Jacobians of Shimura curves and Jacquet-Langlands Correspondences

by

David Frederick Helm

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Committee in charge:

Professor Kenneth A. Ribet, Chair
Professor Bernd Sturmfels
Professor Christina Shannon

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Abstract

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Fix a squarefree integer N, with an even number of prime divisors, and let Γ' be a congruence subgroup of level M, where M is prime to N. For each D dividing N with an even number of prime divisors, we consider the Shimura curve $X^D(\Gamma_0(N/D) \cap \Gamma')$ associated to the indefinite quaternion algebra of discriminant D and $\Gamma_0(N/D) \cap \Gamma'$-level structure, and its Jacobian $J^D(\Gamma_0(N/D) \cap \Gamma')$. Let J^D denote the $\frac{N}{D}$-new subvariety of this Jacobian. (This depends on N and Γ', but as we will be considering the varieties J^D for a fixed N and Γ we suppress this from the notation.)

The Jacquet-Langlands correspondence [JaLa] and Faltings' isogeny theorem [Fa] imply that there are Hecke-equivariant isogenies among the various varieties J^D defined above. However, since the isomorphism of Jacquet-Langlands is noncanonical, this perspective gives no information about the isogenies so obtained beyond their existence. In this paper, we study maps between the varieties J^D in terms of the maps they induce on the character groups of the tori corresponding to the mod p reductions of these varieties for p dividing N. Our characterization of such maps in these terms allows us to describe the possible kernels of maps from J^D to $J^{D'}$, for D dividing D', up to support on a small finite set of maximal ideals of the Hecke algebra. These descriptions are in terms of modules over a Hecke algebra which are explicitly computable. Among other things, this allows us to compute the Tate modules $T_{\mathfrak{m}}J^D$ of J^D at all non-Eisenstein $\mathfrak{m}$ of residue characteristic $l > 3$. These computations have implications for the multiplicities of irreducible Galois representations in the torsion of Jacobians of Shimura curves; one such consequence is a "multiplicity one" result for Jacobians of Shimura curves. As a corollary of this result, we are able to prove that certain quotients of Hecke algebras are Gorenstein, a fact which has applications to proving the modularity of Galois representations [Kh].

To my fianceé, Jennifer Harkness

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Introduction

Let D be a squarefree integer with an even number of prime divisors, and let p and q be distinct primes not dividing D. Let B and B' denote the indefinite quaternion algebras over $\mathbb{Q}$ of discriminants D and Dpq, respectively. The Jacquet-Langlands correspondence ([JaLa], see also Theorem 1.4.1 of this dissertation) gives a noncanonical isomorphism between the space of modular forms for the quaternion algebra B', and the space of newforms for the quaternion algebra B with $\Gamma_0(pq)$-level structure. (More generally, if Γ is a congruence subgroup of level prime to Dpq, then the Jacquet-Langlands correspondence gives an isomorphism between the space of modular forms for B' with Γ-level structure and the space of pq-new forms for B with $\Gamma_0(pq) \cap \Gamma$-level structure.)

The existence of these isomorphisms has consequences for the geometry of modular curves and Shimura curves. In particular, if we let $X^D(\Gamma_0(pq) \cap \Gamma)$ denote the Shimura curve associated to the quaternion algebra B with $\Gamma_0(pq) \cap \Gamma$-level structure, and $X^{Dpq}(\Gamma)$ denote the Shimura curve associated to B' with Γ-level structure, we may consider their Jacobians $J^D(\Gamma_0(pq) \cap \Gamma)$ and $J^{Dpq}(\Gamma)$. Then the Jacquet-Langlands correspondence, together with Eichler-Shimura theory, tells us that the Tate modules of $J^{Dpq}(\Gamma)$ and the Tate modules of the pq-new subvariety of $J^D(\Gamma_0(pq) \cap \Gamma)$ become isomorphic as Galois modules after tensoring with $\mathbb{Q}$. It follows, for instance from [Ri1] or from Faltings' isogeny theorem [Fa], that J^{Dpq} is isogenous to the pq-new subvariety of $J^D(\Gamma_0(pq) \cap \Gamma)$.

Actually producing an isogeny between the two varieties, on the other hand, seems much more difficult. The above argument, while enough to establish the existence of such an isogeny, provides no hint as to how an explicit map might be constructed.

The first results to give more detailed information about the relationship between the two varieties are due to Ribet [Ri5], who established an isomorphism between the character group of the torus associated to the mod p reduction of $J^{Dpq}(\Gamma)$ and a certain subgroup of the character group associated to the mod q reduction of $J^D(\Gamma)$. Using this isomorphism, he later showed [Ri6] that even in the case where $D = 1$ and Γ is trivial, $J_0(pq)_{pq\text{-new}}$ and $J^{pq}(1)$ could have decidedly different structures. In particular, he constructed examples of maximal ideals $\mathfrak{m}$ of the Hecke algebra $\mathbb{T}$ for which $J_0(pq)_{pq\text{-new}}[\mathfrak{m}]$ has dimension two yet $J^{pq}(1)[\mathfrak{m}]$ has dimension four. This imposes conditions on the structure of any isogeny between the two varieties. More precisely, Ribet's argument showed that the dimension of $J^{pq}(1)[\mathfrak{m}]$ depended on the behavior of the mod l Galois representation associated to $\mathfrak{m}$ when restricted to decomposition groups at p and q. Thus any isogeny between the two varieties necessarily encodes all of this arithmetic data.

This dissertation has two goals. First, we give a summary of the arithmetic of modular curves and Shimura curves, with an eye towards explaining what is known about the isogenies discussed above, and developing the tools to study them more closely. This is the focus of chapter 1. We begin with the definitions of the modular and Shimura curves and their Jacobians, the Hecke algebras that act on them, and their old and new subvarieties. Our presentation here is idiosyncratic, for two reasons. The first is that we develop the theory of Shimura curves and modular curves simultaneously. Since the construction of Hecke operators for Shimura curves is similar to the construction for modular curves, we only give the construction for Shimura curves here, and leave it to the reader to determine the analogous construction for modular curves. This is a pattern we follow all throughout chapter 1. We also avoid giving here a theory of q-expansions, as such an exposition would take us far afield and for the most part we can do without this theory. (A notable exception is Proposition 1.4.2). We have thus tried wherever possible to give proofs which do not depend on the q-expansion principle, at the expense of making the argument less standard.

We then develop aspects of the arithmetic theory of modular curves and Shimura curves that are necessary to understand both the Jacquet-Langlands correspondence and the techniques we apply to study the isogenies that arise from it. In particular we develop the theory of level raising, which we will make extensive use of later in section 2.5, and the reduction theory of Jacobians of Shimura curves at certain primes dividing the level. The latter in particular is at the heart of our study of these isogenies.

Our ultimate goal is to give an explicit characterization of the kernels of the various isogenies between Jacobians of Shimura curves that are known to exist because of the Jacquet-Langlands correspondence. This is the subject of chapter 2, in which we approach this problem in a way first suggested by Mazur. Fix a squarefree integer N, prime to the level M of Γ, and for each D dividing N that has an even number of prime divisors, let $J^D = J^D(\Gamma_0(N/D) \cap \Gamma)_{\frac{N}{D}\text{-new}}$. Choose a particular D as above, and let $\mathbb{T}$ be the subalgebra of the endomorphism ring of J^D generated by the Hecke operators T_l for l prime, and the diamond bracket operators $\langle d \rangle$. Repeated applications of the Jacquet-Langlands correspondence show that $\mathbb{T}$ is independent (up to canonical isomorphism) of the choice of J^D used to define it, and therefore acts simultaneously on all of the varieties J^D. Moreover, the abelian varieties J^D are all $\mathbb{T}$-equivariantly isogenous. Rather than searching for a specific isogeny $J^D \to J^{D'}$, we attempt to classify $\mathbb{T}$-equivariant maps between the two varieties, and determine the kernels of all such maps. We do this via a systematic study of the isogeny class of J^1; in particular, we show that if one is willing to work "up to support on S", where S is the set of Eisenstein primes of $\mathbb{T}$, that one can classify all abelian varieties $\mathbb{T}$-equivariantly isogenous to J^1 in terms of isomorphism classes of $\mathbb{T}$-modules. Specifically, to any abelian variety J isogenous to J^1 one can associate a certain $\mathbb{T}$-module $[J]$ which is free over $\mathbb{Z}$ and satisfies $[J] \otimes \mathbb{Q} \cong \mathbb{T} \otimes \mathbb{Q}$. (Such modules were first introduced by Mazur [Ma], who called them "rank one" modules.) After we formally invert isogenies supported on S, and maps of $\mathbb{T}$-modules whose cokernels are supported on S (in a sense that will be made precise in section 2.1), this functor becomes an antiequivalence of categories (Theorem 2.2.11). Moreover, if J and J' are isogenous to J^1, knowing the isomorphism classes of $[J]$ and $[J']$ as $\mathbb{T}$-modules is often enough to determine the kernels of maps $J \to J'$ up to support on S (Proposition 2.2.12). Thus computing $[J]$ and $[J']$ allows us to construct the isomorphism class of J' in terms of the isomorphism class of J, up to isogeny supported on S. In particular one can express the Tate modules of J' at maximal ideals outside S in terms of the Tate modules of J and the modules $[J]$ and $[J']$.

Moreover, we obtain a relationship between the character groups $X_p(J)$ of the tori coming from the mod p reductions of J for p dividing N, and the module $[J]$, for any J isogenous to J^1 (Proposition 2.2.14). In section 2.3 we establish a relationship among the character groups of J^D as D varies. We exploit these two results to compute the modules $[J^D]$ explicitly. The key result we obtain is Theorem 2.4.1, in which we obtain $[J^D]$ for $D = N$ in terms of a certain module of maps between character groups.

Sections 2.4 and 2.5 are devoted to the proof of Theorem 2.4.1. In section 2.4, we introduce a condition, called *controllability*, associated to a prime p dividing N and a maximal ideal $\mathfrak{m}$ of $\mathbb{T}$. Controllability of $\mathfrak{m}$ at p implies that the character group of a certain abelian variety isogenous to J^D becomes free of rank one after completing at $\mathfrak{m}$ (c.f. Lemma 2.4.5); in section 2.4 we exploit this fact to show that Theorem 2.4.1 holds locally at any maximal ideal $\mathfrak{m}$ which is controllable at some prime dividing D.

It remains to handle those maximal ideals that are not controllable at any primes dividing D, which we do in section 2.5. Although such ideals are not at all common, they are much more difficult to handle, and the arguments presented here are of a very technical sort. (The reader who is primarily interested in our results, or who is willing to take this extreme case of Theorem 2.4.1 on faith, is encouraged to read section 2.6 first; once Theorem 2.4.1 is established we have no further need of the results of section 2.5.) Our basic strategy for dealing with these maximal ideals is a level-raising argument. In particular, we fix such a maximal ideal $\mathfrak{m}$, and introduce two primes q_1 and q_2 to the level, in such a way that $\mathfrak{m}$ is controllable at both q_1 and q_2. We establish an analogue of Theorem 2.4.1 in this new setting, using controllability at q_1 and q_2, and then show that it implies the desired result.

In section 2.6 we explore the consequences of Theorem 2.4.1. In particular, we apply Theorem 2.4.1 along with Proposition 2.3.6 to obtain several expressions for $[J^D]$ in terms of $[J^{D'}]$ whenever D' divides D (Theorem 2.6.5, Theorem 2.6.7, and Corollary 2.6.15). In particular, if p is a divisor of D', we find that the $\mathbb{T}$-modules $\mathrm{Hom}(J^{D'}, J^D)$ and $\mathrm{Hom}_{\mathbb{T}}(X_p(J^D), X_p(J^{D'}))$ are isomorphic up to support on S, via the natural map which takes a morphism of varieties to the map it induces on character groups. Via Proposition 2.2.12, this allows us to determine the kernels of all morphisms $J^D \to J^{D'}$, up to support on S. We give several different descriptions of these kernels (Corollaries 2.6.6, 2.6.8, and 2.6.16) of varying degrees of explicitness. The latter two expressions, for instance, are in terms of character groups which can be computed algorithmically [Ko]. The upshot of this is that given the isomorphism class of J^1, one can explicitly compute the isomorphism class of J^D for any D, up to support on S.

This recipe for constructing J^D from J^1 allows us to compute many invariants of J^D in terms of J^1. In particular one can compute the $\mathfrak{m}$-adic Tate modules of J^D in this fashion, for any $\mathfrak{m}$ outside S. In particular one obtains expressions for the dimension of $J^D[\mathfrak{m}]$ in this manner. In this fashion, we obtain a bound on the dimension of $J^D[\mathfrak{m}]$ in terms of the number of primes dividing D at which $\mathfrak{m}$ is not controllable, a significant strengthening of an earlier result of L. Yang [Ya]. An important special case of this bound is when $\mathfrak{m}$ is controllable at every prime and $J^1[\mathfrak{m}]$ has dimension two; in this case $J^D[\mathfrak{m}]$ has dimension two as well—that is, we have "multiplicity one" for J^D at $\mathfrak{m}$. This implies that certain natural quotients of Hecke algebras are Gorenstein, a fact used by Khare [Kh] to construct isomorphisms between Hecke rings and deformation rings.

Chapter 1

Shimura curves and their Jacobians

1.1 Modular curves and Shimura curves

We begin by introducing the primary objects of interest for this dissertation, namely the modular curves $X(\Gamma)$ and the Shimura curves $X^D(\Gamma)$.

Definition 1.1.1 *Let N be a positive integer and H a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$. The group $\Gamma_H(N)$ is the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ consisting of matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for which c is divisible by N and a and d lie in H modulo N.*

Definition 1.1.2 *A* congruence subgroup *of level N is a subgroup of the form $\Gamma_H(N)$ for some positive integer N and subgroup H of $(\mathbb{Z}/N\mathbb{Z})^\times$. Two congruence subgroups will be of particular importance in this exposition. We denote by $\Gamma_0(N)$ the congruence subgroup $\Gamma_H(N)$ with $H = (\mathbb{Z}/N\mathbb{Z})^\times$, and $\Gamma_1(N)$ the congruence subgroup $\Gamma_H(N)$ with H trivial.*

Let Γ be a congruence subgroup of some level N. We can view Γ as a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, and the latter acts on the complex upper half plane $\mathcal{H}$ by fractional linear transformations. The quotient of $\mathcal{H}$ by the action of Γ is a smooth quasi-projective curve over $\mathbb{C}$, which we denote by $Y(\Gamma)$.

We can compactify $Y(\Gamma)$ as follows. Let $\mathcal{H}^* \subset \mathbb{P}^1(\mathbb{C})$ be given by $\mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$. Then the action of Γ preserves $\mathcal{H}^*$, and the quotient of $\mathcal{H}^*$ by Γ is a projective curve $X(\Gamma)$, which is defined over $\mathbb{C}$ and contains $Y(\Gamma)$ in a natural way (see chapter 1 of [Sh] for details.)

The curves $X(\Gamma)$ and $Y(\Gamma)$ possess canonical models as moduli spaces of elliptic curves, which we now describe. Let E be an elliptic curve, and Γ the congruence subgroup corresponding to an integer N and a given subgroup H of $(\mathbb{Z}/N\mathbb{Z})^\times$. Then H acts on the subgroup $E[N]$ of N-torsion points of E by multiplication. This action clearly preserves the points of order N. We define a Γ-level structure on E to be an H-orbit of points of order N in $E[N]$. (In particular, a $\Gamma_1(N)$-level structure on E is a choice of a point of order N on E. On the other hand, fixing a $\Gamma_0(N)$-level structure on E is equivalent to fixing a cyclic subgroup of E of order N.)

The coarse moduli space classifying elliptic curves with Γ-level structure is a regular scheme of relative dimension 1 over $\mathbb{Z}[\frac{1}{N}]$. By abuse of notation, we denote this scheme by $Y(\Gamma)$; over the complex numbers this agrees with our original definition of $Y(\Gamma)$.

We will also need a moduli interpretation for $X(\Gamma)$, as later we will later define morphisms of these curves in terms of the moduli. Although the exact definition of the moduli problem will not concern us, we will give it in detail for completeness' sake.

Finding an analogous moduli interpretation for $X(\Gamma)$ amounts to enlarging our moduli problem to consider certain "degenerate" elliptic curves, which will correspond to the points in the complement of $Y(\Gamma)$ in $X(\Gamma)$. To make this precise, we introduce the Néron polygons:

Definition 1.1.3 *A Néron n-gon, for n a non-negative integer, is a curve C over an algebraically closed field k, such that:*

1. *If $n = 1$, then C is isomorphic to $\mathbb{P}^1$ with two points identified.*
2. *If $n = 2$, then C is isomorphic to two copies of $\mathbb{P}^1$, intersecting transversally at two distinct points.*
3. *For $n > 2$, C is a connected curve with n irreducible components, each of which is isomorphic to $\mathbb{P}^1$, such that any irreducible component intersects exactly two other irreducible components, in one point each, and these intersections are transverse.*

The Néron polygons classify possible degenerations of elliptic curves over algebraically closed fields. Using this notion, Deligne-Rapoport [DeRa], define a "generalized elliptic curve" over a scheme S to be a family of curves E over S whose geometric fibers are either elliptic curves or Néron polygons, together with a section Id of E^{reg} over S and a morphism $+ : E^{\mathrm{reg}} \times E \to E$. (Here E^{reg} is the union of the smooth points in the fibers of E over S.) We also require that the restriction to $E^{\mathrm{reg}} \times E^{\mathrm{reg}}$ of the map $+$ make E^{reg} into a group scheme over S, with identity given by Id, that $+$ defines an action of E^{reg} on E, and that on singular fibers E_s of E the action of an element of E^{reg} induces a cyclic permutation of the irreducible components. (This guarantees that $+$ will restrict to the usual addition law on those fibers of E which are actual elliptic curves, and that Id restricts to the identity on such fibers.)

A Γ-level structure for a generalized elliptic curve E is defined in the obvious way. With these definitions, we take $X(\Gamma)$ to be the coarse moduli space of generalized elliptic curves with Γ-level structure; it is a regular scheme over $\mathbb{Z}[\frac{1}{N}]$.

In addition to the modular curves, we will also be considering the closely related family of Shimura curves. These come from discrete subgroups of maximal orders in indefinite quaternion algebras, in an analogous fashion to the way in which modular curves come from discrete subgroups of $\mathrm{SL}_2(\mathbb{Z})$. In particular, let $D > 1$ be a squarefree integer, with even number of prime divisors, and B a quaternion algebra over $\mathbb{Q}$ of discriminant D. The choice of D determines B up to isomorphism. Fix a maximal order $\mathcal{O}$ in B. Since B has discriminant D, $\mathcal{O} \otimes \mathbb{R}$ is isomorphic to $M_2(\mathbb{R})$, and $\mathcal{O} \otimes \mathbb{Z}_p$ is isomorphic to $M_2(\mathbb{Z}_p)$ for all p not dividing D. We fix a particular choice of isomorphism as above for $\mathbb{R}$ and each prime p dividing D. Finally, let $\mathcal{O}^{1,\times}$ denote the group of units of $\mathcal{O}$ with reduced norm 1.

Once we have fixed these choices, a congruence subgroup Γ of level N, where N is prime to D, determines a subgroup Γ_D of $\mathcal{O}^{1,\times}$ as follows. If Γ comes from a subgroup H of $(\mathbb{Z}/N\mathbb{Z})^\times$, let $\overline{\Gamma}$ be the subgroup of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ consisting of upper triangular matrices in $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ whose diagonal entries lie in H. Then Γ is the preimage of $\overline{\Gamma}$ under the natural map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. The isomorphisms we fixed above determine an isomorphism: $\mathcal{O}/N\mathcal{O} \cong M_2(\mathbb{Z}/N\mathbb{Z})$; the image of $\mathcal{O}^{1,\times}$ under this map is $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. We thus obtain a surjection $\mathcal{O}^{1,\times} \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$; we define Γ_D to be the preimage of $\overline{\Gamma}$ under this surjection.

The isomorphism $\mathcal{O} \otimes \mathbb{R} \to M_2(\mathbb{R})$ induces an injection of $\mathcal{O}^{1,\times}$ into $\mathrm{SL}_2(\mathbb{R})$. Since $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H}$, this induces an action of Γ_D on $\mathcal{H}$. We let $X^D(\Gamma)$ denote the quotient of $\mathcal{H}$ by this action of Γ_D. As was the case with modular curves, this is a smooth algebraic curve over $\mathbb{C}$. Unlike the modular curves, however, this quotient is *compact*; there is no need to add additional points to compactify. Moreover, the resulting curve is independent up to isomorphism of the choice of maximal order $\mathcal{O}$ and the various isomorphisms we fixed above.

The curve $X^D(\Gamma)$ can be thought of as a coarse moduli space for a certain class of abelian varieties which we now describe. A "fake elliptic curve" for $\mathcal{O}$ is an abelian surface A, together with a faithful action of $\mathcal{O}$; that is, an embedding of $\mathcal{O}$ in $\mathrm{End}(A)$. We require that the trace of an element $\tau \in \mathcal{O}$, as an endomorphism of the tangent space of A at the identity, be equal to the reduced trace of τ as an element of $\mathcal{O}$.

For N prime to D, the isomorphisms we fixed above define an isomorphism $\mathcal{O}/N\mathcal{O} \to M_2(\mathbb{Z}/N\mathbb{Z})$. Let π_N denote an element of $\mathcal{O}/N\mathcal{O}$ mapping to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ under this isomorphism. We will use this element to help us define level structures on A.

A Γ-level structure on A, for Γ a congruence subgroup of level N prime to D and $H \subset (\mathbb{Z}/N\mathbb{Z})^\times$, is an H-orbit of points of order N in $\pi_N(A[N])$. Observe that in particular there is a bijection between $\Gamma_0(N)$-level structures on A and subgroups of $A[N]$, of order N^2, which are stable under the action of $\mathcal{O}$.

Remark 1.1.4 The introduction of π_N in this context may seem somewhat odd. It appears here in order to make the resulting moduli problem look more like the corresponding moduli problem for elliptic curves. In particular, for an elliptic curve E, the set $E[N]$ of N-torsion points has order N^2. By contrast, if E is a "fake elliptic curve", then $E[N]$ is a free module over $\mathcal{O}/N\mathcal{O}$, and hence has order N^4. In particular, if we tried to define a $\Gamma_1(N)$-level structure on E as an arbitrary point of order N on E, then the degree of $X^D(\Gamma_1(N))$ over $X^D(\mathrm{SL}_2(\mathbb{Z}))$ would be much larger than the degree of $X(\Gamma_1(N))$ over $X(\mathrm{SL}_2(\mathbb{Z}))$. By contrast, introducing the projection π_N cuts $E[N]$ down to a subgroup of the right size; as a result the two degrees given above agree.

Note that if one takes $D = 1$ in either construction described above (so that $B = M_2(\mathbb{Q})$), the curve one obtains is naturally isomorphic to the modular curve $Y(\Gamma)$, as given a "fake elliptic curve" A for $M_2(\mathbb{Z})$ with a Γ-level structure, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A$ is an elliptic curve with a Γ-level structure. Conversely, given an elliptic curve E with a Γ-level structure P, E^2 has a natural action of $M_2(\mathbb{Z})$, and the set $\{(p, 0) : p \in P\}$ is a Γ-level structure on E^2. (This bijection is further evidence that the introduction of π_N above is the right thing to do.) Thus the modular curves are in some sense the $D = 1$ analogues of Shimura curves. However, since the Shimura curves for $D > 1$ are compact, it is generally more useful to consider the compactified curves $X(\Gamma)$ as "Shimura curves for $D = 1$". We thus take $X^1(\Gamma) = X(\Gamma)$ for any congruence subgroup Γ.

1.2 Jacobians of Shimura curves and Hecke operators

For a particular discriminant D and congruence subgroup Γ of level prime to D, let $J^D(\Gamma)$ denote the Jacobian variety of the curve $X^D(\Gamma)$.

Recall that for a smooth curve C, the Jacobian variety $J(C)$ is an abelian variety whose points over a field k are in canonical bijection with the set of linear equivalence classes of degree zero divisors on C over k. If we have a morphism $f : C \to C'$, then f induces a morphism $f_* : J(C) \to J(C')$ such that

$$f(\Sigma n_i p_i) = \Sigma n_i f(p_i),$$

where $\Sigma n_i p_i$ is a divisor of degree zero on C. We also have a morphism $f^* : J(C') \to J(C)$ given by pullback on Cartier divisors, i.e. such that

$$f^*(p_i) = \Sigma_{x \in f^{-1}(p_i)} e_i(x) d_i(x) x,$$

where e_i is the ramification index of x over p_i and d_i is the degree of the field extension $k(x)/k(p_i)$.

Fix a squarefree integer D which is a product of an even number of primes, congruence subgroup Γ of level N prime to D, and an integer l prime to ND. Then $X^D(\Gamma_0(l) \cap \Gamma)$ can be thought of as the moduli space of "fake elliptic curves" with an $\mathcal{O}$-stable subgroup of order l^2 and a Γ-level structure; that is, as the space of triples (E, C, P) where E is a "fake elliptic curve", C is a $\mathcal{O}$-stable subgroup of $E[l]$, and P is a Γ-level structure on E. (If $D = 1$ then E is a generalized elliptic curve and C is a cyclic subgroup of order l instead. In general, for these constructions, we will give a definition that works for $D > 1$. Making these definitions work for $D = 1$ amounts to replacing "fake elliptic curves" with generalized elliptic curves, and $\mathcal{O}$-stable subgroups of order l^2 with cyclic subgroups of order l. Rather than state this repeatedly in what follows, we will simply assume that $D > 1$ and leave the analogous constructions for $D = 1$ to the reader.)

The moduli problems then gives us two natural "degeneracy" maps α_l and β_l from $X^D(\Gamma_0(l) \cap \Gamma)$ to $X^D(\Gamma)$. In particular, one can either forget C or quotient by C. That is, we define α_l by $\alpha_l(E, C, P) = (E, P)$, and β_l by $\beta_l(E, C, P) = (E/C, P/C)$. We then define the lth Hecke operator T_l on $J^D(\Gamma)$ to be the endomorphism $(\beta_l)_*(\alpha_l)^*$. On divisors we have

$$T_l(E, P) = \Sigma_{C \subset E}(E/C, P/C),$$

where the sum here is over $\mathcal{O}$-stable subgroups C of E, of order l^2.

We may also define Hecke operators T_l for l dividing ND, but the definition is more complicated. We first consider the case in which l divides N. In this setting, let $X^D(l, \Gamma)$ denote the moduli space of triples (E, C, P) where E is a "fake elliptic curve", P is a Γ-level structure, and C is a $\mathcal{O}$-stable subgroup of E of order l^2 which has trivial intersection with P. As above, we have two natural maps α_l,β_l from $X^D(l,\Gamma)$ to $X^D(\Gamma)$, and we define T_l to be $(\beta_l)_*(\alpha_l)^*$ on $J^D(\Gamma)$. On divisors we have

$$T_l(E,P) = \Sigma_{C \subset E; C \cap P = \emptyset}(E/C, P/C).$$

When l divides D there is a unique maximal ideal π_l of $\mathcal{O}$ such that $\pi_l^2 = (l)$. This defines a map $t_l : X^D(\Gamma) \to X^D(\Gamma)$ by $t_l(E,P) = (E/E[\pi_l], P/E[\pi_l])$. We define T_l on $J^D(\Gamma)$ by $T_l = (t_l)_*$.

Finally, we define diamond bracket operators $\langle d \rangle$, for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ on $J^D(\Gamma)$. Observe that for a "fake elliptic curve" E, $(\mathbb{Z}/N\mathbb{Z})^\times$ acts on $E[N]$ by multiplication. We thus have an action of $(\mathbb{Z}/N\mathbb{Z})^\times$ on Γ-level structures on E. We let $\langle d \rangle(E,P) = (E, dP)$, both as an endomorphism of $X^D(\Gamma)$ and $J^D(\Gamma)$. Observe that $\langle d \rangle$ is the identity if $d \in H$, so we can speak of the diamond bracket operators for $d \in (\mathbb{Z}/N\mathbb{Z})^\times/H$. It will occasionally be advantageous to do so; for instance if $\Gamma = \Gamma_0(l) \cap \Gamma'$, and Γ' has level N prime to l, then H contains the kernel of the map $(\mathbb{Z}/Nl\mathbb{Z})^\times \to (\mathbb{Z}/N\mathbb{Z})^\times$, so we can speak of operators $\langle d \rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. In particular the operator $\langle l \rangle$ makes sense from this perspective, even though l is not an element of $(\mathbb{Z}/Nl\mathbb{Z})^\times$.

One can easily verify that these operators all commute with each other. We let $\mathbb{T}^D(\Gamma)$ denote the subalgebra of $\mathrm{End}(J^D(\Gamma))$ generated by the operators T_l for all prime l, and the diamond bracket operators. Since $\mathrm{End}(J^D(\Gamma))$ is a finite, flat $\mathbb{Z}$-algebra, $\mathbb{T}$ is as well.

All of the above operations are defined over $\mathbb{Z}[\frac{1}{N}]$, since the maps α_l and β_l are. We will also need to define one more class of operators on $J^D(\Gamma)$, namely the Atkin-Lehner operators w_n for n dividing N. These depend on a choice of Nth root of unity ζ_N, and are defined only over $\mathbb{Z}[\frac{1}{N}, \zeta_N]$.

In order to construct these operators, we must first introduce the Weil pairing on a "fake elliptic curve" E. On any such E there is a canonical principal polarization θ_E such that the Rosati involution of $\mathrm{End}(E)$ induced by θ_E preserves $\mathcal{O} \subset \mathrm{End}(E)$ and induces the canonical involution $x \to \overline{x}$ of $\mathcal{O}$.

This principal polarization induces a skew-symmetric, bilinear, perfect pairing $\langle -, - \rangle : E[N] \times E[N] \to \mu_N$, where μ_N is the group of Nth roots of unity. The adjoint of an element of x of $\mathcal{O}$ under this pairing is $\overline{x}$. Moreover, the pairing is Galois-equivariant, in the sense that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $\langle \sigma(p), \sigma(q) \rangle = \sigma(\langle p, q \rangle)$.

Now suppose we have a "fake elliptic curve" E with Γ-level structure P. Choosing a point p in P, and taking its orbit under the action of $\mathcal{O}$, we obtain an $\mathcal{O}$-stable subgroup of $E[N]$, of order N^2, which is independent of the choice of p. This subgroup has a unique $\mathcal{O}$-stable subgroup C of order n^2.

We wish to define a Γ-level structure on $E' = E/C$. Recall that this is an H-orbit of points of order N in $\pi_N(E'[N])$, where π_N is any element of $\mathcal{O}$ such that π_N maps to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ under the isomorphism $\mathcal{O}/N\mathcal{O} \to M_2(\mathbb{Z}/N\mathbb{Z})$. Fix a point p in P, and choose a point q of $\pi_N(E[N])$ such that $nq = np$, and $\langle p, q \rangle = \zeta_N^{\frac{N}{n}}$. The properties of the Weil pairing described above make it easy to see that q lies outside of C, and is well-defined up to an element of $\pi_N(C)$. Thus q maps to a point of order N in E/C which depends on p but not q. Moreover, replacing p by dp for some $p \in H$ amounts to replacing q by $d^{-1}q$. Thus the

H-orbit Q of q in E/C is independent of the choice of p and q, and hence a well-defined Γ-level structure on E'.

We define w_n on $X^D(\Gamma)$ to be the map sending (E, P) to the pair (E', Q) constructed above. This induces an endomorphism $(w_n)_*$ of $J^D(\Gamma)$, which (in an abuse of notation) we also denote by w_n. Although these operators are not necessarily defined over $\mathbb{Q}$, they do have nice properties with respect to the action of $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$. In particular suppose σ is an element of $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ such that $\sigma\zeta_N = \zeta_N^d$. Then $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, and $\sigma w_N(x) = \langle d\rangle w_n \sigma(x)$ for any point x on $X^D(\Gamma)$.

Example 1.2.1 The two main special cases of this construction that interest us are the operator w_p on $X^D(\Gamma_0(p) \cap \Gamma)$, where Γ is of level N prime to p, and the operator w_N on $X^D(\Gamma)$. In the first case, a point on $X^D(\Gamma_0(p) \cap \Gamma)$ is a triple (E, C, P), where E is a "fake elliptic curve", C is a $\mathcal{O}$-stable subgroup of order p^2, and P is a Γ-level structure. We have $w_p(E, C, P) = (E/C, E[p]/C, f(P))$, where f is the natural map $E \to E/C$. Note that in this special case, w_p is actually defined over $\mathbb{Q}$, and independent of any choice of root of unity, and that $w_p^2 = \langle p\rangle$.

On the other hand, in the second case we have $w_N(E, P) = (E/C_P, f(Q))$, where C_P is the $\mathcal{O}$-stable subgroup of E generated by a representative of P, f is the natural map $E \to E/C_P$, and Q is the set of points q of order N in $\pi_N(E[N])$ such that $\langle p, q\rangle = \zeta_N$ for some p in P.

It is also straightforward to verify the following interaction between w_N and the degeneracy maps described above:

Lemma 1.2.2 *Let l be a prime, not dividing D, and let α_l and β_l be the maps $X^D(\Gamma_0(l) \cap \Gamma) \to X^D(\Gamma)$ defined above if l does not divide N, or the maps $X^D(l, \Gamma) \to X^D(\Gamma)$ defined above if l does divide N. Then $\alpha_l w_{Nl} = \langle l\rangle w_N \beta_l$, and $\beta_l w_{Nl} = \langle l\rangle w_N \alpha_l$.*

This calculation allows us to determine the effect of conjugation by w_N on the Hecke operators constructed above. Conjugation by w_N can be thought of automorphism of $\mathrm{End}(J^D(\Gamma))$; this automorphism will turn out to be closely related to another automorphism of $J^D(\Gamma)$—the Rosati involution. Recall that for any curve C, the Jacobian $J(C)$ is equipped with a canonical principal polarization θ, that is, an isomorphism $\theta : J(C) \to J(C)^\vee$, where $J(C)^\vee$ is the dual abelian variety of $J(C)$. The Rosati involution of $\mathrm{End}(J(C))$ takes an endomorphism f of $J(C)$ to the endomorphism $f^\circ : \theta^{-1} f^\vee \theta$ of $J(C)$. Moreover, if $f : C \to C'$ is a morphism of curves, and θ and θ' are the canonical polarizations of $J(C)$ and $J(C')$, then we have $\theta' f_* = (f^*)^\vee \theta$. We can use this latter fact to compute the Rosati involution of the various Hecke operators constructed above.

Proposition 1.2.3 *Let x be an element of $\mathbb{T}^D(\Gamma)$. Then $x^\circ = w_N x w_N^{-1}$. In particular, we have $\langle d\rangle^\circ = \langle d\rangle^{-1}$ and $T_l^\circ = \langle l\rangle^{-1} T_l$ for all l prime to N.*

Proof. It suffices to show the first statement for $x = \langle d\rangle$ and $x = T_l$. The automorphism $\langle d\rangle$ is induced by an automorphism of the curve $X^D(\Gamma)$; it follows from the above discussion that we have $\langle d\rangle^\circ = \langle d\rangle^{-1}$. Moreover, our construction of w_N implies immediately that $w_N \langle d\rangle w_N^{-1} = \langle d\rangle^{-1}$, since $\langle p, q\rangle = \langle dp, d^{-1}q\rangle$ for any N-torsion points p and q on a "fake elliptic curve" E. Thus both statements of the proposition hold for $\langle d\rangle$.

If l is a prime dividing D, T_l is again induced by an automorphism of the curve, so $T_l^\circ = T_l^{-1}$. We also have the identity $T_l^2 = \langle l \rangle$, so $T_l^\circ = \langle l \rangle^{-1} T_l$. One checks immediately that $w_N T_l w_N^{-1} = \langle l \rangle^{-1} T_l$.

