{L^2} = \|u_\mathrm{f}\|_{L^2}$. There exist $J \in \mathbb{N}$, $T > 0$, and smooth functions $\{\eta_j\}_{j\le J}$, $\{I_j\}_{j\le J} \subset \mathcal{C}^\infty([0,T],\mathbb{R}^+)$, and $\{a_j\}_{j\le J} \subset \mathcal{C}^\infty([0,T],(0,1))$, with time derivatives bounded by κ in the absolute value, such that*

$$\|\Gamma_0^T u_\mathrm{i} \;-\; u_\mathrm{f}\|_{L^2} \;\le\; \varepsilon,$$

where Γ_s^t denotes the unitary propagator generated by the linear Schrödinger equation (1.1) in the time interval $[s,t] \subset [0,T]$ with the potential V given by (1.5). The functions η_j can be taken constant, and the functions I_j can be taken identically zero at the beginning and the end of the control interval $[0,T]$.

Theorem 2 is stronger than Theorem 1; however, we formulated Theorem 1 separately, as it provides a simpler control protocol.

C. Some bibliography

The controllability of Schrödinger equations of the form (1.1) is commonly studied with potentials $V(t,\cdot) = v(t)B$ for $t \in [0,T] \subset \mathbb{R}^+$. The function v represents the time-dependent intensity of the controlling external field described by the bounded symmetric operator B. Such bilinear Schrödinger equations are known to be not exactly controllable in $L^2((0,1),\mathbb{C})$ when $v \in L^r_{loc}(\mathbb{R}^+,\mathbb{R})$ with $r > 1$ (see Ref. 6). As a consequence, the exact controllability of the equation has been only addressed in suitable subspaces of $L^2((0,1),\mathbb{C})$. To the subject, we refer to Refs. 7–12. The approximate controllability of bilinear quantum systems was proved with Lyapunov techniques in Refs. 13 and 14 and via Lie–Galerkin methods in Refs. 15 and 16.

Adiabatic controls have been studied in quantum mechanics for a long time. Finite-level systems were, for instance, considered in Refs. 17–19 where the controllability via finitely many scalar controls is obtained with some good robustness properties with respect to parameter uncertainties. The mathematical aspects of this type of problem were analyzed in Ref. 20. There, the permutation of eigenmodes for a finite-dimensional quantum system is obtained by applying the classical adiabatic theory to a specific Hamiltonian with a uniformly separated spectrum. The adiabatic controllability of Schrödinger equations with V given by the sum of bilinear controls was studied in Refs. 21–24. In these works, the authors exploited the emergence of conical intersections of eigenvalues in the presence of at least two controls. Through a suitable choice of the adiabatic controls, it is possible to cross the conical intersections and then ensure the global approximate controllability. For other adiabatic controllability results, we refer to Refs. 25 and 26.

In our framework, we can neither create a simple path that links different eigenmodes nor use conical intersections. Indeed, all the eigenvalues related to (1.1) are simple since Ω is a one-dimensional interval (see Proposition 2). Thus, the above adiabatic strategies fail. From this perspective, the main idea of our work could be seen as first creating "almost-intersections" of eigenvalues, then traveling toward them adiabatically, and finally moving from one energy level to another near the almost-intersection. Hence, our method provides a new and different way to perform the global approximate controllability.

Finally, note that the controllability of linear Schrödinger equations with controls on a boundary has been shown, e.g., in Refs. 27 and 28 by the multiplier method, in Refs. 29 and 30 by microlocal analysis, and in Refs. 31–33 with the use of Carleman estimates. Concerning the controllability of various PDEs by moving boundaries, we refer to Refs. 34–38.

II. PRELIMINARIES

We equip the Hilbert space $L^2((0,1),\mathbb{C})$ with the scalar product $\langle \psi_1 \,|\, \psi_2 \rangle_{L^2} = \int_0^1 \overline{\psi_1(x)}\psi_2(x)\,\mathrm{d}x$ and the corresponding norm $\|\cdot\|_{L^2} = \sqrt{\langle \cdot \,|\, \cdot \rangle_{L^2}}$. We consider the classical Sobolev's spaces $H^m((0,1),\mathbb{C})$ for $m \geq 0$ with the standard norms, the space $H_0^1((0,1),\mathbb{C}) = \{u \in H^1((0,1),\mathbb{C}),\ u(0) = u(1) = 0\}$, and the space $H^{-1}((0,1),\mathbb{C})$, which is the dual space of $H_0^1((0,1),\mathbb{C})$ with respect to the L^2-duality. Applying the classical result of Kisyński[39] (see also Refs. 40 and 41), we obtain that Eq. (1.1) is well posed.

Proposition 1. Let V be given by (1.2) with $a \in \mathcal{C}^2([0,T],(0,1))$, $I \in \mathcal{C}^2([0,T],\mathbb{R}^+)$, and $\eta \in \mathcal{C}^2([0,T],\mathbb{R}^+)$. For any $u_0 \in H_0^1((0,1),\mathbb{C})$, Eq. (1.1) has a unique solution $u \in \mathcal{C}^0([0,T],H_0^1((0,1),\mathbb{C})) \cap \mathcal{C}^1([0,T],H^{-1}((0,1),\mathbb{C}))$, and

$$\|u_0\|_{L^2} = \|u(t)\|_{L^2}, \qquad \forall t \in [0,T].$$

The flow of (1.1) can be unitary extended by density on $L^2((0,1),\mathbb{C})$, and if $u_0 \in H^2((0,1),\mathbb{C}) \cap H_0^1((0,1),\mathbb{C})$, then

$$u \in \mathcal{C}^0\big([0,T],H^2((0,1),\mathbb{C}) \cap H_0^1((0,1),\mathbb{C})\big) \cap \mathcal{C}^1([0,T],L^2((0,1),\mathbb{C})).$$

Proof. In our framework, the Hamiltonian associated to the Schrödinger equation [Eq. (1.1)]

$$H(t) := -\partial_{xx}^2 + I(t)\rho^{\eta(t)}(x - a(t)) \tag{2.1}$$

gives a family of positive self-adjoint operators smoothly depending on the time $t \in [0,T]$. Note that

$$u \in H_0^1((0,1),\mathbb{C}) \longmapsto V(t,\cdot)u \in H^{-1}((0,1),\mathbb{C})$$

is of class $\mathcal{C}^2$ with respect to t due to the smoothness of a, η, I, and ρ. This enables us to apply the result of Ref. 39. □

Proposition 2. For any $a \in (0,1)$, $I \geq 0$, and $\eta > 0$, the eigenvalues of $-\partial_{xx}^2 + I\rho^\eta(x-a)$ are simple.

Proof. If $\phi, \psi \in H^2((0,1),\mathbb{C}) \cap H_0^1((0,1),\mathbb{C})$ are two eigenfunctions corresponding to the same eigenvalue λ, then they both satisfy the same second-order differential equation $f''(x) = I\rho^\eta(x-a)f(x) + \lambda f(x)$ with the initial data $\phi(0) = \psi(0) = 0$. Thus, they only differ by a multiplicative factor depending on $\phi'(0)$ and $\psi'(0)$. □

By the classical spectral theory, the eigenvalues of $H(t)$ in (2.1) form smooth curves, and the basis made by the corresponding eigenfunctions can be chosen smooth with respect to t, up to adjusting the phases (see Ref. 3). By Proposition 2, the eigenvalues do not cross, and so they admit the same ordering for all $t \in [0,T]$. Denote by $(\lambda_k(t))_{k\in\mathbb{N}^*}$ the ordered sequence of eigenvalues of $H(t)$, and let $(\phi_k(t))_{k\in\mathbb{N}^*}$ be the Hilbert basis made by the corresponding eigenfunctions, smooth in t. Denote by Γ_s^t the unitary propagator in $L^2((0,1),\mathbb{C})$ generated by (1.1),

as given by Proposition 1. The operator Γ_s^t represents the flow of (1.1), and for any mild solution $u \in L^2((0,1),\mathbb{C})$, we have $\Gamma_s^t u(s) = u(t)$. The classical adiabatic theory, as in Chap. IV of Ref. 42 (see also Refs. 41 and 43), gives us the following proposition:

Proposition 3. Let $N \in \mathbb{N}^*$. For every $\varepsilon > 0$, $\widetilde{\eta} \in \mathcal{C}^2([0,1],\mathbb{R}^+)$, $\widetilde{I} \in \mathcal{C}^2([0,1],\mathbb{R}^+)$, and $\widetilde{a} \in \mathcal{C}^2([0,1],(0,1))$, there exists $T^* > 0$ such that, for every $T \geq T^*$, the following property holds. Take $\eta(t) = \widetilde{\eta}\left(\frac{t}{T}\right)$, $I(t) = \widetilde{I}\left(\frac{t}{T}\right)$, and $a(t) = \widetilde{a}\left(\frac{t}{T}\right)$, then the flow of Eq. (1.1) satisfies

$$\forall\, k \leq N, \quad \exists\, \alpha_k \in \mathbb{C} \ : \ |\alpha_k| = 1 \quad \text{and} \quad \left\| \Gamma_0^T \phi_k(0) - \alpha_k \phi_k(T) \right\|_{L^2} \leq \varepsilon.$$

