# Graphs, matroids and the Hrushovski constructions

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# 0. Background

- Pregeometry/ matroid: set with a closure operation which satisfies exchange.
- Model-theoretic example: algebraic closure in a strongly minimal set.
- Try to get a better understanding of the (pre)geometries appearing in Hushovski's paper 'A new strongly minimal set' (APAL, 1993).
- Hrushovski's question: How many local isomorphism types of flat strongly minimal sets (of countably infinite dimension) are there?

# This talk:

- (with Marco Ferreira) There are countably many local isomorphism types of geometries of strongly minimal sets arising from the examples in Hrushovski's paper.
- Where do Hrushovski's examples appear in matroid theory (– the branch of combinatorics which studies pregeometries)?

# 1. Pregeometries

A pregeometry $\mathcal{X} = (X, \mathrm{cl})$ consists of a set $X$ and a closure operator $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ which satisfies, for $Y, Z \subseteq X$ and $a, b \in X$:

(1) $Y \subseteq \mathrm{cl}(Y)$

(2) $Y \subseteq \mathrm{cl}(Z) \Rightarrow \mathrm{cl}(Y) \subseteq \mathrm{cl}(Z)$

(3) (Exchange) If $a \in \mathrm{cl}(Y \cup \{b\}) \setminus \mathrm{cl}(Y)$ then $b \in \mathrm{cl}(Y \cup \{a\})$

(4) (Finitary) $\mathrm{cl}(Z) = \bigcup\{\mathrm{cl}(Z_0) : Z_0 \subseteq_{fin} Z\}$

Sometimes called a matroid.
If $Y = \mathrm{cl}(Y)$ say $Y$ is closed; note $\mathrm{cl}(Z)$ is closed (by (1,2)).

EXAMPLE: $X$ a vector space; $\mathrm{cl}$ linear span.

# Dimension

$(X, \mathrm{cl})$ a pregeometry.

- $I \subseteq X$ is independent if $\forall a \in I$, $a \notin \mathrm{cl}(I \setminus \{a\})$
- $I$ is a basis of $X$ if it is independent and $\mathrm{cl}(I) = X$
- FACT: If $I$, $J$ are bases then $|I| = |J|$: the dimension of $X$.

If $Y \subseteq X$ we can restrict the closure to $Y$:

$$\mathrm{cl}^Y(Z) = \mathrm{cl}(Z) \cap Y \text{ for } Z \subseteq Y$$

and refer to independence, basis and dimension in $Y$.

Dimension of $Y$: cardinality of a maximal independent subset of $Y$.

# Geometries; localization

DEFINITION: A pregeometry $(X, \mathrm{cl})$ is a geometry if $\mathrm{cl}(\emptyset) = \emptyset$ and singletons are closed.

If $(X, \mathrm{cl})$ is a pregeometry, we obtain a geometry $(\tilde{X}, \tilde{\mathrm{cl}})$ in a canonical way: the relation $x \sim y \Leftrightarrow \mathrm{cl}(x) = \mathrm{cl}(y)$ is an equivalence relation on $X \setminus \mathrm{cl}(\emptyset)$, and we take $\tilde{X}$ to be the set of classes.

Why 'Geometry'? Call closed sets of dimensions 1, 2, 3,.. points, lines, planes, ...

DEFINITION: If $(X, \mathrm{cl})$ is a pregeometry and $Y \subseteq X$, define a closure operation $\mathrm{cl}_Y$ on $X$ by setting $\mathrm{cl}_Y(Z) = \mathrm{cl}(Y \cup Z)$. This gives a pregeometry $(X, \mathrm{cl}_Y)$ called the localization of $(X, \mathrm{cl})$ over $Y$.

Pregeometries $(X, \mathrm{cl})$ and $(X', \mathrm{cl}')$ are locally isomorphic if there exist finite $Y \subseteq X$ and $Y' \subseteq X'$ such that the localizations $(X, \mathrm{cl}_Y)$, $(X', \mathrm{cl}'_{Y'})$ are isomorphic.

# 2. Examples of pregeometries in model theory

(1) $X$ a strongly minimal structure; $\mathrm{cl}$ is algebraic closure. For example:
   - A pure set
   - An infinite vector space
   - An algebraically closed field

(2) Closure associated with a regular type in a stable theory. For example: differential dependence in a differentially closed field of characteristic 0.

(3) Algebraic closure in an o-minimal structure.

(4) (Jonathan Kirby) Exponential closure in an exponential field.

# Homogeneity

Pregeometries arising in Examples (1) and (2) have rich automorphism groups (assuming sufficient saturation).