Suppose finally that l does not divide D. Then T_l is given by $(\beta_l)_*(\alpha_l)^*$, where β_l and α_l are the degeneracy maps constructed above. By the discussion immediately preceding the proposition, it follows that $T_l^\circ = (\alpha_l)_*(\beta_l)^*$. On the other hand, $w_N T_l w_N^{-1}$ $= w_N(\beta_l)_*(\alpha_l)^* w_N^{-1}$. The latter is equal to $(w_N\beta_l)_*(w_N\alpha_l)^*$. By the previous lemma, this is equal to $\langle l \rangle^{-1}(\alpha_l w_{Nl})_*(\langle l \rangle^{-1}\beta_l w_{Nl})^*$. Rearranging terms, and observing that $w_{Nl}^* = w_{Nl}^{-1}$ and $\langle l \rangle$ commutes with α_l and β_l, we find that this is $(\alpha_l)_*(\beta_l)^* = T_l^\circ$, as required. The last statement now follows since if l does not divide N, then $(\alpha_l)_*(\beta_l)^* = \langle l \rangle^{-1}(\beta_l)_*(\alpha_l)^*$, as one can easily check in terms of the expressions for these maps on divisors. $\square$

Corollary 1.2.4 *There is a $\mathbb{T}^D(\Gamma)$-equivariant isomorphism $\psi : J^D(\Gamma) \to J^D(\Gamma)^\vee$, defined over a finite extension of $\mathbb{Q}$, such that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and x in $J^D(\Gamma)$, $\sigma\psi(x) = \chi(\sigma)\psi\sigma(x)$, where $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to (\mathbb{T}^D(\Gamma))^\times$ is the character such that if $\sigma\zeta_N = \zeta_N^d$ for a primitive Nth root of unity ζ_N, then $\chi(\sigma) = \langle d \rangle$.*

Proof. Let θ be the canonical polarization on $J^D(\Gamma)$. The above proposition implies that $\psi = \theta w_N$ is $\mathbb{T}^D(\Gamma)$ equivariant. The identity $\sigma\psi = \chi(\sigma)\psi\sigma$ now follows from the corresponding identity for w_N. $\square$

To conclude this section, we introduce the space $S_2^D(\Gamma)$ of quaternionic cusp forms of weight 2, discriminant D, and level Γ, as the space of holomorphic differentials on $X^D(\Gamma)(\mathbb{C})$. The space $S_2^D(\Gamma)$ is thus naturally isomorphic to the (complex) tangent space of $J^D(\Gamma)$ at the identity. The action of $\mathbb{T}^D(\Gamma)$ on $J^D(\Gamma)$ thus induces an action on $S_2^D(\Gamma)$ as well. (This action can be defined more directly on $S_2^D(\Gamma)$ but it will not be necessary to do so for our purposes.)

For any complex abelian variety A, the tangent space of $A^\vee$ at the identity is (antilinearly) dual to the tangent space of A at the identity. Thus the canonical polarization $\theta : J^D(\Gamma) \to J^D(\Gamma)^\vee$ induces a perfect, $\mathbb{C}$-valued bilinear pairing on $S_2^D(\Gamma)$. Since θ is a polarization, this pairing is Hermitian and positive-definite. Moreover, our computation of the Rosati involution above shows that for l prime to N, T_l commutes with its adjoint with respect to this pairing, as does $\langle d \rangle$. In particular, the matrices giving the action of T_l on the finite-dimensional complex vector space $S_2^D(\Gamma)$ are simultaneously diagonalizable, as l ranges over the set of primes not dividing N. It follows that the subalgebra of $\mathbb{T}^D(\Gamma)$ generated by the operators T_l for l prime to N and $\langle d \rangle$ is a reduced $\mathbb{Z}$-algebra. (Note that by contrast that the whole algebra $\mathbb{T}^D(\Gamma)$ need not be reduced.)

The existence of the bilinear pairing described above has important consequences for the structure of $S_2^D(\Gamma)$ and $\mathbb{T}^D(\Gamma)$. In particular, we have:

Proposition 1.2.5 *Let V be a subspace of $S_2^D(\Gamma)$, which is stable under the action of $\mathbb{T}^D(\Gamma)$ and w_N. Let W be its orthogonal complement under the bilinear pairing induced by the canonical polarization θ of $J^D(\Gamma)$. Then W is also stable under $\mathbb{T}^D(\Gamma)$ and w_N.*

Proof. Let x be an element of $\mathbb{T}^D(\Gamma)$ or the Atkin-Lehner operator w_N. Then x° is adjoint to x, and hence preserves W. But $w_N^\circ = w_N^{-1}$, so w_N preserves W. On the other hand, if x is an element of $\mathbb{T}^D(\Gamma)$, then $x = w_N^{-1}x^\circ w_N$. Since x° and w_N preserve W, so does x. $\square$

1.3 Old and new subspaces and subvarieties

Let Γ be a congruence subgroup of level N, and p a prime not dividing DN. The maps α_p and β_p from $X^D(\Gamma_0(p)\cap\Gamma)$ to $X^D(\Gamma)$ give rise to interactions among Jacobians of Shimura curves of different levels. In particular, one obtains a map:

$$i_p : J^D(\Gamma)^2 \to J^D(\Gamma_0(p)\cap\Gamma),$$

defined by $i_p(x,y) = \alpha^*(x) + \beta^*(y)$ for any points x and y on $J^D(\Gamma)$.

One checks immediately that the map i_p is compatible with the action of the Hecke operators T_l with p prime to l, in the sense that $i_p(T_l x, T_l y) = T_l i_p(x,y)$ for such p. A similar relation holds for the diamond bracket operators. On the other hand, i_p is manifestly *not* compatible with T_p. It will therefore be necessary to distinguish between the endomorphism of $J^D(\Gamma)$ induced by T_p and the corresponding endomorphism of $J^D(\Gamma_0(l)\cap\Gamma)$. We denote the former by τ_p and reserve the symbol T_p for the latter. With this language, we have:

Proposition 1.3.1 *The image of i_p is stable under the action of T_p. In particular, one has $T_p i_p(x,y) = i_p(-y, p\langle p\rangle x + \tau_p y)$.*

Proof. This proposition is originally due to Ribet in the case $D = 1$ and $\Gamma = \Gamma_0(N)$ [Ri5]. The proof we give here does not differ substantially from that one, except that one has to take a little more care with the diamond bracket operators. The general case can be found in [Di].

First, observe that $\alpha_p w_p = \beta_p$, and $\beta_p w_p = \langle p\rangle\alpha_p$, as can easily be checked from their definitions. In particular w_p preserves the image of i_p.

One also has the identity $T_p + w_p = \langle p\rangle(\beta_p)^*(\alpha_p)_*$ on $J^D(\Gamma_0(p)\cap\Gamma)$. (Again, this is easily checked on divisors.) It follows that $T_p + w_p$ preserves the image of i_p (indeed, it maps all of $J^D(\Gamma_0(p)\cap\Gamma)$ into the image of i_p.) Thus T_p preserves the image of i_p as well.

Now consider $T_p i_p(x,y)$. This equals $\langle p\rangle(\beta_p)^*(\alpha_p)_* i_p(x,y) - w_p i_p(x,y)$. We have:

$$w_p i_p(x,y) = w_p\alpha_p^*(x) + w_p\beta_p^*(y) = \beta_p^*(\langle p\rangle x) + \alpha_p^*(y).$$

We also have:

$$\langle p\rangle(\beta_p)^*(\alpha_p)_* i_p(x,y) = \langle p\rangle\beta_p^*(\alpha_p)_*\alpha_p^*(x) + \langle p\rangle\beta_p^*(\alpha_p)_*\beta_p^*(y) = \langle p\rangle\beta_p^*((p+1)x) + \beta_p^*(\tau_p y).$$

(The last equality holds because α has degree $p+1$ as a map $X^D(\Gamma_0(p)\cap\Gamma)\to X^D$, as a "fake elliptic curve" E has exactly $p+1$ $\mathcal{O}$-stable subgroups of order p^2.) Subtracting, we find that

$$T_p i_p(x,y) = \alpha_p^*(-y) + \beta_p^*(p\langle p\rangle x + \tau_p y) = i_p(-y, p\langle p\rangle x + \tau_p y).$$

□

The above result allows us to define an action of $\mathbb{T}^D(\Gamma_0(p)\cap\Gamma)$ on $J^D(\Gamma)^2$. The operators T_l for l prime to p and the diamond bracket operators act diagonally on $J^D(\Gamma)^2$; the action of T_p is given by the matrix $\begin{pmatrix} 0 & -1 \\ p\langle p\rangle & \tau_p \end{pmatrix}$. That this is actually an action of $\mathbb{T}^D(\Gamma_0(p)\cap\Gamma)$ can be seen from the fact that i_p is equivariant with respect to this action.

In particular, the image of i_p is a $\mathbb{T}^D(\Gamma_0(p)\cap\Gamma)$- stable subvariety of $J^D(\Gamma_0(p)\cap\Gamma)$. We call this subvariety the p-old subvariety $J^D(\Gamma_0(l)\cap\Gamma)_{p\text{-old}}$. Its tangent space is equal to $i_p(S_2^D(\Gamma)^2)$, and is called the p-old subspace $S_2^D(\Gamma_0(p)\cap\Gamma)_{p\text{-old}}$ of $S_2^D(\Gamma_0(p)\cap\Gamma)$. The above arguments show that it is stable under the action of $\mathbb{T}^D(\Gamma_0(p)\cap\Gamma)$; it follows from Lemma 1.2.2 that it is stable under w_{Np} as well. Thus the orthogonal complement of the p-old subspace is also stable under the action of the Hecke operators; we call this complement the p-new subspace $S_2^D(\Gamma_0(p)\cap\Gamma)_{p\text{-new}}$. Let $\mathbb{T}^D(\Gamma_0(p)\cap\Gamma)^{p\text{-old}}$ denote the corresponding quotient of the Hecke algebra, i.e., the quotient of $\mathbb{T}^D(\Gamma_0(p)\cap\Gamma)$ that acts faithfully on the p-new subspace.

We may also define the p-new subvariety of $J^D(\Gamma_0(p)\cap\Gamma)$ as follows: let I be the kernel of the map from $\mathbb{T}^D(\Gamma_0(p)\cap\Gamma)$ to its p-old quotient. Then $IS_2^D(\Gamma_0(p)\cap\Gamma)$ is a subspace of the p-new subspace of modular forms (in fact, we will see later that it is equal to the p-new subspace). We take $J^D(\Gamma_0(p)\cap\Gamma)_{p\text{-new}}$ to be $IJ^D(\Gamma_0(p)\cap\Gamma)$. Observe that since $T_p+w_p=\langle p\rangle\beta_p^*(\alpha_p)_*$ maps $J^D(\Gamma_0(p)\cap\Gamma)$ onto its old subvariety, we must have $T_p=-w_p$ on the p-new subvariety.

We will need the following fact, whose proof we defer to the next section:

Lemma 1.3.2 *The tangent space to $J^D(\Gamma_0(p)\cap\Gamma)_{p\text{-new}}$ at the identity is equal to $S_2^D(\Gamma_0(p)\cap\Gamma)_{p\text{-new}}$.*

More generally, let Γ' be a congruence subgroup of level N prime to D, let M be a squarefree integer prime to ND, and take $\Gamma=\Gamma_0(M)\cap\Gamma'$. Let U and R be relatively prime divisors of M. Then we define $J^D(\Gamma)_{U\text{-new},R\text{-old}}$ to be the connected component of the intersection of the varieties $J^D(\Gamma)_{p\text{-new}}$ for p dividing U and $J^D(\Gamma)_{q\text{-old}}$ for q dividing R. We define $S_2^D(\Gamma)_{U\text{-new},R\text{-old}}$ similarly. It follows immediately (once we have proven the previous lemma) that the latter is the tangent space of the former, at the identity.

We may also define quotients of $J^D(\Gamma)$ in this manner. Specifically, we take $J^D(\Gamma)^{U\text{-new},R\text{-old}}$ to be the quotient of $J^D(\Gamma)$ by the sum of the subvarieties $J^D(\Gamma)_{p\text{-old}}$ for p dividing U and $J^D(\Gamma)_{q\text{-new}}$ for q dividing R. Once again the tangent space is isomorphic to $S_2^D(\Gamma)_{U\text{-new},R\text{-old}}$. Moreover, the composition of the inclusion of $J^D(\Gamma)_{U\text{-new},R\text{-old}}$ into $J^D(\Gamma)$ with the surjection of $J^D(\Gamma)$ onto $J^D(\Gamma)^{U\text{-new},R\text{-old}}$ induces the identity map on the tangent spaces, and hence has finite kernel.

1.4 The Jacquet-Langlands correspondence

We can now state the Jacquet-Langlands correspondence, a fundamental result relating Shimura curves coming from quaternion algebras of different discriminants. Basically, this relates the quaternionic modular forms of level N and discriminant Dpq to those quaternionic modular forms of level Npq and discriminant D which are pq-new. More precisely:

Theorem 1.4.1 *(Jacquet-Langlands, Ribet) Let Γ' be a congruence subgroup of level N, D a squarefree product of an even number of primes not dividing N, and p and q two distinct primes not dividing ND. Let $\Gamma=\Gamma_0(pq)\cap\Gamma'$. Then the algebras $\mathbb{T}^D(\Gamma)^{pq\text{-new}}$ and $\mathbb{T}^{Dpq}(\Gamma')$ are naturally isomorphic, via a map that sends an operator T_l in one algebra to* *the corresponding operator T_l in the other (and similarly for U_l and $\langle d\rangle$.) Moreover, the $\mathbb{T}^{Dpq}(\Gamma')\otimes\mathbb{C}$-modules $S_2^D(\Gamma)_{pq\text{-new}}$ and $S_2^{Dpq}(\Gamma')$ are isomorphic.*

Proof. Jacquet and Langlands showed this for the subalgebras of $\mathbb{T}^D(\Gamma)^{pq\text{-new}}$ and $\mathbb{T}^{Dpq}(\Gamma')$ generated by the diamond bracket operators and the Hecke operators T_l for l prime to $DpqN$, using the theory of automorphic forms. As we will see in section 1.8, the results of Ribet [Ri5] imply that it holds for the full Hecke algebra. $\square$

We can use this correspondence to pass from results that are known for classical modular forms to corresponding results for quaternionic modular forms. In particular, we will apply this to the following result of Ribet:

Proposition 1.4.2 *The space $S_2^1(\Gamma)$ is naturally isomorphic to $\mathrm{Hom}_{\mathbb{C}}(\mathbb{T}^1(\Gamma)\otimes\mathbb{C},\mathbb{C})$.*

Proof. This is [Ri2], Theorem 2.2. The proof relies on the theory of q-expansions, which is available only in the $D=1$ case, and therefore does not generalize immediately to $D>1$. $\square$

This result has a number of useful corollaries, which we list below. We would like to obtain analogues of these corollaries for $D>1$.

Corollary 1.4.3 *The algebra $\mathbb{T}^1(\Gamma)\otimes\mathbb{Q}$ is a zero-dimensional Gorenstein ring. In particular, $S_2^1(\Gamma)$ is free of rank one over $\mathbb{T}^1(\Gamma)\otimes\mathbb{C}$.*

Proof. The $\mathbb{T}^D(\Gamma)$-equivariant isomorphism ψ from $J^D(\Gamma)$ to its dual identifies $S_2^D(\Gamma)$ with its $\mathbb{C}$-linear dual in a $\mathbb{T}^D(\Gamma)$ equivariant fashion. Since $S_2^D(\Gamma)$ is the $\mathbb{C}$-linear dual of $\mathbb{T}^D(\Gamma)\otimes\mathbb{C}$, it follows that $\mathbb{T}^D(\Gamma)\otimes\mathbb{C}$ is a self-dual, $\mathbb{C}$-algebra, and a finite dimensional $\mathbb{C}$-vector space. In particular it is a zero-dimensional Gorenstein ring. Since $\mathbb{C}$ is faithfully flat over $\mathbb{Q}$, the first statement follows.

The second statement follows immediately, since we have an isomorphism:

$$\mathrm{Hom}_{\mathbb{C}}(\mathbb{T}^1(\Gamma)\otimes\mathbb{C},\mathbb{C})\cong\mathbb{T}^1(\Gamma)\otimes\mathbb{C}.$$

$\square$

Corollary 1.4.4 *Let $\Gamma=\Gamma_0(M)\cap\Gamma'$, where Γ' has level N and M is squarefree and prime to N. Let U and R be relatively prime divisors of M. Then $S_2^1(\Gamma)_{U\text{-new},R\text{-old}}$ is free of rank one over $\mathbb{T}^1(\Gamma)^{U\text{-new},R\text{-old}}\otimes\mathbb{C}$.*

Proof. We know that $S_2^1(\Gamma)_{U\text{-new},R\text{-old}}$ is a quotient of $S_2^1(\Gamma)$ which is a faithful $\mathbb{T}^1(\Gamma)^{U\text{-new},R\text{-old}}$-module. It is thus generated by a single element over $\mathbb{T}^1(\Gamma)^{U\text{-new},R\text{-old}}$, and the annihilator of this element in $\mathbb{T}^1(\Gamma)^{U\text{-new},R\text{-old}}$ is trivial. $\square$

Proposition 1.4.5 *Let D be a squarefree integer with an even number of prime divisors, Γ' a congruence subgroup of level N prime to D, and M a squarefree integer prime to DN. Let $\Gamma=\Gamma_0(M)\cap\Gamma$, and let U and R be relatively prime divisors of M. Then $S_2^D(\Gamma)_{U\text{-new},R\text{-old}}$ is free of rank one over $\mathbb{T}^D(\Gamma)^{U\text{-new},R\text{-old}}\otimes\mathbb{C}$.*

Proof. This follows immediately from the last result applied to $S_2^1(\Gamma_0(D)\cap\Gamma)_{DU\text{-new},R\text{-old}}$, by repeated applications of Jacquet-Langlands. $\square$

Corollary 1.4.6 *The algebra* $\mathbb{T}^D(\Gamma)^{U\text{-new},R\text{-old}} \otimes \mathbb{Q}$ *is a zero-dimensional Gorenstein ring.*

Proof. The space $S_2^D(\Gamma)^{U\text{-new},R\text{-old}}$ is free of rank one over $\mathbb{T}^D(\Gamma)^{U\text{-new},R\text{-old}} \otimes \mathbb{C}$ and self dual. It follows that $\mathbb{T}^D(\Gamma)^{U\text{-new},R\text{-old}} \otimes \mathbb{C}$ is a zero-dimensional Gorenstein ring. The result follows since $\mathbb{C}$ is faithfully flat over $\mathbb{Q}$. □

Corollary 1.4.7 *The* $\mathbb{Z}$*-rank of* $\mathbb{T}^D(\Gamma)^{U\text{-new},R\text{-old}}$ *equals the dimension of* $J^D(\Gamma)_{U\text{-new},R\text{-old}}$.

We are now in a position to tie up the loose end we had from the previous section:

Corollary 1.4.8 *The tangent space to* $J^D(\Gamma_0(p) \cap \Gamma)_{p\text{-new}}$ *is equal to* $S_2^D(\Gamma_0(p) \cap \Gamma)_{p\text{-new}}$.

Proof. By definition, the tangent space to $J^D(\Gamma_0(p) \cap \Gamma)_{p\text{-new}}$ is $IS_2^D(\Gamma_0(p) \cap \Gamma)$, where I is the kernel of the map from $\mathbb{T}^D(\Gamma_0(p) \cap \Gamma)$ to its p-old quotient. The space $S_2^D(\Gamma_0(p) \cap \Gamma)$ is free of rank one over $\mathbb{T}^D(\Gamma_0(p) \cap \Gamma) \otimes \mathbb{C}$. This space is naturally isomorphic to the product:

$$(\mathbb{T}^D(\Gamma_0(p) \cap \Gamma)^{p\text{-new}} \otimes \mathbb{C}) \times (\mathbb{T}^D(\Gamma_0(p) \cap \Gamma)^{p\text{-old}} \otimes \mathbb{C}).$$

Thus $IS_2^D(\Gamma_0(p) \cap \Gamma)$ is free of rank one over $\mathbb{T}^D(\Gamma_0(p) \cap \Gamma)^{p\text{-new}} \otimes \mathbb{C}$, and contained in $S_2^D(\Gamma_0(p) \cap \Gamma)_{p\text{-new}}$. A dimension count shows that the spaces are equal. □

From this, it follows that the $\mathbb{Z}$-rank of $\mathbb{T}^D(\Gamma)^{U\text{-new},R\text{-old}}$ is equal to the dimension of $J^D(\Gamma)^{U\text{-new},R\text{-old}}$.

1.5 Galois representations

Our next objective is to study the Galois representations associated to $J^D(\Gamma)$. We begin by introducing a construction that we will find very useful in what follows: that of the contravariant Tate module of an abelian variety.

Definition 1.5.1 *Let* G *be a finite, flat group scheme. The Cartier dual* $G^\vee$ *is the group scheme* $\mathrm{Hom}(G, \mathbb{G}_m)$.

Definition 1.5.2 *Let* $\mathbb{T}$ *be a finite, flat, commutative* $\mathbb{Z}$*-algebra, and let* A *be an abelian variety on which* $\mathbb{T}$ *acts faithfully. For* $\mathfrak{m}$ *a maximal ideal of* $\mathbb{T}$*, we define the* $\mathfrak{m}$*-adic contravariant Tate module* $T_\mathfrak{m}A$ *to be the projective limit of the group schemes* $A[\mathfrak{m}^k]^\vee$ *over* k*. Here* $A[\mathfrak{m}^k]$ *is the subgroup of points of* A *killed by* $\mathfrak{m}^k$*, and the transition maps are the Cartier duals of the inclusions* $A[\mathfrak{m}^{k-1}] \to A[\mathfrak{m}^k]$*. (Alternatively, one may define* $T_\mathfrak{m}A$ *to be the module* $\mathrm{Hom}(A[\mathfrak{m}^\infty], \mu_{l^\infty})$*, where* μ_{l^∞} *is the group of* l*-power roots of unity and* l *is the residue characteristic of* $\mathfrak{m}$*.)*

For any $\mathfrak{m} \subset \mathbb{T}^D(\Gamma)$, the contravariant Tate module $T_\mathfrak{m}J^D(\Gamma)$ is a module over the $\mathfrak{m}$-adic completion $\mathbb{T}^D(\Gamma)_\mathfrak{m}$ of $\mathbb{T}^D(\Gamma)$. Moreover, since all of the operators of $\mathbb{T}^D(\Gamma)$ are defined over $\mathbb{Q}$, the Galois action on $T_\mathfrak{m}J^D(\Gamma)$ is $\mathbb{T}^D(\Gamma)$-linear. This furnishes us with a representation of the Galois group of $\mathbb{Q}$ over $\mathbb{T}^D(\Gamma)$.

Proposition 1.5.3 *For any* $\mathfrak{m}$, $T_\mathfrak{m}J^D(\Gamma) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ *is free of rank two over* $\mathbb{T}^D(\Gamma)_\mathfrak{m} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$*, where* l *is the residue characteristic of* $\mathfrak{m}$.

Proof. By a result of Shimura, ([Sh], chapter 8) one has an isomorphism

$$H^1_p(X^D(\Gamma), \mathbb{C}) \cong S^D_2(\Gamma) \oplus \overline{S^D_2(\Gamma)}.$$

Here $H^1_p(X^D(\Gamma), \mathbb{C})$ denotes the singular cohomology of $X^D(\Gamma)$ when $D > 1$, and the image of the singular cohomology with compact support of $Y(\Gamma)$ inside $X(\Gamma)$ when $D = 1$.

It follows that $\mathrm{Hom}(H^1_p(X^D(\Gamma), \mathbb{C}), \mathbb{C})$ is free of rank two over $\mathbb{T}^D(\Gamma) \otimes \mathbb{C}$, as both $\mathrm{Hom}(S^D_2(\Gamma), \mathbb{C})$ and $\mathrm{Hom}(\overline{S^D_2(\Gamma)}, \mathbb{C})$ are. Since $\mathbb{C}$ is faithfully flat over $\mathbb{Q}$, it follows that $\mathrm{Hom}(H^1_p(X^D(\Gamma), \mathbb{Z}) \otimes \mathbb{Q}, \mathbb{Q})$ is free of rank two over $\mathbb{T}^D(\Gamma) \otimes \mathbb{Q}$. But $H^1_p(X^D(\Gamma), \mathbb{Z})_{\mathfrak{m}}$ is the $\mathbb{Z}_l$-dual of $T_{\mathfrak{m}} J^D(\Gamma)$. Thus $H^1_p(X^D(\Gamma), \mathbb{Q})_{\mathfrak{m}}$ is the $\mathbb{Q}_l$-dual of $T_{\mathfrak{m}} J^D(\Gamma) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. It thus follows that $T_{\mathfrak{m}} J^D(\Gamma) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is free of rank two over $\mathbb{T}^D(\Gamma)_{\mathfrak{m}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. □

Choosing a basis for $T_{\mathfrak{m}} J^D(\Gamma) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ gives us a representation $\rho_{\mathfrak{m}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{T}^D(\Gamma)_{\mathfrak{m}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)$. Miraculously, the character of this representation is explicitly computable:

Theorem 1.5.4 *The representation $\rho_{\mathfrak{m}}$ is odd, and unramified at any prime p not dividing DNl. Moreover, one has:*

$$\begin{aligned} \det \rho_{\mathfrak{m}}(\mathrm{Frob}_p) &= p\langle p \rangle^{-1} \\ \mathrm{Tr}\, \rho_{\mathfrak{m}}(\mathrm{Frob}_p) &= \langle p \rangle^{-1} T_p, \end{aligned}$$

where Frob_p *denotes a Frobenius element of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *at p.*

Proof. This result is originally due to Shimura, who stated it in a much different form. For the result in the above form, we refer the reader to [Ca], where it appears as the $k = 2$ case of the lemma at the beginning of section 3.2.1. (We should remark that the module that Carayol works with (and which is standard in the literature) is constructed from the *covariant* Tate module at $\mathfrak{m}$, and is thus dual to the one we present here, so that the usual formulas for the trace and determinant differ slightly from the ones we give above. We deviate from convention because contravariant Tate modules will be much more natural for what we do later.) □

Since the elements of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ conjugate to Frob_p for some p are dense in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, it follows that the characteristic polynomial of $\rho_{\mathfrak{m}}(\sigma)$ has coefficients in $\mathbb{T}_{\mathfrak{m}}$ for any σ. In particular the character of $\rho_{\mathfrak{m}}$ takes values in $\mathbb{T}^D(\Gamma)_{\mathfrak{m}}$. Reducing modulo $\mathfrak{m}$ we obtain the character of a two-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over the finite field $\mathbb{T}^D(\Gamma)/\mathfrak{m}$. Up to semisimplification, this representation will either be the sum of two one-dimensional representations or an irreducible two-dimensional representation. In the latter case the representation is unique up to isomorphism, and we denote it by $\overline{\rho}_{\mathfrak{m}}$. In the former case $\overline{\rho}_{\mathfrak{m}}$ is only defined up to semisimplification.

Definition 1.5.5 *A maximal ideal $\mathfrak{m}$ is called Eisenstein if $\overline{\rho}_{\mathfrak{m}}$ fails to be absolutely irreducible.*

Proposition 1.5.6 *Only finitely many primes of* $\mathbb{T}^D(\Gamma)$ *are Eisenstein.*

Proof. This is well-known; for instance, see [Ri3], Theorem 2.1. In our terms this says that for any minimal prime $\mathfrak{p}$ of $\mathbb{T}^D(\Gamma)$, $\overline{\rho}_{\mathfrak{m}}$ will be absolutely irreducible for all but finitely many $\mathfrak{m}$ containing $\mathfrak{p}$. Since $\mathbb{T}^D(\Gamma)$ has only finitely many minimal primes, the result follows. □

The Jacquet-Langlands correspondence, coupled with the above results, has very important consequences for the geometry of Jacobians of Shimura curves. In particular, one has:

Proposition 1.5.7 *Let D be a squarefree integer with an even number of prime divisors, N an integer prime to D, Γ a congruence subgroup of level N, and p and q two distinct primes not dividing DN. Let $J' = J^{Dpq}(\Gamma)$, and $J = J^D(\Gamma_0(pq) \cap \Gamma)_{pq\text{-new}}$. Then $\mathbb{T} = \mathbb{T}^{Dpq}(\Gamma)$ acts on both J and J', and we have*

$$T_{\mathfrak{m}} J \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \cong T_{\mathfrak{m}} J' \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

as Galois modules.

Proof. Both $T_{\mathfrak{m}} J \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ and $T_{\mathfrak{m}} J' \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ are free of rank two over $\mathbb{T}_{\mathfrak{m}} \otimes_{\mathbb{Z}} \mathbb{Q}_l$, so it suffices to check that they give equivalent Galois representations. But this is immediate since the traces and determinants of any Galois group elements are the same on the two modules. □

This is especially useful when combined with the following result of Faltings:

Theorem 1.5.8 *(Faltings) Let A and B be abelian varieties over $\mathbb{Q}$. Then $\mathrm{Hom}(A, B) \otimes \mathbb{Z}_l \cong \mathrm{Hom}_G(T_l B, T_l A)$, where Hom_G denotes homomorphisms of Galois modules.*

Proof. See [Fa]. □

Corollary 1.5.9 *The abelian varieties J and J' described above are $\mathbb{T}$-equivariantly isogenous.*

Proof. It suffices to show that we have an isomorphism in $\mathrm{Hom}_{\mathbb{T}}(J, J') \otimes \mathbb{Q}$, or equivalently in $\mathrm{Hom}_{\mathbb{T}}(J, J') \otimes \mathbb{Q}_l$. The natural map $\mathrm{Hom}(J, J') \otimes \mathbb{Z}_l \to \mathrm{Hom}_G(T_l J', T_l J)$ sends $\mathbb{T}$-equivariant morphisms to $\mathbb{T}$-equivariant morphisms. By Faltings' result, it thus induces an isomorphism of $\mathrm{Hom}_{\mathbb{T}}(J, J') \otimes \mathbb{Q}_l$ with $\mathrm{Hom}_{\mathbb{T}}(T_l J', T_l J) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. This is isomorphic to the product over all $\mathfrak{m}$ of residue characteristic l of $\mathrm{Hom}(T_{\mathfrak{m}} J', T_{\mathfrak{m}} J) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Since the latter modules all contain isomorphisms, so does the former. □

Note that the method by which we obtain this isogeny is completely nonconstructive. We start with a noncanonical isomorphism between two spaces of modular forms, and use this to show that the traces of two Galois representations are equal. This gives us a noncanonical isomorphism between the two Galois representations, which we then use to show the existence of our isogeny. At no stage is there anything explicit.

The question of what the possible maps between J and J' look like is the fundamental question we seek to answer in this dissertation.

1.6 Level-raising for modular representations

We return to the setting of section 1.3. In particular, let Γ be a congruence subgroup of level N prime to D, and p a prime not dividing ND. For convenience, we assume p is odd. Let J be the variety $J^D(\Gamma)$, and $\tilde{J}$ denote the variety $J^D(\Gamma_0(p) \cap \Gamma)$. Similarly, let $\mathbb{T}$ denote $\mathbb{T}^D(\Gamma)$ and $\tilde{\mathbb{T}}$ denote $J^D(\Gamma_0(p) \cap \Gamma)$. Recall that we have constructed a map $i_p : J^2 \to \tilde{J}^{p\text{-old}}$ that is $\tilde{\mathbb{T}}$-equivariant.

Let $\mathbb{T}_p$ denote the subalgebra of $\mathbb{T}$ generated by the Hecke operators T_l for l prime to p, and the diamond bracket operators. Then $\mathbb{T}_p$ acts diagonally on J^2, and we showed in section 1.3 that the action of T_l on J^2 is the same whether we considered T_l as an element of $\mathbb{T}$ or as an element of $\tilde{\mathbb{T}}^{p\text{-old}}$. (The operator T_p on the other hand, is not; as in section 1.3 we use the symbol τ_p to denote the pth Hecke operator as an element of $\mathbb{T}$, and reserve the symbol T_p for the pth Hecke operator as an element of $\tilde{\mathbb{T}}$.) Since $\tilde{\mathbb{T}}^{p\text{-old}}$ acts faithfully on J^2, it follows that we can view $\mathbb{T}_p$ as a subalgebra of $\tilde{\mathbb{T}}^{p\text{-old}}$. Moreover, since $\tilde{\mathbb{T}}^{p\text{-old}}$ is generated by the operators T_l and the diamond bracket operators, we have $\tilde{\mathbb{T}}^{p\text{-old}} = \mathbb{T}_p[T_p]$. Moreover, we have:

Lemma 1.6.1 *The subalgebra $\mathbb{T}_p$ is all of $\mathbb{T}$.*

Proof. This appears as the lemma on page 461 of [Wi]. Wiles attributes it to Ribet. □

In particular, τ_p lies inside $\mathbb{T}_p$, and $\tilde{\mathbb{T}}^{p\text{-old}} = \mathbb{T}[T_p]$. The expression we obtained for T_p as a two by two matrix of elements of $\mathbb{T}$ shows that the minimal polynomial of T_p over $\mathbb{T}$ is $T_p^2 - \tau_p T_p + p\langle p\rangle$. Thus we have $\tilde{\mathbb{T}}^{p\text{-old}} = \mathbb{T}[T_p]/(T_p^2 - \tau_p T_p + p\langle p\rangle)$. The algebra $\tilde{\mathbb{T}}^{p\text{-old}}$ is therefore a free $\mathbb{T}$-algebra of rank two.

Corollary 1.6.2 *The map $i_p : J^2 \to \tilde{J}$ has finite kernel.*

Proof. It is equivalent to show that i_p is an injection on tangent spaces. More precisely, we show that i_p induces an isomorphism of the tangent space of J^2 at the identity with the tangent space of $\tilde{J}^{p\text{-old}}$ at the identity. This map on tangent spaces is a surjection by definition. Moreover, since the tangent space $S_2^D(\Gamma_0(p)\cap\Gamma)$ of $\tilde{J}^{p\text{-old}}$ is free of rank one over $\tilde{\mathbb{T}}^{p\text{-old}}$ and the tangent space $S_2^D(\Gamma)^2$ of J^2 is free of rank two over $\mathbb{T}$, injectivity follows by a dimension count. □

Remark 1.6.3 This can be (and usually is) shown much more directly by a q-expansion calculation for $D = 1$, and then invoking Jacquet-Langlands to establish it for more general D; we use the more convoluted method presented here because we wish to avoid developing the theory of q-expansions.

Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}$. Since $\tilde{\mathbb{T}}^{p\text{-old}}$ is a free $\mathbb{T}$-algebra of finite rank, we can lift $\mathfrak{m}$ to a maximal ideal $\tilde{\mathfrak{m}}$ of $\tilde{\mathbb{T}}^{p\text{-old}}$. Since $\tilde{\mathbb{T}}^{p\text{-old}}$ is a quotient of $\tilde{\mathbb{T}}$, we can consider $\tilde{\mathfrak{m}}$ as a maximal ideal of $\tilde{\mathbb{T}}$ as well. Observe that the representations $\overline{\rho}_{\mathfrak{m}}$ and $\overline{\rho}_{\tilde{\mathfrak{m}}}$ are isomorphic, as their traces and determinants are "the same" —i.e., they are identified under the natural injection of $\mathbb{T}/\mathfrak{m}$ into $\tilde{\mathbb{T}}/\tilde{\mathfrak{m}}$.

Given our construction of the p-old subvariety, this is not too surprising. What is more surprising is the fact that we can sometimes consider $\tilde{\mathfrak{m}}$ as a maximal ideal of the *p-new* quotient as well! To explain how this can happen, consider the injection: $\tilde{\mathbb{T}} \to \tilde{\mathbb{T}}^{p\text{-old}} \times \tilde{\mathbb{T}}^{p\text{-new}}$. We have seen that after tensoring with $\mathbb{C}$ this becomes an isomorphism. However, this does not imply that the map itself is an isomorphism—only that its image has finite index. Moreover, the action of $\tilde{\mathbb{T}}$ on the cokernel factors through both $\tilde{\mathbb{T}}^{p\text{-old}}$ and $\tilde{\mathbb{T}}^{p\text{-new}}$. Hence any maximal ideal $\tilde{\mathfrak{m}}$ on which the cokernel is supported is simultaneously a maximal ideal of $\tilde{\mathbb{T}}^{p\text{-old}}$ and $\tilde{\mathbb{T}}^{p\text{-new}}$. (In particular, there will be a minimal prime of $\tilde{\mathbb{T}}^{p\text{-old}}$ and a minimal prime of $\tilde{\mathbb{T}}^{p\text{-new}}$ which both specialize to the same maximal ideal. In the language of modular forms this means that there are two modular eigenforms: one in the p-old subspace and one in the p-new subspace, whose eigenvalues under the Hecke operators T_q for q prime to p are all congruent modulo the residue characteristic l of $\tilde{\mathfrak{m}}$. Such congruences between modular forms play a central role in the theory.)