To mimic the dynamics of the model (1.3), we need a very high and sharp potential wall. Hence, we consider the Hamiltonian $H^{I,\eta,a} = -\partial_{xx}^2 + I\rho^\eta(x-a)$ with both I and η very large. We note that, from a spectral point of view, the limit of $H^{I,\eta,a}$ for $I, \eta \to +\infty$ is the Laplacian on a split segment $(0,a) \cup (a,1)$ with the Dirichlet boundary conditions,

$$H^{\infty,a} = -\partial_{xx}^2, \qquad D(H^{\infty,a}) = H^2((0,a)\cup(a,1),\mathbb{C}) \cap H_0^1((0,a)\cup(a,1),\mathbb{C}).$$

To formulate this result properly (Theorem 3), denote by $\left(\lambda_k^{I,\eta,a}\right)_{k\in\mathbb{N}^*}$ and $\left(\phi_k^{I,\eta,a}\right)_{k\in\mathbb{N}^*}$ the ordered sequence of eigenvalues of $H^{I,\eta,a}$ and, respectively, a Hilbert basis of $L^2((0,1),\mathbb{C})$ made by the corresponding eigenfunctions. The spectrum of $H^{\infty,a}$ is composed of the numbers $(\mu_p^l(a))_{p\in\mathbb{N}^*}$ and $(\mu_q^r(a))_{q\in\mathbb{N}^*}$ defined in (1.4); the corresponding eigenfunctions are

$$\varphi_p^l(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{p\pi}{a}x\right) \mathbf{1}_{x\in[0,a]}, \tag{2.2}$$

$$\varphi_q^r(x) = \sqrt{\frac{2}{1-a}} \sin\left(\frac{q\pi}{1-a}(1-x)\right) \mathbf{1}_{x\in[a,1]}, \tag{2.3}$$

where $\mathbf{1}_{x\in J}$ is equal to 1 in $J \subset (0,1)$ and to 0 in $(0,1)\backslash J$. We denote as $(\lambda_k^{\infty,a})_{k\in\mathbb{N}^*}$ and $(\phi_k^{\infty,a})_{k\in\mathbb{N}^*}$ the ordered spectrum of $H^{\infty,a}$ obtained by reordering the eigenvalues given by (1.4) in ascending order and, respectively, a Hilbert basis of $L^2((0,1),\mathbb{C})$ made by the corresponding eigenfunctions given by (2.2) and (2.3). Note that $H^{\infty,a}$ may have multiple eigenvalues, unlike the operator $H^{I,\eta,a}$ (see Proposition 2). More precisely, $H^{\infty,a}$ has at most double eigenvalues appearing if and only if a is rational. The convergence of the spectrum $(\lambda_k^{I,\eta,a})_{k\in\mathbb{N}^*}$ of $H^{I,\eta,a}$ to the spectrum of $H^{\infty,a}$ as $\eta, I \to +\infty$ when I and η are of the same order is described as follows:

Theorem 3. *For each $k \in \mathbb{N}^*$ and $\delta > 0$, there exists a constant $C_{k,\delta} > 0$ [which depends continuously on $a \in (0,1)$ and on the shape of ρ] such that, for all $\eta \geq 1$ and $I \geq 1$ satisfying $\delta\,\eta \leq I \leq \frac{1}{\delta}\eta$, we have*

$$\lambda_k^{\infty,a} - \frac{C_{k,\delta}}{\sqrt{\eta}} \leq \lambda_k^{I,\eta,a} \leq \lambda_k^{\infty,a} + \frac{C_{k,\delta}}{\eta}.$$

If $\lambda_k^{\infty,a}$ is a simple eigenvalue of $H^{\infty,a}$ with a normalized eigenfunction $\phi_k^{\infty,a}$, then there exists $\alpha_k^{I,\eta,a} \in \mathbb{C}$ such that $|\alpha_k^{I,\eta,a}| = 1$ and

$$\|\phi_k^{I,\eta,a} - \alpha_k^{I,\eta,a}\phi_k^{\infty,a}\|_{L^2} \leq \frac{C_{k,\delta}}{\sqrt{\eta}}.$$

If $\lambda_k^{\infty,a} = \lambda_{k+1}^{\infty,a}$ is a double eigenvalue of $H^{\infty,a}$ corresponding to a pair of normalized eigenfunctions $\phi_k^{\infty,a}$ and $\phi_{k+1}^{\infty,a}$, then there exist $\alpha_k^{I,\eta,a} \in \mathbb{C}$ and $\beta_k^{I,\eta,a} \in \mathbb{C}$ such that $|\alpha_k^{I,\eta,a}|^2 + |\beta_k^{I,\eta,a}|^2 = 1$ and

$$\|\phi_k^{I,\eta,a} - \alpha_k^{I,\eta,a}\phi_k^{\infty,a} - \beta_k^{I,\eta,a}\phi_{k+1}^{\infty,a}\|_{L^2} \leq \frac{C_{k,\delta}}{\sqrt{\eta}}.$$

The main tools of the proof are the min-max theorem (Ref. 44, Theorem XIII.1) and estimates of the difference of the Rayleigh quotients associated with $H^{\infty,a}$ and $H^{I,\eta,a}$. The convergence of the eigenvalues and eigenfunctions is quite classical. The main interest of Theorem 3 is to provide quantitative estimates. They could be useful for estimating the time of control and the required sharpness of the wall or for providing control of the error terms. However, this kind of discussion is not central to the present paper. Therefore, we skip the proof and refer the reader to Ref. 5.

III. SHORT-TIME MOVEMENT OF THE HIGH POTENTIAL WALL

In this section, we consider $V(t,x) = I(t)\rho^{\eta(t)}(x-a(t))$ with smooth controls I, η, and a. For large η, we can see V as a very thin (of width η^{-1}) moving wall. When I (which determines the height of the wall) is large, the norm of the map $u \in H_0^1((0,1),\mathbb{C}) \longmapsto V(x,t)u = I(t)\rho^{\eta(t)}(x-a(t))u \in H^{-1}((0,1),\mathbb{C})$ is large. Therefore, even a small displacement of $a(t)$ may, in principle, cause a strong modification of the initial state. However, the following lemma shows that if the initial condition is localized outside of the support interval $\left[a(t)-\frac{1}{\eta}, a(t)+\frac{1}{\eta}\right]$ of the potential V, then the speed with which the solution deviates from the initial condition is uniformly bounded for all η and I.

Lemma 1. Let $\eta(t) \geq \eta_* \in \mathbb{R}^+$ for all $t \in [t_1,t_2]$, and let $u(t)$ be the solution of the Schrödinger equation (1.1) in this interval with the $L^2((0,1),\mathbb{C})$ initial data. Take any function $\psi \in C^1([t_1,t_2], H^2((0,1),\mathbb{C}) \cap H_0^1((0,1),\mathbb{C}))$ that, for every $t \in [t_1,t_2]$, vanishes in $\left[a(t)-\frac{1}{\eta_*}, a(t)+\frac{1}{\eta_*}\right]$ (so $V\psi \equiv 0$). Then, for all $t \in [t_1,t_2]$,

$$\|u(t)-\psi(t)\|_{L^2}^2 \leq \|u(t_1)-\psi(t_1)\|_{L^2}^2 + C(t_2-t_1), \tag{3.1}$$

with C independent of the choice of functions η, I, and a and given by

$$C = 2 \sup_{t\in[t_1,t_2]} \left(\|u(t_1)\|_{L^2}\|\partial_{xx}^2\psi(t)\|_{L^2} + (\|\psi(t)\|_{L^2} + \|u(t_1)\|_{L^2})\|\partial_t\psi(t)\|_{L^2} \right).$$

Proof. As the evolution operator of the Schrödinger equation is unitary, $\|u(t)\|_{L^2}$ is constant in time. Let us, first, assume that $u(t_1) \in D(-\partial_{xx}^2 + V(t_1,\cdot))$, which implies regularity of the solution with respect to time as shown in Proposition 1. As $V(t,\cdot)\psi(t,\cdot) \equiv 0$ for every $t \in [t_1,t_2]$, we have

$$\begin{aligned}
\partial_t \frac{1}{2}\|u(t)-\psi(t)\|_{L^2}^2 &= \partial_t\left(\frac{1}{2}\|\psi(t)\|^2 - \mathrm{Re}(\langle u(t)|\psi(t)\rangle_{L^2})\right) \\
&\leq \|\psi(t)\|_{L^2}\|\partial_t\psi(t)\|_{L^2} + |\langle(-\partial_{xx}^2 + V(t,\cdot))u(t)|\psi(t)\rangle_{L^2}| + \|u(t)\|_{L^2}\|\partial_t\psi(t)\|_{L^2} \\
&\leq |\langle\partial_{xx}^2 u(t)|\psi(t)\rangle_{L^2}| + (\|\psi(t)\|_{L^2} + \|u(t_1)\|_{L^2})\|\partial_t\psi(t)\|_{L^2} \\
&\leq |\langle u(t)|\partial_{xx}^2\psi(t)\rangle_{L^2}| + (\|\psi(t)\|_{L^2} + \|u(t_1)\|_{L^2})\|\partial_t\psi(t)\|_{L^2} \\
&\leq \|u(t_1)\|_{L^2}\|\partial_{xx}^2\psi(t)\|_{L^2} + (\|\psi(t)\|_{L^2} + \|u(t_1)\|_{L^2})\|\partial_t\psi(t)\|_{L^2}.
\end{aligned}$$

