Model-theoretic constructions for ω-categorical structures

David M Evans

School of Mathematics,

UEA, Norwich, UK.

Hattingen, July 2003.

ω-categoricity

NOTATION: L a first-order language; M a countably infinite L-structure.

DEFINITION: M is *ω-categorical* if every countable model of $Th(M)$ is isomorphic to M.

FACTS: (Engeler, Ryll-Nardzewski, Svenonius) Let $G = \mathrm{Aut}(M)$. Then M is ω-categorical iff G has finitely many orbits on M^n (for all $n \in \mathbb{N}$).

Orbits: $\{(ga_1, \ldots, ga_n) : g \in G\}$ for $(a_1, \ldots, a_n) \in M^n$.

If M is ω-categorical then:

G-orbits on M^n correspond to complete n-types over $\emptyset$.

NOTE: If M is ω-categorical, then it is *locally finite*: any finitely generated substructure is finite.

Constructions of ω-categorical structures

1. EXAMPLES IN NATURE:

- Pure set Ω (automorphism group $\mathrm{Sym}(\Omega)$)
- $(\mathbb{Q}, \leq)$ (Cantor's theorem)
- Vector spaces $V(\omega, q)$ over finite fields
- ...

2. NEW STRUCTURES FROM OLD ONES:

- Finite products; covers.
- Any structure interpretable in a ω-categorical structure is ω-categorical. For example:
  - n-sets from a pure set ($[\Omega]^n$ with $\mathrm{Sym}(\Omega)$ as automorphism group)
  - Reducts (mysterious, but interesting)

3. BOOLEAN POWERS:

- Important in, for example, ω-categorical groups.

4. AMALGAMATION METHODS:

- The main focus of this talk.

Amalgamation: the basic Fraïssé construction

A class $\mathcal{C}$ of finite L-structures is an *amalgamation class* if:

- $\mathcal{C}$ has countably many isomorphism types
- $\mathcal{C}$ is closed under substructures
- (Joint embedding) Any two structures in $\mathcal{C}$ can be embedded in a third
- (Amalgamation) If $A, B_1, B_2 \in \mathcal{C}$ and $f_i : A \longrightarrow B_i$ are embeddings there exists $C \in \mathcal{C}$ and embeddings $g_i : B_i \longrightarrow C$ with $g_1 \circ f_1 = g_2 \circ f_2$.

Given this, there exists a chain of structures in $\mathcal{C}$:

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \subseteq M_i \subseteq \cdots$$

such that:

- Every structure in $\mathcal{C}$ is isomorphic to a substructure of some M_i
- If A is a substructure of M_i, and $B \in \mathcal{C}$ and $f : A \longrightarrow B$ is an embedding, then there exixts $j \geq i$ and an embedding $g : B \longrightarrow M_j$ such that $g \circ f$ is the identity on A.

Let $M = \bigcup_{i \in \mathbb{N}} M_i$. Then:

1. M is countable and locally finite
2. Age$(M) = \mathcal{C}$
3. If $A \subseteq M$ is a finite substructure, $B \in \mathcal{C}$ and $f : A \longrightarrow B$ is an embedding, then there exists and embedding $g : B \longrightarrow M$ with $g \circ f$ the identity on A.

Moreover, using a back-and-forth argument:

- Properties 1, 2, 3 determine M up to isomorphism
- ((Ultra-)Homogeneity) Any isomorphism between between finite substructures of M extends to an automorphism of M

Refer to M as the *Fraïssé limit* or *generic structure* of the amalgamation class $\mathcal{C}$.

THEOREM: (R. Fraïssé) A (locally finite) countable structure M is homogeneous iff Age(M) is an amalgamation class.

NOTES: 1. Homogeneous structure M is ω-categorical iff it is locally finite and for each $n \in \mathbb{N}$ there are finitely many isomorphism types of n-generator substructures of M.
2. An ω-categorical structure is homogeneous (in this sense) iff it has QE.