We would like a criterion for when such a situation can occur. This is provided by the following:

Theorem 1.6.4 *(Ribet, Diamond-Taylor) Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbb{T}$, of residue characteristic greater than 3, and suppose that $\tau_p^2 - (p+1)^2\langle p\rangle$ lies in $\mathfrak{m}$. Then there is a lift $\tilde{\mathfrak{m}}$ of $\mathfrak{m}$ to $\tilde{\mathbb{T}}$ such that $\tilde{\mathfrak{m}}$ is p-new; i.e., $\tilde{\mathbb{T}}^{p\text{-new}}_{\tilde{\mathfrak{m}}}$ is nontrivial.*

This theorem is due to Ribet in the case $D = 1$, who proved it without the extra condition on the residue characteristic of $\mathfrak{m}$. We essentially follow his proof. We will need the following deep result, due to Ribet when $D = 1$, and Diamond and Taylor ([DiTa],Theorem 2) in general:

Theorem 1.6.5 *Let K be the kernel of i_p, and $\mathfrak{m}$ a maximal ideal of $\mathbb{T}$ in the support of K. Then either $\mathfrak{m}$ is Eisenstein, or $D > 1$ and $\mathfrak{m}$ has residue characteristic two or three.*

Proof of Theorem 1.6.4: Consider the map $j_p : \tilde{J} \to J^2$ defined by $j_p(x) = ((\alpha_p)_* x, (\beta_p)_* x)$. The composite map $j_p i_p : J^2 \to J^2$ is given by the matrix $\begin{pmatrix} (\alpha_p)_*\alpha_p^* & (\alpha_p)_*\beta_p^* \\ (\beta_p)_*\alpha_p^* & (\beta_p)_*\beta_p^* \end{pmatrix}$. By the definition of the Hecke operator T_p in terms of α_p and β_p this matrix is equal to the matrix $\begin{pmatrix} p+1 & \langle p\rangle^{-1}\tau_p \\ \tau_p & p+1 \end{pmatrix}$.

The map $j_p i_p$ is manifestly equivariant with respect to the action of the Hecke operators T_l for $l \neq p$, but is not compatible with the action of T_p. Nonetheless, one can verify that $j_p i_p = \gamma(T_p^2 - \langle p\rangle)$, where γ is the automorphism of J^2 given by the matrix $\begin{pmatrix} -\langle p\rangle^{-1} & 0 \\ \tau_p\langle p\rangle^{-1} & \langle p\rangle^{-1} \end{pmatrix}$. Thus the kernel of $j_p i_p$ is equal to $J^2[T_p^2 - \langle p\rangle]$, and is in particular stable under the action of $\tilde{\mathbb{T}}$.

On the other hand, the determinant of the matrix $j_p i_p$ is equal to $(p+1)^2 - \langle p\rangle^{-1}\tau_p^2$. By hypothesis, this lies in $\mathfrak{m}$, so $j_p i_p$ kills a nontrivial element x of $J^2[\mathfrak{m}]$. Since the kernel of i_p is supported away from $\mathfrak{m}$, x maps to a nontrivial element of $\tilde{J}$. We will show that x lies in $\tilde{J}_{p\text{-new}}$, and it will then follow immediately that there is a maximal ideal $\tilde{\mathfrak{m}}$ of $\tilde{\mathbb{T}}^{p\text{-new}}$ containing $\mathfrak{m}$, namely any maximal ideal containing the annihilator of x.

The map j_p factors through $\tilde{J}^{p\text{-old}}$. Suppose x mapped to a nonzero element of $\tilde{J}^{p\text{-old}}$. Then we would have an element of $\tilde{J}^{p\text{-old}}$ that mapped to zero in J^2, via j_p, and was supported on $\mathfrak{m}$. But i_p and j_p are dual to each other, since $(\alpha_p)_*$ and $(\alpha_p)^*$ are dual to each other, so this would imply that we had an element of the kernel of i_p which was supported on $\mathfrak{m}$. Since we know this does not happen, x must map to zero in $\tilde{J}^{p\text{-old}}$. Since $\tilde{J}^{p\text{-old}}$ is the quotient of $\tilde{J}$ by $\tilde{J}^{p\text{-new}}$, it follows that x lies in $\tilde{J}^{p\text{-new}}$, as required. □

In fact, this situation occurs quite frequently:

Proposition 1.6.6 *Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of residue characteristic l greater than 3. Then there are infinitely many p such that $\tau_p^2 - (p+1)^2\langle p\rangle$ lies in $\mathfrak{m}$.*

Proof. Let χ be the character such that $\det \rho_{\mathfrak{m}} = \chi\omega$, where ω is the cyclotomic character (i.e., the character $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{T}_{\mathfrak{m}}^{\times}$ which takes a frobenius element Frob_p at p to $p \in \mathbb{T}_{\mathfrak{m}}^{\times}$, for any p prime to DNl). Let σ denote the representation $\overline{\rho}_{\mathfrak{m}} \times \chi$. By the Čebotarev density theorem there are infinitely many primes p such that $\sigma(\mathrm{Frob}_p) = \sigma(c)$, where c is a complex conjugation. Arguments of Diamond [Di] show that p is congruent to -1 mod l, and $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_p)$ has trace zero and determinant -1. Thus τ_p and $p+1$ lie in $\mathfrak{m}$, and the theorem follows. □

1.7 Reduction theory of Jacobians of Shimura curves

The major tool we will need in our study of the geometry of the Jacquet-Langlands correspondence is an understanding of the bad reduction of the Jacobians $J^D(\Gamma)$. In particular we will need to understand the reduction of $X^D(\Gamma_0(p) \cap \Gamma)$ modulo p. Note that the moduli problem we gave in the first section is not well-posed in characteristic p, as "fake elliptic curves" modulo p need not have any p-torsion points at all!. In order to overcome this difficulty we will need to modify the moduli problem slightly, in order to obtain one that makes sense in characteristic p (and still agrees with our original moduli problem in other characteristics.)

In general, to define a moduli problem for $X^D(\Gamma)$ that works for all primes p dividing the level N of Γ is highly nontrivial. In the $D = 1$ case, we refer the reader to Katz-Mazur [KaMa]; analogous results should hold for arbitrary D but have not (to the author's knowledge) been written down. For some work along those lines, see [Bu1].

Fortunately, we need only study the reduction of $X^D(\Gamma_0(p) \cap \Gamma)$ modulo p, where p is prime to the level of Γ. Recall that we had defined a $\Gamma_0(p) \cap \Gamma$-level structure on a "fake elliptic curve" E to be a pair (C, P), where C is an $\mathcal{O}$-stable subgroup of order p^2 on E and P is a Γ-level structure on E. Since the level of Γ is prime to p, there is no trouble defining a Γ-level structure in characteristic p, but as we have already mentioned, if E is a "fake elliptic curve" in characteristic p it might not have any subgroups of order p^2 at all.

To remedy this, observe that a choice of $\mathcal{O}$-cyclic subgroup C of order p^2 on E determines a $\mathcal{O}$-equivariant isogeny $E \to E/C$, of degree p^2. Conversely, if we are not in characteristic p, then given an $\mathcal{O}$-equivariant isogeny $E \to E'$ of degree p^2, we obtain a $\mathcal{O}$-stable subgroup of E of order p^2. Moreover, even in characteristic p, E will possess $\mathcal{O}$-equivariant isogenies of order p^2; these might not correspond to subgroups of E since their kernels might not be reduced.

It turns out that this is the "right" moduli problem to use in characteristic p. In particular, if we take $X^D(\Gamma_0(p) \cap \Gamma)$ to be the coarse moduli space of triples (E, E', ϕ, P), where E and E' are "fake elliptic curves", ϕ is an $\mathcal{O}$-equivariant isogeny $E \to E'$ of degree p^2, and P is a Γ-level structure on E, then $X^D(\Gamma_0(p) \cap \Gamma)$ is a well-defined curve over $\mathbb{Z}[\frac{1}{N}]$, which is smooth over $\mathbb{Z}[\frac{1}{pDN}]$. Moreover, if Γ is sufficiently small (for instance, if Γ is contained in $\Gamma_1(n)$ for some $n > 3$), then $X^D(\Gamma_0(p) \cap \Gamma)$ will actually be a fine moduli space. It is straightforward to verify that the moduli problem so defined coincides with our original moduli problem away from characteristic p.

We will summarize the facts about this moduli problem, and its mod p reduction, shortly. First we make some observations about the behavior of this moduli problem in characteristic p.

Let E be a "fake elliptic curve" over a field of characteristic p. There are two isogenies of degree p^2 naturally associated to E. The first is the (relative) Frobenius morphism $\mathrm{Frob}_E : E \to E^{(p)}$. The second is the Verschiebung map $\mathrm{Ver}_E : E^{(p)} \to E$, which is obtained by factoring the endomorphism "multiplication by p" of E through Frob_E. If E is *supersingular*, then $E = (E^{(p)})^{(p)}$ and Frob_E and $\mathrm{Ver}_{E^{(p)}}$ coincide.

Now assume that Γ is contained in $\Gamma_1(n)$ for some $n > 3$, so that $X^D(\Gamma)$ is a fine moduli space. We can use this to define two morphisms Frob and Ver from $X^D(\Gamma)_{\mathbb{F}_p}$ to $X^D(\Gamma_0(p) \cap \Gamma)_{\mathbb{F}_p}$. In particular we define Frob to be the morphism taking (E, P) to $(E, E^{(p)}, \mathrm{Frob}_E, P)$. Similarly, we define Ver to be the morphism which takes (E, P) to $(E^{(p)}, E, \mathrm{Ver}_E, \mathrm{Frob}_E(P))$. The images of these two maps are two distinct irreducible components of $X^D(\Gamma_0(p) \cap \Gamma)_{\mathbb{F}_p}$, each isomorphic to $X^D(\Gamma)_{\mathbb{F}_p}$, which intersect at the supersingular points.

In fact, this gives a complete description of the characteristic p fiber of $X^D(\Gamma_0(p) \cap \Gamma)$. More precisely, one has:

Theorem 1.7.1 *Suppose that Γ is contained in $\Gamma_1(n)$ for $n > 3$. Then $X^D(\Gamma_0(p) \cap \Gamma)$ is a regular scheme, defined over $\mathbb{Z}[\frac{1}{N}]$, and smooth over $\mathbb{Z}[\frac{1}{pDN}]$. The maps* Frob *and* Ver *from $X^D(\Gamma)_{\mathbb{F}_p}$ to $X^D(\Gamma_0(p) \cap \Gamma)_{\mathbb{F}_p}$ are closed immersions; the union of their images is all of $X^D(\Gamma_0(p) \cap \Gamma)_{\mathbb{F}_p}$. In particular, the latter is the union of two irreducible components* $\mathrm{Frob} = \mathrm{Frob}(X^D(\Gamma)_{\mathbb{F}_p})$ *and* $\mathrm{Ver} = \mathrm{Ver}(X^D(\Gamma)_{\mathbb{F}_p})$, *in which each supersingular point of* Frob *is glued to its Frobenius twist on* Ver. *Moreover, if x is a supersingular point on the special fiber of $X^D(\Gamma_0(p) \cap \Gamma)$ over $\mathbb{F}_p$, then the completion of the strict henselization of the local ring at x is isomorphic to $W(\overline{\mathbb{F}}_p)[[X, Y]]/(XY - p)$, where $W(\overline{\mathbb{F}}_p)$ denotes the ring of Witt vectors of $\overline{\mathbb{F}}_p$, i.e., the integral closure of $\mathbb{Z}_p$ in the maximal unramified extension of $\mathbb{Q}_p$.*

Proof. For $D = 1$, this is due to Deligne-Rapoport [DeRa]. For $D > 1$ this is proven in [Bu1]. □

We wish to remove the requirement that Γ be contained in $\Gamma_1(n)$ for some sufficiently large n. Since the schemes involved only exist as coarse moduli spaces for such n, this will involve comparing a fine moduli stack with its underlying coarse moduli scheme. In particular some singularities may be introduced at the supersingular points on the special fiber. To describe these singularities we will need a little bit of notation. Let x be a point on the special fiber of $X^D(\Gamma_0(p) \cap \Gamma)$, corresponding to a "fake elliptic curve" E with level structure P, and define an integer $k(x)$ to be the number of automorphisms of the pair (E, P) if -1 is not an automorphism of (E, P), or one-half the number of automorphisms of (E, P) if -1 is an automorphism. We have:

Theorem 1.7.2 *The coarse moduli space $X^D(\Gamma_0(p) \cap \Gamma)$ is regular over $\mathbb{Z}[\frac{1}{Np}]$. As a scheme over $\mathbb{Z}[\frac{1}{N}]$, it is regular away from the supersingular points on the special fiber over $\mathbb{F}_p$. If x is a supersingular point, then the completion of strict henselization of the local ring of $X^D(\Gamma_0(p) \cap \Gamma)$ at x is isomorphic to $W(\overline{\mathbb{F}}_p)[[X, Y]]/(XY - p^{k(x)})$. In particular, x is a regular point if and only if $k(x) = 1$.*

Moreover, there are two closed immersions Frob *and* Ver *from $X^D(\Gamma)_{\mathbb{F}_p}$ into $X^D(\Gamma_0(p) \cap \Gamma)_{\mathbb{F}_p}$, such that the union of the images of* Frob *and* Ver *is all of $X^D(\Gamma_0(p) \cap \Gamma)_{\mathbb{F}_p}$,* *and the images of the two maps intersect precisely at the supersingular points of* $X^D(\Gamma_0(p)\cap\Gamma)$.

Proof. This is presumably well-known, but does not seem to be in the literature (except when $D = 1$, where it can be found in [DeRa]). We will therefore give a proof below. As the $D = 1$ case is published, we will assume $D > 1$.

Let X and $\tilde{X}$ denote the coarse moduli schemes $X^D(\Gamma)$ and $X^D(\Gamma_0(p)\cap\Gamma)$, respectively. Let $\mathcal{X}$ and $\tilde{\mathcal{X}}$ denote the corresponding fine moduli stacks. Fix an $r > 3$, and prime to DNp, and let X_r and $\tilde{X}_r$ denote the schemes $X^D(\Gamma_1(r)\cap\Gamma)$ and $X^D(\Gamma_1(r)\cap\Gamma_0(p)\cap\Gamma)$, respectively. Note that X_r and $\tilde{X}_r$ are in fact fine moduli spaces, by the previous theorem.

Now X_r and $\tilde{X}_r$ are étale over $\mathcal{X}$ and $\tilde{\mathcal{X}}$, respectively. It follows that $\mathcal{X}$ and $\tilde{\mathcal{X}}$ are regular, and that $\mathcal{X}$ is smooth over $\mathbb{Z}[\frac{1}{ND}]$, whereas $\tilde{\mathcal{X}}$ is smooth over $\mathbb{Z}[\frac{1}{NDp}]$. Moreover, we have a commutative diagram:

$$\begin{array}{ccc} (X_r)_{\mathbb{F}_p} & \longrightarrow & (\tilde{X}_r)_{\mathbb{F}_p} \\ \downarrow & & \downarrow \\ \mathcal{X}_{\mathbb{F}_p} & \longrightarrow & \tilde{\mathcal{X}}_{\mathbb{F}_p}, \end{array}$$

where the horizontal maps are either Frob or Ver. It follows that Frob and Ver are closed immersions of $\mathcal{X}_{\mathbb{F}_p}$ into $\tilde{\mathcal{X}}_{\mathbb{F}_p}$, that the union of their images is all of $\tilde{\mathcal{X}}_{\mathbb{F}_p}$, and that the intersections of the images of Frob and Ver is exactly the supersingular locus on $\tilde{\mathcal{X}}_{\mathbb{F}_p}$. Moreover, it follows that $\tilde{\mathcal{X}}_{\mathbb{F}_p}$ is nonsingular away from the supersingular points of $\tilde{\mathcal{X}}_{\mathbb{F}_p}$, and that the completion of the strict henselization of the local ring of $\tilde{\mathcal{X}}$ at a supersingular point on the special fiber at p is $W(\overline{\mathbb{F}}_p)[[X,Y]]/(XY-p)$.

There are natural maps from any fine moduli stack to its underlying coarse moduli space, by the universal property of coarse moduli spaces. The results of [DeRa], section I.8, show that the maps $\mathcal{X} \to X$ and $\tilde{\mathcal{X}} \to \tilde{X}$ are ramified precisely at those points x for which $k(x) > 1$. Fix such a point x on $\mathcal{X}$. As in [DeRa], section I.8, the automorphism group G of x acts on the local ring of $\mathcal{X}$ at x, and the natural map: $\pi : \mathcal{X} \to X$ identifies the local ring of X at $\pi(x)$ with the G-invariants of the local ring of $\mathcal{X}$ at x. Since x is a regular point of $\mathcal{X}$, the strict henselization of the local ring of $\mathcal{X}$ at x is isomorphic to $W(\overline{\mathbb{F}}_p)[[X]]$; the same argument as in the proof of [DeRa], VI.6.9 shows that the ring of G-invariants of this ring is $W(\overline{\mathbb{F}}_p)[[X^{k(x)}]]$. Thus $\pi(x)$ is a regular point of $\mathcal{X}$, so $\mathcal{X}$ is regular.

Similarly, if x is a regular point of $\tilde{\mathcal{X}}$, and $\tilde{\pi}$ is the surjection $\tilde{\mathcal{X}} \to \tilde{X}$, then $\tilde{\pi}(x)$ is a regular point of $\tilde{X}$. On the other hand, if x is a supersingular point of $\tilde{\mathcal{X}}$, then the completion of the strict henselization of the local ring of $\tilde{X}$ at $\tilde{\pi}(x)$ is isomorphic to the ring of G-invariants of $W(\overline{\mathbb{F}}_p)[[X,Y]]/(XY-p)$; by the same argument as the proof of [DeRa], VI.6.9, the ring of G-invariants of this ring is isomorphic to $W(\overline{\mathbb{F}}_p)[[X,Y]]/(XY-p^{k(x)})$.

Now consider the morphisms Frob and Ver from $\mathcal{X}_{\mathbb{F}_p}$ to $\tilde{\mathcal{X}}_{\mathbb{F}_p}$. Since any automorphism of a "fake elliptic curve" preserves the kernels of Frobenius and Vershiebung, we have $k(\mathrm{Frob}(x)) = k(x)$, and similarly for Ver. Composing Frob and Ver with $\tilde{\pi}$, we obtain morphisms from $\mathcal{X}_{\mathbb{F}_p}$ to $\tilde{X}_{\mathbb{F}_p}$. The local ring calculations given above then show that these morphisms factor through π, to give morphisms $X_{\mathbb{F}_p} \to \tilde{X}_{\mathbb{F}_p}$. By abuse of notation, we use Frob and Ver to refer to these morphisms as well. The local ring calculations above show that these induce surjections on local rings, and are therefore closed immersions. Moreover, the union of their images is all of $\tilde{X}_{\mathbb{F}_p}$, and these images intersect exactly at points which are images of supersingular points of $\tilde{\mathcal{X}}_{\mathbb{F}_p}$. □

The above model has implications for the "mod p reduction" of $J^D(\Gamma_0(p) \cap \Gamma)$. To make this precise we introduce the concept of a Néron model of an abelian variety.

Definition 1.7.3 *Let J be an abelian variety over $\mathbb{Q}$. The* Néron model $J_{\mathbb{Z}}$ *of J over $\mathbb{Z}$ is a smooth group scheme over $\mathbb{Z}$ whose fiber over $\mathbb{Q}$ is J, which has the following universal property: For any smooth scheme X over $\mathbb{Z}$, and any map $\phi : X_{\mathbb{Q}} \to J$, ϕ extends uniquely to a morphism $X \to J_{\mathbb{Z}}$.*

For a proof of the existence and uniqueness of Néron models, we refer the reader to [Ar]. This definition gives us a well-defined notion of the "mod p reduction of J", namely, the special fiber of the Néron model of J over $\mathbb{F}_p$.

We now present some results showing how the special fiber of the Néron model of the Jacobian of a curve C over $\mathbb{F}_p$ can be described in terms of a minimal regular model $\mathcal{C}$ of C. The results we use can all be found in section 12 of [SGA]. Alternatively, Ribet ([Ri5], section 2) presents a concise summary of just those results we will need.

In particular, suppose that the greatest common divisor of the multiplicities of the irreducible components of the special fiber of $\mathcal{C}$ over $\mathbb{F}_p$ is 1, and that all the singularities of the special fiber are ordinary double points. The normalization of the special fiber of $\mathcal{C}$ can be written as the disjoint union of nonsingular curves D_i over $\mathbb{F}_p$.

Definition 1.7.4 *Let A be an abelian variety over $\mathbb{Q}$. We say A has* semistable reduction *at p if the connected component of the identity in the special fiber at p of the Néron model of A is an extension of an abelian variety by a torus T. We call T the* maximal subtorus *of A modulo p. The* character group $X_p(A)$ *of A at p is the group* $\mathrm{Hom}_{\text{gr-sch}}(T, \mathbb{G}_m)$*, where* $\mathrm{Hom}_{\text{gr-sch}}$ *denotes morphisms of group schemes over $\overline{\mathbb{F}}_p$. It is a free $\mathbb{Z}$-module of rank equal to the dimension of T.*

Let J be the Jacobian of C, and $\mathcal{J}$ its Néron model. In such a setting J will have semistable reduction at p. More precisely, we have:

Proposition 1.7.5 *Let $\mathcal{J}_p^0$ denote the connected component of the identity in the special fiber $\mathcal{J}_p$ of $\mathcal{J}$ over $\mathbb{F}_p$. The maps: $D_i \to \mathcal{C}_{\mathbb{F}_p}$ induce a surjection:*

$$\mathcal{J}_p^0 \to \Pi J(D_i),$$

where $J(D_i)$ is the Jacobian of D_i. The kernel of this map is a torus T, i.e. a group scheme which becomes isomorphic to $\mathbb{G}_m^n$ over $\overline{\mathbb{F}}_p$.

Moreover, the torus can be described explicitly in terms of the special fiber of $\mathcal{C}$. In particular, let G be the graph whose vertices are the components D_i of $\mathcal{C}_{\mathbb{F}_p}$, and whose edges between D_i and D_j are the points of intersection of D_i and D_j. Then one has $T \cong H^1(G, \mathbb{Z}) \otimes \mathbb{G}_m$, where $H^1(G, \mathbb{Z})$ is the integral cohomology of G. In particular the character group $X_p(\mathcal{J})$ of T is $H_1(G, \mathbb{Z})$. (This is [SGA], 12.3.7.)

Moreover, G comes equipped with a natural action of the absolute Galois group of $\mathbb{F}_p$ (coming from the Galois action on the special fiber of $\mathcal{C}$), and the above isomorphism is compatible with this action. In particular, the Galois action on T can be read off of G.

Let us now specialize to the case where the curve under consideration is $X^D(\Gamma_0(p) \cap \Gamma)$. To simplify notation, write $\tilde{X}$ for $X^D(\Gamma_0(p) \cap \Gamma)$, $\tilde{J}$ for $J^D(\Gamma_0(p) \cap \Gamma)$, X for $X^D(\Gamma)$, and $\tilde{X}$ for $X^D(\Gamma_0(p) \cap \Gamma)$. Let $J_{\mathbb{F}_p}$ be the Jacobian of X over $\mathbb{F}_p$ (recall that X is smooth over $\mathbb{F}_p$, so this makes sense), and let $\tilde{J}_{\mathbb{F}_p}$ be the special fiber of the Néron model of $\tilde{J}$ over $\mathbb{F}_p$.

Theorem 1.7.6 *(Ribet) We have an exact sequence:*

$$0 \to T \to \tilde{J}^0_{\mathbb{F}_p} \to J^2_{\mathbb{F}_p} \to 0,$$

where T is a torus. Moreover:

1. *The torus T is stable under the action of the Hecke operators, and the induced action of the Hecke the operators T_l for l prime to p on $J^2_{\mathbb{F}_p}$ is the diagonal action.*
2. *The operator T_p acts on $J^2_{\mathbb{F}_p}$ via the matrix $\begin{pmatrix} 0 & -1 \\ p\langle p\rangle & \tau_p \end{pmatrix}$.*
3. *The group $X_p(\tilde{J})$ is canonically isomorphic (up to a choice of sign) to the group of degree zero divisors on the supersingular points of the special fiber of $\tilde{X}$ over $\mathbb{F}_p$.*
4. *The Frobenius element of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ acts on T as w_p.*

Proof. Proofs of these statements can be found in [Ri5], section 3, when $D = 1$ and $\Gamma = \Gamma_0(N)$. Ribet's argument carries over to the general case without difficulty. We therefore only give a sketch here. Our calculation of the special fiber of $\tilde{X}$ over $\mathbb{F}_p$ shows that a minimal regular model for $\tilde{X}$ can be obtained by blowing up each supersingular point x on the special fiber $k(x) - 1$ times. This has the effect of replacing x by a chain of $k(x) - 1$ copies of the projective line.

The normalization of the special fiber is therefore two copies of $X_{\mathbb{F}_p}$, together with some number of projective lines. The latter have trivial Jacobians, so the "abelian variety" part of $\tilde{J}^0_{\mathbb{F}_p}$ is a product of two copies of the Jacobian of $X_{\mathbb{F}_p}$. The surjection onto these two copies is given by Frob^* on one component and Ver^* on the other. The statements about the Hecke action on the above sequence follows.

As for $X_p(\tilde{J})$, observe that the graph corresponding to $\tilde{X}_{\mathbb{F}_p}$ consists of two vertices corresponding to the two copies of $X_{\mathbb{F}_p}$, joined by one path for each supersingular point of $\tilde{X}_{\mathbb{F}_p}$. Along the path corresponding to x, there are $k(x) - 1$ vertices of valency two. The homology of this graph is just a formal sum of these paths, of degree zero; up to a choice of sign, this is canonically in bijection with the set of degree zero formal integer combinations of supersingular points of $\tilde{X}_{\mathbb{F}_p}$.

Finally, for the Frobenius action, observe that both Frobenius and w_p send a supersingular point $(E, E^{(p)}, \mathrm{Frob}_E, P)$ to $(E^{(p)}, E, \mathrm{Frob}_{E^{(p)}}, \mathrm{Frob}_E(P))$. □

Observe that the rank of $X_p(\tilde{J})$ over $\mathbb{Z}$ is equal to the dimension of $\tilde{J}$ minus twice the dimension of J. This quantity is equal to the dimension of the p-new subvariety of $\tilde{J}$. Moreover, we have:

Theorem 1.7.7 *(Ribet) The action of* $\mathbb{T}^D(\Gamma_0(p)\cap\Gamma)$ *on* $X_p(\tilde{J})$ *factors through* $\mathbb{T}^D(\Gamma_0(p)\cap\Gamma)^{p\text{-new}}$*. Moreover, the action of* $\mathbb{T}^D(\Gamma_0(p)\cap\Gamma)^{p\text{-new}}$ *on* $X_p(\tilde{J})$ *is faithful.*

Proof. This is [Ri5], Theorem 3.10, although Ribet's definition of the Hecke operators differs from the one we give here by a Rosati involution. □

Lemma 1.7.8 *Let* R *be a zero-dimensional Gorenstein* $\mathbb{Q}$*-algebra of finite rank, and* M *a faithful* R*-module, whose dimension as a vector space over* $\mathbb{Q}$ *is the same as that of* R*. Then* M *is free of rank one over* R*.*

Proof. It suffices to show that M contains a submodule that is free of rank one over R. Moreover, since R is zero-dimensional, R is equal to the product of the rings $R_{\mathfrak{p}}$, where $\mathfrak{p}$ is a maximal ideal of R, and similarly M is equal to the product over all $\mathfrak{p}$ of $M_{\mathfrak{p}}$. We may thus assume R is local, with maximal ideal $\mathfrak{p}$. Since R is Gorenstein, the submodule $R[\mathfrak{p}]$ of elements of R killed by $\mathfrak{p}$ is a one-dimensional $R/\mathfrak{p}$-vector space, generated by a single element σ.

Since M is faithful as an R-module, there is some element x of M not killed by σ. Let I be the annihilator of x. Then I is killed by some power of $\mathfrak{p}$, so I intersects $A[\mathfrak{p}]$ nontrivially unless it is the zero ideal. Since $R[\mathfrak{p}]$ is one-dimensional, generated by σ, this is only possible if I contains σ. Since I does not, I must be the zero ideal, so the submodule of M generated by x is free of rank one over R. □

Corollary 1.7.9 *The module* $X_p(\tilde{J})\otimes\mathbb{Q}$ *is free of rank one over* $\mathbb{T}^D(\Gamma_0(p)\cap\Gamma)^{p\text{-new}}\otimes\mathbb{Q}$*.*

Proof. We know that $\mathbb{T}^D(\Gamma_0(p)\cap\Gamma)^{p\text{-new}}\otimes\mathbb{Q}$ is Gorenstein and finite dimensional over $\mathbb{Q}$, and that its action on $X_p(\tilde{J})\otimes\mathbb{Q}$ is faithful. It thus suffices to show that they have the same dimension as $\mathbb{Q}$-vector spaces. But the $\mathbb{Z}$-ranks of both $X_p(\tilde{J})$ and $\mathbb{T}^D(\Gamma_0(p)\cap\Gamma)$ are equal to the dimension of the abelian variety $\tilde{J}^{p\text{-new}}$, so the result follows. □

In addition to the character groups described above, there is one more geometric invariant of the mod p reduction of $\tilde{J}$ which we will need to understand.

Definition 1.7.10 *Let* A *be an abelian variety. The set of connected components of the special fiber of the Néron model of* A *has the natural structure of a finite, flat group scheme, called the* component group of A at p*, and denoted* $\Phi_p(A)$*.*

If A has semistable reduction, one can actually recover the component group of A at p from the character groups of A and $A^\vee$ at p:

Theorem 1.7.11 *([SGA], 11.5.2) Let* A *have semistable reduction. There is a natural bilinear pairing,* $X_p(A)\times X_p(A^\vee)\to\mathbb{Z}$*, on the character groups of* A *and* $A^\vee$*, called the* monodromy pairing *associated to* A*. If* f *is an endomorphism of* A*, then the adjoint of* f *under this pairing is the endomorphism* $f^\vee$ *of* $A^\vee$*. Moreover, the component group* $\Phi_p(A)$ *at* p *is naturally isomorphic to the cokernel of the map* $X_p(A^\vee)\to\operatorname{Hom}(X_p(A),\mathbb{Z})$ *induced by the monodromy pairing.*

The action of the Hecke operators on $\tilde{J}$ makes $X_p(\tilde{J})$ into a module over $\mathbb{T}^D(\Gamma_0(p)\cap\Gamma)$. Moreover, the monodromy pairing on $X_p(\tilde{J})$ can be computed explicitly. Once this is done, one obtains the following result on the component group $\Phi_p(\tilde{J})$:

Theorem 1.7.12 *(Ribet) Let* $\mathfrak{m}$ *be a maximal ideal of* $\mathbb{T}^D(\Gamma_0(p)\cap\Gamma)$ *which is in the support of* $\Phi_p(\tilde{J})$*. Then either* $\mathfrak{m}$ *is Eisenstein, or* $\mathfrak{m}$ *has characteristic dividing* 6*. Moreover, if* Γ *is equal to* $\Gamma_0(N)$ *for some* N*, or* Γ *is contained in* $\Gamma_1(N)$ *for some* $N > 3$*, then* $\mathfrak{m}$ *is Eisenstein.*

Proof. A proof can be found in [Ri7] in the case when $D = 1$; it carries over without change to larger values of D. □

1.8 Ribet's exact sequence

We now turn to a different question about the reduction of Jacobians of Shimura curves- the question of describing the reduction modulo a prime p dividing D. In particular, consider the variety $J^D(\Gamma)$, and let p be a prime dividing D.

Theorem 1.8.1 *The variety* $J^D(\Gamma)$ *has* purely toric *reduction at* p*. In other words, the the special fiber at* p *of the Néron model of* $J^D(\Gamma)$ *is an extension of a torus* T *by the finite group* $\Phi_p(J^D(\Gamma))$*.*

Proof. This is proven in the discussion at the beginning of section 4 of [Ri5], using Cerednik's construction of a minimal regular model for $X^D(\Gamma)$ over $\mathbb{Z}[\frac{1}{N}]$. Ribet assumes that D is the product of two primes here, but his argument works in general. □

In particular, the dimension of T is equal to the dimension of $J^D(\Gamma)$. Thus the rank of $X_p(J^D(\Gamma))$ is equal to the dimension of $J^D(\Gamma)$, and hence is equal to the rank (over $\mathbb{Z}$) of $\mathbb{T}^D(\Gamma)$. Moreover, since $\mathbb{T}^D(\Gamma)$ acts faithfully on $J^D(\Gamma)$, it also acts faithfully on T, and hence acts faithfully on $X_p(J^D(\Gamma))$ as well. By Lemma 1.7.8, it follows that $X_p(J^D(\Gamma)) \otimes \mathbb{Q}$ is free of rank one over $\mathbb{T}^D(\Gamma) \otimes \mathbb{Q}$.

We have introduced the group $X_p(J^D(\Gamma))$ because of the following remarkable result of Ribet, which relates the geometry of Shimura curves of differing discriminants. This result will allow us to prove the Jacquet-Langlands correspondence, stated earlier, and should be thought of as an integral version of the Jacquet-Langlands correspondence:

Theorem 1.8.2 *([Ri5], Theorem 4.1) Let* D *be the product of an even number of primes, and let* p *and* q *be two distinct primes not dividing* D*. Let* Γ *be a congruence subgroup of level* N*, where* N *is prime to* Dpq*. Then there is an exact sequence:*

$$0 \to X_q(J^{Dpq}(\Gamma)) \to X_p(J^D(\Gamma_0(pq) \cap \Gamma)) \to X_p(J^D(\Gamma_0(p) \cap \Gamma)^2) \to 0,$$

which is compatible with the action of the Hecke operators T_l *and the diamond bracket operators* $\langle d \rangle$ *for all* l *and* d*.*

Proof. A proof is given in [Ri5] when $D = 1$. The proof generalizes immediately to arbitrary D once one has the results of the previous section on the mod p reduction of $J^D(\Gamma_0(p) \cap \Gamma)$. □

Here the map

$$X_p(J^D(\Gamma_0(pq) \cap \Gamma)) \to X_p(J^D(\Gamma_0(p) \cap \Gamma)^2)$$

is induced by the map

$$i_q(x,y) = \alpha_q^* x + \beta_q^* y : J^D(\Gamma_0(p)\cap\Gamma)^2 \to J^D(\Gamma_0(pq)\cap\Gamma).$$

On the other hand, the map

$$X_q(J^{Dpq}(\Gamma)) \to X_p(J^D(\Gamma_0(pq)\cap\Gamma))$$

is quite mysterious; for one thing, it is a map between character groups in two different characteristics! Ribet constructs this map by characterizing the components and singular points of the mod q reduction of $X^{Dpq}(\Gamma)$ in terms of modules over an order of certain a quaternion algebra. He obtains a similar characterization of the singular points on the mod p reductions of $X^D(\Gamma_0(pq)\cap\Gamma)$ and $X^D(\Gamma_0(p)\cap\Gamma)$. Amazingly, these characterizations turn out to be closely related, and the upshot is the exact sequence above. The clearest exposition of these ideas can be found in [Ri4].

We are now in a position to give a proof of Theorem 1.4.1. This will require much of the machinery of this chapter, including some results which depend on special cases of Theorem 1.4.1. We will thus have to work inductively on D, being very careful to avoid circular reasoning.

In particular, fix a D, and assume we have proven the Jacquet-Langlands correspondence between discriminants D' and $D'rs$ for all D' and rs such that $D'rs$ divides D. We will then establish the Jacquet-Langlands correspondence between discriminants D and Dpq. This inductive hypothesis implies immediately that $X_p(J^D(\Gamma_0(pq)\cap\Gamma))\otimes\mathbb{C}$ is free of rank one over $\mathbb{T}^D(\Gamma_0(pq)\cap\Gamma)^{p\text{-new}}\otimes\mathbb{C}$, and $X_p(J^D(\Gamma_0(p)\cap\Gamma)^2)\otimes\mathbb{C}$ is free of rank one over $\mathbb{T}^D(\Gamma_0(pq)\cap\Gamma)^{p\text{-new},q\text{-old}}$. On the other hand, we do *not* yet know that $X_q(J^{Dpq}(\Gamma))\otimes\mathbb{C}$ is free of rank one over $\mathbb{T}^{Dpq}(\Gamma)\otimes\mathbb{C}$, as this depends on the Jacquet-Langlands correspondence between D and Dpq.