Now, the validity of estimate (3.1) follows from

$$\|u(t)-\psi(t)\|_{L^2}^2 \leq \|u(t_1)-\psi(t_1)\|_{L^2}^2 + |t_2-t_1| \sup_{t\in[t_1,t_2]} \left|\partial_t\|u(t)-\psi(t)\|_{L^2}^2\right| \quad \text{for all } t \in [t_1,t_2].$$

We conclude the proof by noting that the estimate is extended by density to any $u(t_1) \in L^2((0,1),\mathbb{C})$. □

Recall that we denote by $\lambda_k^{\infty,a(t)}$ the eigenvalues of the Laplacian operator on the split interval $(0,a(t)) \cup (a(t),1)$ for any frozen value of t. For each $k \in \mathbb{N}^*$, we have $\lambda_k^{\infty,a} \in \{\mu_p^l(a)\}_{p\in\mathbb{N}^*}$ or $\lambda_k^{\infty,a} \in \{\mu_q^r(a)\}_{q\in\mathbb{N}^*}$, where μ_p^l and μ_q^r are given by (1.4). The corresponding eigenfunctions are denoted as $\phi_k^{\infty,a(t)}$ and are equal to φ_p^l or φ_q^r defined by (2.2) or (2.3), respectively. The functions φ_p^l and φ_q^r evolve smoothly with a and are mostly localized outside the support interval of V. As shown in Proposition 4, this implies that if we consider any of the functions $\phi_k^{\infty,a(t)}$ as an initial condition for Eq. (1.1), the solution will not deviate far from $\phi_k^{\infty,a(t)}$ on a certain short interval of time, uniformly for all large η and all I.

In particular, if $a_* \in (0,1)$ is a crossing point where $\lambda_k^{\infty,a_*} = \lambda_{k+1}^{\infty,a_*}$ for some $k \in \mathbb{N}^*$, then for any small v, the split Laplacian eigenfunctions ϕ_k^{∞,a_*-v} and $\phi_{k+1}^{\infty,a_*+v}$ correspond to the same mode φ_p^l or φ_q^r. Therefore, if $a(t_1) = a_* - v$ and $a(t_2) = a_* + v$, then the flow of (1.1) on the interval $[t_1,t_2]$ will take the function $\phi_k^{\infty,a(t_1)}$ close to $\phi_{k+1}^{\infty,a(t_2)}$ (and the function $\phi_{k+1}^{\infty,a(t_1)}$ close to $\phi_k^{\infty,a(t_2)}$) when v and (t_2-t_1) are small enough (see Fig. 4).

This yields exactly the permutation of eigenmodes introduced in Theorem 1. The precise statement is given as follows:

Proposition 4. Let $a_* \in (0,1) \cap \mathbb{Q}$. Take $N \in \mathbb{N}^*$ such that $\lambda_{N+1}^{a_*} \neq \lambda_N^{a_*}$ and any $\varepsilon > 0$, $\kappa > 0$. There exists $\eta^* > 0$ such that, for every sufficiently small $\tau > 0$, $v > 0$ and any smooth functions η, I, and a that satisfy $\eta(t) \geq \eta^*$, $I(t) \geq 0$, $a_* - v = a(0) \leq a(t) \leq a(\tau) = a_* + v$, and $|a'(t)| \leq \kappa$ for all $t \in [0,\tau]$, we have

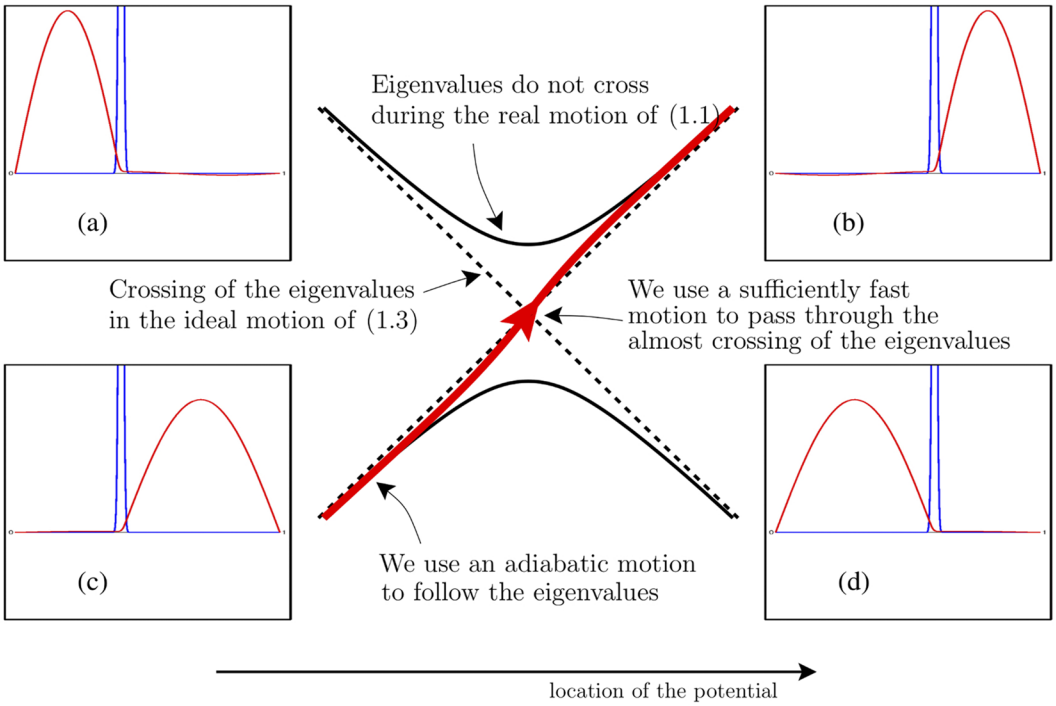

FIG. 4. The evolution of the eigenmode (c) driven by the motion of the potential wall. When the wall moves too slow and it takes a long time to pass the crossing of the eigenvalues of the split Laplacian, the motion is adiabatic and the state is deformed as shown in (d)—it tunnels from one side of the wall V to the other (because the spectral curves of $-\partial_{xx}^2 + V$ do not intersect). On the other hand, when we move the wall faster, the motion goes from the lower spectral curve to the higher as shown by the red line. In such case, the state (c) is deformed to (b) and the state (a) is deformed to (d).

$$\forall k \le N, \quad \|\Gamma_0^\tau \phi_k^{\infty, a_*-\nu} - \phi_{\sigma(k)}^{\infty, a_*+\nu}\|_{L^2} \le \varepsilon,$$

where Γ_s^t *is the flow of Eq.* (1.1) *and* $\sigma = \sigma_{a_*-\nu}^{a_*+\nu}$ *is the permutation defined in* Sec. I.

Proof. Note that the eigenmodes $\phi_k^{\infty,a(t)}$ given by (2.2) and (2.3) are not of class H^2 in the whole interval $(0,1)$, and they do not vanish everywhere in $\left[a(t) - \frac{1}{\eta_*}, a(t) + \frac{1}{\eta_*}\right]$, and so we cannot directly apply Lemma 1 to these functions. We, therefore, modify them near the support interval of V. Choose some $k \le N$ and assume that $\phi_k^{\infty,a_*-\nu}$ is given by (2.2) for some $p \in \mathbb{N}^*$ [the estimates for the case where $\phi_k^{\infty,a_*-\nu}$ is given by (2.3) are similar]. Let χ be a smooth truncation equal to 0 in $[-1, +\infty)$ and to 1 in $(-\infty, -2]$. We introduce $\chi_\alpha(\cdot) = \chi(\cdot/\alpha)$; this function vanishes in $[-\alpha, +\infty)$. Denote

$$\psi_k^{a(t)}(x) := \sqrt{\frac{2}{a(t)}} \sin\left(\frac{p\pi}{a(t)} x\right) \chi_\alpha(x - a(t)). \tag{3.2}$$

Now, $\psi_k^{a(t)}$ is a smooth function vanishing in $\left[a(t) - \frac{1}{\eta_*}, a(t) + \frac{1}{\eta_*}\right]$ when $\alpha \ge \frac{1}{\eta_*}$. Moreover, $\psi_k^{a(t)}$ is close, uniformly for all t, to $\phi_k^{\infty,a(t)} = \sqrt{\frac{2}{a(t)}} \sin\left(\frac{p\pi}{a(t)} x\right) \mathbf{1}_{x \in [0, a(t)]}$ in $L^2((0,1), \mathbb{C})$ when α is small. More precisely,

$$\|\phi_k^{\infty,a} - \psi_k^a\|_{L^2} \le \sqrt{\frac{2}{a} \int_{a-2\alpha}^{a} \sin^2\left(\frac{p\pi}{a} x\right) dx} \le \frac{4\pi}{\sqrt{3}} p \left(\frac{\alpha}{a}\right)^{\frac{3}{2}} = \mathcal{O}(N\alpha^{3/2}). \tag{3.3}$$