DEFINITION: A pregeometry $(X, \mathrm{cl})$ is homogeneous if whenever $Y \subseteq X$ is closed and finite-dimensional and $z_1, z_2 \in X \setminus Y$ there is an automorphism $g$ with $gz_1 = z_2$ and $gy = y$ for all $y \in Y$.

This implies that the automorphism group is transitive on independent sets of the same finite size.

REMARK: A theorem of Cherlin and Zilber (1980's) classifies *locally finite* infinite dimensional homogeneous geometries: they are pure sets or derived from a vector space over a finite field. More precisely, they have the property that some localization over a finite set is *modular*: if $A, B$ are closed subsets then

$$d(A \cup B) = d(A) + d(B) - d(A \cap B),$$

where $d$ denotes dimension.

# 3. The basic Hrushovski construction

ROUGH IDEA: Build a homogeneous pregeometry by amalgamating finite structures, each of which carries a pregeometry; the relevant embeddings between the finite structures should preserve the dimension.

Fix $k \in \mathbb{N} \cup \{\infty\}$ with $k \geq 3$.
Consider structures $(A; R)$ where $A$ is a set and $R \subseteq [A]^{\leq k}$ is a set of finite, non-empty subsets of $A$, each of size at most $k$.

[Think of these as $L_k$-structures where $L_k$ has an $n$-ary relation symbol $R_n$ for each finite $n \leq k$.]

If $B \subseteq A$ let $R[B] = \{r \in R : r \subseteq B\}$ and consider $(B; R[B])$ as a substructure.

If $B \subseteq_{fin} A$ the predimesion of $B$ (in $(A; R)$) is:

$$\delta(B) = |B| - |R[B]|.$$

# The structures

DEFINITION:

(1) $\bar{\mathcal{C}} = \bar{\mathcal{C}}_k$ is the class of structures $(A; R)$ such that $\delta(B) \geq 0$ for all $B \subseteq_{fin} A$.

(2) $\mathcal{C} = \mathcal{C}_k$ is the class of finite structures in $\bar{\mathcal{C}}_k$

(3) If $(A; R) \in \bar{\mathcal{C}}$ and $X \subseteq_{fin} A$ let $d(X) = \min(\delta(B) : X \subseteq B \subseteq_{fin} A)$.

(4) If $(A; R) \in \bar{\mathcal{C}}$ and $X \subseteq_{fin} A$ let $\mathrm{cl}(X) = \{a \in A : d(X \cup \{a\}) = d(X)\}$.

FACT: (Hrushovski) Let $\mathcal{A} = (A; R) \in \bar{\mathcal{C}}$ and $PG(\mathcal{A}) = (A, \mathrm{cl})$. Then $PG(\mathcal{A})$ is a pregeometry and for $X \subseteq_{fin} A$, the dimension of $X$ in the pregeometry is $d(X)$.

# The embeddings

DEFINITION: If $(A; R) \in \bar{\mathcal{C}}$ and $B \subseteq_{fin} A$ write $B \leq A$ to mean $\delta(B) = d(B)$.
(So $\delta(B) \leq \delta(B')$ for all $B \subseteq B' \subseteq_{fin} A$.)
Say that $B$ is self-sufficient in $A$. This can be extended to infinite $B$.
We say that an embedding $f : (A; R) \to (A'; R')$ is self-sufficient if $f(A) \leq A'$.

NOTES:

(1) $(A; R) \in \bar{\mathcal{C}} \Leftrightarrow \emptyset \leq A$.

(2) If $A \leq B \leq C$ then $A \leq C$.

(3) If $X \subseteq B \leq (A; R)$, then $d(X)$ is the same whether computed in $(A; R)$ or in $(B; R[B])$.

(4) If $X \subseteq_{fin} A$ there is $X \subseteq C \subseteq_{fin} A$ with $\delta(C) = d(X)$ and $C \leq A$.

So we have a category $(\bar{\mathcal{C}}, \leq)$ where the maps are self-sufficient embeddings and a functor $PG$ to pregeometries.

# 4. Geometries of the strongly minimal sets

It is easy to see that $(\mathcal{C}_k, \leq)$ has the amalgamation property:

If $f_1 : A \overset{\leq}{\to} B_1$ and $f_2 : A \overset{\leq}{\to} B_2$ are $\leq$-embeddings in $(\mathcal{C}_k, \leq)$, there exist $E \in \mathcal{C}_k$ and $\leq$-embeddings $g_i : B_i \overset{\leq}{\to} E$ with $g_1 \circ f_1 = g_2 \circ f_2$.

We can take $E$ to be the free amalgam of $B_1$ and $B_2$ over $A$: the disjoint union of $B_1$ and $B_2$ over $A$ with relations $R[B_1] \cup R[B_2]$.