EXAMPLES OF AMALGAMATION CLASSES:

1. Finite graphs (- Fraïssé limit is the *random graph*)
2. Finite graphs omitting the complete graph on n vertices (n fixed)
3. Finite digraphs
4. Finite digraphs omitting a given set of tournaments
5. Finite posets
6. Finite distributive lattices
7. Finite groups (- Fraïssé limit is Philip Hall's countable universal locally finite group)

In Examples 1-4 we can take amalgamation to be *free amalgamation*. In all cases apart from 7, the limit is ω-categorical.

Variations on the basic construction

IDEA: Work with a class $\mathcal{K}$ of finite L-structures and a notion:

$$A \sqsubseteq B$$

pronounced 'A is a nicely embedded substructure of B.' Demand the amalgamation property only over *nicely embedded* substructures. More formally, work with $\sqsubseteq$-embeddings $f : A \to B$ - meaning $f(A) \sqsubseteq B$. We'll *assume* that these embeddings include isomorphisms; are closed under composition (- so $\sqsubseteq$ is transitive); and under restriction of the codomain.

Say that $(\mathcal{K}, \sqsubseteq)$ is an *amalgamation class* if:

- $\mathcal{K}$ is closed under $\sqsubseteq$-substructures
- $\mathcal{K}$ has countably many isomorphism types
- (Joint embedding) Any two elements of $\mathcal{K}$ can be $\sqsubseteq$-embedded in a third.
- ($\sqsubseteq$-Amalgamation) If $A, B_1, B_2 \in \mathcal{K}$ and $f_i : A \to B_i$ are $\sqsubseteq$-embeddings, there exist $C \in \mathcal{K}$ and $\sqsubseteq$-embeddings $g_i : B_i \to C$ with $g_1 \circ f_1 = g_2 \circ f_2$.

THEOREM: There is a structure M satisfying:

1. M is the union of a chain $M_1 \sqsubseteq M_2 \sqsubseteq M_3 \sqsubseteq \cdots$ of members of $\mathcal{K}$
2. Any member of $\mathcal{K}$ is isomorphic to a $\sqsubseteq$-substructure of M
3. If $A \sqsubseteq M$ is finite and $f : A \to B \in \mathcal{K}$ is a $\sqsubseteq$-embedding there is a $\sqsubseteq$-embedding $g : B \to M$ with $g \circ f$ the identity on A.

Moreover M is uniquely determined by these properties and any isomorphism between finite $\sqsubseteq$-substructures of M extends to an automorphism of M.

NOTES: 1. We will call M here the *generic structure* for the class $(\mathcal{K}, \sqsubseteq)$.

2. Suppose there are only finitely many isomorphism types of structures in M of any finite size. Suppose also that there is a function $F : \mathbb{N} \to \mathbb{N}$ with the property that if $B \in \mathcal{K}$ and $X \subseteq B$ has size $\leq n$ then there exists $A \sqsubseteq B$ containing X and $|A| \leq F(n)$. Then M is ω-categorical.

EXAMPLE: (Not ω-categorical) Let $\mathcal{K}$ be the class of finite digraphs in which the number of edges coming out of any vertex is at most 2. Write $A \sqsubseteq B$ to mean that there an no edges coming out of A (in B).

(PUZZLE: Take the generic here and forget the direction on the edges. Describe the resulting graph.)

Hrushovski's construction

Work with *graphs*.

Let α be a fixed positive real number. If B is a finite graph define the *predimension* of B as:

$$\delta(B) = |B| - \alpha e(B)$$

where $e(B)$ is the number of edges in B. If $A \subseteq B$ write

$$A \leq B \;\;\Leftrightarrow\;\; \delta(A) < \delta(B_1) \text{ whenever } A \subset B_1 \subseteq B.$$

NOTES: 1. Compare with dimension in a vector space.

