M4P58 ASSESSED PROBLEMS 1 - DUE MONDAY, NOVEMBER 4, 2019

This is the first assessed coursework for M4P58; unlike the discussion problems, all rules for assessed coursework at Imperial apply. It is due Monday, November 4, at 4PM in the undergraduate office, and is worth 5 percent of your total marks for the course.

1. (1 mark) For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{Q})^+$, let $j(\gamma, z) = (cz + d)$. Show that for $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$, and $z \in \mathbb{H}$, we have $j(\gamma', \gamma z)j(\gamma, z) = j(\gamma'\gamma, z)$. Conclude that for any $f : \mathbb{H} \to \mathbb{C}$, one has $(f|_{k,\gamma'})|_{k,\gamma} = f|_{k,\gamma'\gamma}$.

2a. (2 marks) Using the formula for the weighted sum of the zeros of a modular form of weight k, show that if f is any modular form of weight k and k is 2 mod 4, then f has a zero at i. Show similarly that if k is not divisible by 3 then f has a zero at ρ.

2b. (2 marks) Now prove the same claims as in 2a using only the modular transformation law.

3. (1 mark) Express E_{12} as a polynomial in E_4 and E_6. Recall that we have the q-expansions:

$$\begin{aligned} E_4 &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \\ E_6 &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \\ E_{12} &= 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n \end{aligned}$$

4. (2 marks) Let $z = e^{\frac{2\pi i}{12}}$. Find an element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that γz is in the standard fundamental domain for SL_2.

5. (2 marks) Show that any element of $\mathrm{SL}_2(\mathbb{Z})$ of finite order has order $1, 2, 3, 4$, or 6. As a harder question you may want to try (this will not be assessed), give a set of representatives for the conjugacy classes of finite order in $\mathrm{SL}_2(\mathbb{Z})$.