To apply Lemma 1, we need to estimate the derivatives of $\psi_k^{a(t)}$ when α is small. Note that the first and second derivatives of χ_α are of order $\frac{1}{\alpha}$ and $\frac{1}{\alpha^2}$, respectively, and are supported in $[a(t) - 2\alpha, a(t) - \alpha]$. Thus, we have

$$\int_0^1 \left|\sin\left(\frac{p\pi}{a(t)}x\right)\partial_{xx}^2\chi_\alpha(x-a(t))\right|^2 \mathrm{d}x \le \frac{p^2\pi^2\|\partial_{xx}^2\chi\|_\infty^2}{\alpha^4\, a(t)^2}\int_{a(t)-\alpha}^{a(t)-2\alpha}(a(t)-x)^2\,\mathrm{d}x \;=\; \mathcal{O}(N^2\alpha^{-1}),$$

and similarly, $\int_0^1|\cos(\frac{p\pi}{a(t)}x)\partial_x\chi_\alpha(x-a(t))|^2\mathrm{d}x = \mathcal{O}(\alpha^{-1})$ and $\int_0^1|\sin(\frac{p\pi}{a(t)}x)\chi_\alpha(x-a(t))|^2\mathrm{d}x = \mathcal{O}(N^2\alpha^3)$. This gives

$$\begin{aligned}\left\|\partial_{xx}^2\psi_k^{a(t)}\right\|_{L^2} \;\le\; \sqrt{\frac{2}{a(t)}}\Bigg[&\left\|\sin\left(\frac{p\pi}{a(t)}\cdot\right)\,\partial_{xx}^2\chi_\alpha(\cdot-a(t))\right\|_{L^2} + 2\frac{p\pi}{a(t)}\left\|\cos\left(\frac{p\pi}{a(t)}\cdot\right)\,\partial_x\chi_\alpha(\cdot-a(t))\right\|_{L^2}\\ &+\frac{p^2\pi^2}{a(t)^2}\left\|\sin\left(\frac{p\pi}{a(t)}\cdot\right)\chi_\alpha(\cdot-a(t))\right\|_{L^2}\Bigg] = \mathcal{O}(N\alpha^{-1/2}+N^3\alpha^{3/2}).\end{aligned} \tag{3.4}$$

By similar computations,

$$\left\|\partial_t\psi_k^{a(t)}\right\|_{L^2} = \mathcal{O}\left(N\sup_{t\in[0,t]}|a'(t)|\right) = \mathcal{O}(N\kappa). \tag{3.5}$$

The L^2-norm of ψ_k^a is controlled by $\mathcal{O}(1+N\alpha^{3/2})$ due to (3.3). Now, by Lemma 1, if $u(\tau) = \Gamma_0^\tau\phi_k^{\infty,a_*-v}$, then

$$\|u(\tau)-\psi_k^{a(\tau)}\|_{L^2}^2 \le \|\phi_k^{\infty,a_*-v}-\psi_k^{a_*-v}\|_{L^2}^2 + N\tau\,\mathcal{O}(\alpha^{-1/2}+N^2\alpha^{3/2}+(2+N\alpha^{3/2})\kappa)$$

by (3.4) and (3.5). Using (3.3), we can then estimate

$$\begin{aligned}\|u(\tau)-\phi_{\sigma(k)}^{\infty,a(\tau)}\|_{L^2} &\le \|u(\tau)-\psi_k^{a(\tau)}\|_{L^2} + \|\psi_k^{a(\tau)}-\phi_{\sigma(k)}^{\infty,a(\tau)}\|_{L^2}\\ &\le \|\phi_k^{\infty,a_*-v}-\psi_k^{a_*-v}\|_{L^2} + \sqrt{N\tau}\,\mathcal{O}\left(\sqrt{\alpha^{-1/2}+N^2\alpha^{3/2}+(2+N\alpha^{3/2})\kappa}\,\right) + \|\psi_k^{a_*+v}-\phi_{\sigma(k)}^{\infty,a_*+v}\|_{L^2}\\ &= \mathcal{O}(N\alpha^{3/2}) + \sqrt{N\tau}\,\mathcal{O}\left(\sqrt{\alpha^{-1/2}+N^2\alpha^{3/2}+(2+N\alpha^{3/2})\kappa}\,\right).\end{aligned} \tag{3.6}$$

Choose $\alpha \ll (\varepsilon/N)^{2/3}$. Set $\eta_* > 0$ sufficiently large so that $\alpha \ge \frac{1}{\eta_*}$. Then, the above computations are valid for any choice of the function $\eta(t)$ bounded by η_* from below for all $t\in[0,\tau]$. Therefore, if we take

$$\tau \ll \frac{\varepsilon^2}{N(\alpha^{-1/2}+N^2\alpha^{3/2}+(2+N\alpha^{3/2})\kappa)},$$

estimate (3.6) yields the claim. Note that the only restriction on v (the range of displacement of the potential wall) is given by

$$v \le \sup_{t\in[0,t]}|a'(t)|\tau \le \kappa\tau,$$

and so the result holds true for all sufficiently small v. □

Remark. From (3.6), we may estimate the behavior of the parameters in the statement of Proposition 4. For example, for $N\in\mathbb{N}^*$ and $\kappa>0$ fixed, when the error ε is small, the sharpness of the potential η_* is at least $\mathcal{O}(\epsilon^{-\frac{2}{3}})$. Hence, τ and v have to be at most $\mathcal{O}(\epsilon^{\frac{7}{3}})$. For $\epsilon>0$ and $\kappa>0$ fixed, when we consider a large number N of frequencies, then η_* is at least $\mathcal{O}(N^{\frac{2}{3}})$. Thus, τ and v have to be at most $\mathcal{O}(N^{-2})$.

IV. PROOF OF THEOREMS 1 AND 2

A. Proof of Theorem 1

In order to obtain the permutation of eigenmodes described in Theorem 1, we use the following control path (see also Fig. 2). Fix some very large η_* and, first, start to adiabatically increase I from zero to a very large value I_*, thus obtaining a thin and high potential wall located at a_i. The evolution of the eigenstates during this stage is adiabatic, as stated in Proposition 3. Next, we move the wall to the location a_f. In this step, the potential $V(x,t) = I_*\rho^{\eta_*}(x-a(t))$ has a large fixed amplitude and a small fixed width $(\eta_*)^{-1}$. The movement of the wall alternates adiabatic motion (by Proposition 3) and short-time level crossings (by Proposition 4). Finally, we adiabatically decrease I to zero, up to the extinction of the potential.

Crossings of eigenvalues and the corresponding permutations. With no loss of generality, assume $a_{\mathrm{i}} < a_{\mathrm{f}}$. We take any $a(t)$ growing from a_{i} to a_{f}, such that $|a'(t)| \le \kappa$ for all t (where κ is the bound from Theorem 1).

The operator $H^{\infty,a}$ [the Laplacian with the Dirichlet boundary conditions on the split interval $(0,a)\cup(a,1)$, as introduced in Sec. II] corresponds to the ideal motion described by the formal equation (1.3). Recall that its eigenvalues $(\lambda_k^{\infty,a})_{k\in\mathbb{N}^*}$ are grouped into two families $(\mu_p^l(a))_{p\in\mathbb{N}^*}$ and $(\mu_q^r(a))_{q\in\mathbb{N}^*}$ given by (1.4). Specifically, for $N\in\mathbb{N}^*$ given in the statement of Theorem 1, there exist $P,Q\in\mathbb{N}$ such that

$$(\lambda_k^{a_{\mathrm{i}}})_{k\le N} \;=\; \left(\mu_p^l(a_{\mathrm{i}})\right)_{p\le P} \;\cup\; \left(\mu_q^r(a_{\mathrm{i}})\right)_{q\le Q}.$$

When a moves from a_{i} to a_{f}, the spectral curves belonging to $\left(\mu_p^l(a)\right)_{p\le P}$ or $\left(\mu_q^r(a)\right)_{q\le Q}$ can cross a finite number of the spectral curves of $H^{\infty,a}$ at a finite number of locations $(a_{*,j})_{j\le j^*}$ with $j^*\in\mathbb{N}^*$. Order the crossing points so that

$$a_{\mathrm{i}} \;<\; a_{*,1} \;<\; \cdots \;<\; a_{*,j^*} \;<\; a_{\mathrm{f}}$$

(as a_{i} and a_{f} are irrational, no crossing can happen at the end points $a=a_{\mathrm{i}}$ or $a=a_{\mathrm{f}}$). Denote by $M\in\mathbb{N}^*$ the smallest number such that any of the curves $(\mu_p^l(a))_{p\le P}$ and $(\mu_q^r(a))_{q\le Q}$ cross only eigenvalues in $(\lambda_k^{\infty,a})_{k\le M}$ when a moves from a_{i} to a_{f}. For any irrational a and b in $[a_{\mathrm{i}},a_{\mathrm{f}}]$, we define the permutation σ_a^b as follows:

- if $\lambda_k^{\infty,a}$ equals to $\mu_p^l(a)=\frac{p^2\pi^2}{a^2}$ for some p, then $\sigma_a^b(k)$ is the index such that $\mu_p^l(b)=\frac{p^2\pi^2}{b^2}=\lambda_{\sigma_a^b(k)}^b$; and
- if $\lambda_k^{\infty,a}$ equals to $\mu_q^r(a)=\frac{q^2\pi^2}{(1-a)^2}$ for some q, then $\sigma_a^b(k)$ is the index such that $\mu_q^r(b)=\frac{q^2\pi^2}{(1-b)^2}=\lambda_{\sigma_a^b(k)}^b$.