2. There is a related notion $A \leq^* B$: have $\leq$ rather than $<$.

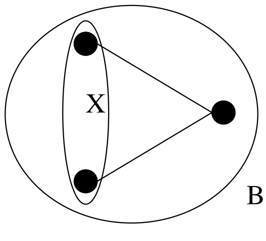
LEMMA: 1. If $A \leq B \leq C$, then $A \leq C$.

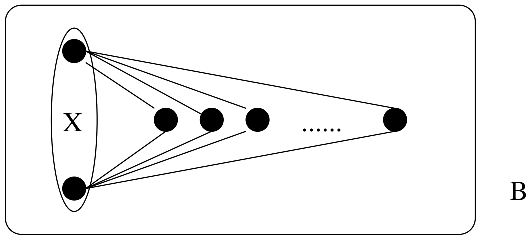
2. If $X \subseteq B$ and $A \leq B$, then $A \cap X \leq X$.

3. If $X \subseteq B$, then $\bigcap\{A : X \subseteq A \leq B\} \leq B$.

Call the set in 3. the *closure* of X in B.

EXAMPLE: Take $\alpha = 1/2$. In each case B is the closure of the two points in X:





DEFINITION: Let $\mathcal{K}_0$ consist of finite graphs A with $\emptyset \leq A$: i.e. for every non-empty subgraph A_1 of A we have $|A_1| - \alpha e(A_1) > 0$.

LEMMA: $(\mathcal{K}_0, \leq)$ is an amalgamation class.

Proof. Show that if $A \leq B_1, B_2 \in \mathcal{K}_0$ then the free amalgam E of B_1 and B_2 over A is in $\mathcal{K}_0$ and $B_1, B_2 \leq E$. If $F \subseteq E$ then F is the free amalgam over $F \cap A$ of $F \cap B_1$ and $F \cap B_2$ and $F \cap A \leq F \cap B_i$. So the only calculation we really need is:

$$\delta(E) = \delta(B_1) + \delta(B_2) - \delta(A) > \delta(B_1) > 0$$

assuming we're not in a trivial case where $A = B_1$ or $A = B_2$. $\square$

The generic for $(\mathcal{K}_0, \leq)$ is **not** ω-categorical. The size of the closure of k points is not bounded by a function of k.

IDEA... for obtaining ω-categoricity:
Take a continuous, increasing bijection $F : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ with $F(x) \to \infty$ as $x \to \infty$. Let $\mathcal{K}_F$ consist of all finite graphs B with