First, observe that the Hecke-equivariance of Ribet's exact sequence means that $X_q(J^{Dpq}(\Gamma))$ acquires an action of $\mathbb{T}^D(\Gamma_0(pq)\cap\Gamma)$, such that the actions of T_l considered as an element of $\mathbb{T}^D(\Gamma_0(pq)\cap\Gamma)$ coincides with the action of T_l as an element of $\mathbb{T}^{Dpq}(\Gamma)$. Since the action of $\mathbb{T}^{Dpq}(\Gamma)$ on $X_q(J^{Dpq,Dpq})$ is faithful, we can view $\mathbb{T}^{Dpq}(\Gamma)$ as the subalgebra of $\mathrm{End}(X_q(J^{Dpq,Dpq}))$ generated by the operators T_l and $\langle l\rangle$. In this fashion we realize $\mathbb{T}^{Dpq}(\Gamma)$ as a quotient of $\mathbb{T}^D(\Gamma_0(pq)\cap\Gamma)$.

Tensoring Ribet's exact sequence with $\mathbb{C}$, we obtain an exact sequence:

$$0 \to M_{Dpq} \to M_D^{p\text{-new}} \to M_D^{p\text{-new},q\text{-old}} \to 0,$$

where $M_D^{p\text{-new}}$ (resp. $M_D^{p\text{-new},q\text{-old}}$) is free of rank one over $\mathbb{T}^D(\Gamma_0(pq)\cap\Gamma)^{p\text{-new}}\otimes\mathbb{C}$ (resp. $\mathbb{T}^D(\Gamma_0(pq)\cap\Gamma)^{p\text{-new},q\text{-old}}\otimes\mathbb{C}$). The fact that Ribet's exact sequence is compatible with the action of the Hecke operators implies that we can view this exact sequence as an exact sequence of $\mathbb{T}^D(\Gamma_0(pq)\cap\Gamma)^{p\text{-new}}\otimes\mathbb{C}$-modules. Since $\mathbb{T}^D(\Gamma_0(pq)\cap\Gamma)^{p\text{-new}}\otimes\mathbb{C}$ is isomorphic to the product of $\mathbb{T}^D(\Gamma_0(pq)\cap\Gamma)^{pq\text{-new}}\otimes\mathbb{C}$ and $\mathbb{T}^D(\Gamma_0(pq)\cap\Gamma)^{p\text{-new},q\text{-old}}\otimes\mathbb{C}$, the above exact sequence splits. In particular M_{Dpq} is free of rank one as a $\mathbb{T}^D(\Gamma_0(pq)\cap\Gamma)^{pq\text{-new}}\otimes\mathbb{C}$-module.

Let $\mathbb{T}^{\mathrm{univ}}$ denote the free $\mathbb{Z}$-algebra generated by the symbols T_l for l prime and $\langle d\rangle$ for $d\in(\mathbb{Z}/N\mathbb{Z})^\times$. Then we have surjections of $\mathbb{T}^{\mathrm{univ}}$ onto $\mathbb{T}^{Dpq}(\Gamma)$ and $\mathbb{T}^D(\Gamma_0(pq)\cap\Gamma)^{pq\text{-new}}$. Since both of the latter act on M_{Dpq}, we obtain two actions of $\mathbb{T}^{\mathrm{univ}}$ on M_{Dpq}; since

Ribet's exact sequence is compatible with the action of the Hecke operators, these actions *coincide*. In particular, the kernel of the map $\mathbb{T}^{\mathrm{univ}} \to \mathbb{T}^{Dpq}(\Gamma)$ is equal to the kernel of the map $\mathbb{T}^{\mathrm{univ}} \to \mathbb{T}^{D}(\Gamma_0(pq) \cap \Gamma)^{pq\text{-new}}$. The surjection of $\mathbb{T}^{\mathrm{univ}}$ onto $\mathbb{T}^{Dpq}(\Gamma)$ thus factors through $\mathbb{T}^{D}(\Gamma_0(pq) \cap \Gamma)^{pq\text{-new}}$ and induces an isomorphism of the latter with $\mathbb{T}^{Dpq}(\Gamma)$. This isomorphism sends T_l considered as an element of one algebra to T_l considered as an element of the other. It follows that M_{Dpq} is free of rank one over $\mathbb{T}^{Dpq}(\Gamma) \otimes \mathbb{C}$.

Finally, since $S_2^{Dpq}(\Gamma)$ is free of rank one over $\mathbb{T}^{Dpq}(\Gamma) \otimes \mathbb{C}$, and $S_2^{D}(\Gamma_0(pq) \cap \Gamma)^{pq\text{-new}}$ is free of rank one over $\mathbb{T}^{D}(\Gamma_0(pq) \cap \Gamma)^{pq\text{-new}} \otimes \mathbb{C}$, these spaces are isomorphic as well (although not canonically). $\square$

Remark 1.8.3 Ribet's exact sequence is thus a stronger result than the Jacquet-Langlands correspondence. Unlike the Jacquet-Langlands correspondence, it gives a canonical map between geometric objects attached to the spaces of modular forms. It also gives information about the action of the Hecke operators T_p for p dividing the level, unlike the original result of Jacquet-Langlands. It will therefore play a very prominent role in our computation of the kernels of morphisms between Jacobians of Shimura curves of varying discriminant.

Chapter 2

Jacquet-Langlands isogenies

We are now in a position to begin our study of the isogenies between the abelian varieties considered in the last chapter. As explained in the introduction, we will do so by studying the category of abelian varieties with a $\mathbb{T}^D(\Gamma)$ action which are $\mathbb{T}^D(\Gamma)$-equivariantly isogenous to $J^D(\Gamma)$, and relating this category to a certain category of modules over $\mathbb{T}^D(\Gamma)$. Before we begin this study, we explain some conventions and notation which we use in this setting.

One thing we will need is a generalization of the concepts of new subvarieties and quotients to an arbitrary abelian variety with a $\mathbb{T}^D(\Gamma)$ action. We construct these subvarieties and quotients via the $\mathbb{T}^D(\Gamma)$ action. In particular, we take the U-new subvariety of J to be the connected component of the subvariety of J annihilated by the kernel I of the map $\mathbb{T}^D(\Gamma) \to \mathbb{T}^D(\Gamma)^{U\text{-new}}$, and the U-new quotient of J to be J/IJ. The U-old subvariety and quotient of J are constructed in a similar manner. It is straightforward to check that this construction agrees with the previous definition of $J^D(\Gamma)^{U\text{-new}}$ and $J^D(\Gamma)_{U\text{-new}}$.

We have the following relationship between the subvarieties and quotients defined above and duality of abelian varieties:

Proposition 2.0.4 *Suppose J is isogenous to $J^D(\Gamma)$. Then have $(J_{U\text{-new}})^\vee \cong (J^\vee)^{U\text{-new}}$.*

Proof. Consider the exact sequence:

$$0 \to J_{U\text{-new}} \to J \to A \to 0.$$

Taking duals, we obtain an exact sequence:

$$0 \to A^\vee \to J^\vee \to (J_{U\text{-new}})^\vee \to 0.$$

(Here the action of $\mathbb{T}$ on $J^\vee$ is induced by duality of abelian varieties.) It thus suffices to show that we can identify $A^\vee$ with $IJ^\vee$, where I is the kernel of the map $\mathbb{T}^D(\Gamma) \to \mathbb{T}^D(\Gamma)^{U\text{-new}}$. Since I annihilates $(J_{U\text{-new}})^\vee$, we must have $IJ^\vee \subset A$. Counting ranks of abelian varieties, we find that the two must be equal. $\square$

The connection between abelian varieties with $\mathbb{T}^D(\Gamma)$ action and modules for $\mathbb{T}^D(\Gamma)$ mean that we will often be considering modules over a finite, flat $\mathbb{Z}$-algebra $\mathbb{T}$. If M is a $\mathbb{T}$-module, and $\mathfrak{m}$ is a maximal ideal of $\mathbb{T}$, then $M_{\mathfrak{m}}$ denotes the *completion* of M at $\mathfrak{m}$.

If M is free as a $\mathbb{Z}$-module, then M^* denotes the $\mathbb{Z}$-dual $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, with the $\mathbb{T}$-module structure induced from M. If M is free as a $\mathbb{Z}_l$-module for some l, on the other hand, then by abuse of notation we write M^* for $\mathrm{Hom}_{\mathbb{Z}_l}(M, \mathbb{Z}_l)$. The advantage of this notation is that $(M_{\mathfrak{m}})^* \cong (M^*)_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of $\mathbb{T}$.

The $\mathbb{T}$-modules we are most interested in are all torsion free over $\mathbb{Z}$. Unfortunately, the tensor product of two $\mathbb{T}$-modules without $\mathbb{Z}$-torsion often has $\mathbb{Z}$-torsion, which we wish to ignore. Therefore we adopt the convention that if M and M' are torsion free $\mathbb{T}$-modules or $\mathbb{T}_{\mathfrak{m}}$-modules, then $M \otimes M'$ denotes their tensor product (over $\mathbb{T}$ or $\mathbb{T}_{\mathfrak{m}}$) *modulo $\mathbb{Z}$-torsion.* It can be verified that this operation satisfies the universal property of the tensor product in the category of $\mathbb{T}$-modules without $\mathbb{Z}$-torsion. Note that if I and I' are ideals of $\mathbb{T}$, then $I \otimes I' \cong II'$ under this convention, as the natural map sending $x \otimes x'$ to xx' is a surjection by construction, and the modules in question have the same rank over $\mathbb{Z}$. Thus the kernel is finite, and hence trivial since under our convention $I \otimes I'$ is torsion free.

We will find the following observation about these two operations to be useful in what follows:

Lemma 2.0.5 *Let M and N be finitely generated, torsion free $\mathbb{T}$-modules. We have natural isomorphisms:*

1. $M^* \cong \mathrm{Hom}_{\mathbb{T}}(M, \mathbb{T}^*)$
2. $(M \otimes N)^* \cong \mathrm{Hom}_{\mathbb{T}}(M, N^*)$
3. $\mathrm{Hom}_{\mathbb{T}}(M, N)^* \cong M \otimes N^*$.

Proof. We have $M^* \cong \mathrm{Hom}_{\mathbb{T}}(\mathbb{T}, M^*)$. Taking $\mathbb{Z}$-duals, we find that the latter is naturally isomorphic to $\mathrm{Hom}_{\mathbb{T}}(M, \mathbb{T}^*)$. In particular, $(M \otimes N)^*$ is isomorphic to $\mathrm{Hom}_{\mathbb{T}}(M \otimes N, \mathbb{T}^*)$, which in turn is canonically isomorphic to $\mathrm{Hom}_{\mathbb{T}}(M, \mathrm{Hom}_{\mathbb{T}}(N, \mathbb{T}^*))$, and the second identity follows. The last identity follows from the previous one by replacing N with N^* and dualizing. □

2.1 Morphisms "up to support on S"

In what follows, we will often find it convenient to work in categories of abelian varieties in which certain morphisms have been formally inverted. In particular, we will fix a finite flat $\mathbb{Z}$-algebra $\mathbb{T}$ and an abelian variety $J_{/\mathbb{Q}}$ on which $\mathbb{T}$ acts faithfully.

Definition 2.1.1 *Let $\mathcal{A}$ be the category whose objects consist of abelian varieties J' with an action of $\mathbb{T}$, such that there exists a $\mathbb{T}$-equivariant isogeny $\phi : J \to J'$ defined over $\mathbb{Q}$, and whose morphisms are $\mathbb{T}$-equivariant maps of abelian varieties defined over $\mathbb{Q}$.*

We are interested in the behavior of maps in this category up to support on a certain finite set S of maximal ideals of $\mathbb{T}$. In order to exclude phenomena that happen solely at maximal ideals inside S, we wish to formally invert morphisms in $\mathcal{A}$ which are isomorphisms at all $\mathfrak{m}$ outside S.

This motivates us to consider the category $\mathcal{A}_S$ of "abelian varieties up to S-isogeny", whose objects are the same as the objects of $\mathcal{A}$, but whose morphisms include formal inverses for those isogenies with kernel supported on S. In order to make this precise, we invoke the following lemma:

Lemma 2.1.2 *Let S be a finite set of maximal ideals of $\mathbb{T}$. Then there is an element σ_S of $\mathbb{T}$ such that for any maximal ideal $\mathfrak{m}$ of $\mathbb{T}$, $\sigma_S \in \mathfrak{m}$ if and only if $\mathfrak{m} \in S$. In particular, $U = \operatorname{Spec} \mathbb{T} - S$ is affine, and equal to $\operatorname{Spec} \mathbb{T}[\frac{1}{\sigma_S}]$.*

Proof. It suffices to show this when $\mathbb{T}$ is reduced, as if σ_S^{red} is an element of $\mathbb{T}^{\mathrm{red}}$ (the reduced quotient of $\mathbb{T}$) with the desired property, then any lift of σ_S^{red} to $\mathbb{T}$ will have the desired property.

Suppose $\mathbb{T}$ is reduced, and let $\hat{\mathbb{T}}$ be its normalization. Then $\hat{\mathbb{T}}$ is a product of maximal orders in number fields, and so its Picard group is finite. In particular, for each maximal ideal $\mathfrak{m}$ of $\mathbb{T}$, we can find an element $\hat{\sigma}_{\mathfrak{m}}$ of $\hat{\mathbb{T}}$ such that $\hat{\sigma}_{\mathfrak{m}}$ lies in $\mathfrak{m}\hat{\mathbb{T}}$ but does not lie in $\mathfrak{m}'\hat{\mathbb{T}}$ for any $\mathfrak{m}'$ different from $\mathfrak{m}$.

Lemma 2.1.3 *For n sufficiently large, $\hat{\sigma}_{\mathfrak{m}}^n$ lies in $\mathbb{T}$.*

Proof. Let N be the index of $\mathbb{T}$ in $\hat{\mathbb{T}}$. It suffices to check this statement locally at each maximal ideal of $\mathbb{T}$ containing N. Since $\hat{\sigma}_{\mathfrak{m}}$ lies in $\mathfrak{m}\hat{\mathbb{T}}$, for some n $\sigma_{\mathfrak{m}}^n$ lies in $N\hat{\mathbb{T}}_{\mathfrak{m}}$ and therefore lies in $\mathbb{T}_{\mathfrak{m}}$. Fix a maximal ideal $\mathfrak{m}'$ of $\mathbb{T}$ distinct from $\mathfrak{m}$ but containing N. Then $\hat{\sigma}_{\mathfrak{m}}$ lies in $\mathbb{T}_{\mathfrak{m}'}^{\times}$. Let n be the order of $(\mathbb{T}_{\mathfrak{m}'}/N\mathbb{T}_{\mathfrak{m}'})^{\times}$. Then $\hat{\sigma}_{\mathfrak{m}}^n$ lies in $1 + N\mathbb{T}_{\mathfrak{m}'} \subset u\mathbb{T}_{\mathfrak{m}'}$, as required. □

We now resume the proof of Lemma 2.1.2. Let $\sigma_{\mathfrak{m}} = \hat{\sigma}_{\mathfrak{m}}^n$. Then $\sigma_{\mathfrak{m}}$ lies in $\mathfrak{m}$, but does not lie in $\mathfrak{m}'$ for any $\mathfrak{m}'$ distinct from $\mathfrak{m}$. Taking σ_S equal to the product of the $\sigma_{\mathfrak{m}}$ gives us a σ_S with the desired properties. □

Let $\mathbb{T}_S = \mathbb{T}[\frac{1}{\sigma_S}]$. We define $\mathcal{A}_S$ to be the category whose objects are the objects of $\mathcal{A}$, but whose morphisms are given by $\operatorname{Hom}_{\mathcal{A}_S}(A_1, A_2) = \operatorname{Hom}_{\mathcal{A}}(A_1, A_2) \otimes_{\mathbb{T}} \mathbb{T}_S$. (The $\mathbb{T}$-module structure on $\operatorname{Hom}_{\mathcal{A}}(A_1, A_2)$ is defined by composition either on the left or the right; since morphisms in $\mathcal{A}$ are $\mathbb{T}$-equivariant it does not matter which.)

This has the desired effect of ignoring behavior supported on S, as any morphism whose kernel is supported on S factors through a sufficiently high power of σ_S, and hence is an isomorphism in $\mathcal{A}_S$. We will make use of this construction in the next section.

Observe that since σ_S is a unit in $\mathbb{T}_{\mathfrak{m}}$ for any $\mathfrak{m}$ outside S, any morphism $A_1 \to A_2$ in $\mathcal{A}_S$ induces a well defined map $A_1[\mathfrak{m}^\infty] \to A_2[\mathfrak{m}^\infty]$ which sends $x \in A_1[\mathfrak{m}^\infty]$ to $\sigma_S^{-n}(\sigma_S^n f)(x)$, for n sufficiently large to make σ_S^n a morphism in $\mathcal{A}$.

It will be convenient to speak of the "kernel" of a map in $\mathcal{A}_S$, even though strictly speaking this notion is not well defined. It is possible to make it precise "up to support on S", however; for any morphism $f : A_1 \to A_2$ in $\mathcal{A}_S$, we can fix an n such that $\sigma_S^n f$ is a morphism in $\mathcal{A}$, and take $\ker_S f = \ker \sigma_S^n f$. This is independent of n up to a finite group supported on S.

Definition 2.1.4 *Let B be a subvariety of A_1 which is stable under $\mathbb{T}$ and closed under addition and inverses, and $f : A_1 \to A_2$ a morphism in $\mathcal{A}_S$. Then $\ker_S f = B$ "up to support on S" if for some n, $\sigma_S^n f$ is a morphism in $\mathcal{A}$, $\ker \sigma_S^n f$ contains B, and the group $(\ker \sigma_S^n f)/B$ is finite and supported on S. (Equivalently, $\ker_S f = B$ "up to support on S" if $B[\mathfrak{m}^\infty] = \ker f : A_1[\mathfrak{m}^\infty] \to A_2[\mathfrak{m}^\infty]$ for all $\mathfrak{m}$ outside S.)*

2.2 The isogeny class of an abelian variety with $\mathbb{T}$-action

Our goal is to study the isogeny class of $J^D(\Gamma)_{pq\text{-new}}$. Because what we prove in this section holds in much more generality than just for $J^D(\Gamma)_{pq\text{-new}}$, we work in the following setting: let J be an abelian variety, defined over $\mathbb{Q}$, and suppose that J comes equipped with the action of a finite, flat $\mathbb{Z}$-algebra $\mathbb{T}$ such that the endomorphisms of J coming from elements of $\mathbb{T}$ are all defined over $\mathbb{Q}$.

If $\mathfrak{m}$ is a maximal ideal of $\mathbb{T}$ of residue characteristic l, let $T_{\mathfrak{m}}J$ denote the $\mathfrak{m}$-adic *contravariant* Tate module of J at $\mathfrak{m}$. Recall that this is the inverse limit of $J[\mathfrak{m}^k]^\vee$ over k, where $(-)^\vee$ denotes Cartier duality. Thus $T_{\mathfrak{m}}J$ is naturally isomorphic to $\mathrm{Hom}(J[\mathfrak{m}^\infty], \mu_{l^\infty})$. If $\alpha \in T_{\mathfrak{m}}J$ and $x \in J[\mathfrak{m}^\infty]$, we denote by $\alpha(x)$ the root of unity obtained by evaluating the element of $\mathrm{Hom}(J[\mathfrak{m}^\infty], \mu_{l^\infty})$ which corresponds to α at the element x. For a morphism $f : A_1 \to A_2$ of abelian varieties with $\mathbb{T}$-action, we let $f_{\mathfrak{m}}$ denote the induced map $T_{\mathfrak{m}}A_2 \to T_{\mathfrak{m}}A_1$.

The action of $\mathbb{T}$ on J induces an action of $\mathbb{T} \otimes \mathbb{Q}$ on the singular cohomology $H^1(J(\mathbb{C}), \mathbb{Q})$ of J; we assume that this makes $H^1(J(\mathbb{C}), \mathbb{Q})$ into a free $\mathbb{T} \otimes \mathbb{Q}$-module, of rank two. In particular, this means that the action of $\mathbb{T}$ on J is *faithful*.

Since there are natural isomorphisms $H^1(J(\mathbb{C}), \mathbb{Z})_{\mathfrak{m}} \cong T_{\mathfrak{m}}J$, $T_{\mathfrak{m}}J \otimes \mathbb{Q}$ is free of rank two as a $\mathbb{T}_{\mathfrak{m}} \otimes \mathbb{Q}$-module. The action of $G_{\mathbb{Q}}$ on $T_{\mathfrak{m}}J \otimes \mathbb{Q}$ gives us a two-dimensional representation $\rho_{\mathfrak{m}}$ of $G_{\mathbb{Q}}$ over $\mathbb{T}_{\mathfrak{m}} \otimes \mathbb{Q}$. We impose the further hypothesis that $\mathrm{Tr}\,\rho_{\mathfrak{m}}(g)$ and $\det \rho_{\mathfrak{m}}(g)$ (a priori elements of $\mathbb{T}_{\mathfrak{m}} \otimes \mathbb{Q}$), lie in $\mathbb{T}_{\mathfrak{m}}$ for all $\mathfrak{m}$ and g. We further assume that the subalgebra of $\mathbb{T}_{\mathfrak{m}}$ generated by the elements $\mathrm{Tr}\,\rho_{\mathfrak{m}}(g)$ and $\det \rho_{\mathfrak{m}}(g)$ as g varies is *reduced* for all $\mathfrak{m}$.

Finally, we assume that there exists a $\mathbb{T}$-equivariant map $\psi : J \to J^\vee$, defined over a finite extension K of $\mathbb{Q}$, and a character χ of $\mathrm{Gal}(K/\mathbb{Q})$, taking values in $\mathbb{T}^\times$, such that ψ has finite kernel, and for all $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, we have $\sigma\psi(x) = \chi(\sigma)\psi(\sigma(x))$ for all x in J.

Remark 2.2.1 The motivation for these assumptions comes from the example in which $J = J^D(\Gamma)$ for some discriminant D and congruence subgroup Γ of level N, and $\mathbb{T}$ is the Hecke algebra $\mathbb{T}^D(\Gamma)$. In this case we have shown (in the proof of Proposition 1.5.3) that $H^1(J(\mathbb{C}), \mathbb{Q})$ is free of rank two over $\mathbb{T} \otimes \mathbb{Q}$. Moreover, for a prime l not dividing ND, $\mathrm{Tr}\,\rho_{\mathfrak{m}}(\mathrm{Frob}_l)$ is equal to $\langle l\rangle^{-1}T_l \in \mathbb{T}$, and $\det \rho_{\mathfrak{m}}(\mathrm{Frob}_l) = l\langle l\rangle^{-1}$ (Theorem 1.5.4). Since the elements Frob_l for l prime to N are dense in $G_{\mathbb{Q}}$, and $\mathbb{T}_{\mathfrak{m}}$ is complete, the subalgebra of $\mathbb{T}_{\mathfrak{m}}$ generated by the traces and determinants of elements of $G_{\mathbb{Q}}$ is generated by the aforementioned $\langle l\rangle$ and T_l. We saw in section 1.2 that this algebra is reduced. Finally, the map ψ described above is constructed in Corollary 1.2.4. It follows that the hypotheses of this section are satisfied in this case.

The existence of the map ψ described above has strong consequences for the structure of $\mathbb{T}$. In particular, ψ induces an isomorphism $T_{\mathfrak{m}}J \otimes \mathbb{Q} \to (T_{\mathfrak{m}}J)^* \otimes \mathbb{Q}$. Since the former is free of rank two over $\mathbb{T}_{\mathfrak{m}} \otimes \mathbb{Q}$, and the latter is isomorphic to $(\mathbb{T}_{\mathfrak{m}}^* \otimes \mathbb{Q})^2$, it follows that $\mathbb{T}_{\mathfrak{m}}^* \otimes \mathbb{Q}$ is free of rank one over $\mathbb{T}_{\mathfrak{m}} \otimes \mathbb{Q}$. In particular, $\mathbb{T} \otimes \mathbb{Q}$ is a zero-dimensional Gorenstein ring.

Following the analysis of the Tate modules of modular Jacobians in [Ca], we fix a maximal ideal $\mathfrak{m}$ of $\mathbb{T}$, and let $\mathbb{T}'_{\mathfrak{m}}$ denote the subalgebra of $\mathbb{T}_{\mathfrak{m}}$ generated by the traces and determinants of $\rho_{\mathfrak{m}}(g)$ for $g \in G_{\mathbb{Q}}$. By our hypothesis, this is a reduced subalgebra of $\mathbb{T}_{\mathfrak{m}}$. We can thus consider the normalization $\hat{\mathbb{T}}'_{\mathfrak{m}}$ of $\mathbb{T}'_{\mathfrak{m}}$; this will be a product of discrete valuation rings $\mathcal{O}_i$, with corresponding fields of fractions K_i.

For each i, $T_{\mathfrak{m}}J \otimes_{\mathbb{T}'_{\mathfrak{m}}} K_i$ is free of rank two over the Artinian K_i-algebra $\mathbb{T}_{\mathfrak{m}} \otimes_{\mathbb{T}'_{\mathfrak{m}}} K_i$; we consider the representation $\rho_{\mathfrak{m},i}$ of $G_{\mathbb{Q}}$ over $\mathbb{T}_{\mathfrak{m}} \otimes_{\mathbb{T}'_{\mathfrak{m}}} K_i$ obtained in this manner. Since the traces and determinants of $\rho_{\mathfrak{m}}$ lie in $\mathbb{T}'_{\mathfrak{m}}$, the traces and determinants of $\rho_{\mathfrak{m},i}$ lie in $\mathcal{O}_i$ for all i. Thus $\rho_{\mathfrak{m},i}$ can be considered as a representation of $G_{\mathbb{Q}}$ over $\mathcal{O}_i$.

Let k_i denote the residue field of $\mathcal{O}_i$; then we may consider the mod $\mathfrak{m}$ reduction $\overline{\rho}_{\mathfrak{m},i} : G_{\mathbb{Q}} \to \mathrm{GL}_2(k_i)$ of $\rho_{\mathfrak{m},i}$. For $g \in G$, $\mathrm{Tr}\,\overline{\rho}_{\mathfrak{m},i}(g)$ is equal to the reduction of $\mathrm{Tr}\,\rho_{\mathfrak{m}}(g)$ modulo $\mathfrak{m}$, and is therefore independent of i. Thus the semisimplification of $\overline{\rho}_{\mathfrak{m},i}$ is independent of i as well.

With this definition, we make one final assumption on J, namely, that for all but finitely many $\mathfrak{m}$, the representations $\overline{\rho}_{\mathfrak{m},i}$ are absolutely irreducible. For such $\mathfrak{m}$, $\overline{\rho}_{\mathfrak{m},i}$ is independent of i and we denote it simply by $\overline{\rho}_{\mathfrak{m}}$.

Remark 2.2.2 It should be pointed out here that in the case in which J is a modular Jacobian, the $\overline{\rho}_{\mathfrak{m}}$ constructed above is the Cartier dual of the representation usually called $\overline{\rho}_{\mathfrak{m}}$ in the literature. (This is because the latter is defined in terms of the $\mathfrak{m}$-torsion of J, whereas the representations we consider are defined in terms of $T_{\mathfrak{m}}J$ instead.)

Fix a finite set S of maximal ideals of $\mathbb{T}$, containing all $\mathfrak{m}$ for which one (and hence all) of the $\overline{\rho}_{\mathfrak{m},i}$ is not absolutely irreducible. The following result is a straightforward generalization of [Ca], Theorem 3:

Proposition 2.2.3 *For all $\mathfrak{m}$ outside S, there exists a $G_{\mathbb{Q}}$-module $V_{\mathfrak{m}}$, free of rank two over $\mathbb{T}_{\mathfrak{m}}$, such that $V_{\mathfrak{m}} \otimes \mathbb{Q}$ and $T_{\mathfrak{m}}J \otimes \mathbb{Q}$ are isomorphic as $G_{\mathbb{Q}}$-modules.*

Proof. Consider the action of $G_{\mathbb{Q}}$ on $\hat{V}'_{\mathfrak{m}} = (\hat{\mathbb{T}}'_{\mathfrak{m}})^2$ in which $G_{\mathbb{Q}}$ acts on each factor $(\mathcal{O}_i)^2$ of $\hat{\mathbb{T}}'_{\mathfrak{m}}$ by $\rho_{\mathfrak{m},i}$. This gives a representation of $G_{\mathbb{Q}}$ in $\mathrm{GL}_2(\hat{\mathbb{T}}'_{\mathfrak{m}})$ whose character agrees with $\rho_{\mathfrak{m}}$. By [Ca], Theorem 2, this representation is realizable over $\mathbb{T}'_{\mathfrak{m}}$; i.e., there is a $G_{\mathbb{Q}}$-module $V'_{\mathfrak{m}}$ free of rank two over $\mathbb{T}'_{\mathfrak{m}}$ such that the character of the $G_{\mathbb{Q}}$-action agrees with that of $\rho_{\mathfrak{m}}$. Then $V_{\mathfrak{m}} = V'_{\mathfrak{m}} \otimes_{\mathbb{T}'_{\mathfrak{m}}} \mathbb{T}_{\mathfrak{m}}$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$, and $V_{\mathfrak{m}} \otimes \mathbb{Q}$ is equivalent to $\rho_{\mathfrak{m}}$ by [Ca], Theorem 1. □

Our goal is to study $\mathbb{T}$-equivariant isogenies of J which are defined over $\mathbb{Q}$, up to support on S. The first step in this study is to understand the structure of $T_{\mathfrak{m}}J$ for $\mathfrak{m}$ outside S.

Lemma 2.2.4 *For all but finitely many maximal ideals $\mathfrak{m}$ of $\mathbb{T}$, $T_{\mathfrak{m}}J$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$.*

Proof. Since we have assumed that $H^1(J(\mathbb{C}), \mathbb{Q})$ is free of rank two over $\mathbb{T} \otimes \mathbb{Q}$, we can choose two elements of $H^1(J(\mathbb{C}), \mathbb{Z})$ that generate $H^1(J(\mathbb{C}), \mathbb{Q})$ over $\mathbb{T} \otimes \mathbb{Q}$. These two elements give us a map $\mathbb{T}^2 \to H^1(J(\mathbb{C}), \mathbb{Z})$, with finite cokernel. Localizing this map at any $\mathfrak{m}$ on which the cokernel has no support gives the desired isomorphism $\mathbb{T}_{\mathfrak{m}}^2 \cong T_{\mathfrak{m}}J$. □

We will see (Lemma 2.2.6, below) that the Tate modules at such $\mathfrak{m}$ are particularly well-behaved. It will thus be convenient for us to replace J with an isogenous abelian variety, which we call $J^{\min}$, for which $T_{\mathfrak{m}}J^{\min}$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$ for all $\mathfrak{m}$ outside S. The following lemma allows us to do so.

Lemma 2.2.5 *There exists an abelian variety $J^{\min}$, with a $\mathbb{T}$-action satisfying the above hypotheses, such that J and $J^{\min}$ are $\mathbb{T}$-equivariantly isogenous (over $\mathbb{Q}$), and $T_{\mathfrak{m}}J^{\min}$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$ for all $\mathfrak{m}$ outside S.*

Proof. Let S' be the set of $\mathfrak{m}$ outside S at which $T_{\mathfrak{m}}J$ is not free of rank two; then S' is finite. For each $\mathfrak{m}$ in S, fix an isomorphism $V_{\mathfrak{m}} \otimes \mathbb{Q} \cong T_{\mathfrak{m}}J \otimes \mathbb{Q}$. Multiplying this isomorphism by a sufficiently large $n \in \mathbb{Z}$, we may assume that it maps $V_{\mathfrak{m}}$ into $T_{\mathfrak{m}}J$, with finite index. Let $W_{\mathfrak{m}} \subset J[\mathfrak{m}^\infty]$ be the finite subset of the $\mathfrak{m}$-divisible group annihilated by the image of $V_{\mathfrak{m}}$ in $T_{\mathfrak{m}}J$, under the natural pairing $J[\mathfrak{m}^\infty] \times T_{\mathfrak{m}}J \to \mu_{l^\infty}$. If we take $J^{\min}$ to be the quotient of J by the sum of the $W_{\mathfrak{m}}$, we find that the map $T_{\mathfrak{m}}J^{\min} \to T_{\mathfrak{m}}J$ identifies $T_{\mathfrak{m}}J^{\min}$ with the image of $V_{\mathfrak{m}}$ for all $\mathfrak{m}$ in S', and is an isomorphism for $\mathfrak{m}$ outside S'; in either case $T_{\mathfrak{m}}J^{\min}$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$. □

We fix, for the remainder of this section, a $J^{\min}$ as above.

Lemma 2.2.6 *Let M be a $G_{\mathbb{Q}}$-stable submodule of $T_{\mathfrak{m}}J^{\min}$, of finite index. If $\mathfrak{m}$ lies outside S, then $M = IT_{\mathfrak{m}}J^{\min}$ for some ideal I of $\mathbb{T}$.*

Proof. We use induction on the length of a maximal $G_{\mathbb{Q}}$-stable filtration of $T_{\mathfrak{m}}J^{\min}/M$. If this length is zero, then I is the unit ideal and the result is clear. In general, let $M = M_n \subset M_{n-1} \subset \ldots \subset M_0 = T_{\mathfrak{m}}J^{\min}$ be a maximal $G_{\mathbb{Q}}$-stable chain of submodules. By induction we may assume that $M_{n-1} = I'T_{\mathfrak{m}}J^{\min}$ for some I'.

Consider $\mathfrak{m}M_{n-1}+M$. This is $G_{\mathbb{Q}}$-stable, contains M, and is contained in M_{n-1}. It is thus equal to either M or M_{n-1}. Suppose it were equal to M_{n-1}. Then $M/\mathfrak{m}M_{n-1}$ would equal $M_{n-1}/\mathfrak{m}M_{n-1}$, which is impossible since then M would contain a set of generators for M_{n-1} by Nakayama's lemma. Thus $\mathfrak{m}M_{n-1} + M = M$, so M contains $\mathfrak{m}M_{n-1}$.

Let $V_{\overline{\rho}_{\mathfrak{m}}}$ be a two-dimensional $(\mathbb{T}/\mathfrak{m})$ vector space on which $G_{\mathbb{Q}}$ acts via $\overline{\rho}_{\mathfrak{m}}$. As a $G_{\mathbb{Q}}$-module, $M_{n-1}/\mathfrak{m}M_{n-1}$ is isomorphic to $(I'/\mathfrak{m}I') \otimes_{\mathbb{T}/\mathfrak{m}} V_{\overline{\rho}_{\mathfrak{m}}}$, where $G_{\mathbb{Q}}$ acts trivially on $I'/\mathfrak{m}I'$. Let V be the image of M in $M_{n-1}/\mathfrak{m}M_{n-1}$. Since V is $G_{\mathbb{Q}}$-invariant, and $V_{\overline{\rho}_{\mathfrak{m}}}$ is irreducible, V is given by $\hat{V} \otimes V_{\overline{\rho}_{\mathfrak{m}}}$ for some subspace $\hat{V}$ of $I'/\mathfrak{m}I'$. Let I be the preimage of $\hat{V}$ in I'. Then $IT_{\mathfrak{m}}J = M$, since both contain $\mathfrak{m}M_{n-1}$ and are isomorphic to V modulo $\mathfrak{m}M_{n-1}$. □

This result has important consequences for $\mathbb{T}$-equivariant maps from $J^{\min}$. In particular, for any abelian varieties A_1, A_2 which are $\mathbb{T}$-equivariantly isogenous to J over $\mathbb{Q}$, let $\mathrm{Hom}(A_1, A_2)$ (resp. $\mathrm{End}(A_1)$) denote the $\mathbb{T}$-module of $\mathbb{T}$-equivariant maps: $A_1 \to A_2$ (resp. $\mathbb{T}$-equivariant endomorphisms of A_1) which are defined over $\mathbb{Q}$. Then:

Corollary 2.2.7 *The cokernel of the natural map $\mathbb{T} \to \mathrm{End}(J^{\min})$ is supported on S.*

Proof. Fix $\mathfrak{m}$ outside S; we show that the natural map $\mathbb{T}_{\mathfrak{m}} \to \mathrm{End}(J^{\min})_{\mathfrak{m}}$ is an isomorphism. We have an isomorphism $\mathrm{End}(J^{\min})_{\mathfrak{m}} \cong \mathrm{End}_G(T_{\mathfrak{m}}J^{\min})$, by [Fa], Theorem 4. $\mathrm{End}_G(T_{\mathfrak{m}}J^{\min})$ is clearly faithful over $\mathbb{T}_{\mathfrak{m}}$, and finitely generated. It thus suffices to show that $\mathrm{End}_G(T_{\mathfrak{m}}J^{\min}) \otimes \mathbb{T}/\mathfrak{m}$ is 1-dimensional. Observe that $\mathrm{End}_G(J^{\min}[\mathfrak{m}])$ is 1-dimensional, by Schur's lemma and the (absolute) irreducibility of $\overline{\rho}_{\mathfrak{m}}$. We construct an injection of $\mathrm{End}_G(T_{\mathfrak{m}}J^{\min}) \otimes \mathbb{T}/\mathfrak{m}$ into $\mathrm{End}_G(J^{\min}[\mathfrak{m}])$.