Let $\nu_*>0$ be small enough so that $a_{\mathrm{i}}<a_{*,1}-\nu_*$, $a_{*,j}+\nu_*<a_{*,j+1}-\nu_*$ for $j=1,\dots,j^*-1$ and $a_{*,j^*}+\nu_*<a_{\mathrm{f}}$. As $(a_{*,j})_{j\le j^*}$ correspond to the crossing point of the N first eigenvalues, we have

$$\forall k\le N,\quad \sigma_{a_{\mathrm{i}}}^{a_{\mathrm{f}}}(k) \;=\; \sigma_{a_{*,j^*}-\nu_*}^{a_{*,j^*}+\nu_*}\circ\cdots\circ\sigma_{a_{*,1}-\nu_*}^{a_{*,1}+\nu_*}(k).$$

Fixing the parameters. Let $\varepsilon'<\varepsilon/(4j^*+3)$, where $\varepsilon>0$ is the small error introduced in Theorem 1. We apply Proposition 4 to each moment $a=a_{*,j}$ where the crossings of eigenvalues occur. We obtain some large η_* and we can choose, for every $j\le j^*$, a distance $\nu_j<\nu_*$, a time τ_j, and a path $a_j\in\mathcal{C}^\infty([0,\tau_j],[a_{*,j}-\nu_j,a_{*,j}+\nu_j])$ satisfying

$$\|a'\|_{L^\infty([0,\tau],\mathbb{R})}\le\kappa,\qquad a_j^{(m)}(0)=a_j^{(m)}(\tau_j)=0,\qquad \forall m\in\mathbb{N}^*, \tag{4.1}$$

such that

$$\forall\eta\ge\eta_*,\quad \forall k\le N,\quad \|\Gamma_0^\tau\phi_k^{\infty,a_{*,j}-\nu_j}-\phi_{\sigma_j(k)}^{\infty,a_{*,j}+\nu_j}\|_{L^2}\le\varepsilon'. \tag{4.2}$$

Next, we apply Theorem 3 for each $a=a_{*,j}\pm\nu_j$ and for $k\le M$. We obtain that, maybe with a larger η_*, estimate (4.2) is also guaranteed for $\eta=\eta_*$ and for some sufficiently large $I=I_*$,

$$\forall k\le M,\ \exists\alpha_k^{I_*,\eta_*,j\pm}\in\mathbb{U},\quad \|\phi_k^{I_*,\eta_*,a_j\pm\nu_j}-\alpha_k^{I_*,\eta_*,j\pm}\phi_k^{\infty,a_j\pm\nu_j}\|_{L^2}\le\varepsilon', \tag{4.3}$$

where $\mathbb{U}=\{z\in\mathbb{C},\ |z|=1\}$.

The initial "vertical" adiabatic motion. Now, we start to construct the potential $V(x,t)=I(t)\rho^{\eta(t)}(x-a(t))$. First, we fix $a(t)$ to be constantly equal to a_{i}, $\eta(t)=\eta_*$, and choose $I(t)$ to be a smooth function going from 0 to I_*, with the derivative $I'(t)$ compactly supported in $(0,T_1)$ and satisfying $|I'(t)|<\kappa$ for some sufficiently large $T_1>0$. Then, we apply Proposition 3 and obtain an adiabatic evolution lasting up to the time $t=T_1$ such that

$$\forall k\le N,\quad \exists\alpha_k\in\mathbb{U},\quad \|\Gamma_0^{T_1}\sqrt{2}\sin(k\pi\cdot)-\alpha_k\phi_k^{I_*,\eta_*,a_{\mathrm{i}}}\|_{L^2}\le\varepsilon'. \tag{4.4}$$

The initial "horizontal" adiabatic motion. Next, we fix $I(t)$ to be a constant, $I=I_*$, and choose $a(t)$ to be a smooth monotone function going from a_{i} to $a_{*,1}-\nu_1$, with the derivative a' compactly supported in $(0,T_2)$ for a sufficiently large $T_2>0$. We apply again Proposition 3 and obtain an adiabatic motion lasting the time T_2 such that

$$\forall k\le N,\quad \exists\alpha_k\in\mathbb{U},\quad \|\Gamma_0^{T_2}\phi_k^{I_*,\eta_*,a_{\mathrm{i}}}-\alpha_k\phi_k^{I_*,\eta_*,a_{*,1}-\nu_1}\|_{L^2}\le\varepsilon'. \tag{4.5}$$

As above, we choose T_2 sufficiently large so that $\|a'\|_{L^\infty((0,T_2),\mathbb{R})} \le \kappa$. Now, we concatenate the obtained path with the previous one. Note that the concatenation is still smooth, since the derivatives vanish near the joint point. Moreover, Γ_s^t is unitary, and so the L^2-errors propagate without change. Gathering (4.4) and (4.5), we obtain the paths $a(t)$, $I(t)$, and $\eta(t)$, all smooth in $[0, T_1 + T_2]$, satisfying

$$\forall k \le N, \quad \exists \alpha_k \in \mathbb{U}, \quad \|\Gamma_0^{T_1+T_2}\sqrt{2}\sin(k\pi\cdot) - \alpha_k \phi_k^{I_*,\eta_*,a_{*,1}-\nu_1}\|_{L^2} \le 2\varepsilon'. \tag{4.6}$$

In addition, we still have $\|I'\|_{L^\infty((0,T_1+T_2),\mathbb{R})} \le \kappa$, $\|\eta'\|_{L^\infty((0,T_1+T_2),\mathbb{R})} \le \kappa$, and $\|a'\|_{L^\infty((0,T_1+T_2),\mathbb{R})} \le \kappa$.

The first short-time crossing. In the previous step, we stopped at $a_{*,1} - \nu_1$, just before the first crossing point of the ideal eigenvalues $\lambda_k^{\infty,a}$. At this point, we can no longer follow the perfectly adiabatic motion, as explained in Sec. III (see Fig. 4). Thus, we use a quasi-adiabatic path $a_1(t)$, lasting a time T_3, to go from $a_{*,1} - \nu_1$ to $a_{*,1} + \nu_1$ so that conditions (4.1) and (4.2) are satisfied. We concatenate this path to the previous ones. Once again, the evolution operator is unitary and the error term from (4.6) is transmitted without amplification. Due to (4.1)–(4.3), the concatenated control satisfies

$$\forall k \le N, \quad \exists \alpha_k \in \mathbb{U}, \quad \|\Gamma_0^{T_1+T_2+T_3}\sqrt{2}\sin(k\pi\cdot) - \alpha_k \phi_{\sigma_1(k)}^{I_*,\eta_*,a_{*,1}+\nu_1}\|_{L^2} \le 5\varepsilon'.$$

Again, we keep $\|I'\|_{L^\infty((0,T_1+T_2+T_3),\mathbb{R})} \le \kappa$, $\|\eta'\|_{L^\infty((0,T_1+T_2+T_3),\mathbb{R})} \le \kappa$, and $\|a'\|_{L^\infty((0,T_1+T_2+T_3),\mathbb{R})} \le \kappa$.

Iteration of the process. We repeat the strategy presented above for every crossing point. Thus, we proceed as follows for every $1 \le j \le j^*$. First, we move adiabatically between $a_{*,j} + \nu_j$ and $a_{*,j+1} - \nu_{j+1}$, which add an error ε', like in (4.5). This motion preserves the ordering of the eigenvalues. The duration of the motions is chosen large enough in order to control the speed of the wall's motion with the parameter κ. Second, we have the quasi-adiabatic dynamics satisfying (4.1)–(4.3) while moving from $a_{*,j+1} - \nu_{j+1}$ to $a_{*,j+1} + \nu_{j+1}$. This motion adds an error $3\varepsilon'$ and applies the permutation σ_{j+1} to the eigenmodes. Finally, we move adiabatically between $a_{*,j^*} + \nu_{j^*}$ and a_f as in the first "horizontal" adiabatic motion by controlling again the speed of the wall's motion. We obtain that, for a suitable time T, there exist smooth controls $I(t)$, $\eta(t) \equiv \eta_*$, and $a(t)$ such that

$$\begin{aligned} &\|\eta'\|_{L^\infty((0,T),\mathbb{R})} \le \kappa, \qquad \|\eta'\|_{L^\infty((0,T),\mathbb{R})} \le \kappa, \qquad \|a'\|_{L^\infty((0,T),\mathbb{R})} \le \kappa, \\ &\text{and } \forall k \le N, \quad \exists \alpha_k \in \mathbb{U}, \quad \|\Gamma_0^T\sqrt{2}\sin(k\pi\cdot) - \alpha_k \phi_{\sigma(k)}^{I_*,\eta_*,a_f}\|_{L^2} \le (4j^*+2)\varepsilon', \end{aligned}$$

where $\sigma = \sigma_{j^*} \circ \cdots \circ \sigma_1$ is exactly the permutation $\sigma_{a_i}^{a_f}$ in Theorem 1.