$$\delta(A) \geq F(|A|)$$

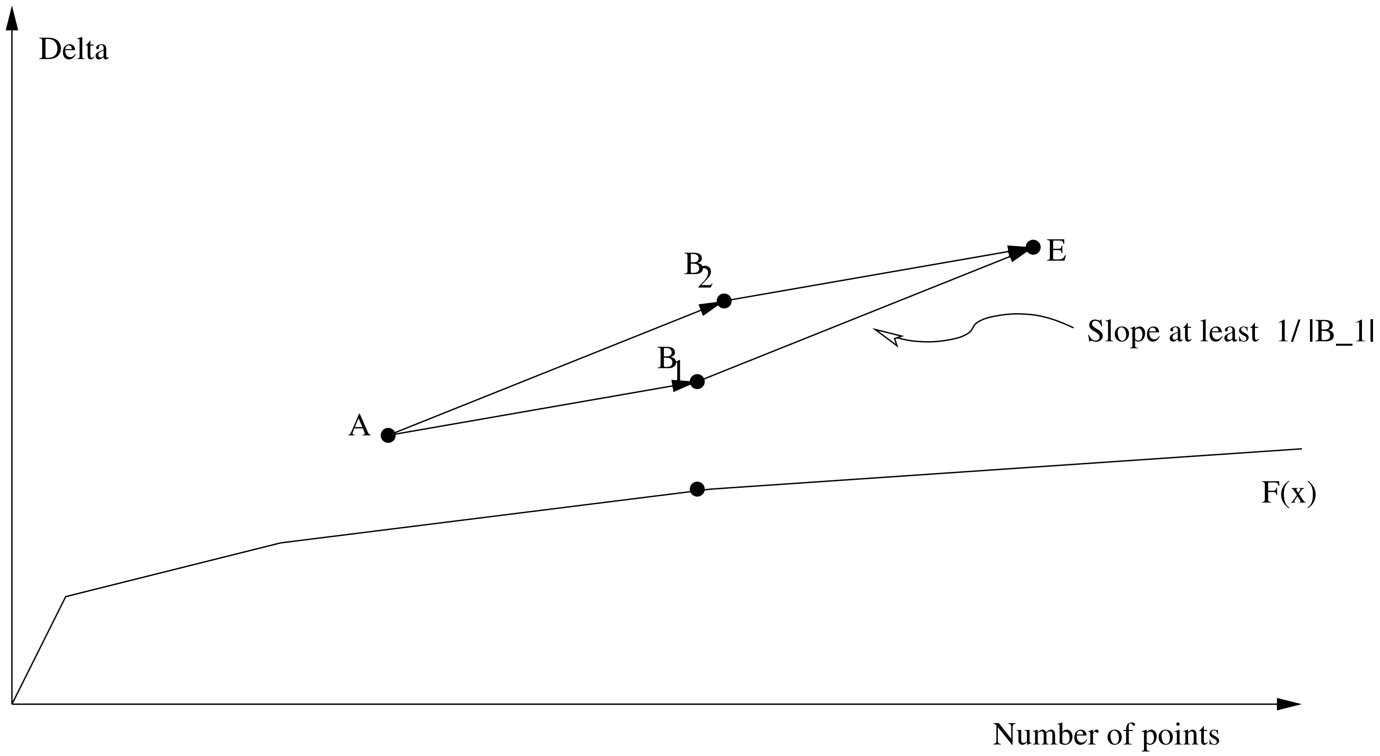
for all $A \subseteq B$.

OBSERVATION: If $X \subseteq B \in \mathcal{K}_F$ then the closure of X in B has size $\leq F^{-1}(\delta(X))$.

So **if** $(\mathcal{K}_F, \leq)$ has the amalgamation property, **then** it is an amalgamation class **and** the generic structure M_F is ω-categorical.

How you choose F to obtain the amalgamation property depends on the α used to define the predimension.

EXAMPLES: 1. (Rational α; Hrushovski, 1988) Suppose $\delta(A) = 2|A| - e(A)$. Choose F right-differentiable (e.g. piecewise linear), with right derivative $F'(x)$ non-increasing and $F'(x) \leq 1/x$.



2. (Irrational α of 'infinite index'; Hrushovski, 1988) Choice of F is more subtle.

Model-theoretic properties: ω-categorical case

1. (E. Hrushovski, 1988) Take α an appropriate irrational and a suitable F. The generic M_F is ω-categorical, stable, but not superstable. (-Counterexample to Lachlan's Conjecture).

2. (E. Hrushovski, 1997) Take α rational and F growing sufficiently slowly. The generic M_F is ω-categorical, supersimple of finite SU-rank and not one-based.

3. (M. E. Pantano, 1995) Take α rational. By letting F grow slowly, we can can obtain algebraic closure growing as fast as we like in M_F.

4. Can work with relations of higher arity to obtain multiply transitive structures in all of the above.

5. By suitable choice of $F(x)$ for small x we can ensure that, for example, the smallest cycle in M_F is a 5-cycle. This is the only known way of constructing an ω-categorical connected graph whose smallest cycle is a 5-cycle and whose automorphism group is transitive on pairs of adjacent vertices.

6. If $(\mathcal{K}_F, \leq)$ is a free amalgamtion class, then M_F does not have the strict order property (- it is $NSOP_4$).

OPEN PROBLEM: Can algebraic closure grow arbitrarily quickly in stable ω-categorical structures? (In a finite language?)

STRANGE PROBLEM: Is there a suitable choice of F for all α (- so irrational α not of infinite index)?

