ON HAMILTONIAN SYSTEMS WITH HOMOCLINIC SADDLE CURVES

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As we know, for Hamiltonian systems the existence of a structurally stable homoclinic curve of an equilibrium state of saddle type is a typical phenomenon. This is connected with the fact that since the equilibrium state and its stable ($W^s$) and unstable ($W^u$) manifolds lie on one energy level then $W^s$ and $W^u$ may intersect transversally along homoclinic curves. Hence one may expect that the set of all trajectories of a Hamiltonian system lying entirely in a neighborhood of a homoclinic curve or a bouquet of homoclinic curves of an equilibrium state of saddle type has a reasonable, if not complete, description. For the case of a saddle-focus homoclinic curve this case was considered by Devaney [1]. He established that the set of trajectories lying in an energy level of a saddle-focus has a description in terms of symbolic dynamics with countably many symbols.

It is interesting (see [2]) that this description is completely analogous to the description of the structure of a neighborhood of a structurally stable Poincaré homoclinic curve [3].

In this note we consider the case when the equilibrium state is a saddle.

Assume that a system $X$ with a Hamiltonian $H \in C^3$ in a domain $D \subseteq \mathbb{R}^{2n}$, $n \geq 2$, has an equilibrium state $0$. Let $\pm \lambda_1, \ldots, \pm \lambda_n$ be the roots of the characteristic equation of $O$. Assume that $O$ is a saddle, i.e., $0 < \lambda_1 < \operatorname{Re} \lambda_i$, $i = 2, \ldots, n$. Near the saddle the vector field is written in the form $\dot{x} = -\lambda_1 x + \cdots$, $\dot{y} = -Ay + \cdots$, $\dot{u} = \lambda_1 u + \cdots$, $\dot{v} = A^T v + \cdots$, where $x \in \mathbb{R}^1$, $y \in \mathbb{R}^{n-1}$, $u \in \mathbb{R}^1$, $v \in \mathbb{R}^{n-1}$.

Spec $A = \{\lambda_2, \ldots, \lambda_n\}$, and the dots denote terms of order higher than one. $W^u$ is tangent at $0$ to the plane $u = 0$, $v = 0$, and $W^s$ is tangent to the plane $x = 0$, $y = 0$. We denote by $W^ss$ ($W^uu$) the stable (unstable) nonleading $(n-1)$-dimensional saddle manifold. $W^ss$ is tangent to the $y$-axis, and $W^uu$ is tangent to the $v$-axis. $W^ss$ divides $W^s$ into two parts: $W^s_+$ and $W^s_-$. Similarly, we have $W^u = W^uu \cup W^u_+ \cup W^u_-$. We shall assume that $W^u_-$ approaches $W^uu$ from the domain $u > 0$, and $W^u_+$ approaches $W^ss$ from the domain $x > 0$. Let us assume that $W^u$ and $W^s$ intersect transversally along $m$ homoclinic trajectories $\Gamma_1, \ldots, \Gamma_m$ not lying in $W^uu$ and $W^ss$. The latter means that the $\Gamma_i$ enter the saddle and leave it tangentially to the leading directions, the $x$ and $u$ axes respectively. Let us number the $\Gamma_i$ so that

$$
\begin{align*}
\bigcup_{i=1}^{m_1+m_2} \Gamma_i & \subseteq W^s_+ \cap W^u_+, \\
\bigcup_{i=m_1+1}^{m_1+m_2+m_3} \Gamma_i & \subseteq W^s_+ \cap W^u_-, \\
\bigcup_{i=m_1+m_2+1}^{m_1+m_2+m_3+m_4} \Gamma_i & \subseteq W^s_- \cap W^u_+, \\
\bigcup_{i=m_1+m_2+m_3+1}^{m_1+m_2+m_3+m_4+m} \Gamma_i & \subseteq W^s_- \cap W^u_-,
\end{align*}
$$

$$m_1 + m_2 + m_3 + m_4 = m.
$$

We shall assume that $m_1 \neq 0$.  

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(\textsuperscript{1}) In the case of general smooth dynamical systems with a homoclinic saddle-focus curve we have also a complicated structure in the behavior of the trajectories [4], and on the bifurcation surface of such systems there is a dense structural instability.
Let $H = 0$ be the level containing $O$. We denote by $X_h$ the restriction of the system to the level $H = h$. Let $V$ be a small neighborhood of the bouquet $\Gamma_1 \cup \cdots \cup \Gamma_m \cup O$. We denote by $\Omega_h$ the set of trajectories of $X_h$ lying entirely in $V$.

**Theorem 1.** For a sufficiently small $V$ and a sufficiently small $h_0 > 0$ depending on $V$, the following assertions are true:

1) $\Omega_0 = \{\Gamma_1, \ldots, \Gamma_m, O\}$.