Final "vertical" adiabatic motion. With fixed $a(t) \equiv a_f$, and $\eta(t) = \eta_*$, we adiabatically decrease I to 0 with the speed slower than κ. The whole concatenated control path gives us Theorem 1 as

$$\forall k \le N, \quad \exists \alpha_k \in \mathbb{U}, \quad \|\Gamma_0^T\sqrt{2}\sin(k\pi\cdot) - \alpha_k\sqrt{2}\sin(\sigma(k)\pi\cdot)\|_{L^2} \le (4j^*+3)\varepsilon' < \varepsilon.$$

□

B. Realizing an arbitrary finite permutation

It is clear that the arguments in the Proof of Theorem 1 can be generalized to the case of the potential given by

$$V(t,x) = \sum_{j=1}^{J} I_j(t)\rho^{\eta_j(t)}(x - a_j(t)) \tag{4.7}$$

with several moving walls that "almost split" the interval. While Theorem 1 allows us to realize a specific permutation $\sigma_{a_i}^{a_f}$ to the eigenmodes, it is not clear that permutations are reachable by combining several permutations of this specific type. However, introducing several potential walls in (4.7), we can reach any permutation of any given number of the modes.

Proposition 5. Take any permutation $\sigma : \mathbb{N}^* \to \mathbb{N}^*$. For all $N \in \mathbb{N}^*$ and $\varepsilon > 0$, there exist $J \in \mathbb{N}$, $T > 0$, and smooth functions $I_j \in \mathcal{C}^\infty([0,T],\mathbb{R}^+)$, $\eta_j \in \mathcal{C}^\infty([0,T],\mathbb{R}^+)$, and $a_j \in \mathcal{C}^\infty([0,T],(0,1))$, $j = 1,\dots,J$, such that the following holds. Let Γ_s^t be the unitary propagator generated by the Schrödinger equation (1.1) in the time interval $[s,t] \subset [0,T]$ with the potential V defined by (4.7). Then, for all $k \le N$, there exists $\alpha_k \in \mathbb{C}$ with $|\alpha_k| = 1$ such that

$$\left\| \Gamma_0^T \sin(k\pi x) - \alpha_k \sin(\sigma(k)\pi x) \right\|_{L^2} \le \varepsilon.$$

Proof. Since the main arguments of the Proof of Theorem 1 still hold for V given by (4.7), it is enough to consider the ideal problem—free particle in the interval $[0,1]$ completely split into several subintervals. We show that any permutation σ of $\mathbb{N}^*$ can be realized by changing the locations $a_j(t)$ of the splittings. We examine the first N eigenvalues. Let $J = \max\{N, \sigma(1), \dots, \sigma(N)\} - 1$.

The initial "vertical" adiabatic motion. At $t = 0$, we split the interval $[0,1]$ into $(J+1)$ subintervals $[0,a_1(0)] \cup [a_1(0),a_2(0)] \cup \cdots \cup [a_J(0),1]$. We choose $a_j(0)$ such that the intervals have decreasing lengths that are very close. More precisely, we assume $|a_1(0)-0| > |a_2(0)-a_1(0)| > \ldots > |1-a_J(0)| > \frac{1}{2}|a_1(0)-0|$. When we increase adiabatically the intensity I_j of the potential walls from 0 to a large value $I = I_*$, the first mode becomes localized in the first interval $[0,a_1(0)]$, the second mode in the second interval, and so on. Note that since $|1-a_J(0)| > \frac{1}{2}|a_1(0)-0|$, the first mode corresponding to any subinterval corresponds to an eigenvalue lower than the one corresponding to the second mode in any of the subintervals.

The "horizontal" quasi-adiabatic motion. Using the quasi-adiabatic motion as in Theorem 1, we change the location of the walls in such a way that $[0,a_1(T)]$ has the $\sigma(1)$-th length among the intervals, $[a_1(T),a_2(T)]$ has the $\sigma(2)-$th length, and so on, until $[a_{N-1}(T),a_N(T)]$. For $j > N$, we make the length of $[a_N(T),a_{N+1}(T)]$ have order p among the lengths of all the intervals, where p is the least integer in $\mathbb{N}^*\backslash\{\sigma(1),\ldots,\sigma(N)\}$, the interval $[a_{N+1}(T),a_{N+2}(T)]$ gets the qth length with q being the least integer in $\mathbb{N}^*\backslash\{\sigma(1),\ldots,\sigma(N),p\}$, and so on. Moreover, we require (as above) that the ratio of any two lengths is strictly less than 2. As a consequence, the second eigenvalue in the longest subinterval is higher than the first eigenvalue in all the subintervals.

The final "vertical" adiabatic motion. We adiabatically remove the walls by decreasing I to 0. The mode localized in the first subinterval corresponds to the $\sigma(1)-$th mode, i.e., $\sin(\sigma(1)\pi x)$, the mode localized in the second subinterval becomes $\sin(\sigma(2)\pi x)$, and so on. Thus, our quasi-adiabatic control has changed the modes $\sin(k\pi x)$ for $k \le N$ into the modes $\sin(\sigma(k)\pi x)$, up to error terms as small as we want. □

C. Proof of Theorem 2

The construction of Proposition 5 can be used to prove Theorem 2. It is enough to show that by applying a control of the type of (4.7), we can move the initial state $A\sin(\pi\cdot)$ arbitrarily L^2-close to any given state u_f with norm $A/\sqrt{2}$. Indeed, by reversing time, this would imply that starting from any state u_i, we can drive the system arbitrarily L^2-close to the state $\sqrt{2}\|u_i\|\sin(\pi\cdot)$. By combining both motions, we can start from u_i and drive the system arbitrarily close to u_f if both states have the same L^2-norm.

Hence, let us show that given any set of real numbers c_k such that $\sum_k c_k^2 = 1$ and any set of $\alpha_k \in \mathbb{U} = \{z \in \mathbb{C}, |z| = 1\}$, $k = 1,\ldots,N$, we can drive the initial state $\sin(\pi\cdot)$ arbitrarily close to

$$u = \sum_{k=1}^{N} c_k\alpha_k \sin(k\pi\cdot) \tag{4.8}$$

(since we work up to a small error, we may choose N large and assume that u is only supported by the first N modes).

We start with the control described in Proposition 5 that sends the first mode to the Nth mode as in Fig. 5 and modify the speed with which the potential walls move in order to reach the vicinity of the state u defined by (4.8) with the given values of c_k and arbitrary α_k, $k = 1,\ldots,N$. After that, we tune the phases.

First, split $[0,1]$ into N subintervals with decreasing lengths, the largest subinterval being less than twice longer than the smallest one. Then, grow adiabatically the potential walls at the boundary points between these intervals. Next, move each $a_j(t)$, $j = 1,\ldots,J$, to decrease the length of the first subinterval and to increase the length of the others. During this process, the line of eigenvalues starting from the lowest mode of the initial problem crosses the lines of eigenvalues corresponding to the modes $2,\ldots,N$ of the initial problem, one by one, exactly in this order (see Fig. 5) (note that these lines do not cross with each other). Before the crossing with the line of the second mode, the solution is close to the lowest energy mode ψ_1. By the adiabatic theorem, this will remain the case if we move the walls extremely slowly. On the other hand, if we move the walls faster and make a short-time crossing as in Theorem 1, then after the crossing the solution will get close to the

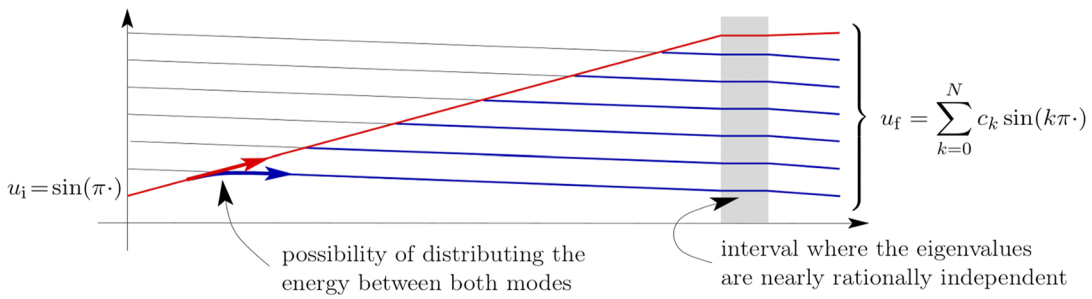

FIG. 5. We start with $u_i = \sin(\pi\cdot)$, split the interval into N subintervals, and slowly modify their lengths such that the first eigenvalue of the split Laplacian crosses the next $N-1$ ones. At each crossing, we can choose the speed of the wall motion in the range between the one described in Sec. III, which would lead to complete crossing, and a much slower one that would make the system change adiabatically. This allows us to create a superposition of the crossing modes with any given amplitudes. After that, we can stop changing parameters for some time and wait until the evolution of the autonomous Schrödinger equation sends the phases α_k sufficiently close to their target values.

second (in the order of the increase in energy) mode ψ_2 (see Sec. III). By continuity, we can choose an intermediate speed in such a way that the solution after the crossing gets close to a superposition of these modes, namely, to $c_1\ \alpha_1\ \psi_1 + \sqrt{1-c_1^2}\ \alpha_2\ \psi_2$ with some $\alpha_1, \alpha_2 \in \mathbb{U}$ and c_1 as in (4.8).