Model-theoretic properties: the unconstrained case

- $\delta(B) = |B| - \alpha e(B)$
- $A \leq B$ iff $\delta(A) < \delta(B_1)$ for all $A \subset B_1 \subseteq B$
- $\mathcal{K}_0$: $\emptyset \leq A$
- $(\mathcal{K}_0, \leq)$-generic: M_0
- $A \leq^* B$ iff $\delta(A) \leq \delta(B_1)$ for all $A \subseteq B_1 \subseteq B$
- $\mathcal{K}_0^*$: $\emptyset \leq^* A$
- $(\mathcal{K}_0^*, \leq^*)$-generic M_0^*

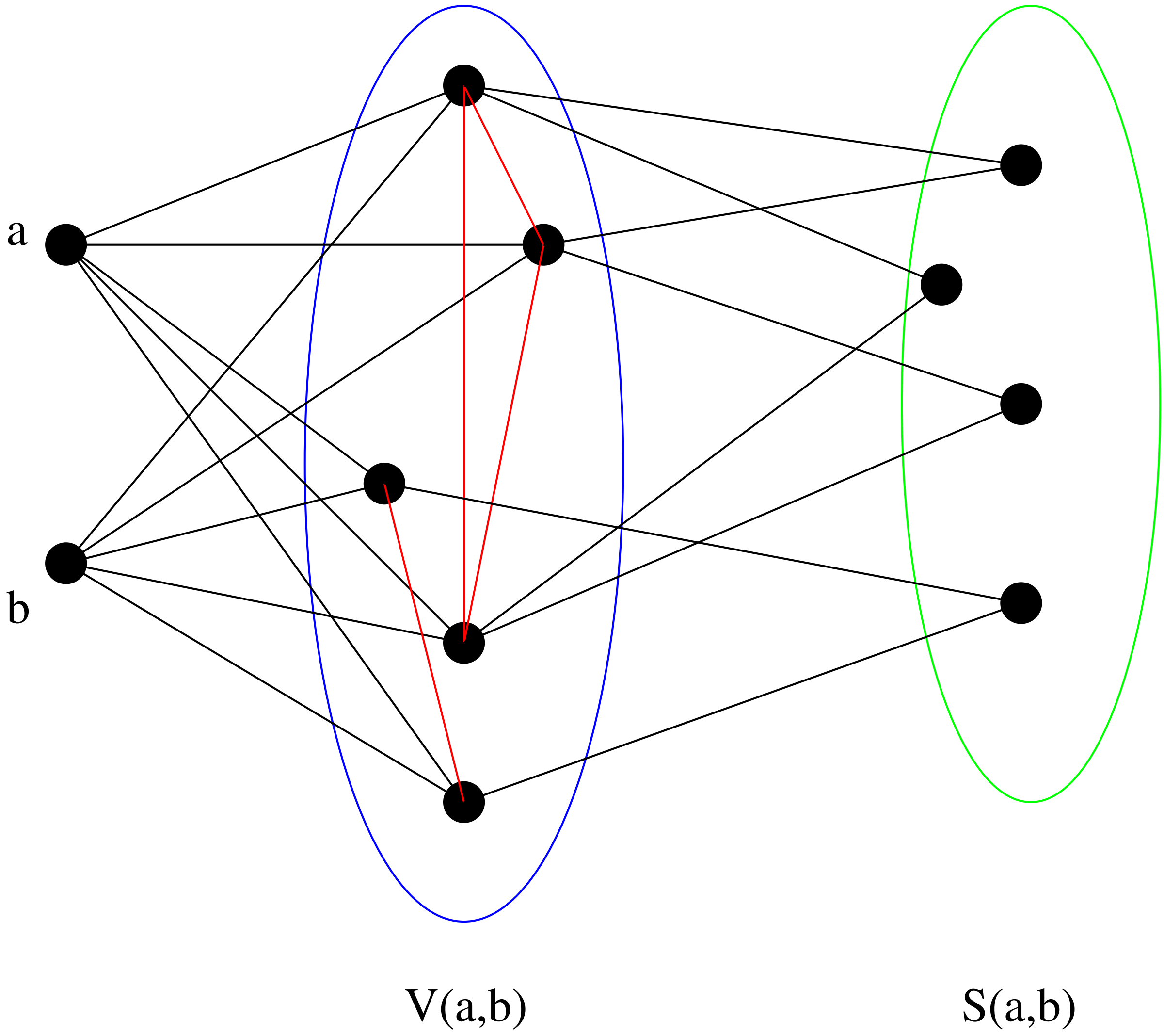
NOTE: If α is irrational then $\leq$ and $\leq^*$ coincide.