2) If $m_3 = m_4 = 0$, then $X_h\vert_{\Omega_h}$ for $h \in (0, h_0)$ is topologically equivalent to a suspension of a Bernoulli scheme of $m_1$ symbols (if $m_1 = 1$ then $\Omega_h$ consists of one saddle cycle); and for $h \in (-h_0, 0)$, if $m_2 > 0$ it is equivalent to a suspension of a Bernoulli scheme of $m_2$ symbols, while if $m_2 = 0$ then $\cdot \Omega_h = \emptyset$.

3) If $m_3 \neq 0$ or $m_4 \neq 0$ then $X_h\vert_{\Omega_h}$ for $h \in (-h_0, 0) \setminus \{0\}$ is topologically equivalent to a suspension of a topological Markov chain (TMC) given in the case $h > 0$ by the transition matrix

$$
\begin{pmatrix}
  m_1 & m_2 \\
  m_4 & m_3
\end{pmatrix},
$$

and in the case $h < 0$ by

$$
\begin{pmatrix}
  m_2 & m_1 \\
  m_3 & m_4
\end{pmatrix}.
$$

**Remark 1.** Clearly, for $m \geq 3$ at least one of the graphs given by the matrices has a vertex belonging to at least two cycles. Therefore, for $m \geq 3$ the system $X$ has a complicated structure.

**Remark 2.** It is possible to number the edges of the graphs so that a periodic trajectory of the TMC $\{i_1, \ldots, i_k\}$, $i_j \in \{1, \ldots, m\}$, corresponds to a periodic trajectory of the system $X_h$ homotopic in $V$ to the product $\Gamma_{i_1} \cdots \Gamma_{i_k}$.

We consider below a simple case of a bouquet of countably many homoclinic curves. Assume that $X$ has a saddle periodic motion $L$ in the energy level of a saddle. Then, as we know, $X$ has a one-parameter family $L_h$ of saddle periodic motions, with $L_0 = L$. Let us assume that $W^u(O)$ and $W^s(L)$ intersect transversally along a trajectory $\Gamma_1$, and $W^s(L)$ also intersect transversally along a trajectory $\Gamma_2$. Assume that $\Gamma_1 \not= W^u(O)$ and $\Gamma_2 \not= W^s(O)$ ($\Gamma_1 \subset W^u_s$ and $\Gamma_2 \subset W^s_l$). Let us take a small neighborhood $V$ of the contour $\Gamma_1 \cup \Gamma_2 \cup L \cup O$. Its fundamental group has two generators: we choose $L$ as one of the generators, and we choose the second arbitrarily and denote it by $S$. Let us denote by $\Omega_h$ the set of trajectories of $X_h$ lying entirely in $V$.

**Theorem 2.** For a sufficiently small $V$ and a small $h_0 > 0$ depending on $V$, the following assertions are true:

1) $\Omega_0 = \{\Gamma_1, \Gamma_2, L, O\} \cup (\bigcup_{i \geq h_0} \{\gamma_i\})$, where $i_0$ is an integer and $\gamma_i$ is a trajectory homoclinic to $O$, homotopic to $SL^i$ in $V$.

2) If $h \in (-h_0, 0)$, then $\Omega_h = \{L_h\}$.

3) If $h \in (0, h_0)$, then $X_h\vert_{\Omega_h}$ is topologically equivalent to a suspension of a Bernoulli scheme of two symbols $L$ and $S$; moreover, a periodic trajectory of the Bernoulli scheme $\{i_1, \ldots, i_k\}$ corresponds to a periodic trajectory of the system $X_h$ homotopic in $V$ to the product $i_1 \cdots i_k$.

Let us consider now the case when there are two saddles $O_1$ and $O_2$ in the level $H = 0$. We assume that $W^u(O_1) \cup W^u(O_2)$ intersects transversally in the level $H = 0$ with $W^s(O_1) \cup W^s(O_2)$ along $m$ trajectories $\Gamma_1, \ldots, \Gamma_m$ not lying in $W^s(O_1) \cup W^s(O_2) \cup W^u(O_1) \cup W^u(O_2)$. We set $W^s_u(O_1) = W^s_u(O_1), W^s_s(O_1) = W^s_s(O_1)$.

(2) See [6]–[8] for representation of a TMC by a multigraph and a transition matrix.
$W_3^{s(u)} = W_+^{s(u)}(O_2)$, and $W_4^{s(u)} = W_-^{s(u)}(O_2)$. Let $m_{i,j}$ be the number of trajectories from the array $\Gamma_1, \ldots, \Gamma_m$ lying in $W_i^u \cap W_j^s$, $i, j \in \{1, 2, 3, 4\}$, and put $\sum_{i,j} m_{i,j} = m$. We denote by $\Omega_h$ the set of trajectories of the system $X_h$ lying entirely in a small neighborhood $V$ of the contour $\Gamma_1 \cup \cdots \cup \Gamma_m \cup O_1 \cup O_2$.