Since the equation is linear, we can trace the evolution of the modes ψ_1 and ψ_2 separately. By construction, there will be no further crossings for ψ_1, while the line of eigenvalues corresponding to ψ_2 will cross the line corresponding to ψ_3. As before, by tuning the speed of crossing, we can keep an energy c_2 on the mode ψ_2 and drive the remaining part on ψ_3. One repeats this procedure until the last crossing. Then, the solution gets close to $\sum_{k=1}^N c_k \tilde{\alpha}_k \psi_k$ with exactly the same coefficients c_k as in (4.8), but with uncontrolled coefficients $\tilde{\alpha}_k \in \mathbb{U}$.

It remains to tune the phases. We can always arrange the above procedure in such a way that at the end of it the lengths $L_1, \dots, L_N$ of the subintervals into which the potential walls divide the interval $(0,1)$ were sufficiently close to be rationally independent. Since the walls are very high, the first eigenvalues are close to $\lambda_k = (\pi/L_k)^2$, and so they are close to a set of rationally independent numbers.

Recall that we can make this approximation as good as we want if we make the potential wall sufficiently thin and high. Thus, if we just wait for a time t without changing the potential, the solution will get close to $\sum_{k=1}^N c_k \alpha_k e^{-it\lambda_k} \psi_k$. Due to the closeness to rational independence, the flow $t \mapsto (e^{-it\lambda_k})_{k\le N}$ is almost dense in the torus $\mathbb{T}^N$, and we can tune the phases α_k to any $\hat{\alpha}_k \in \mathbb{U}$ prescribed in advance. Note that the upper bound on the waiting time t is independent of the initial and target values of α_k, $k = 1, \dots, N$, and depends only on the desired accuracy of the approximation.

Now, we slowly decrease the height of the potential walls up to the extinction of the potential. This is an adiabatic process that transforms (up to a small error) the modes ψ_k into $\sin(\pi k \cdot)$ while keeping c_k constant. The phases acquire a shift, i.e., the solution gets close to $\sum_{k=1}^N c_k \hat{\alpha}_k e^{i\theta_k} \psi_k$ for some real θ_k, $k = 1, \dots, N$. Importantly, the phase shifts θ_k do not depend on $\hat{\alpha}_k$, and so by choosing $\hat{\alpha}_k = e^{-i\theta_k}\alpha_k$, we obtain the desired result. □

V. A PERMUTATION LEADING TO AN EXPONENTIAL GROWTH OF ENERGY

As we remarked, the potential V can be taken zero at the beginning and the end of the time interval $[0, T]$ in Theorem 1, and so the permutation described in this theorem can be applied as many times as we want. Let us demonstrate that this, typically, should lead to an exponential growth in energy. Specifically, if we start with an eigenstate ψ_k with the eigenvalue λ_k, then after applying n times the permutation $\sigma_{a_i}^{a_f}$, we will find the system at the eigenstate ψ_{k_n} with $k_n = (\sigma_{a_i}^{a_f})^n(k)$. The claim is that the energy λ_{k_n} behaves as

$$\lambda_{k_n} \sim e^{nr} \lambda_k$$

for some $r(a_i, a_f) > 0$, for a typical initial condition $k_0 = k$ and typical values of a_i and a_f.

Proving this (after a proper rigorous reformulation) could be a non-trivial task. Still, it is not hard to build explicit examples of the exponential growth (see Ref. 4). The argument behind the general claim is as follows: Note that for any permutation σ of the set of natural numbers, its trajectory $k_n = \sigma^n(k_0)$ is either looped or tends to infinity both at forward and backward iterations. The exponential growth claim means that for the permutations $\sigma_{a_i}^{a_f}$, most of the trajectories are not looped and the averaged value of $\ln \frac{k_{n+1}}{k_n}$ is strictly positive typically (as $\lambda_k \sim k^2$, the exponential growth of the energy is equivalent to the exponential growth of the eigenstate number k).

In order to estimate the average value of the increment in $\ln k$, we recall that at each irrational a, the eigenstates of the Dirichlet–Laplacian in the split interval $(0, a) \cup (a, 1)$ are divided into two groups, the left eigenstates are supported in $(0, a)$, and the right eigenstates are supported in $(a, 1)$. Order the eigenstates by their energy λ_k and introduce the indicator sequence: $\xi^a(k) = 1$ if the eigenstate ψ_k is left, and $\xi^a(k) = -1$ if ψ_k is right. The two sequences ξ^{a_i} and ξ^{a_f} completely determine the permutation $\sigma_{a_i}^{a_f}$. Indeed, if at some a, the state ψ_k is left and acquires the number m when we order the left states by the increase in energy, then there are exactly m left and $(k-m)$ right states with the energies not exceeding λ_k, and so $S^a(k) := \xi^a(1) + \dots + \xi^a(k) = 2m - k$. If ψ_k is a right state with the number n in its group, then there are exactly n right and $(k-n)$ left states with the energies not exceeding λ_k, and so $S^a(k) = k - 2n$. These two formulas can be rewritten as

$$m \text{ or } n = \frac{1}{2}(k + \xi^a(k) S^a(k)).$$

After splitting the interval $(0,1)$ at $a = a_i$, the state ψ_k becomes left if $\xi^{a_i}(k) = 1$ or right if $\xi^{a_i}(k) = -1$. When a changes from a_i to a_f, the left states remain left and the right states remain right, and the corresponding number m or n stays constant, implying that the number k changes to $\tilde{k} = \sigma_{a_i}^{a_f}$ according to the rule

$$k + \xi^{a_i}(k) S^{a_i}(k) = \tilde{k} + \xi^{a_i}(k) S^{a_f}(\tilde{k}). \tag{5.1}$$

One needs a proper analysis of the dynamics of k defined by this formula, but we just make a heuristic assumption that ξ^{ai} and ξ^{af} are sequences of independent, identically distributed random variables. Let $\beta \in (0,1)$ be the probability of $\xi^{a_i} = 1$, and let $\gamma \in (0,1)$

be the probability of $\sigma^{a_f} = 1$. Then, in the limit of large k, we may substitute $S^{a_i}(k) \sim (2\beta - 1)k$ and $S^{a_f}(\bar{k}) \sim (2\gamma - 1)\bar{k}$ in (5.1), which gives

$$\bar{k} = \begin{cases} \dfrac{\beta}{\gamma}k + o(k) & \text{with probability } \beta \\ \dfrac{1-\beta}{1-\gamma}k + o(k) & \text{with probability } 1-\beta. \end{cases}$$

It follows that in the limit of large k,

$$r := \mathbb{E}(\ln \bar{k} - \ln k) = \beta \ln \frac{\beta}{\gamma} + (1-\beta)\ln \frac{1-\beta}{1-\gamma}.$$

This quantity is strictly positive if $\beta \neq \gamma$, giving the claimed exponential growth. It would be interesting to do a more rigorous analysis of the dynamics generated by permutations of eigenvalues due to this or other cyclic processes.

The process described by Eq. (1.1), which we study in this paper, follows the "ideal" permutation $\sigma_{a_i}^{a_f}$ only approximately, and only until the eigenstate number k is smaller than a certain fixed number N. This means that one can repeat this process until the corresponding eigenvalue remains significantly lower than I_*, the maximal intensity of the potential $V(x,t)$. Thus, until this moment, we can expect the exponential growth of the energy at the repeated application of the control cycle of Theorem 1. This lets us to estimate as $\mathcal{O}(\ln I_*)$ the number of the control cycles sufficient to transform a low energy state to the state with energy of order I_*. As the speed with which we change I is bounded from above, and we have $I = 0$ at the beginning of each cycle, the duration of one cycle is $\mathcal{O}(I_*)$. Thus, we can transform a low energy eigenstate to the state of energy of order I_* in a time of order $I_* \ln I_*$.