The usual Fraïssé-style argument allows us to show that there is a generic structure for $(\mathcal{C}_k, \leq)$, denoted by $\mathcal{M}_k = (M; R) \in \bar{\mathcal{C}}$ :

- $M = \bigcup_{i<\omega} A_i$ where $A_0 \leq A_1 \leq A_2 \leq \ldots$ are in $\mathcal{C}_k$
- if $A \leq A_i$ and $A \leq B \in \mathcal{C}_k$, there is $j \geq i$ and $f : B \overset{\leq}{\to} A_j$ with $f|_A = id$.

These properties determine $\mathcal{M}_k$ up to isomorphism.

# Various facts

(1) $\mathcal{M}_k$ is $\omega$-stable (of Morley rank $\omega$).

(2) ('Collapse') Hrushovski defines subclasses $(\mathcal{C}_k(\mu), \leq)$ of $(\mathcal{C}_k, \leq)$ which are amalgamation classes and whose generic structures $D_k(\mu)$ are strongly minimal (and non-isomorphic for different $\mu$); the dimension function $d$ is given by algebraic closure.

(3) Other variations (with $k = 3$) in Hrushovski's paper give $2^{\aleph_0}$ strongly minimal sets of countably infinite dimension whose pregeometries are non-isomorphic.

The situation appears to be chaotic.

However, Hrushovski asks whether the pregeometries in (3) are locally isomorphic, and whether there is more than one local isomorphism type of geometry of a (countable, saturated) strongly minimal set arising here.

# Local isomorphism types

**Theorem (DE and Marco Ferreira)**

(1) If $k > \ell \geq 3$ then $PG(\mathcal{M}_k)$ and $PG(\mathcal{M}_\ell)$ are not locally isomorphic.
(2) The pregeometries of the strongly minimal sets $D_k(\mu)$ are all isomorphic to $PG(\mathcal{M}_k)$.
(3) The pregeometries of the strongly minimal sets in Fact 3 are locally isomorphic to $PG(\mathcal{M}_3)$.
(4) All other countable, saturated strongly minimal sets in Hrushovski's paper have pregeometries locally isomorphic to $PG(\mathcal{M}_k)$ for some $k$.

(1, 2) are from Marco Ferreira's PhD thesis (for $k = 3$ in (2)); (3,4) are joint work of MF and DE using an argument which gives a different proof of 2.

# Flatness

Hrushovski isolates the following dimension-theoretic property for a pregeometry.

DEFINITION Suppose $(A, \mathrm{cl})$ is a pregeometry with dimension function $d$ and $F_1, \ldots, F_s$ are finite-dimensional closed subsets of $A$. If $\emptyset \neq S \subseteq \{1, \ldots, s\}$ let $F_S = \bigcap_{i \in S} F_i$. We say that $(A, \mathrm{cl})$ is flat if for all such $F_1, \ldots, F_s$:

$$d(\bigcup_{i=1}^{s} F_i) \leq \sum_{\emptyset \neq S} (-1)^{|S|+1} d(F_S).$$

NOTES:

(1) Compare with the inclusion-exclusion principle (eg. if $d(X) = |X|$ for all $X$).

(2) $d(F_1 \cup F_2) \leq d(F_1) + d(F_2) - d(F_1 \cap F_2)$ always holds.

(3) If $(A; R) \in \bar{\mathcal{C}}$ then its pregeometry $PG(A; R)$ is flat (Hrushovski).

# 5. Some results on matroids.

Take $k = \infty$.

The following can be deduced from results in the matroid theory literature.

**Theorem 1**

The matroids of the form $PG(A; R)$ for $(A; R) \in \mathcal{C}$ are the *strict gammoids* (or *cotransversal matroids*).

**Theorem 2**

If $(A, \mathrm{cl})$ is a finite flat pregeometry, then there exists $R \subseteq \mathcal{P}(A)$ such that $(A; R) \in \mathcal{C}$ and $(A, \mathrm{cl}) = PG(A; R)$.

Theorem 1 identifies the finite pregeometries in Hrushovski's construction with a class of matroids studied in the early 1970's. Theorem 2 is a converse to Hrushovski's observation that these pregeometries are flat. Thus these pregeometries are characterised by a property of their dimension function.

# Theorem 1: Transversal matroids

DEFINITION: If $A$ is a finite set and $R$ a set of non-empty subsets of $A$, a transversal of $(A; R)$ is an injective function $t : R \to A$ with $t(r) \in r$ for all $r \in R$. Abusing terminology, we say that the image $t(R)$ is a transversal.

Hall's Marriage Theorem:

$$(A; R) \text{ has a transversal } \Leftrightarrow |\bigcup R'| \geq |R'| \;\; \forall R' \subseteq R.$$

It is easy to show the latter holds iff $(A; R) \in \mathcal{C}$.