The exact sequence

$$0 \to \mathfrak{m}T_{\mathfrak{m}}J^{\min} \to T_{\mathfrak{m}}J^{\min} \to J^{\min}[\mathfrak{m}]^\vee \to 0$$

induces an exact sequence

$$0 \to \mathrm{Hom}_G(T_{\mathfrak{m}}J^{\min}, \mathfrak{m}T_{\mathfrak{m}}) \to \mathrm{End}_G(T_{\mathfrak{m}}J^{\min}) \to \mathrm{Hom}_G(T_{\mathfrak{m}}J^{\min}, J^{\min}[\mathfrak{m}]^{\vee}),$$

and the latter is isomorphic (via Cartier duality) to $\mathrm{End}_G(J^{\min}[\mathfrak{m}])$. It thus suffices to show that $\mathrm{Hom}_G(T_{\mathfrak{m}}J^{\min}, \mathfrak{m}T_{\mathfrak{m}}) = \mathfrak{m}\,\mathrm{End}_G(T_{\mathfrak{m}}J^{\min})$. The former clearly contains the latter. Let $\phi \in \mathrm{Hom}_G(T_{\mathfrak{m}}J^{\min}, \mathfrak{m}T_{\mathfrak{m}})$; by Lemma 2.2.6, the image of ϕ is isomorphic to $IT_{\mathfrak{m}}J^{\min}$ for some $I \subset \mathfrak{m}$. Since $IT_{\mathfrak{m}}J^{\min} \cong T_{\mathfrak{m}}J^{\min}$ and $T_{\mathfrak{m}}J^{\min}$ is free over $\mathbb{T}_{\mathfrak{m}}$, I is principal; say generated by σ. Thus ϕ/σ is a well defined endomorphism of $T_{\mathfrak{m}}J^{\min}$. Since $\sigma \in \mathfrak{m}$, ϕ lies in $\mathfrak{m}\,\mathrm{End}_G(T_{\mathfrak{m}}J^{\min})$ as required. □

Corollary 2.2.8 *Let $\phi : J^{\min} \to A$ be a $\mathbb{T}$-equivariant isogeny. Let $I = \mathrm{Ann}(\ker\phi)$. Then $(\ker\phi)_{\mathfrak{m}} = J^{\min}[I]_{\mathfrak{m}}$ for all $\mathfrak{m}$ outside S.*

Proof. The map ϕ induces an exact sequence

$$0 \to T_{\mathfrak{m}}A \to T_{\mathfrak{m}}J^{\min} \to (\ker\phi)^{\vee}_{\mathfrak{m}} \to 0,$$

In particular it maps $T_{\mathfrak{m}}A$ to a $\mathbb{T}$-stable submodule of $T_{\mathfrak{m}}J^{\min}$. By Lemma 2.2.6, we can find an ideal I' of $\mathbb{T}$ such that the image of $T_{\mathfrak{m}}A$ is $I'_{\mathfrak{m}}T_{\mathfrak{m}}J^{\min}$ for all $\mathfrak{m}$ outside S.

Now $[\ker\phi]^{\vee}_{\mathfrak{m}} = T_{\mathfrak{m}}J^{\min}/I'T_{\mathfrak{m}}J^{\min}$ for all $\mathfrak{m}$ outside S, by the above exact sequence. Thus $I'_{\mathfrak{m}} = I_{\mathfrak{m}}$ for all $\mathfrak{m}$ outside S. Since $T_{\mathfrak{m}}J^{\min}/IT_{\mathfrak{m}}J^{\min} = J^{\min}[I]^{\vee}_{\mathfrak{m}}$, the result follows. □

Note that the above result implies that the isomorphism class of $I_{\mathfrak{m}}$ does not depend on the isogeny ϕ, but only on the variety A. In fact, this is true globally, as well as locally, as long as one works "up to support on S". To see this, fix a ϕ and I as above. Then for $\sigma \in I$, $\ker\phi \subset J^{\min}[I] \subset J^{\min}[\sigma]$. Thus the map $\sigma : J^{\min} \to J^{\min}$ factors through ϕ; that is, $\sigma = \varphi\phi$ for a unique map $\varphi : J' \to J^{\min}$. Let f to be the map which associates to each σ the corresponding map φ.

Proposition 2.2.9 *The map $f : I \to \mathrm{Hom}(A, J^{\min})$ is injective, and $f_{\mathfrak{m}}$ is an isomorphism for all $\mathfrak{m}$ outside S.*

Proof. The map f must be injective, as if σ induced the zero map $J' \to J^{\min}$, then the action of $\mathbb{T}$ on $J^{\min}$ could not be faithful. It remains to show that $f_{\mathfrak{m}}$ is surjective for $\mathfrak{m}$ outside S.

Let φ be an element of $\mathrm{Hom}(A, J^{\min})$. Then the composition $\varphi\phi$ lies in $\mathrm{End}(J^{\min})$. By Corollary 2.2.7, the inclusion $\mathbb{T}_{\mathfrak{m}} \to \mathrm{Hom}(J^{\min}, J^{\min})_{\mathfrak{m}}$ is an isomorphism. Thus we can choose an element σ of $\mathbb{T}_{\mathfrak{m}}$ corresponding to $\varphi\phi$; since $J^{\min}[I]_{\mathfrak{m}}$ is contained in the kernel of $\varphi\phi$ by Corollary 2.2.8, σ lies in $I_{\mathfrak{m}}$. Moreover, $f_{\mathfrak{m}}(\sigma) = \varphi$ by construction. □

Corollary 2.2.10 *If A is $\mathbb{T}$-equivariantly isogenous to J, and $\mathfrak{m}$ lies outside S, then the natural map $T_{\mathfrak{m}}J^{\min} \otimes \mathrm{Hom}(A, J^{\min})_{\mathfrak{m}} \to T_{\mathfrak{m}}A$ is an isomorphism. In particular, $T_{\mathfrak{m}}A \cong \mathrm{Hom}(J', J^{\min})^2_{\mathfrak{m}}$ as $\mathbb{T}_{\mathfrak{m}}$-modules.*

Proof. Since $T_{\mathfrak{m}}J^{\min}$ is free over $\mathbb{T}_{\mathfrak{m}}$, $T_{\mathfrak{m}}J^{\min} \otimes \mathrm{Hom}(A, J^{\min})_{\mathfrak{m}}$ is torsion free. It thus suffices to show that the natural map is surjective, as the modules in question have the same $\mathbb{Z}$-rank.

Fix an isogeny $\phi : J^{\min} \to A$. Then ϕ identifies $T_{\mathfrak{m}}A$ with $IT_{\mathfrak{m}}J^{\min}$, where $I = \mathrm{Ann}(\ker\phi)$. Let $x \in T_{\mathfrak{m}}A$; then ϕ identifies x with an element y of $IT_{\mathfrak{m}}J^{\min}$; we write y

as a sum of elements of the form $\sigma_i y_i$ with $\sigma_i \in I$ and $y_i \in T_{\mathfrak{m}} J^{\min}$. Since $\sigma_i \in I$, the map $\sigma_i : J^{\min} \to J^{\min}$ factors through ϕ, i.e., we have maps φ_i such that $\sigma_i = \varphi_i \phi$. Then $\sigma_i y_i = \varphi_i(\phi(y_i))$, so $\phi(y_i) \otimes \varphi_i$ maps to $\sigma_i y_i$ under the natural map. Thus each of the $\sigma_i y_i$ are in the image, so y is in the image as well. The second statement follows immediately since $T_{\mathfrak{m}} J^{\min}$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$. □

Thus to any A which is $\mathbb{T}$-equivariantly isogenous to J we can associate the module $\mathrm{Hom}(A, J^{\min})$. This module is torsion free as a $\mathbb{Z}$-module, and $\mathrm{Hom}(A, J^{\min}) \otimes \mathbb{Q}$ is free of rank one as a $\mathbb{T} \otimes \mathbb{Q}$-module by Proposition 2.2.9. Following the previous section, we let $\mathcal{A}$ be the category whose objects consist of abelian varieties A such that there exists a $\mathbb{T}$-equivariant isogeny $\phi : J \to A$ defined over $\mathbb{Q}$, and whose morphisms are $\mathbb{T}$-equivariant maps of abelian varieties defined over $\mathbb{Q}$. Then $A \mapsto \mathrm{Hom}(A, J^{\min})$ is a well defined functor $\mathcal{A} \to \mathcal{M}$, where $\mathcal{M}$ denotes the category of finitely generated $\mathbb{T}$-modules M such that M is torsion free over $\mathbb{Z}$ and $M \otimes \mathbb{Q}$ is free of rank one over $\mathbb{T} \otimes \mathbb{Q}$.

Let $\mathbb{T}_S$ be the $\mathbb{T}$-algebra defined in the previous section, and consider the category $\mathcal{A}_S$ defined therein. The functor defined above induces a functor $\mathcal{A}_S \to \mathcal{M}_S$, where $\mathcal{M}_S$ is the category of "rank one" $\mathbb{T}_S$-modules, i.e., the category of finitely generated $\mathbb{T}_S$-modules that are torsion free over $\mathbb{Z}$ and become free of rank one over $\mathbb{T}_S \otimes \mathbb{Q}$.

Theorem 2.2.11 *The functor* $A \mapsto [A] = \mathrm{Hom}(A, J^{\min})$ *is an antiequivalence of categories* $\mathcal{A}_S \to \mathcal{M}_S$.

Proof. We must show that the functor is full, faithful, and dense. Given an $M \in \mathcal{M}$, we can find an injection $M \to \mathbb{T}$. Let I be its image. Then by Proposition 2.2.9, we have a map $I \to \mathrm{Hom}(J^{\min}/J^{\min}[I], J^{\min})$ whose cokernel is supported on S. In particular, $\mathrm{Hom}(J^{\min}/J^{\min}[I], J^{\min})$ is isomorphic to M in $\mathcal{M}_S$. Thus the functor is dense.

It remains to show that the functor is fully faithful. Since $\mathrm{Hom}_{\mathcal{A}_S}(A_1, A_2)$ and $\mathrm{Hom}_{\mathcal{M}_S}([A_2], [A_1])$ are $\mathbb{T}_S$-modules, it suffices to show that the map $\mathrm{Hom}(A_1, A_2)_{\mathfrak{m}} \to \mathrm{Hom}([A_2], [A_1])_{\mathfrak{m}}$ is an isomorphism for any $\mathfrak{m}$ outside S and any $A_1, A_2 \in \mathcal{A}$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(A_1, A_2)_{\mathfrak{m}} & \longrightarrow & \mathrm{Hom}([A_2], [A_1])_{\mathfrak{m}} \\ \downarrow & & \downarrow \\ \mathrm{Hom}(T_{\mathfrak{m}} A_2, T_{\mathfrak{m}} A_1) & \longrightarrow & \mathrm{Hom}(T_{\mathfrak{m}} J^{\min} \otimes [A_2], T_{\mathfrak{m}} J^{\min} \otimes [A_1]) \end{array},$$

in which the bottom horizontal map is induced by the isomorphisms $[A_1] \otimes T_{\mathfrak{m}} J^{\min} \to T_{\mathfrak{m}} A_1$ and $[A_2] \otimes T_{\mathfrak{m}} J^{\min} \to T_{\mathfrak{m}} A_2$ of Corollary 2.2.10. Note that each map appearing in this diagram is injective, as the modules in question have no $\mathbb{Z}_l$-torsion and each map becomes an isomorphism after tensoring with $\mathbb{Q}$. Moreover, the left-hand vertical map and the bottom horizontal map are isomorphisms. Thus the right-hand vertical map is surjective, and hence an isomorphism. It follows that the upper horizontal map is an isomorphism as well. □

In light of this result, we will primarily be interested in objects and morphisms in $\mathcal{A}_S$ and $\mathcal{M}_S$ for the remainder of this chapter. Rather than clutter our notation by writing "$\otimes_{\mathbb{T}} \mathbb{T}_S$" repeatedly, we adopt the following convention: if M and N are objects of $\mathcal{M}$, an S-morphism: $M \to N$ is a map of $\mathbb{T}_S$-modules $M \otimes_{\mathbb{T}} \mathbb{T}_S \to N \otimes_{\mathbb{T}} \mathbb{T}_S$. An S-isomorphism

$M \to N$ is an S-morphism that is an isomorphism in $\mathcal{M}_S$, and M and N are said to be S-isomorphic if there exists an S-isomorphism between them. (In this case, we write $M \cong_S N$.)

The formalism we have introduced gives information about isogenies in $\mathcal{A}$, up to support on S, but it does so in terms of our choice of $J^{\min}$, which was somewhat arbitrary. Our next goal is to obtain a result that does not involve this choice. Let us assume that we have objects A_1 and A_2 in $\mathcal{A}$; then we can study the maps $A_1 \to A_2$ by studying maps from $[A_2]$ to $[A_1]$. In general $[A_2]$ and $[A_1]$ depend on $J^{\min}$. However, if we have a $\mathbb{T}$-module M such that $[A_2] \cong_S [A_1] \otimes M$ for one choice of $J^{\min}$, then it is easy to check that this relationship is independent of the choice of $J^{\min}$. (Changing $J^{\min}$ amounts to twisting $[A_2]$ and $[A_1]$ by the same locally free $\mathbb{T}$-module.) In this case we can characterise maps from A_1 to A_2 in terms of M. Recall that we have adopted the convention that if M and M' are objects of $\mathcal{M}$, then $M \otimes M'$ is taken to mean the usual tensor product modulo $\mathbb{Z}$-torsion (and thus also lies in $\mathcal{M}$).

If we have an S-morphism $h : [A_1]\otimes M \to [A_2]$, then h induces an S-morphism map $\hat{h} : M \to \mathrm{Hom}([A_1], [A_2])$, by $\hat{h}(m)(x) = h(m\otimes x)$ for $x \in [A_1], m \in M$. By Theorem 2.2.11, we can consider $\hat{h}$ to be a map $M \to \mathrm{Hom}(A_2, A_1)$. This gives us a map $g : \mathrm{Hom}(A_1, A_2) \to \mathrm{Hom}(M, \mathrm{End}(A_1))$, defined by $g(\phi)(m) = \hat{h}(m) \circ \phi$, for $\phi : A_1 \to A_2$.

For an element ϕ of $\mathrm{Hom}_{\mathcal{A}_S}(A_1, A_2)$, let $I_{\phi,S} \subset \mathrm{End}(A_1) \otimes_{\mathbb{T}} \mathbb{T}_S$ be the ideal generated by the image of $M \otimes_{\mathbb{T}} \mathbb{T}_S$ under $g(\phi)$, and let I_ϕ be the ideal $I_{\phi,S} \cap \mathrm{End}(A_1)$.

Proposition 2.2.12 *The map g defined above is an S-isomorphism. In particular, g induces an isomorphism $\mathrm{Hom}_{\mathcal{A}_S}(A_1, A_2) \to \mathrm{Hom}_{\mathcal{M}_S}(M, \mathrm{End}(A_1))$. Moreover, if ϕ is an element of $\mathrm{Hom}_{\mathcal{A}_S}(A_1, A_2)$, then $\ker_S \phi = A_1[I_\phi]$ "up to support on S."*

Proof. We construct an inverse to g in $\mathcal{M}_S$. We have a sequence of natural S-morphisms:

$$\begin{array}{lcl} \mathrm{Hom}(M, \mathrm{End}(A_1)) & \to & \mathrm{Hom}(M, \mathrm{End}([A_1])) \overset{\otimes[A_1]}{\to} \mathrm{Hom}(M \otimes [A_1], \mathrm{End}([A_1]) \otimes [A_1]) \\ & \overset{h}{\to} & \mathrm{Hom}([A_2], [A_1]) \quad \to \quad \mathrm{Hom}(A_1, A_2), \end{array}$$

where the unlabeled morphisms come from Theorem 2.2.11. It is straightforward to check that this provides an inverse to g.

Now fix an $\phi \in \mathrm{Hom}_{\mathcal{A}_S}(A_1, A_2)$, and an $\mathfrak{m}$ outside S, and define I_ϕ as in the statement of the theorem. Since $[A_1]_\mathfrak{m}\otimes M_\mathfrak{m}$ maps onto $[A_2]_\mathfrak{m}$, the map $T_\mathfrak{m}A_1 \otimes M_\mathfrak{m} \to T_\mathfrak{m}A_2$ defined by sending $\beta\otimes m$ to $(\hat{h}(m))_\mathfrak{m}(\beta)$, is surjective. (The latter map can be obtained from the former by tensoring with $T_\mathfrak{m}J^{\min}$ and exploiting the isomorphism $T_\mathfrak{m}J' \cong T_\mathfrak{m}J^{\min} \otimes [J']$ for $J' = A_1$ or A_2.)

Thus we can write any $\alpha \in T_\mathfrak{m}A_2$ as a sum of terms of the form $(\hat{h}(m_i))_\mathfrak{m}(\beta_i)$ for some m_i in $M_\mathfrak{m}$ and β_i in $T_\mathfrak{m}A_1$. But by our construction of I_f, $\phi_\mathfrak{m}(\hat{h}(m_i))_\mathfrak{m}(\beta_i) = g(\phi)(m_i)\beta_i$ is in $I_\phi T_\mathfrak{m}A_1$ for all i, so $\phi_\mathfrak{m}$ maps α into $I_\phi T_\mathfrak{m}A_1$. Hence the image of $\phi_\mathfrak{m}$ is contained in $I_{\phi,\mathfrak{m}}T_\mathfrak{m}A_1$.

Conversely, observe that by construction, every element of I_ϕ factors through ϕ, since any element of I_ϕ is in the image of $g(\phi)$ and the definition of $g(\phi)$ involved composition with ϕ. Thus the image of $\phi_\mathfrak{m}$ contains $I_\phi T_\mathfrak{m}A_1$.

It follows that $\phi_{\mathfrak{m}} T_{\mathfrak{m}} A_2 = I_\phi T_{\mathfrak{m}} A_1$ for all $\mathfrak{m}$ outside S. Since the former is the annihilator of $\ker \phi : A_1[\mathfrak{m}^\infty] \to A_2[\mathfrak{m}^\infty]$ and the latter is the annihilator of $A_1[\mathfrak{m}^\infty][I_\phi]$, we have $\ker_S \phi = A_1[I_\phi]$ up to support on S. □

The behavior of certain invariants of J under isogeny can be studied using this formalism. In particular, let p be a prime, and suppose that J has purely toric reduction at p. Then the special fiber of the Néron model of J at p is an extension of a finite group scheme by a torus T, and we can consider the character group $X_p(J) = \mathrm{Hom}_{\text{gr-sch}}(T, \mathbb{G}_m)$ of T. More generally, any object A of $\mathcal{A}$ will also have purely toric reduction at p, and we can consider its character group $X_p(A)$.

Lemma 2.2.13 *The character group $X_p(A)$ is a "rank one" $\mathbb{T}$-module.*

Proof. By construction, $X_p(A)$ has finite rank over $\mathbb{Z}$, and is torsion free as a $\mathbb{Z}$-module. It thus suffices to show that $X_p(A) \otimes \mathbb{Q}$ is free of rank one as a $\mathbb{T} \otimes \mathbb{Q}$-module. But $\mathbb{T}$ acts faithfully on $X_p(A)$, and the rank of $X_p(A)$ is equal to the dimension of J, and hence to the rank of $\mathbb{T}$ over $\mathbb{Z}$. The result thus follows by Lemma 1.7.8 □

If A_1 and A_2 are two objects in $\mathcal{A}$, and $\phi : A_1 \to A_2$ is a $\mathbb{T}$-equivariant map defined over $\mathbb{Q}$, then ϕ induces a natural map $X_p(A_2) \to X_p(A_1)$. By abuse of notation, we denote this map by ϕ as well. (It will always be clear from the context whether we mean a morphism of abelian varieties or the map it induces on the character groups.)

Proposition 2.2.14 *Let $A \in \mathcal{A}$. The natural map $X_p(J^{\min}) \otimes \mathrm{Hom}(A, J^{\min}) \to X_p(A)$ defined by $\phi \otimes x \mapsto \phi(x)$ is an isomorphism in $\mathcal{M}_S$.*

Proof. Let $T_l A$ denote the *co*variant l-adic Tate module of A, and $(T_l A)^t$ be the submodule of $T_l A$ defined in [SGA], chapter 5. By Grothendieck's orthogonality theorem, $(T_l A)^t$ is the orthogonal complement of a certain submodule $(T_l A^\vee)^f$ of $T_l A^\vee$ under the Weil pairing; in particular if $l^n x \in (T_l A)^t$ for some x in $T_l A$ and $n \in \mathbb{Z}$, then $x \in (T_l A)^t$ as well. It follows that the map $\mathrm{Hom}(T_l A, \mathbb{Z}_l) \to \mathrm{Hom}((T_l A)^t, \mathbb{Z}_l)$ is surjective. Localizing at $\mathfrak{m}$, we obtain a surjection $T_{\mathfrak{m}} A \to \mathrm{Hom}((T_l A)^t, \mathbb{Z}_l)_{\mathfrak{m}}$, where as usual $T_{\mathfrak{m}} A$ denotes the $\mathfrak{m}$-adic contravariant Tate module. Since $(T_l A)^t \cong \mathrm{Hom}(X_p(A), \mathbb{Z}_l(1))$, by the results of [SGA], chapter 5, this gives us a functorial surjection $T_{\mathfrak{m}} A \to X_p(A)_{\mathfrak{m}}$. Similarly, we obtain a functorial surjection $T_{\mathfrak{m}} J^{\min} \to X_p(J^{\min})_{\mathfrak{m}}$.

These surjections fit into the commutative diagram:

$$\begin{array}{ccc} T_{\mathfrak{m}} J^{\min} \otimes \mathrm{Hom}(A, J^{\min})_{\mathfrak{m}} & \longrightarrow & T_{\mathfrak{m}} A \\ \downarrow & & \downarrow \\ X_p(J^{\min})_{\mathfrak{m}} \otimes \mathrm{Hom}(A, J^{\min})_{\mathfrak{m}} & \longrightarrow & X_p(A)_{\mathfrak{m}}, \end{array}$$

in which the horizontal maps are the natural ones, the right hand vertical map is the surjection defined above, and the left hand vertical map is the surjection defined above tensored with the identity. By Corollary 2.2.10, the upper horizontal map is surjective; since both vertical maps are surjections the lower horizontal map must be surjective as well. □

We will also have use for the following result, which follows immediately:

Corollary 2.2.15 *Let $\mathfrak{m}$ be a maximal ideal outside S, and suppose that $X_p(J^{\min})_{\mathfrak{m}}$ is free of rank one over $\mathbb{T}_{\mathfrak{m}}$. Then for any $A \in \mathcal{A}$, we have $T_{\mathfrak{m}}A \cong (X_p(A)_{\mathfrak{m}})^2$.*

Proof. By the proposition, and the fact that $X_p(A)_{\mathfrak{m}}$ is free of rank one over $\mathbb{T}_{\mathfrak{m}}$, we have a natural isomorphism: $X_p(A)_{\mathfrak{m}} \cong [A]_{\mathfrak{m}}$. Since $T_{\mathfrak{m}}A \cong T_{\mathfrak{m}}J^{\min} \otimes [A]_{\mathfrak{m}}$, and $T_{\mathfrak{m}}J^{\min}$ is free of rank two, the result follows. □

Finally, we wish to interpret duality of abelian varieties in terms of the formalism we have constructed. Recall that we have a character $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{T}^{\times}$, and a $\mathbb{T}$-equivariant map $\psi : J \to J^{\vee}$, such that $\sigma\psi(x) = \chi(\sigma)\psi(\sigma(x))$ for all x in J. Let $\mathcal{A}_{\chi}$ denote the category of abelian varieties which are $\mathbb{T}$-equivariantly isogenous to $J^{\vee}$, and whose morphisms morphisms of abelian varieties that are $\mathbb{T}$-equivariant and defined over $\mathbb{Q}$. Duality of abelian varieties yields a functor $A \mapsto A^{\vee}$ from $\mathcal{A}$ to $\mathcal{A}_{\chi}$.

Consider also the category $\mathcal{A}_{1,\chi}$, whose objects are triples (A, A_{χ}, φ), where $A \in \mathcal{A}$, $A_{\chi} \in \mathcal{A}_{\chi}$, and ψ is a $\mathbb{T}$-equivariant isomorphism $A \to A_{\chi}$, defined over K, such that $\sigma\varphi = \chi(\sigma)\varphi\sigma$, for all $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$. A morphism in $\mathcal{A}_{1,\chi}$ from (A, A_{χ}, φ) to $(A', A'_{\chi}, \varphi')$ is a pair of morphisms $A \to A'$ (in $\mathcal{A}$) and $A_{\chi} \to A'_{\chi}$ (in $\mathcal{A}_{\chi}$) which makes the obvious diagram commute. There are natural forgetful functors $\mathcal{A}_{1,\chi} \to \mathcal{A}$ and $\mathcal{A}_{1,\chi} \to \mathcal{A}_{\chi}$.

Lemma 2.2.16 *The functors $\mathcal{A}_{1,\chi} \to \mathcal{A}$ and $\mathcal{A}_{1,\chi} \to \mathcal{A}_{\chi}$ are equivalences of categories.*

Proof. The functor $\mathcal{A}_{1,\chi}$ is clearly fully faithful; it suffices to show that it is dense. Fix an $A \in \mathcal{A}$; then it is enough to show that we have an $A' \in \mathcal{A}_{\chi}$ and a map $\varphi : A \to A'$ with the properties described above.

Fix an isogeny $\phi : J \to A$. Multiplying by a sufficiently large integer n, we may assume that $\ker\phi$ contains the kernel of the map $\psi : J \to J^{\vee}$. Let $A' = J^{\vee}/\psi(\ker\phi)$. The natural map $J \to A'$ induced by ψ has the same kernel as ϕ, and thus descends to an isomorphism φ from A to A'. Since ψ is defined over K, so is φ, and we have $\sigma\varphi = \chi(\sigma)\varphi\sigma$ since the corresponding relation holds for ψ. It follows that (A, A', φ) is an object of $A_{1,\chi}$ mapping to A.

The proof for the functor $\mathcal{A}_{1,\chi} \to \mathcal{A}_{\chi}$ is similar. □

The upshot is that $\mathcal{A}$ and $\mathcal{A}_{\chi}$ are equivalent categories. In particular the above equivalences induce a functor $A \mapsto A_{\chi}$ from $\mathcal{A}$ to $\mathcal{A}_{\chi}$ and the inverse functor $A' \mapsto A'_{\chi^{-1}}$, and these functors are equivalences of categories. (These functors can be thought of as "twisting" an abelian variety A by the character χ or its inverse. In particular A and A_{χ} become isomorphic over the splitting field of χ, via an isomorphism φ which satisfies $\sigma\varphi = \chi(\sigma)\varphi\sigma$ for all σ in $\mathrm{Gal}(K/\mathbb{Q})$.) This allows us to defined a natural duality operation on $\mathcal{A}$, by taking $A^{\dagger} = (A^{\vee})_{\chi^{-1}}$.

This operation interacts with our equivalence of categories $\mathcal{A}_S \to \mathcal{M}_S$ as follows:

Proposition 2.2.17 *The $\mathbb{T}_S$-module $[(J^{\min})^{\dagger}]^*$ is locally free of rank one. If L denotes its inverse as a $\mathbb{T}_S$-module, then we have $[A^{\dagger}] \cong [A]^* \otimes L$ for any $A \in \mathcal{A}$.*

Proof. The Weil pairing induces an isomorphism, $T_{\mathfrak{m}}(J^{\min})^{\vee} \cong \mathrm{Hom}(T_{\mathfrak{m}}J^{\min}, \mathbb{Z}_l(1))$. Since $T_{\mathfrak{m}}(J^{\min})^{\vee} \cong T_{\mathfrak{m}}(J^{\min})^{\dagger}$ as $\mathbb{T}_{\mathfrak{m}}$-modules (but not Galois modules), it follows that $T_{\mathfrak{m}}(J^{\min})^{\dagger}$ is isomorphic to $\mathrm{Hom}(T_{\mathfrak{m}}J^{\min}, \mathbb{Z}_l(1))$. As $T_{\mathfrak{m}}J^{\min}$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$, it follows that

$(T_{\mathfrak{m}}(J^{\min})^\dagger)^*$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$. By Corollary 2.2.10, $[(J^{\min})^\dagger]^*$ is free of rank one over $\mathbb{T}_{\mathfrak{m}}$.

Now for any $A \in \mathcal{A}$, we have

$$[A^\dagger] \cong_S \mathrm{Hom}(A^\dagger, J^{\min}) \cong_S \mathrm{Hom}((J^{\min})^\dagger, A) \cong_S \mathrm{Hom}([A], [(J^{\min})^\dagger]).$$

We have $[(J^{\min})^\dagger] \cong \mathbb{T}_S^* \otimes L$, so $\mathrm{Hom}([A], [(J^{\min})^\dagger]) \cong_S [A]^* \otimes L$ by Lemma 2.0.5. □

2.3 The character group of the pq-new subvariety

We now return to the situation considered in chapter 1. Fix a squarefree integer N, with an even number of prime divisors, and let Γ be a congruence subgroup of level M for some integer prime to N. For R a divisor of N, and D a divisor of R which has an even number of prime divisors, we let $J^{D,R}$ denote the abelian variety $J^D(\Gamma_0(N/D) \cap \Gamma)_{\frac{R}{D}\text{-new}}$. Let $\mathbb{T}^R$ denote the subalgebra of $\mathrm{End}(J^{1,R})$ generated by the Hecke operators T_l for l prime and the diamond bracket operators, and let $\mathcal{A}^R$ denote the category of abelian varieties defined over $\mathbb{Q}$ which are $\mathbb{T}^R$-equivariantly isogenous to $J^{1,R}$. Then for each D as above, $J^{D,R}$ is an object of $\mathcal{A}^R$. Since we will be primarily interested in $\mathcal{A}^N$, we let $J^D = J^{D,N}$ for any divisor D of N which has an even number of prime divisors. Similarly, we take $\mathcal{A} = \mathcal{A}^N$, $\mathbb{T} = \mathbb{T}^N$, and so forth. Let S^R be the set of maximal ideals of $\mathbb{T}^R$ which are either Eisenstein, or of residue characteristic 2 or 3, and take $S = S^N$.

As mentioned in Remark 2.2.1, the hypotheses (and hence the results) of the previous section hold for $\mathcal{A}^R$. We fix, once and for all, a $J_1^{\min}$ inside $\mathcal{A}^1$ such that $T_{\mathfrak{m}} J_1^{\min}$ is free of rank two over $\mathbb{T}^1_{\mathfrak{m}}$ for all non-Eisenstein $\mathfrak{m}$. If we then take $J_R^{\min} = (J_1^{\min})_{R\text{-new}}$, then $T_{\mathfrak{m}} J_R^{\min}$ is free of rank two over $\mathbb{T}^R$ for all non-Eisenstein maximal ideals $\mathfrak{m}$ of $\mathbb{T}^R$. We can thus consider the functors $J \to [J]_R = \mathrm{Hom}(J, J_R^{\min})$ for $J \in \mathcal{A}^R$, and use them to study isogenies between the abelian varieties $J^{D,R}$ for various D. By abuse of notation, we often denote $[J]_R$ simply by $[J]$ if it is clear from the context which category J belongs to.

$J^{1,R}$ (and hence every variety in $\mathcal{A}^R$) has purely toric reduction at any prime p dividing R. Thus we can consider the character groups of $J^{D,R}$ at such primes. We then have relationships between these character groups and the module $[J^{D,R}]$, given by Proposition 2.2.14. Thus, if we can compute the character groups $X_p(J^{D,R})$, we can hope to obtain information about maps between varieties of this form from this data. We carry out this program in the remainder of this paper.

Let R' be an integer dividing N and divisible by R. By [CoSt], Theorem 8.2, the injection $J^{D,R'} \to J^{D,R}$ induces an injection on the maximal tori of the special fibers at p, and hence a surjection $X_p(J^{D,R}) \to X_p(J^{D,R'})$. Moreover, both $X_p(J^{D,R})$ and $X_p(J^{D,R'})$ are free $\mathbb{Z}$-modules, and there are isomorphisms $X_p(J^{D,R}) \otimes \mathbb{Q} \cong \mathbb{T}^R \otimes \mathbb{Q}$ and $X_p(J^{D,R'}) \otimes \mathbb{Q} \cong \mathbb{T}^{R'} \otimes \mathbb{Q}$. This actually suffices to determine the isomorphism class of $X_p(J^{D,R'})$ in terms of $X_p(J^{D,R})$.

More generally, let A be any finite flat $\mathbb{Z}$-algebra, and suppose that $A \otimes \mathbb{Q}$ is Gorenstein. (This is easily seen to hold for $A = \mathbb{T}^R$, for instance by considering the monodromy pairing on $X_p(J^{R,R}) \otimes \mathbb{Q}$, which is free of rank one over $\mathbb{T}^R \otimes \mathbb{Q}$.) Let A' be a quotient of A which is flat over $\mathbb{Z}$. Let X be a finitely generated R-module such that $X \otimes \mathbb{Q}$ is free of rank one over $A \otimes \mathbb{Q}$, and let Y be a quotient of X such that Y is a faithful A'-module. Suppose that X and Y are both flat over $\mathbb{Z}$.

Lemma 2.3.1 *Suppose X and Y are as above, and let L be the kernel of the map $X \to Y$. For an ideal I of A, let $I^\perp$ be the annihilator of I in A. (This notation is due to Emerton [Em].) Let K be the kernel of the map $A \to A'$. Then $L = X[K^\perp]$, and we have $Y \cong X \otimes_A A'$ (recall our convention, in which we take the tensor product to be the usual tensor product modulo $\mathbb{Z}$-torsion).*

Proof. We begin by showing that K has finite index in $K^{\perp\perp}$. (Note that $K^{\perp\perp}$ trivially contains K.) Since $\hat{A} = A \otimes \mathbb{Q}$ is a zero-dimensional Gorenstein ring, we fix an isomorphism $\hat{A} \cong \mathrm{Hom}_{\mathbb{Q}}(\hat{A}, \mathbb{Q})$. For any ideal I of $\hat{A}$, this isomorphism identifies $I^\perp$ with $\mathrm{Hom}_{\mathbb{Q}}(\hat{A}/I, \mathbb{Q})$. In particular $\dim_{\mathbb{Q}} I^\perp = \dim_{\mathbb{Q}} \hat{A} - \dim_{\mathbb{Q}} I$, and so $\dim_{\mathbb{Q}} I^{\perp\perp} = \dim_{\mathbb{Q}} I$. Since I is contained in $I^{\perp\perp}$, the two must be equal. It follows that K is contained in $K^{\perp\perp}$ with finite index, as required.

We have an exact sequence:

$$0 \to L \to X \to Y \to 0.$$

Since Y is an A'-module, K annihilates Y, so $KX \subset L$. Now $KX \subset X[K^\perp]$, by definition of $K^\perp$. Moreover, since $X \otimes \mathbb{Q}$ is free over $\hat{A}$, $X[K^\perp] \otimes \mathbb{Q} = \hat{X}[K^\perp] = X^{\perp\perp} \otimes \mathbb{Q}$. Thus $X[K^\perp]$ contains $K^{\perp\perp}X$ with finite index, and hence contains KX with finite index. Let n be the index of KX in $X[K^\perp]$. Suppose x in $X[K^\perp]$ maps to a nonzero element of Y. Then nx maps to the zero element of Y, so the image of x is a nontrivial $\mathbb{Z}$-torsion element of Y. Since we assumed Y was flat over $\mathbb{Z}$, we must have $X[K^\perp] \subset L$.