Let us consider an arbitrary integer square matrix $Q$. If all the entries of some row of the matrix are zero we remove from $Q$ this row and the column with the same index. We repeat this process until we obtain a matrix having a nonzero element in each row. We denote this matrix by $\hat{Q}$.

**Theorem 3.** For a sufficiently small $V$ and a small $h_0 > 0$ depending on $V$, the following assertions are true:

1) $\Omega_0 = \{\Gamma_1, \ldots, \Gamma_m, O_1, O_2\}$.

2) If $h \in (-h_0, h_0) \setminus \{0\}$, then $X_h|_{\Omega_h}$ is topologically equivalent to a suspension of a TMC given for $h > 0$ by the matrix $\hat{Q}_1$, and for $h < 0$ by the matrix $\hat{Q}_2$, where $Q_1$ and $Q_2$ are respectively

\[
\begin{pmatrix}
  m_{11} & m_{12} & m_{13} & m_{14} \\
  m_{21} & m_{22} & m_{23} & m_{24} \\
  m_{31} & m_{32} & m_{33} & m_{34} \\
  m_{41} & m_{42} & m_{43} & m_{44}
\end{pmatrix}
\]  and  

\[
\begin{pmatrix}
  m_{11} & m_{12} & m_{13} & m_{14} \\
  m_{22} & m_{23} & m_{24} & m_{25} \\
  m_{32} & m_{33} & m_{34} & m_{35} \\
  m_{42} & m_{43} & m_{44} & m_{45}
\end{pmatrix}
\]

This situation is not structurally stable. In this connection we consider a one-parameter family $X_{\mu}$ of dynamical systems with Hamiltonian $H_{\mu} \in C^3$. We assume that $H_{\mu}(O_1) = \mu$ and $H_{\mu}(O_2) = -\mu$. Let us denote by $\Omega_{h_{\mu}}$ the set of trajectories of $X_{\mu}|_{H_{\mu}=h}$ lying entirely in $V$. By the symmetry of the problem we may restrict ourselves to the case $\mu > 0$.

**Theorem 4.** For a sufficiently small $V$ and small $h_0 > 0$ and $\mu_0 \in (0, h_0)$, depending on $V$, if $\mu \in (0, \mu_0)$, $|h| < h_0$, and $|h| \neq \mu$, then $X_h|_{\Omega_{h_{\mu}}}$ is topologically equivalent to a suspension of a TMC given for $h \in (\mu, h_0)$ by the matrix $\hat{Q}_1$, for $h \in (-h_0, -\mu)$ by the matrix $\hat{Q}_2$, and for $h \in (-\mu, \mu)$ by the matrix $\hat{Q}_3$, where

\[
\hat{Q}_3 = \begin{pmatrix}
  m_{21} & m_{11} & m_{31} & m_{41} \\
  m_{22} & m_{12} & m_{32} & m_{42} \\
  m_{23} & m_{13} & m_{33} & m_{43} \\
  m_{24} & m_{14} & m_{34} & m_{44}
\end{pmatrix}
\]

For $h = \mu \ (h = -\mu)$, $\Omega_{h_{\mu}}$ contains a bouquet of homoclinic curves of the saddle $O_1 \ (O_2)$. $X_h|_{\Omega_{h_{\mu}}}$ is equivalent to a suspension of a TMC given for $h = \mu$ by the matrix $\hat{Q}_4$, and for $h = -\mu$ by the matrix $\hat{Q}_5$, where we have identified two trajectories:

\[
\cdots(m + 1)(m + 1)(m + 1) \cdots \text{ and } (m + 2)(m + 2)(m + 2) \cdots.
\]

Here $Q_4$ and $Q_5$ are respectively

\[
\begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & m_{31} + m_{32} & m_{41} + m_{42} \\
  m_{13} + m_{23} & 0 & m_{33} & m_{43} \\
  m_{14} + m_{24} & 0 & m_{34} & m_{44}
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
  m_{21} & m_{11} & 0 & m_{31} + m_{41} \\
  m_{22} & m_{12} & 0 & m_{32} + m_{42} \\
  m_{23} + m_{24} & m_{23} + m_{14} & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}
\]

in the case $h = \mu \ (h = -\mu)$ we have denoted by $(m + 1)$ and $(m + 2)$ the edges from the vertex 1 (the vertex 4) into itself and from the vertex 2 (the vertex 3) into itself.
To the pair of identified trajectories in the suspension there correspond the saddle $O_1$ for $h = \mu$ and the saddle $O_2$ for $h = -\mu$.\(^{(3)}\) To the trajectory $\{i_j\}_{j=0}^{\infty}$ of the TMC, where $i_j = m + 1$ for $j < 0$, $i_j \in \{1, \ldots, m\}$ for $j = 1, \ldots, k$, and $i_j = m + 2$ for $j > k$, there corresponds a homoclinic trajectory to the saddle, homotopic in $V'$ to the product $\Gamma_{i_1} \cdots \Gamma_{i_k}$.

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\(^{(3)}\)See [6] for suspensions including an equilibrium state.