THEOREM: (Edmonds and Fulkerson, 1965) Suppose $(A; R) \in \mathcal{C}$. Then the transversals of $(A; R)$ form the bases of a pregeometry on $A$, called the *transversal matroid* of $(A; R)$.

This is not the matroid $PG(A; R)$.

# Theorem 1: Duality

THEOREM: (Whitney, 1935) If $(A, \mathrm{cl})$ is a finite pregeometry, there is a pregeometry $(A, \mathrm{cl}^*)$ whose bases are the complements of the bases of $(A, \mathrm{cl})$. This is called the *dual pregeometry*.

THEOREM: If $(A; R) \in \mathcal{C}$, then $PG(A; R)$ is the dual of the transversal matroid of $(A; R)$.

This can be read off from results of Ingleton-Piff and McDiarmid in the 1970's.

J H Mason (1972) defines a class of matroids known as the *strict gammoids*; Ingleton and Piff show that these are the duals of the transversal matroids. So we get Theorem 1.

# Strict gammoids

GIVEN:
$\Gamma = (A; D)$: finite directed graph; vertices $A$, directed edges $D$.
$B \subseteq A$.

In the strict gammoid on $A$ determined by these, a subset $C \subseteq A$ is independent iff it is *linked* to a subset of $B$: this means that there is a set of disjoint directed paths with the vertices in $C$ as initial nodes and whose terminal nodes are in $B$.

Suppose $(A; R) \in \mathcal{C}$ and $t : R \to A$ is a transversal with image $A \setminus B$. Define a directed graph $\Gamma$ on $A$ with directed edges $\{(t(r), c) : r \in R,\ c \in r,\ c \neq t(r)\}$. Then it can be shown that $PG(A; R)$ is the strict gammoid given by $\Gamma$ and $B$.

# Theorem 2: Mason's $\alpha$-function

DEFINITION: Supopose $(A, \mathrm{cl})$ is a finite pregeometry with dimension function $d$. We define $\alpha(X)$ for $X$ a union of closed sets by the following formula:

$$\alpha(X) = |X| - d(X) - \sum_F \alpha(F)$$

where $F$ ranges over the closed subsets of $A$ which are properly contained in $X$.

REMARK: Think of this as first being defined for closed sets by induction on the dimension.

## Lemma

Suppose $(A, \mathrm{cl})$ is a finite pregeometry and $X \subseteq A$ is a union of closed sets. Let $F_1, \ldots, F_s$ be the closed sets properly contained in $X$. Then

$$-\alpha(X) = d(X) + \sum_{S \neq \emptyset} (-1)^{|S|} d(F_S).$$

# Theorem 2: Mason's theorem

## Theorem

The following are equivalent for a finite matroid $(A, \mathrm{cl})$:

(1) $\alpha(X) \geq 0$ whenever $X \subseteq A$ is a union of closed sets.

(2) There is an $\alpha$-transversal of the closed sets of $(A, \mathrm{cl})$.

(3) There is a set $R$ of non-empty subsets of $A$ such that $(A; R) \in \mathcal{C}$ and $PG(A; R) = (A, \mathrm{cl})$.

Moreover, we can choose $R$ in (3) to be a set of subsets of size $\leq k$ iff $\alpha(F) = 0$ for all closed sets $F$ with $d(F) \geq k$.

The first part here is due to J. H. Mason (1972). Theorem 2 follows from ths and the previous lemma. The 'Moreover' part is DE (2011).

This gives a dimension-theoretic characterization of the pregeometries $PG(\mathcal{C}_k)$.

## Question

If $\mathcal{A} = (A, \mathrm{cl})$ is a flat pregeometry, does there exist $(A; R) \in \bar{\mathcal{C}}$ with $\mathcal{A} = PG(A; R)$?

So this is asking whether Theorem 2 holds when $A$ is infinite.

# Further Questions.

(1) Is the pregeometry of a (countable, infinite dimensional, non-disintegrated) flat strongly minimal set locally isomorphic to $PG(\mathcal{M}_k)$ (for some $k$)?

(1)' What are the countable, infinite dimensional, flat homogeneous geometries with infinitely many points on a line?

(2) Let $(\mathcal{P}_k, \trianglelefteq_k)$ be the image under the forgetful functor $PG$ of $(\mathcal{C}_k, \leq)$ in the category of finite pregeometries and pregeometry embeddings. This is a subcategory which has the amalgamation property. What are the amalgamation subclasses of $(\mathcal{P}_k, \trianglelefteq_k)$?

(3) Is there a more natural hypothesis which leads to flatness?