Since $X \otimes \mathbb{Q}$ is generated by a single element over $A \otimes \mathbb{Q}$, $Y \otimes \mathbb{Q}$ is generated by a single element over $A' \otimes \mathbb{Q}$; since Y is faithful over A', $Y \otimes \mathbb{Q}$ is free of rank one over $A' \otimes \mathbb{Q}$. We thus have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L \otimes \mathbb{Q} & \longrightarrow & A \otimes \mathbb{Q} & \longrightarrow & A' \otimes \mathbb{Q} & \longrightarrow & 0. \end{array}$$

Since the kernel of the map $A \otimes \mathbb{Q} \to A' \otimes \mathbb{Q}$ is killed by $K^\perp$, so is $L \otimes \mathbb{Q}$. Hence L is killed by $K^\perp$, so $L = X[K^\perp]$. In particular L contains KX with finite index. Hence $Y \cong X/L$ is isomorphic to X/KX, modulo $\mathbb{Z}$-torsion in X/KX. In other words, Y is isomorphic to $X \otimes_A A'$. □

We record the following useful corollary before we proceed:

Corollary 2.3.2 *Let X, Y, A, A', and K be as above. Then $Y^* \cong X^*[K]$. In particular $X^*[K] \cong (X \otimes_A A')^*$ for any X and K as above.*

Proof. We have an exact sequence

$$0 \to L \to X \to Y \to 0.$$

Taking duals gives us the sequence:

$$0 \to Y^* \to X^* \to L^* \to 0;$$

the result now follows by the previous lemma. □

The upshot of Lemma 2.3.1 is the following corollary:

Corollary 2.3.3 *Let R and R' be divisors of N, and suppose R divides R'. For any p dividing R, we have $X_p(J^{D,R'}) \cong X_p(J^{D,R}) \otimes_{\mathbb{T}^R} \mathbb{T}^{R'}$.*

We now fix two distinct primes p and q dividing N. Let D be a divisor of N, which has an even number of prime divisors but is *not* divisible by p or q. The previous result, together with Theorem 1.8.2, establishes a relationship between the character groups $X_p(J^{D,Dpq})$ and $X_q(J^{Dpq,Dpq})$. In our setting, Theorem 1.8.2 immediately implies:

Theorem 2.3.4 *There is a $\mathbb{T}^D$-equivariant exact sequence*

$$0 \to X_q(J^{Dpq,Dpq}) \to X_p(J^{D,D}) \to X_p(J^D(\Gamma_0(N/Dq) \cap \Gamma))^2 \to 0.$$

Dualizing this exact sequence shows that $X_q(J^{Dpq,Dpq})^*$ is a quotient of $X_q(J^{D,D})^*$ which is faithful as a $\mathbb{T}^{Dpq}$-module and has no $\mathbb{Z}$-torsion. This is almost what we need; it remains to compare $X_q(J^{D,D})^*$ with $X_q(J^{D,D})$. We do this via the monodromy pairing on $X_q(J^{D,D})$. This is a bilinear pairing: $X_q(J^{D,D}) \times X_q((J^{D,D})^\vee) \to \mathbb{Z}$, for which the adjoint of T_l acting on $J^{D,D}$ is $T_l^\vee$ acting on $(J^{D,D})^\vee$. In particular, if we identify $J^{D,D}$ with its dual via the canonical polarization, then the adjoint of T_l is T_l°.

We would prefer to have a pairing on $X_p(J^{D,D})$ for which the Hecke operators are self-adjoint. To obtain one, we identify $J^{D,D}$ with $(J^{D,D})^\vee$ via the map ψ of Corollary 1.2.4, over a finite extension of $\mathbb{Q}$. In the language of the previous section, we have the duality operation $A \to A^\dagger$ on $\mathcal{A}_D$, and ψ defines an isomorphism $J^{D,D} \cong (J^{D,D})^\dagger$. Moreover, $(J^{D,D})^\dagger$ is isomorphic to $(J^{D,D})^\vee$ over some finite extension of $\mathbb{Q}$. Since the character groups at q of $(J^{D,D})^\dagger$ and $(J^{D,D})^\vee$ are the same whether we consider the varieties over $\mathbb{Q}$ or over a finite extension, the map ψ induces an isomorphism $X_q(J^{D,D})^\vee \cong X_q(J^{D,D})$ (as $\mathbb{T}^D$-modules but not Galois modules). Since ψ is $\mathbb{T}^D$-equivariant, in the sense that $T_l^\vee\psi = \psi T_l$, we can view the monodromy pairing as a $\mathbb{T}^D$-equivariant bilinear pairing on $X_q(J^{D,D})$; i.e., as a $\mathbb{T}^D$-equivariant map $X_q(J^{D,D}) \to X_q(J^{D,D})^*$. The cokernel of this map is the component group of $J^{D,D}$ at q.

Proposition 2.3.5 *The component group of $J^{D,D}$ at q is supported on S^D.*

Proof. Since S^D contains the Eisenstein maximal ideals, and those maximal ideals of residue characteristic 2 and 3, this is just Theorem 1.7.12. □

With this in hand, we can now show:

Proposition 2.3.6 *We have $X_p(J^{D,Dpq}) \cong_{S^{Dpq}} X_q(J^{Dpq,Dpq})^*$.*

Proof. Corollary 2.3.3 identifies $X_p(J^{D,Dpq})$ with $X_p(J^{D,D}) \otimes_{\mathbb{T}^D} \mathbb{T}^{Dpq}$. On the other hand, the $\mathbb{Z}$-dual of the exact sequence of Theorem 1.8.2 identifies $X_q(J^{Dpq,Dpq})^*$ with the module $X_p(J^{D,D})^* \otimes_{\mathbb{T}^D} \mathbb{T}^{Dpq}$ by Lemma 2.3.1. Since $X_p(J^{D,D}) \cong_{S^D} X_p(J^{D,D})^*$ by the above discussion, it follows that $X_p(J^{D,Dpq}) \cong_{S^{Dpq}} X_q(J^{Dpq,Dpq})^*$. □

We will also have need of a compatibility property which the above isomorphisms possess. Let p and p' be two distinct primes dividing D, take $D' = \frac{D}{pp'}$, and choose distinct primes q_1 and q_2 dividing N but not D. Let $\underline{q} = q_1 q_2$.

Lemma 2.3.7 *Let $f : J^{D'\underline{q},D'\underline{q}} \to J^{D',D'}$ and $g : J^{D\underline{q},D\underline{q}} \to J^{D,D}$ be maps such that there is a commutative diagram:*

$$\begin{array}{ccc} X_p(J^{D\underline{q},D\underline{q}}) & \longrightarrow & X_{p'}(J^{D'\underline{q},D'\underline{q}}) \\ {\scriptstyle g^\dagger}\big\downarrow & & \big\downarrow{\scriptstyle f^\dagger} \\ X_p(J^{D,D}) & \longrightarrow & X_{p'}(J^{D',D'}) \end{array}$$

in which the the horizontal maps are those of Theorem 1.8.2. Let $\overline{f}$ denote the restriction of f to the pp'-new subvariety $J^{D'\underline{q},D\underline{q}}$ of $J^{D'\underline{q},D'\underline{q}}$. Then we have a commutative diagram:

$$\begin{array}{ccc} X_p(J^{D\underline{q},D\underline{q}}) & \longrightarrow & X_{p'}(J^{D'\underline{q},D\underline{q}})^* \\ {\scriptstyle g^\dagger}\big\downarrow & & \big\downarrow{\scriptstyle \overline{f}^*} \\ X_p(J^{D,D}) & \longrightarrow & X_{p'}(J^{D',D})^*, \end{array}$$

in which the horizontal maps are the isomorphisms constructed in Proposition 2.3.6.

Proof. Consider the diagram:

$$\begin{array}{ccccccc} X_p(J^{D\underline{q},D\underline{q}}) & \longrightarrow & X_{p'}(J^{D'\underline{q},D'\underline{q}}) & \longrightarrow & X_{p'}(J^{D'\underline{q},D'\underline{q}})^* & \longleftarrow & X_{p'}(J^{D'\underline{q},D\underline{q}})^* \\ {\scriptstyle g^\dagger}\big\downarrow & & {\scriptstyle f^\dagger}\big\downarrow & & \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle \overline{f}^*} \\ X_p(J^{D,D}) & \longrightarrow & X_{p'}(J^{D',D'}) & \longrightarrow & X_{p'}(J^{D',D'})^* & \longleftarrow & X_{p'}(J^{D',D})^*. \end{array}$$

Here the left-hand square is that appearing in the hypothesis of the lemma, and commutes by hypothesis. The middle square commutes because $f^\dagger$ and f are adjoints under the monodromy pairing. The right-hand square is the $\mathbb{Z}$-dual of the diagram:

$$\begin{array}{ccc} X_{p'}(J^{D',D}) & \longrightarrow & X_{p'}(J^{D',D'}) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \overline{f}} \\ X_{p'}(J^{D'\underline{q},D\underline{q}}) & \longrightarrow & X_{p'}(J^{D'\underline{q},D'\underline{q}}), \end{array}$$

which clearly commutes since $\overline{f}$ is the restriction of f to the pp'-new subvariety, and the horizontal maps are just inclusions of the pp'-new subvarieties into the corresponding variety.

The isomorphisms: $X_p(J^{D,D}) \to X_{p'}(J^{D',D})^*$ and $X_p(J^{D\underline{q},D\underline{q}}) \to X_{p'}(J^{D'\underline{q},D\underline{q}})^*$ are obtained as the compositions of the two leftmost horizontal map with the "inverse" of the rightmost horizontal map (bearing in mind that the image of the rightmost horizontal map is the same as the image of the composition of the two rightmost), in the corresponding row of the above diagram. The commutativity of the desired square thus follows immediately. □

We will also need the following result about character groups in section 2.5.

Proposition 2.3.8 *Let q_1 and q_2 be two distinct primes dividing N but not D. Let $Y_{q_1} = X_{q_1}(J^{Dq_1q_2,Dq_1q_2})$ and let $X_{q_2} = X_{q_2}(J^D(\Gamma_0(N/Dq_1) \cap \Gamma))^2$. Then there is a natural S^D-isomorphism of $\mathbb{T}^D$-modules:*

$$Y_{q_1}^*/Y_{q_1} \cong_{S^D} X_{q_2}/(T_{q_1}^2 - \langle q_1 \rangle)X_{q_2}.$$

(Here Y_{q_1} is considered as a submodule of $Y_{q_1}^*$ via the monodromy pairing.)

Proof. In [Ri5], Ribet constructs an exact sequence:

$$0 \to K \to X_{q_2}/(T_{q_1}^2 - \langle q_1 \rangle)X_{q_2} \to Y_{q_1}^*/Y_{q_1} \to 0,$$

where K and C are supported on S^D, in the case $D = 1$ and $\Gamma = \Gamma_0(N)$. As with Theorem 1.8.2, his arguments carry over to our case once one invokes Theorem 1.7.6 in place of the Deligne-Rapoport theorem. □

2.4 Induced maps on character groups

In light of the character group computations of the previous section, we might hope to study morphisms in $\mathcal{A}^N$ by studying the maps they induce on character groups. The main tool we will use to do this is the following result, whose proof will occupy this section as well as the next.

Theorem 2.4.1 *Fix D dividing N. For any prime p dividing D, the natural map $[J^{D,D}] \to \mathrm{Hom}(X_p(J_D^{\min}), X_p(J^{D,D}))$ (which sends a map $J_D^{\min} \to J^{D,D}$ to the induced map on character groups at p) is an S^D-isomorphism.*

We will prove this theorem by constructing isomorphisms:

$$[J^{D,D}]_{\mathfrak{m}} \to \mathrm{Hom}(X_p(J_D^{\min}), X_p(J^{D,D}))_{\mathfrak{m}}$$

for each $\mathfrak{m}$ outside S^D. Although the isomorphisms we construct in this way are in some sense natural, showing any sort of compatibility among them seems difficult. The following lemma provides us a way around this difficulty:

Lemma 2.4.2 *Let M and N be $\mathbb{T}_{\mathfrak{m}}^D$-modules, flat over $\mathbb{Z}$, and suppose that $M \otimes \mathbb{Q}$ and $N \otimes \mathbb{Q}$ are free of rank one over $\mathbb{T}_{\mathfrak{m}}^D \otimes \mathbb{Q}$. Suppose we have a $\mathbb{T}_{\mathfrak{m}}^D$ module H, with no $\mathbb{Z}$-torsion, and an isomorphism $f : M \otimes H \to N$. Then the map $M \otimes \mathrm{Hom}(M, N) \to N$ given by evaluation is an isomorphism. Moreover, there is a natural map $g : H \to \mathrm{Hom}(M, N)$ such that the diagram:*

$$\begin{array}{ccc} M \otimes H & \xrightarrow{f} & N \\ {\scriptstyle 1\otimes g}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ M \otimes \mathrm{Hom}(M,N) & \longrightarrow & N \end{array}$$

commutes, where the right-hand vertical map is the identity. Finally, if $H \cong \mathrm{Hom}(M, N)$, *then* g *is an isomorphism.*

Proof. The natural map g takes an element h of H to the map $m \mapsto f(h \otimes m)$. The commutativity of the above diagram is trivial. Moreover, the upper horizontal map is an isomorphism by hypothesis, and the right-hand vertical map is the identity. It follows that the lower horizontal map is surjective, and hence (by counting ranks over $\mathbb{Z}$) an isomorphism. (Recall that $M \otimes \mathrm{Hom}(M, N)$ is $\mathbb{Z}$ torsion free by convention.)

It remains to prove that if $H \cong \mathrm{Hom}(M, N)$, then the natural map given above is an isomorphism. Composing the inverse of the isomorphism $H \to \mathrm{Hom}(M, N)$ with g, we obtain an endomorphism σ of $\mathrm{Hom}(M, N)$, such that $g(H) \subset \mathrm{Hom}(M, N)$ is equal to $\sigma \mathrm{Hom}(M, N)$. It thus suffices to show that σ is a unit in $\mathrm{End}(\mathrm{Hom}(M, N))$. The map $M \otimes \mathrm{Hom}(M, N) \cong N$ induces an inclusion of $\mathrm{End}(\mathrm{Hom}(M, N))$ into $\mathrm{End}(N)$. It follows that the image of $M \otimes H$ in N, under the composition of the left-hand vertical map and the bottom horizontal map, is equal to σN. By the commutativity of the diagram, σ is a unit in $\mathrm{End}(N)$. Suppose σ were not a unit in $\mathrm{End}(\mathrm{Hom}(M, N))$. Then there would be a maximal ideal $\mathfrak{m}_1$ of $\mathrm{End}(\mathrm{Hom}(M, N))$ containing σ. Since $\mathrm{End}(\mathrm{Hom}(M, N))$ is contained in $\mathrm{End}(N)$ with finite index, there is a maximal ideal $\hat{\mathfrak{m}}_1$ of $\mathrm{End}(N)$ containing $\mathfrak{m}_1$. In particular, $\hat{\mathfrak{m}}_1$ would contain σ, and so σ would not be a unit in $\mathrm{End}(N)$. Thus σ is not a unit in $\mathrm{End}(\mathrm{Hom}(M, N))$, as required. □

The usefulness of this lemma can be seen in the following corollary:

Corollary 2.4.3 *Let* $J \in \mathcal{A}^D$, *p be a prime dividing R,* $\mathfrak{m}$ *a maximal ideal of* $\mathbb{T}^R$ *outside* S_D, *and suppose that* $[J]_{\mathfrak{m}} \cong \mathrm{Hom}(X_p(J_D^{\min}), X_p(J))_{\mathfrak{m}}$. *Then the natural map* $[J]_{\mathfrak{m}} \to \mathrm{Hom}(X_p(J_D^{\min}), X_p(J))_{\mathfrak{m}}$ *is an isomorphism.*

Proof. We have a natural isomorphism: $X_p(J_D^{\min})_{\mathfrak{m}} \otimes [J]_{\mathfrak{m}} \to X_p(J)_{\mathfrak{m}}$. By Lemma 2.4.2, this induces a map $[J]_{\mathfrak{m}} \to \mathrm{Hom}(X_p(J_D^{\min}), X_p(J))_{\mathfrak{m}}$; the latter is easily seen to be the natural map $[J]_{\mathfrak{m}} \to \mathrm{Hom}(X_p(J_D^{\min}), X_p(J))_{\mathfrak{m}}$. Since the two modules are isomorphic, this map is an isomorphism by Lemma 2.4.2. □

In particular, in order to prove Theorem 2.4.1, it suffices to construct, for each $\mathfrak{m}$ outside S^D, an isomorphism $[J^{D,D}]_{\mathfrak{m}} \to \mathrm{Hom}(X_p(J_D^{\min}), X_p(J^{D,D}))_{\mathfrak{m}}$. We begin by doing this for maximal ideals $\mathfrak{m}$ with a certain property.

Definition 2.4.4 *Suppose D divides N, and has an even number of prime divisors, and let* $\mathfrak{m}$ *be a maximal ideal of* $\mathbb{T}^D$ *such that the representation* $\overline{\rho}_{\mathfrak{m}}$ *is absolutely irreducible. Then* $\mathfrak{m}$ *is* controllable *at p (for some p dividing N) if one of the following conditions holds:*

1. $\overline{\rho}_{\mathfrak{m}}$ *is not finite at p, or*

2. $\overline{\rho}_{\mathfrak{m}}$ *is unramified at p,* $p \neq l$, *and* $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_p)$ *is not a scalar, or*

3. $p = l$, *and* $l \neq 2$, *or*

4. $p = l = 2$, *and the restriction of* $\overline{\rho}_{\mathfrak{m}}$ *to a decomposition group at* 2 *is not contained in the scalar matrices.*

The point of this definition is the following lemma:

Lemma 2.4.5 *Suppose that $\mathfrak{m}$ is controllable at p. Then $X_p(J_D^{\min})_{\mathfrak{m}}$ is free of rank one over $\mathbb{T}^D_{\mathfrak{m}}$.*

Proof. This is essentially Mazur's principle. By Nakayama's lemma, and the fact that $X_p(J_D^{\min})_{\mathfrak{m}}$ is faithful over $\mathbb{T}^D_{\mathfrak{m}}$, it suffices to show that $X_p(J_D^{\min})/\mathfrak{m}X_p(J_D^{\min})$ has dimension one. We will prove the lemma in this form.

Let $J_D^{\min}[\mathfrak{m}]^f$ denote the maximal $\mathbb{T}^D$-stable subgroup of $J_D^{\min}[\mathfrak{m}]$ which is finite at p, i.e., which extends to a finite, flat group scheme W over $\mathbb{Z}_p$. Then we have an injection $W_{\mathbb{F}_p} \to (J_D^{\min})_{\mathbb{F}_p}[\mathfrak{m}]$, where $(J_D^{\min})_{\mathbb{F}_p}$ is the special fiber of the Néron model of $J_D^{\min}$ over $\mathbb{F}_p$. Since W is flat over $\mathbb{Z}_p$, the dimension of $W_{\mathbb{F}_p}$ and the dimension of the general fiber of W over $\mathbb{T}/\mathfrak{m}$ agree. Since the latter is at most two, and is exactly two if and only if $J_D^{\min}[\mathfrak{m}]$ is finite at p, it follows that $(J_D^{\min})_{\mathbb{F}_p}[\mathfrak{m}]$ has dimension at most two, with equality only if $(J_D^{\min})[\mathfrak{m}]$ is finite at p.

Now, since $(J_D^{\min})_{\mathbb{F}_p}$ is an extension of the torus $\mathrm{Hom}(X_p(J_D^{\min}), \mathbb{G}_m)$ by a finite group, and $X_p(J_D^{\min})/\mathfrak{m}X_p(J_D^{\min})$ has dimension at least one, we have

$$1 \leq \dim X_p(J_D^{\min})/\mathfrak{m}X_p(J_D^{\min}) \leq 2,$$

with equality on the right if and only if $J_D^{\min}[\mathfrak{m}]$ is finite at p. In particular, if $\overline{\rho}_{\mathfrak{m}} = \mathrm{Hom}(J_D^{\min}[\mathfrak{m}], \mu_l)$ is not finite at p, then $J_D^{\min}[\mathfrak{m}]$ is not finite at p, and so $X_p(J_D^{\min})/\mathfrak{m}X_p(J_D^{\min})$ has dimension one.

We may thus assume that $\overline{\rho}_{\mathfrak{m}}$, and hence $J_D^{\min}[\mathfrak{m}]$, is finite at p. Suppose further that $X_p(J_D^{\min})/\mathfrak{m}X_p(J_D^{\min})$ has dimension two. Then we are in the following situation: $J_D^{\min}[\mathfrak{m}]$ extends to a finite flat group scheme W over $\mathbb{Z}_p$, and the special fiber of W over $\mathbb{F}_p$ is the multiplicative group scheme $\mathrm{Hom}(X_p(J_D^{\min})/\mathfrak{m}X_p(J_D^{\min}), \mu_l)$. In particular, the Cartier dual $W^\vee$ of W is an étale group scheme over $\mathbb{Z}_p$. Since $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the general fiber of $W^\vee$ by $\overline{\rho}_{\mathfrak{m}}$, it follows that $\overline{\rho}_{\mathfrak{m}}$ is unramified at p. In particular, the image of a decomposition group at p under $\overline{\rho}_{\mathfrak{m}}$ is determined by the action of Frob_p on the special fiber of $W^\vee = X_p/\mathfrak{m}X_p$.

Suppose first that $p \neq l$. Then Frob_p acts on $X_p(J_D^{\min})/\mathfrak{m}X_p(J_D^{\min})$ via the Atkin-Lehner operator w_p^{-1}, as $X_p(J_D^{\min})$ can be identified with a subspace of $X_p(J^{1,D})$, on which Frob_p acts by w_p^{-1} by Theorem 1.7.6. Since w_p acts on $J^{1,D}$ as the scalar $-T_p$ (as $J^{1,D}$ is p-new), it follows that $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_p)$ is a scalar. In particular, $\mathfrak{m}$ cannot be controllable at p in this case.

Suppose finally that $p = l$. Since the determinant of $\overline{\rho}_{\mathfrak{m}}$ is the mod l cyclotomic character times an even character of finite order, $\overline{\rho}_{\mathfrak{m}}$ is ramified at p unless $p = l = 2$. This contradicts the fact that $W^\vee$ is étale. If $p = l = 2$, then we find as before that the action of Frob_p on $W^\vee = X_p/\mathfrak{m}X_p$ is by scalars, and hence $\mathfrak{m}$ again cannot be controllable at p. □

When this criterion is satisfied for some prime dividing D, we may easily construct the desired isomorphism:

Corollary 2.4.6 *Let p divide D, and suppose $\mathfrak{m}$ is not Eisenstein and controllable at at least one prime dividing D. Then $[J^{D,D}]_{\mathfrak{m}} \cong \mathrm{Hom}(X_p(J_D^{\min}), X_p(J^{D,D}))_{\mathfrak{m}}$ for any $\mathfrak{m}$ outside S^D. In particular, the natural map: $[J^{D,D}]_{\mathfrak{m}} \to \mathrm{Hom}(X_p(J_D^{\min}), X_p(J^{D,D}))_{\mathfrak{m}}$ is an isomorphism.*

Proof. If $\mathfrak{m}$ is controllable at p, then $X_p(J_D^{\min})_{\mathfrak{m}}$ is free of rank one over $\mathbb{T}_{\mathfrak{m}}$, and $X_p(J^{D,D})_{\mathfrak{m}} \cong [J^{D,D}]_{\mathfrak{m}} \otimes X_p(J_D^{\min})_{\mathfrak{m}}$. The result follows immediately.

If $\mathfrak{m}$ is not controllable at p, there is another prime q dividing D at which $\mathfrak{m}$ is controllable. Then we have a sequence of isomorphisms:

$$\begin{array}{rcll} \mathrm{Hom}(X_p(J_D^{\min}), X_p(J^{D,D}))_{\mathfrak{m}} & \cong & \mathrm{Hom}(X_p(J_D^{\min}), X_q(J^{\frac{D}{pq},D})^*)_{\mathfrak{m}} & \text{(Proposition 2.3.6)} \\ & \cong & (X_p(J_D^{\min}) \otimes X_q(J^{\frac{D}{pq},D}))^*_{\mathfrak{m}} & \text{(Lemma 2.0.5)} \\ & \cong & (X_p(J_D^{\min}) \otimes [J^{\frac{D}{pq},D}])^*_{\mathfrak{m}} & \text{(controllability at } q) \\ & \cong & X_p(J^{\frac{D}{pq},D})^*_{\mathfrak{m}} & \text{(Proposition 2.2.14)} \\ & \cong & X_q(J^{D,D})_{\mathfrak{m}} & \text{(Proposition 2.3.6)} \\ & \cong & [J^{D,D}]_{\mathfrak{m}} & \text{(controllability at } q). \end{array}$$

This proves the first statement; the second statement follows by Corollary 2.4.3. □

It remains to deal with those maximal ideals $\mathfrak{m}$ that are not controllable at *any* prime dividing D. We do so in the next section, by "raising the level". Specifically, we introduce two auxiliary primes q_1 and q_2, prime to NM, such that $\mathfrak{m}$ lifts to a maximal ideal of $\tilde{\mathfrak{m}}$ of $\mathbb{T}^D(\Gamma_0(q_1q_2N/D) \cap \Gamma)$ which is *new* at q_1 and q_2 and *controllable* at these primes. Then $\tilde{\mathfrak{m}}$ descends to a maximal ideal of $\mathbb{T}^{Dq_1q_2}(\Gamma_0(N/D) \cap \Gamma)$ (because $\tilde{\mathfrak{m}}$ is new) and an analogue of Corollary 2.4.6 holds for the abelian variety $J^{Dq_1q_2}(\Gamma_0(N/D) \cap \Gamma)$ at $\tilde{\mathfrak{m}}$, as $\tilde{\mathfrak{m}}$ is controllable at q_1. We then exploit a geometric relationship between this abelian variety and $J^{D,D}$ to prove the desired result for $J^{D,D}$.

2.5 Raising the level

The key to the level raising argument is the following result:

Lemma 2.5.1 *Let D be the product of an even number of distinct primes, and Γ a congruence subgroup of level M prime to D. Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of the Hecke algebra $\mathbb{T}^D(\Gamma)$, of residue characteristic $l > 3$. Then there exists a prime q, not dividing $2NDl$, such that:*

1. *$\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_q)$ is not a scalar, and*
2. *There is a maximal ideal $\tilde{\mathfrak{m}}$ of $\mathbb{T}^D(\Gamma_0(q) \cap \Gamma)$ such that $\tilde{\mathfrak{m}}$ is q-new and the representations $\overline{\rho}_{\mathfrak{m}}$ and $\overline{\rho}_{\tilde{\mathfrak{m}}}$ are isomorphic.*

Proof. Proposition 1.6.6 gives us a q such that $T_q^2 - (q+1)^2\langle q\rangle$ lies in $\mathfrak{m}$. The construction of this q shows us that $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_q)$ has order two, trace zero, and determinant -1; in particular it is not a scalar. The result thus follows from Theorem 1.6.4. □

Fix a q and $\tilde{\mathfrak{m}}$ as above. Then $\tilde{\mathfrak{m}}$ is q-old; it descends to an ideal of $\mathbb{T}^D(\Gamma_0(q)\cap\Gamma)^{q\text{-old}}$. We have seen that this latter ring is isomorphic to $\mathbb{T}[T_q]/(T_q^2 - \tau_qT_q + q\langle q\rangle)$, and that $\mathfrak{m}$ is identified with $\tilde{\mathfrak{m}} \cap \mathbb{T}$ under this isomorphism. (Here τ_q denotes the element T_q of $\mathbb{T}$, which must be distinguished from T_q as an element of $\mathbb{T}^D(\Gamma_0(q) \cap \Gamma)$ as in section 1.3.)

Also recall that $\mathbb{T}^D(\Gamma_0(q)\cap\Gamma)^{q\text{-old}}$ acts on $J^D(\Gamma)^2$, where T_q acts via the endomorphism $\begin{pmatrix} 0 & -1 \\ q\langle q\rangle & \tau_q \end{pmatrix}$ and that the isogeny $i_q : J^D(\Gamma)^2 \to J^D(\Gamma_0(q) \cap \Gamma)_{q\text{-old}}$ is compatible with this action.

Proposition 2.5.2 *Let q and $\tilde{\mathfrak{m}}$ be chosen as above. Then*

$$\dim J^D(\Gamma_0(q)\cap\Gamma)[\tilde{\mathfrak{m}}] = \dim J^D(\Gamma_0(q)\cap\Gamma)_{q\text{-new}}[\tilde{\mathfrak{m}}] = \dim J^D(\Gamma_0(q)\cap\Gamma)_{q\text{-old}}[\tilde{\mathfrak{m}}].$$

Proof. To make the notation more concise, let $\tilde{J} = J^D(\Gamma_0(q)\cap\Gamma)$, $\tilde{J}_{\text{new}} = \tilde{J}_{q\text{-new}}$, and $\tilde{J}_{\text{old}} = \tilde{J}_{q\text{-old}}$. Let $J = J^D(\Gamma)$. The two degeneracy maps $J \to \tilde{J}$ induce an isogeny $(J^2) \to \tilde{J}_{\text{old}}$. This isogeny is Hecke equivariant provided one takes care to define the proper Hecke action on J^2. In particular the Hecke operators T_n for n prime to q act diagonally on J^2, but T_q does *not*; its action is given in [Di]. By Theorem 1.6.5, the kernel of this isogeny is Eisenstein; in particular it is supported away from $\tilde{\mathfrak{m}}$.

Let $\tilde{J}^{\text{old}}$ be the q-old *quotient* of $\tilde{J}$ (and thus the dual of $\tilde{J}_{\text{old}}$), and consider the sequence of maps:

$$J^2 \xrightarrow{i_q} \tilde{J} \xrightarrow{j_q} J^2,$$

where (as in section 1.6) j_q is the map $x \to ((\alpha_q)_*(x), (\beta_q)_*(x))$. The composition of these maps is given by the matrix $\begin{pmatrix} q+1 & \langle q\rangle^{-1}T_q \\ T_q & q+1 \end{pmatrix}$, and is in particular equal to $T_q^2 - \langle q\rangle$ times an automorphism of J^2. Observe that the image of i_q lies in $\tilde{J}_{\text{old}}$, and that the kernel of j_q contains the kernel of the map $\tilde{J} \to \tilde{J}^{\text{old}}$.

Let $\psi_{\text{old}} : \tilde{J}_{\text{old}} \to \tilde{J}^{\text{old}}$ be the composition of the natural inclusion of $\tilde{J}_{\text{old}}$ in $\tilde{J}$ with the surjection of $\tilde{J}$ onto $\tilde{J}^{\text{old}}$. Its kernel is equal to the intersection of $\tilde{J}_{\text{old}}$ with the kernel of the map $\tilde{J} \to \tilde{J}^{\text{old}}$; the latter is precisely $\tilde{J}_{\text{new}}$. Since the intersection of $\tilde{J}_{\text{old}}$ with $\tilde{J}_{\text{new}}$ is finite, and $\tilde{J}^{\text{old}}$ and $\tilde{J}_{\text{old}}$ have the same dimension, ψ_{old} is surjective.

The map $J^2 \to \tilde{J}_{\text{old}}$ induced by i_q is an isogeny (and hence surjective). The map $j_q : \tilde{J} \to J^2$ is equal to $\theta^{-1}(i_q)^\vee\tilde{\theta}$, where θ and $\tilde{\theta}$ are the canonical polarizations on J and $\tilde{J}$, and is therefore also surjective. Since $j_q i_q : J^2 \to J^2$ factors as

$$J^2 \to \tilde{J}_{\text{old}} \xrightarrow{\psi_{\text{old}}} \tilde{J}^{\text{old}} \to J^2,$$

it follows that the map $j_q i_q$ is a surjective endomorphism of J^2. Since $j_q i_q$ is equal to $T_q^2 - \langle q\rangle$ up to an automorphism of J^2, we immediately see that the map $T_q^2 - \langle q\rangle : \tilde{J}_{\text{old}} \to \tilde{J}_{\text{old}}$ is surjective as well. Finally, observe that since the kernel of i_q is supported away from $\tilde{\mathfrak{m}}$, it follows that the $\tilde{\mathfrak{m}}$-primary part of $\ker\psi_{\text{old}}$ is equal to the $\tilde{\mathfrak{m}}$-primary part of $J_{\text{old}}[T_q^2 - \langle q\rangle]$.

The operator T_q acts on $\tilde{J}_{\text{new}}$ as $-w_q$, where w_q is the Atkin-Lehner operator. Since $w_q^2 = \langle q\rangle$, it follows that $T_q^2 - \langle q\rangle$ annihilates $\tilde{J}_{\text{new}}$. The endomorphism of $\tilde{J}$ induced by this map thus factors through $\tilde{J}^{\text{old}}$; its image therefore lies in the q-old subvariety $\tilde{J}_{\text{old}}$. Since $(T_q^2 - \langle q\rangle)\tilde{J}_{\text{old}}$ is equal to $\tilde{J}_{\text{old}}$, this image must be all of $\tilde{J}_{\text{old}}$.

Now consider the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{J}_{\text{old}}\cap\tilde{J}_{\text{new}} & \longrightarrow & \tilde{J}_{\text{old}}\oplus\tilde{J}_{\text{new}} & \longrightarrow & \tilde{J} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle T_q^2-\langle q\rangle} & & \downarrow{\scriptstyle T_q^2-\langle q\rangle} & & \downarrow{\scriptstyle T_q^2-\langle q\rangle} & & \\
0 & \longrightarrow & \tilde{J}_{\text{old}}\cap\tilde{J}_{\text{new}} & \longrightarrow & \tilde{J}_{\text{old}}\oplus\tilde{J}_{\text{new}} & \longrightarrow & \tilde{J} & \longrightarrow & 0.
\end{array}$$

The Snake lemma then yields an exact sequence

$$0 \to \tilde{J}_{\text{old}}\cap\tilde{J}_{\text{new}} \to \tilde{J}_{\text{old}}[T_q^2 - \langle q\rangle] \oplus \tilde{J}_{\text{new}} \to \tilde{J}[T_q^2 - \langle q\rangle] \to \tilde{J}_{\text{old}}\cap\tilde{J}_{\text{new}} \to \tilde{J}_{\text{new}}.$$

The final map in this sequence is simply the inclusion $\tilde{J}_{\mathrm{old}} \cap \tilde{J}_{\mathrm{new}} \to \tilde{J}_{\mathrm{new}}$, and hence injective, so this yields a short exact sequence

$$0 \to \tilde{J}_{\mathrm{old}} \cap \tilde{J}_{\mathrm{new}} \to \tilde{J}_{\mathrm{old}}[T_q^2 - \langle q \rangle] \oplus \tilde{J}_{\mathrm{new}} \to \tilde{J}[T_q^2 - \langle q \rangle] \to 0.$$

Taking $\tilde{\mathfrak{m}}^\infty$-torsion then yields

$$0 \to (\tilde{J}_{\mathrm{old}} \cap \tilde{J}_{\mathrm{new}})_{\tilde{\mathfrak{m}}} \to \tilde{J}_{\mathrm{old}} J[T_q^2 - \langle q \rangle]_{\tilde{\mathfrak{m}}} \oplus \tilde{J}_{\mathrm{new}}[\tilde{\mathfrak{m}}^\infty] \to \tilde{J}[T_q^2 - \langle q \rangle, \tilde{\mathfrak{m}}^\infty] \to 0.$$

The discussion above showed that $J_{\mathrm{old}}[T_q^2 - \langle q \rangle]_{\tilde{\mathfrak{m}}}$ lies in the kernel of ψ_{old}, and hence lies in $J_{\mathrm{new}}[T_q^2 - \langle q \rangle]$. It follows that $(J_{\mathrm{old}} \cap J_{\mathrm{new}})_{\tilde{\mathfrak{m}}} = J_{\mathrm{old}}[T_q^2 - \langle q \rangle]_{\tilde{\mathfrak{m}}}$, and hence that $\tilde{J}_{\mathrm{new}}[\tilde{\mathfrak{m}}^\infty]$ is isomorphic to $\tilde{J}[T_q^2 - \langle q \rangle, \tilde{\mathfrak{m}}^\infty]$. Since $T_q^2 - \langle q \rangle$ lies in $\tilde{\mathfrak{m}}$ (since $\tilde{\mathfrak{m}}$ is q-new), it follows that $\tilde{J}_{\mathrm{new}}[\tilde{\mathfrak{m}}] = \tilde{J}[\tilde{\mathfrak{m}}]$.