1. (J. Baldwin and S. Shelah, 1997; S. Shelah and J. Spencer, 1988) If $0 < \alpha < 1$ is irrational, then $Th(M_0)$ is stable and has the finite model property. It is the almost-sure theory of finite graphs on n vertices with edge probablilty $1/n^\alpha$ (as $n \to \infty$).
2. (E. Hrushovski, 1988) If α is rational then $Th(M_0^*)$ is ω-stable (of infinite Morley rank).
3. (DE, 2003; related earlier work of M. Pourmahdian) Take $\alpha = 1/2$. Then $Th(M_0)$ is undecidable and has the strict order property.

Sketch of 3.

Work with $\delta(A) = 2|A| - e(A)$.

IDEA: Already observed that closure of a pair of points can be arbitrarily large (by taking vertices adjacent to both vertices in the pair). Use this to encode finite graphs (Γ, E) into these closures in a uniform way.



This encodes the graph Γ (-marked in red) as a graph A_Γ (-edges in black). We have $A_\Gamma \in \mathcal{K}_0$ and all vertices of A_Γ are in the closure of a, b.

Let $\chi(a,b)$ denote the L-formula which says that this picture is accurate (- so $V(a,b)$ the set of vertices adjacent to a,b has no edges in it etc.). If $A \in \mathcal{K}_0$ and $A \models \chi(a,b)$, then we interpret a graph in A with vertex set $V(a,b)$ and edges determined by $S(a,b)$.

Given any first-order sentence ϕ in the language of graphs we can write down an L-formula $\theta(a,b)$ such that for any graph Γ:

$$\Gamma \models \phi \Leftrightarrow A_\Gamma \models \theta(a,b).$$

THEOREM: With this notation, there is a finite graph Γ which is a model of ϕ iff $M_0 \models \exists a, b(\chi(a,b) \wedge \theta(a,b))$.

Proof: ($\Rightarrow$:) Use $A_\Gamma \leq M_0$.

($\Leftarrow$:) Take such a, b. The closure in M_0 of a, b is finite, so the graph interpreted in M_0 by $V(a,b)$ and $S(a,b)$ is finite. By construction of θ it is a model of ϕ. $\Box$

This gives undecidability of $Th(M_0)$ by Trakhtenbrot's Theorem.

For the strict order property note that we can construct a family of finite graphs in which arbitrarily large finite linear orders are uniformly interpretable. Translating this into the A_Γ, and using compactness, there is a model of $Th(M_0)$ in which an infinite linear order is interpretable (using two parameters).

PROBLEM: Does $Th(M_0)$ have the finite model property?

Some references:

Peter J. Cameron, Oligomorphic Permutation Groups, Cambridge University Press, 1990.

Gregory L. Cherlin, The Classification of Countable Homogeneous Directed Graphs and Countable Homogeneous n-tournaments, Memoirs of the AMS, number 621, 1998.

David M. Evans, ‘Examples of $\aleph_0$-categorical structures’, *in* Automorphisms of First-Order Structures, edited by Richard Kaye and Dugald Macpherson, Oxford Science Publications, 1994.

Frank O. Wagner, ‘Relational structures and dimensions’, *in* Automorphisms of First-Order Structures, edited by Richard Kaye and Dugald Macpherson, Oxford Science Publications, 1994.