In fact, no unbounded exponential growth of the energy is possible in the setting of (1.1), or for any periodic in time, smooth potential $V(x,t)$ in a bounded domain. Indeed, assume that $\psi(t)$ is a solution of the Schrödinger equation with $\psi(0)$ concentrated close to the kth eigenmode. Until our permutation process stays valid, after n cycles, $\psi(nT)$ is concentrated close to the $\sigma^n(k)$–th eigenmode. This implies, in particular, that $\|\psi(nT)\|_{H^1}$ grows like $\mathcal{O}(\sigma^n(k))$. However, it is proved in Ref. 45 (see also Ref. 46) that the H^s–norm of the solution of a periodic Schrödinger equation has at most a polynomial growth rate for any $s > 0$. Even when the ideal permutation σ yields an exponential growth of the eigenmodes, we will move away from this trajectory after a finite time. The realization of the unbounded exponential growth needs to use a more singular evolution like, for example, the splitting process (1.3).

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

REFERENCES

[1] J. E. Avron and A. Elgart, "Adiabatic theorem without a gap condition," Commun. Math. Phys. **203**(2), 445–463 (1999).
[2] M. Born and V. A. Fock, "Beweis des adiabatensatzes," Z. Phys. A **51**(3-4), 165–180 (1928).
[3] T. Kato, *Perturbation Theory for Linear Operators, Classics in Mathematics*, Reprint of the Corrected Printing of the 2nd edition 1980 (Springer-Verlag, 1995).
[4] D. Turaev, "Exponential energy growth due to slow parameter oscillations in quantum mechanical systems," Phys. Rev. E **93**, 050203(R) (2016).
[5] A. Duca, R. Joly, and D. Turaev, "Permuting quantum eigenmodes by a quasi-adiabatic motion of a potential wall," arXiv:1912.07425 (2019).
[6] J. M. Ball, J. E. Marsden, and M. Slemrod, "Controllability for distributed bilinear systems," SIAM J. Control Optim. **20**(4), 575–597 (1982).
[7] K. Beauchard, "Local controllability of a 1-D Schrödinger equation," J. Math. Pures Appl. **84**(7), 851–956 (2005).
[8] K. Beauchard and C. Laurent, "Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control," J. Math. Pures Appl. **94**(5), 520–554 (2010).
[9] A. Duca, "Simultaneous global exact controllability in projection," Dyn. Partial Differ. Equations **17**(3), 275–306 (2020).
[10] A. Duca, "Controllability of bilinear quantum systems in explicit times via explicit control fields," Int. J. Control (published online 2019).
[11] M. Morancey and V. Nersesyan, "Simultaneous global exact controllability of an arbitrary number of 1D bilinear Schrödinger equations," J. Math. Pures Appl. **103**(1), 228–254 (2015).
[12] M. Morancey, "Simultaneous local exact controllability of 1D bilinear Schrödinger equations," Ann. Inst. H. Poincaré Anal. Non Linéaire **31**(3), 501–529 (2014).
[13] M. Mirrahimi, "Lyapunov control of a quantum particle in a decaying potential," Ann. Inst. H. Poincaré Anal. Non Linéaire **26**(5), 1743–1765 (2009).
[14] V. Nersesyan, "Global approximate controllability for Schrödinger equation in higher Sobolev norms and applications," Ann. Inst. H. Poincaré Anal. Non Linéaire **27**(3), 901–915 (2010).
[15] U. Boscain, M. Caponigro, and M. Sigalotti, "Multi-input Schrödinger equation: Controllability, tracking, and application to the quantum angular momentum," J. Differ. Equations **256**(11), 3524–3551 (2014).
[16] N. Boussaïd, M. Caponigro, and T. Chambrion, "Weakly coupled systems in quantum control," IEEE Trans. Autom. Control **58**(9), 2205–2216 (2013).
[17] S. Guerin, L. P. Yatsenko, and H. R. Jauslin, "Topology of adiabatic passage," Phys. Rev. A **65**(4), 043407 (2002).

[18]E. A. Shapiro, V. Milner, and M. Shapiro, "Complete transfer of populations from a single state to a preselected superposition of states using piecewise adiabatic passage: Theory," Phys. Rev. A **79**(2), 023422 (2009).
[19]N. V. Vitanov, T. Halfmann, B. W. Shore, and K. Bergmann, "Laser-induced population transfer by adiabatic passage techniques," Annu. Rev. Phys. Chem. **52**, 763–809 (2001).
[20]Z. Leghtas, A. Sarlette, and P. Rouchon, "Adiabatic passage and ensemble control of quantum systems," J. Phys. B: At., Mol. Opt. Phys. **44**(15), 154017 (2011).
[21]R. Adami, and U. Boscain, "Controllability of the Schrödinger equation via intersection of eigenvalues," in *Proceedings of the 44th IEEE Conference on Decision and Control* (IEEE, 2005), pp. 1080–1085.
[22]N. Augier, U. Boscain, and M. Sigalotti, "Adiabatic ensemble control of a continuum of quantum systems," SIAM J. Control Optim. **56**(6), 4045–4068 (2018).
[23]U. V. Boscain, F. Chittaro, P. Mason, and M. Sigalotti, "Adiabatic control of the Schrödinger equation via conical intersections of the eigenvalues," IEEE Trans. Autom. Control **57**(8), 1970–1983 (2012).
[24]U. Boscain, J.-P. Gauthier, F. Rossi, and M. Sigalotti, "Approximate controllability, exact controllability, and conical eigenvalue intersections for quantum mechanical systems," Commun. Math. Phys. **333**(3), 1225–1239 (2015).
[25]J.-M. Coron and E. Trélat, "Global steady-state controllability of one-dimensional semilinear heat equations," SIAM J. Control Optim. **43**(2), 549–569 (2004).
[26]J.-M. Coron and E. Trélat, "Global steady-state stabilization and controllability of 1D semilinear wave equations," Commun. Contemp. Math. **08**(4), 535–567 (2006).
[27]J.-L. Lions, *Contrôle des Systèmes Distribués Singuliers*, Méthodes Mathématiques de l'Informatique Vol. 13 (Gauthier-Villars, Montrouge, 1983).
[28]E. Machtyngier, "Exact controllability for the Schrödinger equation," SIAM J. Control Optim. **32**(1), 24–34 (1994).
[29]N. Burq, "Contrôle de l'équation de Schrödinger en présence d'obstacles strictement convexes," in *Journées Équations aux Dérivées Partielles (Saint Jean de Monts, 1991)* (École Polytechnique, Palaiseau, 1991), p. 14.
[30]G. Lebeau, "Contrôle de l'équation de Schrödinger," J. Math. Pures Appl. **71**(3), 267–291 (1992).
[31]L. Baudouin and A. Mercado, "An inverse problem for Schrödinger equations with discontinuous main coefficient," Appl. Anal. **87**(10-11), 1145–1165 (2008).
[32]I. Lasiecka and R. Triggiani, "Optimal regularity, exact controllability and uniform stabilization of Schrödinger equations with Dirichlet control," Differ. Integr. Equations **5**(3), 521–535 (1992).
[33]A. Mercado, A. Osses, and L. Rosier, "Inverse problems for the Schrödinger equation via Carleman inequalities with degenerate weights," Inverse Probl. **24**(1), 015017 (2008).
[34]K. Ammari, A. Bchatnia, and K. El Mufti, "A remark on observability of the wave equation with moving boundary," J. Appl. Anal. **23**(1), 43–51 (2017).
[35]K. Ammari, A. Bchatnia, and K. El Mufti, "Stabilization of the wave equation with moving boundary," Eur. J. Control **39**, 35–38 (2018).
[36]K. Beauchard, "Controllability of a quantum particle in a 1D variable domain," ESAIM Control Optim. Calc. Var. **14**(1), 105–147 (2008).
[37]K. Beauchard and J.-M. Coron, "Controllability of a quantum particle in a moving potential well," J. Funct. Anal **232**(2), 328–389 (2006).
[38]J. Dittrich, P. Duclos, and N. Gonzalez, "Stability and instability of the wave equation solutions in a pulsating domain," Rev. Math. Phys. **10**(07), 925–962 (1998).
[39]J. Kisyński, "Sur les opérateurs de Green des problèmes de Cauchy abstraits," Studia Math. **23**, 285–328 (1964).
[40]H. Tanabe, *Equations of Evolution*, Monographs and Studies in Mathematics Vol. 6 (Pitman (Advanced Publishing Program), Boston, Massachusetts; London, 1979), translated from the Japanese by N. Mugibayashi and H. Haneda.
[41]S. Teufel, *Adiabatic Perturbation Theory in Quantum Dynamics*, Lecture Notes in Mathematics Vol. 1821 (Springer-Verlag, Berlin, 2003).
[42]F. Bornemann, *Homogenization in Time of Singularly Perturbed Mechanical Systems*, Lecture Notes in Mathematics Vol. 1687 (Springer-Verlag, Berlin, 1998).
[43]J. Schmid, "Adiabatic theorems for general linear operators with time-dependent domains," Rev. Math. Phys. **31**(5), 1950014 (2018).
[44]M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators* (Academic Press, 1978).
[45]J. Bourgain, "On growth of Sobolev norms in linear Schrödinger equations with smooth time-dependent potential," J. Anal. Math. **77**, 315–348 (1999).
[46]J.-M. Delort, "Growth of Sobolev norms of solutions of linear Schrödinger equations on some compact manifolds," Int. Math. Res. Not. **12**, 2305–2328 (2010).