It remains to show that $\tilde{J}_{\mathrm{old}}[\tilde{\mathfrak{m}}] = \tilde{J}[\tilde{\mathfrak{m}}]$. Suppose $\tilde{J}[\tilde{\mathfrak{m}}]$ has dimension $2n$. Consider the connected component $\tilde{J}_q^0$ of the special fiber of the Néron model of $\tilde{J}$ at q. Since $\tilde{J}$ has semistable reduction at q, it fits into an exact sequence

$$0 \to \mathrm{Hom}(X_q(\tilde{J}), \mathbb{G}_m) \to \tilde{J}_q^0 \to \tilde{J}^2 \to 0.$$

Since the natural isogeny $J^2 \to \tilde{J}_{\mathrm{old}}$ has kernel supported away from $\mathfrak{m}$, it suffices to show that $J^2[\tilde{\mathfrak{m}}]$ has dimension $2n$.

The new subvariety $\tilde{J}_{\mathrm{new}}$ has purely toric reduction at q, so the special fiber of its Néron model at q is an extension of the torus $\mathrm{Hom}(X_q(\tilde{J}_{\mathrm{new}}), \mathbb{G}_m)$ by the finite group $\Phi_q(\tilde{J}_{\mathrm{new}})$. As $\overline{\rho}_{\mathfrak{m}}$ is unramified at q, and $\tilde{J}_{\mathrm{new}}[\tilde{\mathfrak{m}}]$ has dimension $2n$, $(\tilde{J}_{\mathrm{new}})_q[\tilde{\mathfrak{m}}]$ has dimension $2n$. We have an exact sequence:

$$0 \to \mathrm{Hom}(X_q(\tilde{J}_{\mathrm{new}}), \mathbb{G}_m)[\tilde{\mathfrak{m}}] \to (\tilde{J}_{\mathrm{new}})_q[\tilde{\mathfrak{m}}] \to \Phi_q(\tilde{J}_{\mathrm{new}})[\tilde{\mathfrak{m}}].$$

Moreover, since $\tilde{\mathfrak{m}}$ is controllable at q ($\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_q)$ is not a scalar), it follows by Lemma 2.4.5 and Corollary 2.2.15 that $T_{\tilde{\mathfrak{m}}}\tilde{J}_{\mathrm{new}} \cong X_q(\tilde{J}_{\mathrm{new}})^2_{\tilde{\mathfrak{m}}}$. In particular, $X_q(\tilde{J}_{\mathrm{new}})/\tilde{\mathfrak{m}}X_q(\tilde{J}_{\mathrm{new}})$ has dimension n. The above exact sequence then shows that $\Phi_q(\tilde{J}_{\mathrm{new}})[\tilde{\mathfrak{m}}]$ has dimension at least n. (We will see later that it must have dimension exactly n.)

The inclusion $\tilde{J}_{\mathrm{new}} \to \tilde{J}$ induces an injection on the special fibers $(J_{\mathrm{new}})_q$ and $\tilde{J}_q$ of the corresponding Néron models, and in particular an isomorphism $X_q(\tilde{J}_{\mathrm{new}}) \to X_q(J)$. Since the group of connected components $\Phi_q(\tilde{J})$ is Eisenstein (c.f. Proposition 2.3.5), $\Phi_q(\tilde{J}_{\mathrm{new}})[\tilde{\mathfrak{m}}]$ maps into the connected component of $\tilde{J}_q$. Since the map on special fibers is injective, the image of $\Phi_q(\tilde{J}_{\mathrm{new}})[\tilde{\mathfrak{m}}]$ in $\tilde{J}_q$ has trivial intersection with the torus, and hence maps injectively into J^2.

We thus obtain an injection of $\Phi_q(\tilde{J}_{\mathrm{new}})[\tilde{\mathfrak{m}}]$ into $J^2[\tilde{\mathfrak{m}}]$. The latter is a direct sum of two dimensional subspaces V_i on which Frobenius acts via $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_q)$. Since Frob_q acts as a scalar on $\Phi_q(\tilde{J}_{\mathrm{new}})$, the image of $\Phi_q(\tilde{J}_{\mathrm{new}})$ in each V_i has dimension at most one. As $\Phi_q(\tilde{J}_{\mathrm{new}})[\tilde{\mathfrak{m}}]$ has dimension at least n, it follows that $J^2[\tilde{\mathfrak{m}}]$ has dimension at least $2n$. But $\dim J^2[\tilde{\mathfrak{m}}] = \dim \tilde{J}_{\mathrm{old}}[\tilde{\mathfrak{m}}]$, and the latter is less than $\dim \tilde{J}[\tilde{\mathfrak{m}}]$ which is $2n$. Thus $\dim \tilde{J}_{\mathrm{old}}[\tilde{\mathfrak{m}}] = \dim \tilde{J}[\tilde{\mathfrak{m}}]$, as required. □

We now return to the setting of the previous section; we have fixed a D dividing N which is the product of an even number of primes, a maximal ideal $\mathfrak{m}$, and a prime p

dividing D, and we wish to show that $[J^{D,D}]_{\mathfrak{m}} \cong \mathrm{Hom}(X_p(J^{\min}), X_p(J^{D,D}))_{\mathfrak{m}}$. Let p' be a prime dividing D other than p, and let $D' = \frac{D}{pp'}$.

Now we choose two primes q_1 and q_2 as in Lemma 2.5.1. The choice of these primes determines several abelian varieties. In particular, let $\tilde{J}^{D',D'}$ denote $J^{D'}(\Gamma_0(q_1q_2N/D') \cap \Gamma)$, and $\tilde{J}^{D,D}$ denote $J^D(\Gamma_0(q_1q_2N/D) \cap \Gamma)$. (These varieties arise from $J^{D',D'}$ and $J^{D,D}$ by raising the level at q_1 and q_2.) We will also need to consider the varieties $\tilde{J}^{D'q_1q_2,D'q_1q_2} = J^{D'q_1q_2}(\Gamma_0(N/D') \cap \Gamma)$ and $\tilde{J}^{Dq_1q_2,Dq_1q_2} = J^{Dq_1q_2}(\Gamma_0(N/D) \cap \Gamma)$. Let $\tilde{\mathcal{A}}^{D'}, \tilde{\mathcal{A}}^{D}, \tilde{\mathcal{A}}^{D'q_1q_2}$, and $\tilde{\mathcal{A}}^{Dq_1q_2}$ denote the categories of abelian varieties over $\mathbb{Q}$ which are Hecke-equivariantly isogenous to $\tilde{J}^{D',D'}, \tilde{J}^{D,D}, \tilde{J}^{D'q_1q_2,D'q_1q_2}$, and $\tilde{J}^{Dq_1q_2,Dq_1q_2}$, respectively. Also consider the corresponding categories $(\tilde{\mathcal{A}}^{D'})^{\mathrm{old}}$ and $(\tilde{\mathcal{A}}^{D})^{\mathrm{old}}$, associated to the q_1q_2-old quotients of $\tilde{J}^{D'}$ and $\tilde{J}^{D}$, respectively. We adopt all of the notational conventions introduced in section 2.3 for these varieties as well; for instance, $\tilde{J}^{D,Dq_1q_2}$ will denote the q_1q_2-new subvariety of $\tilde{J}^{D,D}$. Finally, let $\tilde{\mathfrak{m}}$ be a maximal ideal of $\tilde{\mathbb{T}}^{D'}$ whose associated Galois representation is isomorphic to $\overline{\rho}_{\mathfrak{m}}$.

The four degeneracy maps $\phi_1, \phi_{q_1}, \phi_{q_2}$, and $\phi_{q_1q_2}$ from $J^{D',D'}$ to $\tilde{J}^{D',D'}$ induce an isogeny $(J^{D',D'})^4 \to (\tilde{J}^{D',D'})_{\mathrm{old}}$, where the subscript old denotes the q_1q_2-old subvariety. The action of T_{q_1} and T_{q_2} on $(J^{D',D'})^4$ is given by the matrices:

$$T_{q_1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ q_1\langle q_1\rangle & \tau_{q_1} & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & q_1\langle q_1\rangle & \tau_{q_1} \end{pmatrix}; \qquad T_{q_2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ q_2\langle q_2\rangle & 0 & \tau_{q_2} & 0 \\ 0 & q_2\langle q_2\rangle & 0 & \tau_{q_2} \end{pmatrix},$$

where τ_{q_1} and τ_{q_2} denote the endomorphisms of $J^{D',D'}$ induced by the q_1th and q_2th Hecke operators. The diamond bracket operators, and the Hecke operators T_n for n prime to q_1q_2, act diagonally. We thus have

$$(\tilde{\mathbb{T}}^{D'})^{\mathrm{old}} \cong \mathbb{T}^{D'}[T_{q_1}, T_{q_2}]/(T_{q_1}^2 - \tau_{q_1}T_{q_1} + q_1\langle q_1\rangle, T_{q_2}^2 - \tau_{q_2}T_{q_2} + q_2\langle q_2\rangle).$$

In particular $(\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}$ is a free $\mathbb{T}^{D'}$-algebra of rank 4.

More generally, given any abelian variety J in $\mathcal{A}^{D'}$ or $\mathcal{A}^{D}$, we can form the abelian variety J^4 on which $(\mathbb{T}^{D'})^{\mathrm{old}}$ acts via the matrices given above. In this situation, we have:

Lemma 2.5.3 *Let A_1 and A_2 be abelian varieties in $\mathcal{A}^{D'}$ or $\mathcal{A}^{D}$. Then*

$$\mathrm{Hom}_{(\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}}(A_1^4, A_2^4) \cong \mathrm{Hom}(A_1, A_2) \otimes_{\mathbb{T}^{D'}} (\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}.$$

Proof. A map $A_1^4 \to A_2^4$ is given by a four by four matrix of maps $A_1 \to A_2$. For such a map to be $(\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}$- equivariant, it must commute with the matrices defining the action of T_{q_1} and T_{q_2}. It is then straightforward to check that every such matrix can be written in the form $a^4 + T_{q_1}b^4 + T_{q_2}c^4 + T_{q_1}T_{q_2}d^4$ for maps a, b, c, and d from A_1 to A_2. (Here $a^4 : (A_1)^4 \to (A_2)^4$ is the map obtained by evaluating the map a at each entry of the four-tuple defining a point of $(A_1)^4$.) That all maps of the above form commute with T_{q_1} and T_{q_2} is clear. □

Recall that we have fixed abelian varieties $J_{D'}^{\min}$ and $J_{D}^{\min}$ in $\mathcal{A}^{D'}$ and $\mathcal{A}^{D}$ which satisfy "multiplicity one" at all maximal ideals outside $S^{D'}$ or S^{D}. We will use the above results to make "compatible" choices of varieties $J^{\min}$ for each of the other categories involved. The key is the following lemma:

Lemma 2.5.4 *Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^{D'}$ or $\mathbb{T}^D$, and $\tilde{\mathfrak{m}}$ a maximal ideal of $(\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}$ (or $(\tilde{\mathbb{T}}^D)^{\mathrm{old}}$) above $\mathfrak{m}$. Then $\dim_{\mathbb{T}^{D'}/\mathfrak{m}} J[\mathfrak{m}] = \dim_{(\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}/\tilde{\mathfrak{m}}}(J^4)[\tilde{\mathfrak{m}}]$ (and analogously for $\mathbb{T}^D$ and $(\tilde{\mathbb{T}}^D)^{\mathrm{old}}$).*

Proof. The set $\{1, T_{q_1}, T_{q_2}, T_{q_1}T_{q_2}\}$ is a basis for $(\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}$ over $\mathbb{T}^{D'}$. This basis gives rise to a dual basis for $\mathrm{Hom}_{\mathbb{T}^{D'}}((\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}, \mathbb{T}^{D'})$, and the matrices giving the action of T_{q_1} and T_{q_2} with respect to this dual basis are the same as the matrices giving the action of T_{q_1} and T_{q_2} on $(J^{D'})^4$. It follows that $(J^{D'})^4[\mathfrak{m}]$ is isomorphic to $\mathrm{Hom}_{\mathbb{T}^{D'}}((\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}, J^{D'}[\mathfrak{m}])$. Therefore, $(J^{D'})^4[\tilde{\mathfrak{m}}]$ is isomorphic to $\mathrm{Hom}_{\mathbb{T}^{D'}/\mathfrak{m}}((\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}/\tilde{\mathfrak{m}}, J^{D'}[\mathfrak{m}])$. The result follows immediately, and the proof for J^D is exactly the same. □

It follows from this that $(J^{\min}_{D'})^4$ satisfies "multiplicity one" at every non-Eisenstein maximal ideal of $(\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}$ of residue characteristic greater than 3. By Theorem 2.2.11, we can find an abelian variety $\tilde{J}^{\min}_{D'}$ in $\tilde{\mathcal{A}}^{D'}$ such that $\tilde{J}^{\min}_{D'}$ satisfies "multiplicity one" at every non-Eisenstein maximal ideal of $\tilde{\mathbb{T}}^{D'}$ of residue characteristic greater than 3, and such that $(\tilde{J}^{\min}_{D'})_{\mathrm{old}} = (J^{\min}_{D'})^4$. As usual, we take $\tilde{J}^{\min}_{D} = (\tilde{J}^{\min}_{D'})_{pp'\text{-new}}$, and similarly for $\tilde{J}^{\min}_{D'q_1q_2}$ and $\tilde{J}^{\min}_{Dq_1q_2}$. Finally, to $(\tilde{\mathcal{A}}^{D'})^{\mathrm{old}}$ and $(\tilde{\mathcal{A}}^{D})^{\mathrm{old}}$ we associate $\tilde{J}^{\min}_{D',\mathrm{old}} = (J^{\min}_{D'})^4$ and $\tilde{J}^{\min}_{D,\mathrm{old}} = (J^{\min}_{D})^4$, respectively. For each category $\tilde{\mathcal{A}}$ in the set

$$\{\tilde{\mathcal{A}}^{D'}, \tilde{\mathcal{A}}^{D}, \tilde{\mathcal{A}}^{D'q_1q_2}, \tilde{\mathcal{A}}^{Dq_1q_2}, (\tilde{\mathcal{A}}^{D'})^{\mathrm{old}}, (\tilde{\mathcal{A}}^{D})^{\mathrm{old}}\}$$

the above choice of abelian variety gives a functor $[-]_{\tilde{\mathcal{A}}}$ from $\tilde{\mathcal{A}}$ to the category of "rank one" modules over the corresponding Hecke algebra.

Lemma 2.5.5 *In this setting we have:*

1. *Let $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}^{D'}$ or $\tilde{\mathcal{A}}^{D}$. Then for any $J \in \tilde{\mathcal{A}}$, $[J_{\mathrm{old}}]_{\tilde{\mathcal{A}}^{\mathrm{old}}} \cong [J]_{\tilde{\mathcal{A}}} \otimes_{\tilde{\mathbb{T}}^{D'}} (\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}$.*
2. *Let $\mathcal{A} = \mathcal{A}^{D'}$ (resp. $\mathcal{A}^D$.) Let $\tilde{\mathcal{A}}$ be $\tilde{\mathcal{A}}^{D'}$ (resp. $\tilde{\mathcal{A}}^{D}$). Then for any $J \in \mathcal{A}$, $[J^4]_{\tilde{\mathcal{A}}^{\mathrm{old}}} = [J]_{\mathcal{A}} \otimes_{\mathbb{T}^{D'}} (\tilde{\mathbb{T}}^{Dp})^{\mathrm{old}}$.*

Proof. We have a natural map $[J]_{\tilde{\mathcal{A}}} \to [J_{\mathrm{old}}]_{\tilde{\mathcal{A}}^{\mathrm{old}}}$ defined by restricting an element of $[J]$ to its q_1q_2-old subvariety. Locally at any $\tilde{\mathfrak{m}}$ outside $S_{\tilde{\mathcal{A}}}$, this fits into a commutative diagram:

$$\begin{array}{ccc}
T_{\tilde{\mathfrak{m}}} J^{\min}_{\tilde{\mathcal{A}}} \otimes [J]_{\tilde{\mathcal{A}}} & \longrightarrow & T_{\tilde{\mathfrak{m}}} J \\
\downarrow & & \downarrow \\
T_{\tilde{\mathfrak{m}}} J^{\min}_{\tilde{\mathcal{A}}^{\mathrm{old}}} \otimes [J^{\mathrm{old}}]_{\tilde{\mathcal{A}}^{\mathrm{old}}} & \longrightarrow & T_{\tilde{\mathfrak{m}}} J^{\mathrm{old}}.
\end{array}$$

Since J^{old} is a subvariety of J, the right-hand vertical map is surjective. Since the horizontal maps are isomorphisms, the left-hand vertical map is surjective as well. Since $T_{\tilde{\mathfrak{m}}} J^{\min}_{\tilde{\mathcal{A}}^{\mathrm{old}}}$ is free of rank two, it follows that the natural map $([J]_{\tilde{\mathcal{A}}})_{\tilde{\mathfrak{m}}} \to ([J_{\mathrm{old}}]_{\tilde{\mathcal{A}}^{\mathrm{old}}})_{\tilde{\mathfrak{m}}}$ is surjective. Statement (1) thus follows by Lemma 2.3.1. Statement (2) is immediate from the previous lemma. □

With these technicalities out of the way, we begin our study of the geometry of these varieties. First of all, we have:

Lemma 2.5.6 *There are non-canonical isomorphisms $[\tilde{J}^{D,Dq_1q_2}]_{\tilde{\mathfrak{m}}} \cong [\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}}$ and $[\tilde{J}^{D',D'q_1q_2}]_{\tilde{\mathfrak{m}}} \cong [\tilde{J}^{D'q_1q_2,D'q_1q_2}]_{\tilde{\mathfrak{m}}}$.*

Proof. We construct these isomorphisms for $\tilde{J}^{D,Dq_1q_2}$ and $\tilde{J}^{Dq_1q_2,Dq_1q_2}$; the other case is nearly identical. Since $\tilde{J}^{Dq_1q_2,Dq_1q_2}$ is self-dual, we have $[\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}} \cong [\tilde{J}^{Dq_1q_2,Dq_1q_2}]^*_{\tilde{\mathfrak{m}}}$. Since $\tilde{\mathfrak{m}}$ is controllable at q_1, we have an isomorphism $[\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}} \cong X_{q_1}(\tilde{J}^{Dq_1q_2,Dq_1q_2})_{\tilde{\mathfrak{m}}}$ by Proposition 2.2.14 and Lemma 2.4.5. By Proposition 2.3.6, we have an isomorphism $X_{q_1}(\tilde{J}^{Dq_1q_2,Dq_1q_2})^*_{\tilde{\mathfrak{m}}} \cong X_{q_2}(\tilde{J}^{D,Dq_1q_2})_{\tilde{\mathfrak{m}}}$. Combining this with the previous two isomorphisms we find $[\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}} \cong X_{q_2}(\tilde{J}^{D,Dq_1q_2})_{\tilde{\mathfrak{m}}}$. As $\tilde{\mathfrak{m}}$ is controllable at q_2, $X_{q_2}(\tilde{J}^{D,Dq_1q_2})_{\tilde{\mathfrak{m}}}$ is isomorphic to $[\tilde{J}^{D,Dq_1q_2}]_{\tilde{\mathfrak{m}}}$ by Proposition 2.2.14 and Lemma 2.4.5. □

Proposition 2.5.7 *We have* $\dim \tilde{J}^{Dq_1q_2,Dq_1q_2}[\tilde{\mathfrak{m}}] = \dim \tilde{J}^{D,D}[\tilde{\mathfrak{m}}]$.

Proof. By Lemma 2.5.6, $\dim \tilde{J}^{Dq_1q_2,Dq_1q_2}[\tilde{\mathfrak{m}}] = \dim \tilde{J}^{D,Dq_1q_2}[\tilde{\mathfrak{m}}]$. As $\tilde{J}^{D,Dq_1q_2}$ is a subvariety of $\tilde{J}^{D,D}$, $\dim \tilde{J}^{D,Dq_1q_2}[\tilde{\mathfrak{m}}] \leq \dim \tilde{J}^{D,D}[\tilde{\mathfrak{m}}]$. We therefore have $\dim \tilde{J}^{Dq_1q_2,Dq_1q_2}[\tilde{\mathfrak{m}}] \leq \dim \tilde{J}^{D,D}[\tilde{\mathfrak{m}}]$, so it suffices to show that $\dim \tilde{J}^{Dq_1q_2,Dq_1q_2}[\tilde{\mathfrak{m}}] \geq \dim \tilde{J}^{D,D}[\tilde{\mathfrak{m}}]$. Let Y_{q_1} denote the character group of $\tilde{J}^{Dq_1q_2,Dq_1q_2}[\tilde{\mathfrak{m}}]$ at q_1, and X_{q_2} denote the character group of $J^D(\Gamma_0(q_2N/D) \cap \Gamma)^2$ at q_2. Since $\tilde{\mathfrak{m}}$ is not Eisenstein and has residue characteristic greater than 3, Proposition 2.3.8 implies that $(Y^*_{q_1}/Y_{q_1})_{\tilde{\mathfrak{m}}} \cong (X_{q_2}/(T^2_{q_1} - \langle q_1 \rangle)X_{q_2})_{\tilde{\mathfrak{m}}}$. In particular $\dim Y^*_{q_1}/\tilde{\mathfrak{m}}Y^*_{q_1} \geq \dim X_{q_2}/\tilde{\mathfrak{m}}X_{q_2}$.

We will relate $Y^*_{q_1}/\tilde{\mathfrak{m}}Y^*_{q_1}$ to $\tilde{J}^{Dq_1q_2,Dq_1q_2}[\tilde{\mathfrak{m}}]$, and $X_{q_2}/\tilde{\mathfrak{m}}X_{q_2}$ to $\tilde{J}^{D,D}[\tilde{\mathfrak{m}}]$. The key is controllability of $\tilde{\mathfrak{m}}$ at q_1 and q_2.

Since $\tilde{\mathfrak{m}}$ is controllable at q_1, $[\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}}$ is isomorphic to $(Y_{q_1})_{\tilde{\mathfrak{m}}}$ by Proposition 2.2.14 and Lemma 2.4.5. Thus the Tate module $T_{\tilde{\mathfrak{m}}}\tilde{J}^{Dq_1q_2,Dq_1q_2}$ is isomorphic to $(Y_{q_1})^2_{\tilde{\mathfrak{m}}}$. It follows that $\tilde{J}^{Dq_1q_2,Dq_1q_2}[\tilde{\mathfrak{m}}]^\vee \cong (Y_{q_1}/\tilde{\mathfrak{m}}Y_{q_1})^2$. Since $\tilde{J}^{Dq_1q_2,Dq_1q_2}$ is self-dual, we also have $[\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}} \cong [\tilde{J}^{Dq_1q_2,Dq_1q_2}]^*_{\tilde{\mathfrak{m}}}$. In particular $(Y_{q_1})_{\tilde{\mathfrak{m}}}$ is also self-dual; that is, $(Y_{q_1})_{\tilde{\mathfrak{m}}} \cong (Y^*_{q_1})_{\tilde{\mathfrak{m}}}$. We thus find that

$$\dim \tilde{J}^{Dq_1q_2,Dq_1q_2}[\tilde{\mathfrak{m}}] = 2\dim Y_{q_1}/\tilde{\mathfrak{m}}Y_{q_1} = 2\dim (Y_{q_1})^*/\tilde{\mathfrak{m}}(Y_{q_1})^*.$$

Combining this result with the above inequality, we see that it suffices to show that $\dim \tilde{J}^{D,D}[\tilde{\mathfrak{m}}] = 2\dim X_{q_2}/\tilde{\mathfrak{m}}X_{q_2}$. Moreover, by Corollary 2.2.15, $(X_{q_2})^2_{\tilde{\mathfrak{m}}}$ is isomorphic to $T_{\tilde{\mathfrak{m}}}J^D(\Gamma_0(q_2N/D) \cap \Gamma)$, since $\tilde{\mathfrak{m}}$ is controllable at q_2. Thus $\dim J^D(\Gamma_0(q_2N/D) \cap \Gamma)[\tilde{\mathfrak{m}}] = 2\dim X_{q_2}/\tilde{\mathfrak{m}}X_{q_2}$. By [DiTa], Theorem 2, the isogeny $J^D(\Gamma_0(q_2N/D) \cap \Gamma)^2 \to (\tilde{J}^{D,D})_{q_2\text{-old}}$ has kernel supported away from $\tilde{\mathfrak{m}}$, so $\dim(\tilde{J}^{D,D})_{q_2\text{-old}}[\tilde{\mathfrak{m}}] = 2\dim X_{q_2}/\tilde{\mathfrak{m}}X_{q_2}$. But by Proposition 2.5.2, $\dim(\tilde{J}^{D,D})_{q_2\text{-old}}[\tilde{\mathfrak{m}}] = \dim \tilde{J}^{D,D}[\tilde{\mathfrak{m}}]$, so the result follows. □

If we fix an isomorphism: $[\tilde{J}^{D',D'q_1q_2}]_{\tilde{\mathfrak{m}}} \to [\tilde{J}^{D'q_1q_2,D'q_1q_2}]_{\tilde{\mathfrak{m}}}$, we obtain by Theorem 2.2.11 an element of $\mathrm{Hom}(\tilde{J}^{D'q_1q_2,D'q_1q_2}, \tilde{J}^{D',D'q_1q_2})_{\tilde{\mathfrak{m}}}$. Composing with the natural inclusion of $\tilde{J}^{D',D'q_1q_2}$ into $\tilde{J}^{D',D'}$, we obtain a map f in $\mathrm{Hom}(\tilde{J}^{D'q_1q_2,D'q_1q_2}, \tilde{J}^{D',D'})_{\tilde{\mathfrak{m}}}$ which induces a surjection $[\tilde{J}^{D',D'}]_{\tilde{\mathfrak{m}}} \to [\tilde{J}^{D'q_1q_2,D'q_1q_2}]_{\tilde{\mathfrak{m}}}$.

Lemma 2.5.8 *There is a map g in* $\mathrm{Hom}(\tilde{J}^{Dq_1q_2,Dq_1q_2}, \tilde{J}^{D,D})_{\tilde{\mathfrak{m}}}$ *such that the "dual" map $g^\dagger$ fits in the commutative diagram:*

$$\begin{array}{ccc}
X_p(\tilde{J}^{Dq_1q_2,Dq_1q_2})_{\tilde{\mathfrak{m}}} & \longrightarrow & X_{p'}(\tilde{J}^{D'q_1q_2,D'q_1q_2})_{\tilde{\mathfrak{m}}} \\
{\scriptstyle g^\dagger}\big\downarrow & & \big\downarrow{\scriptstyle f^\dagger} \\
X_p(\tilde{J}^{D,D})_{\tilde{\mathfrak{m}}} & \longrightarrow & X_{p'}(\tilde{J}^{D',D'})_{\tilde{\mathfrak{m}}},
\end{array}$$

in which the horizontal maps are those in the exact sequence of Theorem 1.8.2, the left-hand vertical map is induced by $g^\dagger$, and the right-hand vertical map is induced by $f^\dagger$.

Proof. As the upper horizontal map in the above diagram identifies $X_p(\tilde{J}^{Dq_1q_2,Dq_1q_2})_{\tilde{\mathfrak{m}}}$ with $X_{p'}(\tilde{J}^{D,D})[I]$, where I is the kernel of the map $\tilde{\mathbb{T}} \to \tilde{\mathbb{T}}^{q_1q_2\text{-new}}$, and similarly for the lower horizontal map, it is clear that there is a map $X_p(\tilde{J}^{Dq_1q_2,Dq_1q_2})_{\tilde{\mathfrak{m}}} \to X_p(\tilde{J}^{D,D})_{\tilde{\mathfrak{m}}}$ making the above diagram commute; the only thing we have to check is that it is induced by an element h of $\mathrm{Hom}(\tilde{J}^{D,D}, \tilde{J}^{Dq_1q_2,Dq_1q_2})_{\tilde{\mathfrak{m}}}$. We may then take $g = h^\dagger$.

In particular it suffices to show that the natural map:

$$\mathrm{Hom}((\tilde{J}^{D,D})^{q_1q_2\text{-new}}, \tilde{J}^{Dq_1q_2,Dq_1q_2})_{\tilde{\mathfrak{m}}} \to \mathrm{Hom}(X_p(\tilde{J}^{Dq_1q_2,Dq_1q_2}), X_p((\tilde{J}^{D,D})^{q_1q_2\text{-new}}))_{\tilde{\mathfrak{m}}}$$

is an isomorphism. (Or, equivalently, that the natural map

$$\mathrm{Hom}([\tilde{J}^{Dq_1q_2,Dq_1q_2}], [(\tilde{J}^{D,D})^{q_1q_2\text{-new}}])_{\tilde{\mathfrak{m}}} \to \mathrm{Hom}(X_p(\tilde{J}^{Dq_1q_2,Dq_1q_2}), X_p((\tilde{J}^{D,D})^{q_1q_2\text{-new}}))_{\tilde{\mathfrak{m}}}$$

is an isomorphism).

By Lemma 2.5.6, the modules $[\tilde{J}^{D,Dq_1q_2}]_{\tilde{\mathfrak{m}}}$ and $[\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}}$ are isomorphic. It follows that there exists an injection of $[\tilde{J}^{D,Dq_1q_2}]$ into $[\tilde{J}^{Dq_1q_2,Dq_1q_2}]$ whose cokernel is supported away from $\tilde{\mathfrak{m}}$. By Theorem 2.2.11, this implies that there is an isogeny $\phi : \tilde{J}^{Dq_1q_2,Dq_1q_2} \to \tilde{J}^{D,Dq_1q_2}$ whose kernel is supported away from $\tilde{\mathfrak{m}}$. Then $\phi^\dagger$ is an isogeny from $(\tilde{J}^{Dq_1q_2,Dq_1q_2})^\dagger$ to $(\tilde{J}^{D,Dq_1q_2})^\dagger$ whose kernel is supported away from $\tilde{\mathfrak{m}}$. But $\tilde{J}^{Dq_1q_2,Dq_1q_2}$ is self-dual, and $(\tilde{J}^{D,Dq_1q_2})^\dagger = (\tilde{J}^{D,D}_{q_1q_2\text{-new}})^\dagger \cong (\tilde{J}^{D,D})^{q_1q_2\text{-new}}$, so we can consider $\phi^\dagger$ as an isogeny from $(\tilde{J}^{D,D})^{q_1q_2\text{-new}}$ to $\tilde{J}^{Dq_1q_2,Dq_1q_2}$, whose kernel is supported away from $\tilde{\mathfrak{m}}$. Viewed in this way, $\phi^\dagger$ induces isomorphisms between $[\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}}$ and $[(\tilde{J}^{D,D})^{q_1q_2\text{-new}}]_{\tilde{\mathfrak{m}}}$, as well as between $X_p((\tilde{J}^{D,D})^{q_1q_2\text{-new}})_{\tilde{\mathfrak{m}}}$ and $X_p(\tilde{J}^{Dq_1q_2,Dq_1q_2})_{\tilde{\mathfrak{m}}}$.

Under these identifications, the natural map:

$$\mathrm{Hom}([\tilde{J}^{Dq_1q_2,Dq_1q_2}], [(\tilde{J}^{D,D})^{q_1q_2\text{-new}}])_{\tilde{\mathfrak{m}}} \to \mathrm{Hom}(X_p(\tilde{J}^{Dq_1q_2,Dq_1q_2}), X_p((\tilde{J}^{D,D})^{q_1q_2\text{-new}}))_{\tilde{\mathfrak{m}}}$$

is identified with the map:

$$\mathrm{End}([\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}}) \to \mathrm{End}(X_p(\tilde{J}^{Dq_1q_2,Dq_1q_2})_{\tilde{\mathfrak{m}}}).$$

It thus suffices to show that the latter is an isomorphism. We have a sequence of isomorphisms:

$$\begin{aligned} \mathrm{End}(X_p(\tilde{J}^{Dq_1q_2,Dq_1q_2}_{\tilde{\mathfrak{m}}})) &\cong \mathrm{End}(X_p(J^{\min}_{Dq_1q_2})_{\tilde{\mathfrak{m}}} \otimes [\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}}) \\ &\cong \mathrm{Hom}([\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}}, \mathrm{Hom}(X_p(J^{\min}_{Dq_1q_2}), X_p(\tilde{J}^{Dq_1q_2,Dq_1q_2}))_{\tilde{\mathfrak{m}}}) \\ &\cong \mathrm{Hom}([\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}}, [\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}}). \end{aligned}$$

(The first isomorphism comes from Proposition 2.2.14; the second from the adjointness of Hom and tensor product, and the third from Corollary 2.4.6, using the fact that $\tilde{\mathfrak{m}}$ is controllable at q_1.) It follows by Lemma 2.4.2 that the natural map in question is an isomorphism. □

We are now in a position to prove:

Proposition 2.5.9 *The natural map $[\tilde{J}^{D,D}]_{\tilde{\mathfrak{m}}} \to \mathrm{Hom}(X_p(\tilde{J}^{\min}_D), X_p(\tilde{J}^{D,D}))_{\tilde{\mathfrak{m}}}$ is an isomorphism.*

Proof. For conciseness of notation, we abbreviate q_1q_2 by $\underline{q}$, and write X_p for $X_p(\tilde{J}_D^{\min})$ and $\overline{X}_p$ for $X_p(\tilde{J}_{D\underline{q}}^{\min})$. There is a natural surjection $\pi_p : X_p \to \overline{X}_p$ induced by the inclusion of $\tilde{J}_{D\underline{q}}^{\min}$ into $\tilde{J}_D^{\min}$.

We have a sequence of isomorphisms:

$$\mathrm{Hom}(X_p, X_p(\tilde{J}^{D,D}))^*_{\tilde{\mathfrak{m}}} \to (X_p)_{\tilde{\mathfrak{m}}} \otimes X_p(\tilde{J}^{D,D})^*_{\tilde{\mathfrak{m}}} \to (X_p)_{\tilde{\mathfrak{m}}} \otimes X_{p'}(\tilde{J}^{D',D})_{\tilde{\mathfrak{m}}},$$

where the first isomorphism comes from Lemma 2.0.5 and the second from Proposition 2.3.6. We also obtain isomorphisms:

$$\mathrm{Hom}(\overline{X}_p, X_p(\tilde{J}^{D\underline{q},D\underline{q}}))^*_{\tilde{\mathfrak{m}}} \to (\overline{X}_p)_{\tilde{\mathfrak{m}}} \otimes X_p(\tilde{J}^{D\underline{q},D\underline{q}})^*_{\tilde{\mathfrak{m}}} \to (\overline{X}_p)_{\tilde{\mathfrak{m}}} \otimes X_{p'}(\tilde{J}^{D'\underline{q},D\underline{q}})_{\tilde{\mathfrak{m}}},$$

in the same fashion. By Lemma 2.3.7 these fit into a commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}(X_p, X_p(\tilde{J}^{D,D}))^*_{\tilde{\mathfrak{m}}} & \longrightarrow & (X_p)_{\tilde{\mathfrak{m}}} \otimes X_p(\tilde{J}^{D,D})^*_{\tilde{\mathfrak{m}}} & \longrightarrow & (X_p)_{\tilde{\mathfrak{m}}} \otimes X_{p'}(\tilde{J}^{D',D})_{\tilde{\mathfrak{m}}} \\
{\scriptstyle (g^\dagger)^*}\downarrow & & {\scriptstyle \pi_p\otimes(g^\dagger)^*}\downarrow & & \downarrow{\scriptstyle \pi_p\otimes\overline{f}} \\
\mathrm{Hom}(\overline{X}_p, X_p(\tilde{J}^{D\underline{q},D\underline{q}}))^*_{\tilde{\mathfrak{m}}} & \longrightarrow & (\overline{X}_p)_{\tilde{\mathfrak{m}}} \otimes X_p(\tilde{J}^{D\underline{q},D\underline{q}})^*_{\tilde{\mathfrak{m}}} & \longrightarrow & (\overline{X}_p)_{\tilde{\mathfrak{m}}} \otimes X_{p'}(\tilde{J}^{D'\underline{q},D\underline{q}})_{\tilde{\mathfrak{m}}}.
\end{array}$$

In particular, the right-hand vertical map is surjective, since the kernel of f is supported away from $\tilde{\mathfrak{m}}$, so the left-hand vertical map is surjective as well. This map fits into a commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}(X_p, X_p(\tilde{J}^{D,D}))^*_{\tilde{\mathfrak{m}}} & \longrightarrow & [\tilde{J}^{D,D}]^*_{\tilde{\mathfrak{m}}} \\
{\scriptstyle (g^\dagger)^*}\downarrow & & {\scriptstyle (g^\dagger)^*}\downarrow \\
\mathrm{Hom}(\overline{X}_p, X_p(\tilde{J}^{D\underline{q},D\underline{q}}))_{\tilde{\mathfrak{m}}} & \to & [\tilde{J}^{D\underline{q},D\underline{q}}]^*_{\tilde{\mathfrak{m}}},
\end{array}$$

where the horizontal maps are the $\mathbb{Z}_l$-duals of the canonical maps

$$[\tilde{J}^{D,D}] \to \mathrm{Hom}(X_p, X_p(\tilde{J}^{D,D})) \quad \text{and} \quad [\tilde{J}^{D\underline{q}}] \to \mathrm{Hom}(\overline{X}_p, X_p(\tilde{J}^{D\underline{q},D\underline{q}})).$$

By Corollary 2.4.6, the bottom horizontal map is an isomorphism.

Since the left-hand vertical map is a surjection, the right-hand vertical map is as well. Moreover, since both $\tilde{J}^{D,D}$ and $\tilde{J}^{D\underline{q},D\underline{q}}$ are self-dual, we have $[\tilde{J}^{D,D}]_{\tilde{\mathfrak{m}}} \cong [\tilde{J}^{D,D}]^*_{\tilde{\mathfrak{m}}}$ and similarly for $[\tilde{J}^{D\underline{q},D\underline{q}}]$. By Proposition 2.5.7, and the fact that $2\dim[\tilde{J}^{D,D}] \otimes \tilde{\mathbb{T}}/\tilde{\mathfrak{m}} = \dim \tilde{J}^{D,D}[\tilde{\mathfrak{m}}]$ (and similarly for $\tilde{J}^{D\underline{q},D\underline{q}}$,) it follows that the right-hand vertical map becomes an isomorphism after tensoring with $\tilde{\mathbb{T}}/\tilde{\mathfrak{m}}$. Thus, after tensoring with $\tilde{\mathbb{T}}/\tilde{\mathfrak{m}}$, the bottom horizontal map and the right-hand vertical maps are isomorphisms, and the left-hand vertical map is surjective. It follows that the upper horizontal map becomes surjective after tensoring with $\tilde{\mathbb{T}}/\tilde{\mathfrak{m}}$. By Nakayama's lemma, this implies that the upper horizontal map is surjective; since both of the modules in the top row have the same $\mathbb{Z}_l$-rank and no $\mathbb{Z}_l$-torsion, the upper horizontal map is an isomorphism. The result follows immediately. □

It remains to pass from $\tilde{J}^{D,D}$ to $(\tilde{J}^{D,D})^{\mathrm{old}}$, and then to $J^{D,D}$.

Proposition 2.5.10 *The natural map:* $[(\tilde{J}^{D,D})^{\mathrm{old}}]_{\tilde{\mathfrak{m}}} \to \mathrm{Hom}(X_p(\tilde{J}^{\min}_{D,\mathrm{old}})_{\tilde{\mathfrak{m}}}, X_p((\tilde{J}^{D,D})^{\mathrm{old}})_{\tilde{\mathfrak{m}}})$ *is an isomorphism.*

Proof. Let ϕ denote the surjection $\tilde{J}^{D,D} \to (\tilde{J}^{D,D})^{\mathrm{old}}$. Composition with ϕ induces a map

$$[(\tilde{J}^{D,D})^{\mathrm{old}}] = \mathrm{Hom}((\tilde{J}^{D,D})^{\mathrm{old}}, \tilde{J}^{\min}_{D,\mathrm{old}}) \to \mathrm{Hom}(\tilde{J}^{D,D}, \tilde{J}^{\min}_{D,\mathrm{old}}).$$

Since every map: $\tilde{J}^{D,D} \to \tilde{J}^{\min}_{D,\mathrm{old}}$ factors through the q_1q_2-old quotient, this map is an isomorphism. The inclusion of $\tilde{J}^{\min}_{D,\mathrm{old}}$ into $\tilde{J}^{\min}_{D}$ thus induces a map $[(\tilde{J}^{D,D})^{\mathrm{old}}] \to [\tilde{J}^{D,D}]$. Since $\tilde{J}^{\min}_{D,\mathrm{old}}$ is the connected component of the identity in the subvariety of $\tilde{J}^{\min}_{D}$ annihilated by the kernel I of the map $\tilde{\mathbb{T}}^D \to (\tilde{\mathbb{T}}^D)^{\mathrm{old}}$, this map identifies $[(\tilde{J}^{D,D})^{\mathrm{old}}]$ with the submodule of $[\tilde{J}^{D,D}]$ killed by I. It follows that the dual map $[\tilde{J}^{D,D}] \to [(\tilde{J}^{D,D})^{\mathrm{old}}]$ is surjective.

We have a commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}(X_p(\tilde{J}^{\min}_{D})_{\tilde{\mathfrak{m}}}, X_p(\tilde{J}^{D,D})_{\tilde{\mathfrak{m}}})^* & \longrightarrow & [\tilde{J}^{D,D}]_{\tilde{\mathfrak{m}}} \\
\downarrow & & \downarrow \\
\mathrm{Hom}(X_p(\tilde{J}^{\min}_{D,\mathrm{old}})_{\tilde{\mathfrak{m}}}, X_p((\tilde{J}^{D,D})^{\mathrm{old}})_{\tilde{\mathfrak{m}}})^* & \longrightarrow & [(\tilde{J}^{D,D})^{\mathrm{old}}]_{\tilde{\mathfrak{m}}},
\end{array}$$

in which the vertical maps are induced by the surjection $\tilde{J}^{D,D} \to (\tilde{J}^{D,D})^{\mathrm{old}}$, and the horizontal maps are the $\mathbb{Z}_l$-duals of the maps taking a morphism of abelian varieties to the induced map on character groups.

By the previous proposition, the upper horizontal map is an isomorphism; the right-hand vertical map is surjective. It follows that the lower horizontal map is surjective, and hence an isomorphism. □

The following technical lemma is now all that we need in order to establish Theorem 2.4.1 at $\mathfrak{m}$.

Lemma 2.5.11 *There is an isogeny $(\tilde{J}^{D,D})^{\mathrm{old}} \to (J^{D,D})^4$ whose kernel is supported away from $\tilde{\mathfrak{m}}$.*

Proof. It is equivalent to show that $[(J^{D,D})^4]_{\tilde{\mathfrak{m}}} \cong [(\tilde{J}^{D,D})^{\mathrm{old}}]_{\tilde{\mathfrak{m}}}$. The isogeny $(J^{D,D})^4 \to \tilde{J}^{D,D}_{\mathrm{old}}$ has kernel supported away from $\tilde{\mathfrak{m}}$ by [DiTa], Theorem 2, so the dual isogeny $(\tilde{J}^{D,D})^{\mathrm{old}} \to ((J^{D,D})^4)^{\dagger}$ has kernel supported away from $\tilde{\mathfrak{m}}$ as well. In particular,

$$[(\tilde{J}^{D,D})^{\mathrm{old}}]_{\tilde{\mathfrak{m}}} \cong [((J^{D,D})^4)^{\dagger}]_{\tilde{\mathfrak{m}}} \cong [(J^{D,D})^4]^*_{\tilde{\mathfrak{m}}},$$

so it suffices to show that $[(J^{D,D})^4]_{\tilde{\mathfrak{m}}}$ is self-dual.

We begin by showing that $(\tilde{\mathbb{T}}^D_{\tilde{\mathfrak{m}}})^* \cong (\mathbb{T}^D_{\mathfrak{m}})^* \otimes \tilde{\mathbb{T}}^D_{\tilde{\mathfrak{m}}}$. Observe that $\tilde{\mathbb{T}}^D$ is free of rank four over $\mathbb{T}^D$, with basis $\{1, T_{q_1}, T_{q_2}, T_{q_1}T_{q_2}\}$. This choice of basis allows us to identify both $(\tilde{\mathbb{T}}^D)^*$ and $(\mathbb{T}^D)^* \otimes \tilde{\mathbb{T}}^D$ with $((\mathbb{T}^D)^*)^4$ as $\mathbb{T}^D$-modules. T_{q_1} and T_{q_2} act via the matrices

$$T_{q_1} = \begin{pmatrix} \tau_{q_1} & 1 & 0 & 0 \\ -q_1\langle q_1\rangle & 0 & 0 & 0 \\ 0 & 0 & \tau_{q_1} & 1 \\ 0 & 0 & -q_1\langle q_1\rangle & 0 \end{pmatrix}; \qquad T_{q_2} = \begin{pmatrix} \tau_{q_2} & 0 & 1 & 0 \\ 0 & \tau_{q_2} & 0 & 1 \\ -q_2\langle q_2\rangle & 0 & 0 & 0 \\ 0 & -q_2\langle q_2\rangle & 0 & 0 \end{pmatrix}$$

on the latter, and via the transposes of these matrices on the former. After tensoring with $\mathbb{Z}_l$, conjugation with the diagonal matrix

$$P = \begin{pmatrix} q_1\langle q_1\rangle q_2\langle q_2\rangle & 0 & 0 & 0 \\ 0 & -q_2\langle q_2\rangle & 0 & 0 \\ 0 & 0 & -q_1\langle q_1\rangle & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

sends the above matrices to their transposes. Thus P induces an isomorphism of $(\mathbb{T}^D)^* \otimes \tilde{\mathbb{T}}^D \otimes \mathbb{Z}_l$ with $(\tilde{\mathbb{T}}^D)^* \otimes \mathbb{Z}_l$. Localizing at $\tilde{\mathfrak{m}}$ then gives the desired isomorphism.

Now we have:

$$[(J^{D,D})^4]^*_{\tilde{\mathfrak{m}}} \cong \mathrm{Hom}([J^{D,D}]_{\mathfrak{m}} \otimes \tilde{\mathbb{T}}^D_{\tilde{\mathfrak{m}}}, (\tilde{\mathbb{T}}^D)^*_{\tilde{\mathfrak{m}}}) \cong \mathrm{Hom}([J^{D,D}]_{\mathfrak{m}} \otimes \tilde{\mathbb{T}}^D_{\tilde{\mathfrak{m}}}, (\mathbb{T}^D)^*_{\mathfrak{m}} \otimes \tilde{\mathbb{T}}^D_{\tilde{\mathfrak{m}}}).$$

The latter is just $[J^{D,D}]^*_{\mathfrak{m}} \otimes \tilde{\mathbb{T}}^D_{\tilde{\mathfrak{m}}}$, which is isomorphic to $[J^{D,D}]_{\mathfrak{m}} \otimes \tilde{\mathbb{T}}^D_{\tilde{\mathfrak{m}}} \cong [(J^{D,D})^4]_{\tilde{\mathfrak{m}}}$, as required. □

Proof of Theorem 2.4.1 It suffices to prove that the map

$$[J^{D,D}]_{\mathfrak{m}} \to \mathrm{Hom}(X_p(J_D^{\min})_{\mathfrak{m}}, X_p(J^{D,D})_{\mathfrak{m}})$$

is an isomorphism. By Lemma 2.5.11 and Proposition 2.5.10, the map

$$[J^{D,D}]^4_{\tilde{\mathfrak{m}}} \to \mathrm{Hom}(X_p((J_D^{\min})^4)_{\tilde{\mathfrak{m}}}, X_p((J^{D,D})^4)_{\tilde{\mathfrak{m}}})$$

is an isomorphism. It is easy to check that this latter map is just the first map

$$[J^{D,D}]_{\mathfrak{m}} \to \mathrm{Hom}(X_p(J_D^{\min})_{\mathfrak{m}}, X_p(J^{D,D})_{\mathfrak{m}})$$

tensored over $\mathbb{T}^D_{\mathfrak{m}}$ with $\tilde{\mathbb{T}}^D_{\tilde{\mathfrak{m}}}$. Since $\tilde{\mathbb{T}}^D_{\tilde{\mathfrak{m}}}$ is free over $\mathbb{T}^D_{\mathfrak{m}}$, the result is immediate. □

2.6 Main results

Having successfully proven Theorem 2.4.1, we will now make use of it to compute $[J^D]$ for D dividing N. Recall that for each D that divides N and has an even number of prime divisors, we have constructed a locally-free $\mathbb{T}^D_{S^D}$-module L_D such that $[(J_D^{\min})^\dagger] \cong_{S^D} (\mathbb{T}^D)^* \otimes L_D$, which is unique up to isomorphism (Proposition 2.2.17.) We have shown that $[J^{D,D}] \cong_{S^D} [J^{D,D}]^* \otimes L_D$.

Proposition 2.6.1 *Let p and q be two distinct primes dividing D. Then we have*

$$[J^{D,D}] \cong_{S^D} [J^{\frac{D}{pq},D}] \otimes X_p(J_D^{\min}) \otimes X_q(J_D^{\min}) \otimes L_D.$$

Proof. We have a sequence of isomorphisms:

$$\begin{aligned} [J^{D,D}] &\cong_{S^D} [J^{D,D}]^* \otimes L_D && \text{(self-duality of } J^{D,D}) \\ &\cong_{S^D} \mathrm{Hom}(X_p(J_D^{\min}), X_p(J^{D,D}))^* \otimes L_D && \text{(Theorem 2.4.1)} \\ &\cong_{S^D} X_p(J_D^{\min}) \otimes X_p(J^{D,D})^* \otimes L_D && \text{(Lemma 2.0.5)} \\ &\cong_{S^D} X_p(J_D^{\min}) \otimes X_q(J^{\frac{D}{pq},D}) \otimes L_D && \text{(Proposition 2.3.6)} \\ &\cong_{S^D} X_p(J_D^{\min}) \otimes X_q(J_D^{\min}) \otimes [J^{\frac{D}{pq},D}] \otimes L_D && \text{(Proposition 2.2.14),} \end{aligned}$$

which immediately proves the claim. ☐

To extend this to a relationship between $[J^{D,R}]$ and $[J^{Dpq,R}]$ for R divisible by Dpq, we use the following lemma:

Lemma 2.6.2 *Let D R, and R' be divisors of N such that D has an even number of prime divisors, and such that $D|R|R'$. Then $[J^{D,R'}] \cong_{S^{R'}} [J^{D,R}] \otimes_{\mathbb{T}^R} \mathbb{T}^{R'}$.*

Proof. There is a natural map $[J^{D,R}] \to [J^{D,R'}]$ given by taking an element of $[J^{D,R}]$, treating it as an element of $\mathrm{Hom}(J^{D,R}, J_R^{\min})$, and restricting it to $J^{D,R'} \subset J^{D,R}$. (The image of $J^{D,R'}$ will always land in $J_{R'}^{\min} \subset J_R^{\min}$.) By Lemma 2.3.1, it suffices to show the surjectivity of this map at $\mathfrak{m}$ outside $S_{R'}$. Let $\hat{\mathfrak{m}}$ be the preimage of $\mathfrak{m}$ in $\mathbb{T}^R$. We have a commutative diagram:

$$\begin{array}{ccc}
T_{\hat{\mathfrak{m}}} J_R^{\min} \otimes [J^{D,R}] & \longrightarrow & T_{\mathfrak{m}} J_{R'}^{\min} \otimes [J^{D,R'}]_{\mathfrak{m}} \\
\big\downarrow & & \big\downarrow \\
T_{\hat{\mathfrak{m}}} J^{D,R} & \longrightarrow & T_{\mathfrak{m}} J^{D,R'}
\end{array}$$

in which the vertical maps are isomorphisms. The lower horizontal map is induced by an inclusion of varieties and is therefore surjective. The upper horizontal map is the restriction map we wish to study, tensored with $T_{\hat{\mathfrak{m}}} J_R^{\min}$. The surjectivity of the lower horizontal map means that the upper horizontal map is surjective as well. But since $T_{\hat{\mathfrak{m}}} J_R^{\min}$ is free of rank two over $\mathbb{T}_{\hat{\mathfrak{m}}}^R$, this implies that the map $[J^{D,R}]_{\hat{\mathfrak{m}}} \to [J^{D,R'}]_{\mathfrak{m}}$ is surjective as well. ☐

We also need to relate L_R with $L_{R'}$ for R dividing R'; since we have chosen our $J_R^{\min}$ compatibly, we have the following:

Lemma 2.6.3 *If R divides R', then $L_{R'} \cong_{S^{R'}} L_R \otimes_{\mathbb{T}^R} \mathbb{T}^{R'}$.*

Proof. The property $[(J_{R'}^{\min})^\dagger] \cong_{S^{R'}} (\mathbb{T}^{R'})^* \otimes L_{R'}$ characterises R' up to $S^{R'}$-isomorphism. Since $J_{R'}^{\min}$ is the $\frac{R'}{R}$-new subvariety of $J_R^{\min}$, $J_{R'}^{\min}$ is the connected component of $J_R^{\min}[I]$, where I is the kernel of the map $\mathbb{T}^R \to \mathbb{T}^{R'}$. Thus $[(J_{R'}^{\min})^\dagger] = \mathrm{Hom}((J_{R'}^{\min})^\dagger, J_{R'}^{\min})$ is isomorphic to $\mathrm{Hom}((J_{R'}^{\min})^\dagger, J_R^{\min})[I]$. The injection $J_{R'}^{\min} \to J_R^{\min}$ dualizes to a surjection $(J_R^{\min})^\dagger \to (J_{R'}^{\min})^\dagger$; moreover, any morphism of $(J_R^{\min})^\dagger$ that is killed by I factors through $(J_{R'}^{\min})^\dagger$. Thus $\mathrm{Hom}((J_{R'}^{\min})^\dagger, J_R^{\min})[I] \cong_{S^{R'}} \mathrm{Hom}((J_R^{\min})^\dagger, J_R^{\min})[I] = [(J_R^{\min})^\dagger][I]$.

It follows that $[(J_{R'}^{\min})^\dagger] \cong [(J_R^{\min})^\dagger][I]$. Dualizing, and invoking Corollary 2.3.2, we find that $L_{R'}^{-1} \cong L_R \otimes_{\mathbb{T}^R} \mathbb{T}^{R'}$, as required. ☐

Putting these together, and passing to $\mathcal{A}^N$, we find:

Proposition 2.6.4 *Let D divide N and suppose D has an even number of prime divisors. For any p and q dividing D, we have:*

$$[J^D] \cong_{S^D} [J^{\frac{D}{pq}}] \otimes X_p(J^{\min}) \otimes X_q(J^{\min}) \otimes L_N.$$

Proof. Tensor the isomorphism of Proposition 2.6.1 with $\mathbb{T}^N$ and apply Lemmas 2.6.2 and 2.6.3. ☐

Working inductively, one immediately proves:

Theorem 2.6.5 *For any D dividing N with exactly $2k$ prime divisors, we have:*

$$[J^D] \cong_S [J^1] \otimes (L_N)^{\otimes k} \otimes \bigotimes_{p|D} X_p(J^{\min}).$$

Putting this together with Proposition 2.2.12, we immediately obtain:

Corollary 2.6.6 *Let D and D' be divisors of N, divisible by $2n$ and $2m$ primes, respectively, such that D divides D'. Let M be the module*

$$(L_N)^{\otimes m-n} \otimes \bigotimes_{p|\frac{D'}{D}} X_p(J^{\min}).$$

Then there is an S-isomorphism:

$$g : \mathrm{Hom}(J^D, J^{D'}) \to \mathrm{Hom}(M, \mathrm{End}(J^D))$$

such that if $\phi \in \mathrm{Hom}_S(J^D, J^{D'})$, and $I_\phi \subset \mathrm{End}(J^D)$ is the ideal generated by image of M under $g(\phi) : M \to \mathrm{End}(J^D)$, then $\ker_S \phi = J^D[I_\phi]$ up to support on S.

This result determines the S-isomorphism class of $J^{D'}$ in terms of J^D; in particular, if one has J^D, and one wishes to construct an abelian variety S-isomorphic to $J^{D'}$, one merely has to find an ideal I of $\mathbb{T}$ (or, more generally, of $\mathrm{End}(J_1)$ isomorphic to M), and construct $J^D/J^D[I]$. Unfortunately, although M is independent of the choice of $J^{\min}$, the various factors in the product defining M are not, making M difficult to compute. One can remedy this, however, by introducing a suitable "multiplicity one" hypothesis.

In particular, we expand our set S to include those maximal ideals for which $J^{1,1} = J^1(\Gamma_0(N) \cap \Gamma)$ fails to satisfy "multiplicity one"; that is, we take S_+^R to be the set of $\mathfrak{m} \subset \mathbb{T}^R$ such that either $\mathfrak{m}$ is in S^R or $J^1(\Gamma_0(N) \cap \Gamma)[\mathfrak{m}]$ has dimension greater than two. (The latter is expected to be a very rare occurrence; in particular it cannot occur if l is prime to $2NM$ [Ti]. Moreover, the only known counterexamples are in residue characteristic 2 [Ki].) As usual we take $S_+ = S_+^N$.

The advantage of enlarging our set of "bad primes" to S_+ is that we can take $J_R^{\min} = J^{1,R}$ for all R; the "multiplicity one" condition we have imposed above makes this a valid choice. Moreover, we can compute L_N in this setting. Since $J^{1,1} \cong (J^{1,1})^\dagger$, and $[J^{1,1}] \cong_{S_+^1} \mathbb{T}^1$, we have $\mathbb{T}^1 \cong_{S_+^1} L^1 \otimes (\mathbb{T}^1)^*$. We thus recover the result that $\mathbb{T}^1_{S_+^1}$ is Gorenstein, and find that $L^1 \cong_{S_+^1} ((\mathbb{T}^1_{S_+^1})^*)^{-1}$. Lemma 2.6.3 then determines L_N explicitly. Putting this together, we find:

Theorem 2.6.7 *For any D dividing N and divisible by exactly $2k$ primes, we have:*

$$[J^D] \cong_{S_+} [J^1] \otimes (L_N)^{\otimes k} \otimes \bigotimes_{p|D} X_p(J^1).$$

Corollary 2.6.8 *Let D and D' be divisors of N, divisible by exactly $2n$ and $2m$ primes, respectively, such that D divides D'. Let M be the module*

$$(L_N)^{\otimes m-n} \otimes \bigotimes_{p|\frac{D'}{D}} X_p(J^1).$$

Then there is an S_+-isomorphism:

$$g : \mathrm{Hom}(J^D, J^{D'}) \to \mathrm{Hom}(M, \mathrm{End}(J^D))$$

such that if $\phi \in \mathrm{Hom}_{S_+}(J^D, J^{D'})$, and $I_\phi \subset \mathrm{End}(J^D)$ is the ideal generated by image of M under $g(\phi) : M \to \mathrm{End}(J^D)$, then $\ker_{S_+} \phi = J^D[I_\phi]$ up to support on S_+.

Remark 2.6.9 The previous two results, although valid over a smaller subset of $\mathrm{Spec}\,\mathbb{T}$, have the advantage that the character groups appearing within them are explicitly computable, via the techniques in [Ko] together with Corollary 2.3.3. It is thus possible, given the isomorphism class of J^1, to *effectively compute* an abelian variety which is S_+-isomorphic to J^D for any D one desires.

The following result, a strengthening of a result proved by Yang [Ya] in the case in which $\overline{\rho}_{\mathfrak{m}}$ is ramified at at least half of the primes dividing D, is an immediate corollary.

Corollary 2.6.10 *Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}$ outside S, and let k be the number of primes dividing D at which $\mathfrak{m}$ is not controllable. Then $\dim J^D[\mathfrak{m}] \leq 2^k \dim J^1[\mathfrak{m}]$.*

Proof. Since $J^D[\mathfrak{m}] \cong (T_{\mathfrak{m}}J^D/\mathfrak{m}T_{\mathfrak{m}}J^D)^\vee$, and similarly for $J_1[\mathfrak{m}]$, it suffices to show that $\dim T_{\mathfrak{m}}J^D/\mathfrak{m}T_{\mathfrak{m}}J^D \leq 2^k \dim J^1/\mathfrak{m}T_{\mathfrak{m}}J^1$. Moreover, $T_{\mathfrak{m}}J^D \cong [J^D]_{\mathfrak{m}} \otimes T_{\mathfrak{m}}J^{\min}$, and similarly for $T_{\mathfrak{m}}J^1$, so it suffices to show that $\dim[J^D]/\mathfrak{m}[J^D] \leq 2^k \dim[J^1]/\mathfrak{m}[J^1]$. By Nakayama's lemma, this is equivalent to showing that the size of a minimal generating set for $[J^D]_{\mathfrak{m}}$ is at most 2^k times the size of a minimal generating set for $[J^1]_{\mathfrak{m}}$. But we have $[J^D]_{\mathfrak{m}} \cong [J^1]_{\mathfrak{m}} \otimes \bigotimes_{p|D} X_p(J^{\min})_{\mathfrak{m}}$. If $\mathfrak{m}$ is controllable at p, $X_p(J^{\min})_{\mathfrak{m}}$ is free of rank one, and can be ignored. If $\mathfrak{m}$ is not controllable at p, the surjection $T_{\mathfrak{m}}J^{\min} \to X_p(J^{\min})_{\mathfrak{m}}$ shows that $X_p(J^{\min})/\mathfrak{m}X_p(J^{\min})$ has dimension at most two. In particular, $X_p(J^{\min})_{\mathfrak{m}}$ is generated by a two-element set, so tensoring with $X_p(J^{\min})_{\mathfrak{m}}$ increases the size of a minimal generating set by at most a factor of two. The result follows. □

Considering the special case in which $\mathfrak{m}$ is controllable at *every* prime dividing D yields a "multiplicity one" result for Jacobians of Shimura curves:

Corollary 2.6.11 *Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}$ lying outside S. If $\mathfrak{m}$ is controllable at every prime dividing D, then $J^D[\mathfrak{m}]$ and $J^1[\mathfrak{m}]$ have the same dimension. In particular, if $\mathfrak{m}$ satisfies "multiplicity one" for $J^{1,1} = J^1(\Gamma_0(N) \cap \Gamma)$, then $\dim J^D[\mathfrak{m}] = 2$.*

Proof. By Theorem 2.6.5, and the fact that $X_p(J^{\min})_{\mathfrak{m}}$ is free for every p dividing D, we have $[J^D]_{\mathfrak{m}} \cong [J^1]_{\mathfrak{m}}$. It follows that $T_{\mathfrak{m}}J^D \cong T_{\mathfrak{m}}J^1$, and hence that $J^D[\mathfrak{m}] \cong J^1[\mathfrak{m}]$. If $\mathfrak{m}$ satisfies "multiplicity one" for the full Jacobian $J^{1,1}$, then it also satisfies "multiplicity one" for J^1, as the latter is just the N-new subvariety of the former. In particular $J^1[\mathfrak{m}]$ has dimension two and the result follows. □

Remark 2.6.12 Although we have been working throughout with maximal ideals $\mathfrak{m}$ outside S (and hence of residue characteristic 2 or 3), the preceding corollary holds in more generality. In particular, if $\Gamma = \Gamma_0(M)$ or if Γ contains $\Gamma_1(r)$, for $r \geq 4$, then the preceding corollary holds for all non-Eisenstein $\mathfrak{m}$. To establish this, one observes observe that the only places in the above argument in which we need the maximal ideals of residue characteristic 2 or 3 to lie in S are in Proposition 2.3.5 and in the level-raising arguments of section 2.5. Since the latter are unnecessary for an $\mathfrak{m}$ which is already controllable at every prime dividing D, and Proposition 2.3.5 holds even in residue characteristics 2 and 3 when $\Gamma = \Gamma_0(M)$ or $\Gamma_1(M)$ [Ri7], all of the results we have obtained above hold locally at $\mathfrak{m}$. This is enough to establish the "multiplicity one" result above.

Corollary 2.6.13 *Let Γ be a congruence subgroup of level M, and U a squarefree integer. Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbb{T}^1(\Gamma_0(U)\cap\Gamma)$, and suppose that $\mathfrak{m}$ is controllable at every prime dividing U. Suppose further that either*

1. *the residue characteristic of $\mathfrak{m}$ is greater than 3, or*
2. *$\Gamma = \Gamma_0(M)$, or*
3. *Γ is contained in $\Gamma_0(r)$ for some $r > 3$.*

Then $\mathbb{T}^1(\Gamma_0(U)\cap\Gamma)^{U\text{-new}}_{\mathfrak{m}}$ is a Gorenstein ring.

Proof. Let D be equal to U if U has an even number of prime divisors; if U has an odd number of prime divisors then fix a prime p dividing U and let $D = \frac{U}{p}$. Consider the abelian variety $J = J^D(\Gamma_0(\frac{U}{D})\cap\Gamma)$. It follows from the preceding corollary that $J[\mathfrak{m}]$ has dimension two.

If the number of divisors of U is even, then $D = U$. By Jacquet-Langlands, $\mathbb{T}^1(\Gamma_0(U)\cap\Gamma)^{U\text{-new}}_{\mathfrak{m}} \cong \mathbb{T}^D(\Gamma)_{\mathfrak{m}}$. It thus suffices to show that $\mathbb{T}^D(\Gamma)^*_{\mathfrak{m}}$ is free of rank one over $\mathbb{T}^D(\Gamma)_{\mathfrak{m}}$. Since $J[\mathfrak{m}]$ has dimension two, $T_{\mathfrak{m}}J$ is free of rank two over $\mathbb{T}^D(\Gamma)_{\mathfrak{m}}$. Since J is self-dual, $(T_{\mathfrak{m}}J)^*$ is isomorphic to $T_{\mathfrak{m}}J$, and is thus also free of rank two over $\mathbb{T}^D(\Gamma)_{\mathfrak{m}}$. It follows that $(\mathbb{T}^D(\Gamma)^2)^*_{\mathfrak{m}}$ is free of rank two over $\mathbb{T}^D(\Gamma)_{\mathfrak{m}}$, and therefore (by Nakayama, for instance), $\mathbb{T}^D(\Gamma)^*_{\mathfrak{m}}$ is free of rank one over $\mathbb{T}^D(\Gamma)_{\mathfrak{m}}$.

Suppose the number of divisors of U is odd. By the same argument as in the proof of Lemma 2.4.5, $X_p(J)_{\mathfrak{m}}$ is free of rank one over $\mathbb{T}^D(\Gamma_0(p)\cap\Gamma)^{p\text{-new}}_{\mathfrak{m}}$. By Jacquet-Langlands the latter is isomorphic to $\mathbb{T}^1(\Gamma_0(U)\cap\Gamma)^{U\text{-new}}_{\mathfrak{m}}$. It thus suffices to show that $X_p(J)_{\mathfrak{m}}$ is isomorphic to $X_p(J)^*_{\mathfrak{m}}$. Since J is self-dual, the monodromy pairing on J induces a map $X_p(J) \to X_p(J)^*$. The cokernel of this map is the component group of J at p, and hence not supported at $\mathfrak{m}$ by Theorem 1.7.12. We thus obtain the desired isomorphism $X_p(J)_{\mathfrak{m}} \to X_p(J)^*_{\mathfrak{m}}$. □

We also have the following alternative characterization of the relationship between $[J^D]$ and $[J^{D'}]$ for D dividing D':

Proposition 2.6.14 *Let D and D' be divisors of N, each with an even number of prime divisors, and suppose that D divides D'. Then*

$$\mathrm{Hom}(J^{D',D'}, J^{D,D'}) \cong_{S^{D'}} \mathrm{Hom}(X_p(J^{D,D'}), X_p(J^{D',D'}))$$

for any p dividing D'.

Proof. We have an isomorphism

$$X_p(J^{D,D'}) \cong_{S^{D'}} X_p(J^{\min}_{D'}) \otimes [J^{D,D'}].$$

Thus

$$\mathrm{Hom}(X_p(J^{D,D'}), X_p(J^{D',D'})) \cong_{S^{D'}} \mathrm{Hom}([J^{D,D'}], \mathrm{Hom}(X_p(J^{\min}_{D'}), X_p(J^{D',D'}))),$$

because of the above isomorphism and the adjointness of Hom and tensor products. By Theorem 2.4.1, the latter module is naturally $S^{D'}$-isomorphic to $\mathrm{Hom}([J^{D,D'}], [J^{D',D'}])$, and hence to $\mathrm{Hom}(J^{D',D'}, J^{D,D'})$ by Theorem 2.2.11. □

Corollary 2.6.15 *There is an isomorphism:*

$$[J^{D',D'}] \cong_{S^{D'}} [J^{D,D'}] \otimes \mathrm{Hom}(X_p(J^{D,D'}), X_p(J^{D',D'})).$$

Proof. By Theorem 2.6.5, we have an M such that

$$[J^{D',D'}] \cong_{S^{D'}} M \otimes [J^{D,D'}].$$

By Lemma 2.4.2, it follows that $[J^{D',D'}] \cong_{S^{D'}} \mathrm{Hom}([J^{D,D'}], [J^{D',D'}]) \otimes [J^{D,D'}]$. The result follows by Proposition 2.6.14. □

Corollary 2.6.16 *Fix D and D' as above, and let p divide D'. Let M be the module*

$$\mathrm{Hom}(X_p(J^{D,D'}), X_p(J^{D',D'})) \otimes_{\mathbb{T}^{D'}} \mathbb{T}.$$

Then $[J^{D'}] = M \otimes [J^D]$, and there is an S-isomorphism:

$$g : \mathrm{Hom}(J^D, J^{D'}) \to \mathrm{Hom}(M, \mathrm{End}(J^D))$$

such that if $\phi \in \mathrm{Hom}_S(J^D, J^{D'})$, and $I_\phi \subset \mathrm{End}(J^D)$ is the ideal generated by image of M under $g(\phi) : M \to \mathrm{End}(J^D)$, then $\ker_S \phi = J^D[I_\phi]$ up to support on S.

Proof. Immediate from Corollary 2.6.15, Lemma 2.6.2, and Proposition 2.2.12. □

2.7 Further questions

These results leave several questions unanswered. The most obvious of these is whether or not the set S can be made smaller. At the moment, a maximal ideal $\mathfrak{m}$ can be in S either because it is Eisenstein or because it has residue characteristic 2 or 3.

It seems likely that in many cases the non-Eisenstein maximal ideals of residue characteristic 2 and 3 may be removed from S. At the moment there are two places in the argument that require the residue characteristic to be greater than 3. The first is that component groups of Jacobians of some Shimura curves can have support at non-Eisenstein primes of residue characteristic 2 or 3, which interferes with the proof of Proposition 2.3.6. This does not happen, however, if Γ either has the form $\Gamma_0(M)$ for some M, or if Γ contains $\Gamma_1(r)$ for some $r \geq 4$ [Ri7]. Thus in many cases this does not pose a problem.

The more serious difficulty occurs in section 2.5, where we rely on [DiTa], Theorem 2, which only holds in residue characteristic greater than 3. However, we have the luxury of choosing the primes at which we raise the level, so we need only find two primes q_1 and q_2 for which Theorem 2 holds locally at $\mathfrak{m}$ (and which satisfy our other requirements.) It seems very likely that this will always be possible. It is thus reasonable to conjecture that (when Γ satisfies one of the conditions given above) Theorem 2.4.1 holds even when S contains only Eisenstein primes.

The non-Eisenstein condition, on the other hand, is much more serious. All of the techniques of section 2.2, and Proposition 2.3.6, fail to hold at Eisenstein maximal ideals. Thus applying the techniques of this paper in this case seems hopeless.

We have also left open the question of whether or not there exist *canonical* maps between the varieties in question. If one is interested in looking for such things, one place to start would be to try to construct canonical maps between the modules constructed in section 2.6.

Finally, Corollary 2.6.11 provides a sufficient condition for "multiplicity one" for Jacobians of Shimura curves. It is interesting to ask if this condition is necessary as well. This would hold if one could establish a converse to Lemma 2.4.5. A special case of such a result appears in [Ri6], where Ribet constructs character groups which are not free of rank one at specific maximal ideals of the Hecke algebra.

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