EXERCISES: REPRESENTATIONS OF GL_2

Here the groups G, $B = TN$, $M = NS$ are as in the lectures, and $\overline{G}$, $\overline{B} = \overline{TN}$, $\overline{MS}$ are the corresponding subgroups of $\mathrm{GL}_2(\mathbb{F}_q)$. All representations are over the complex numbers unless otherwise specified.

1. Let $\overline{\psi}$ be a nontrivial character of $\overline{N}$.

1a. Show that conjugation by S acts transitively on the nontrivial characters of $\overline{N}$.

1b. Show that the restriction of $\mathrm{Ind}_{\overline{N}}^{\overline{M}} \overline{\psi}$ to $\overline{N}$ is the direct sum of all the nontrivial characters of $\overline{N}$.

1c. Conclude that $\mathrm{Ind}_{\overline{N}}^{\overline{M}} \overline{\psi}$ is irreducible of dimension $q - 1$ and independent of $\overline{\psi}$. Show that every other irreducible representation of $\overline{M}$ is one-dimensional, and arises as the composition of a character of $\overline{S}$ with the surjection $\overline{M} \to \overline{S}$.

1d. Show that for any characters χ_1 and χ_2 of $\mathbb{F}_q^\times$, the restriction of $\mathrm{Ind}_{\overline{B}}^{\overline{G}} \chi_1 \otimes \chi_2$ to $\overline{M}$ is the direct sum of two characters and the representation $\mathrm{Ind}_{\overline{N}}^{\overline{M}} \overline{\psi}$.

1e. Verify the claim from lectures that $\mathrm{Ind}_{\overline{B}}^{\overline{G}} \chi_1 \otimes \chi_2$ is irreducible if χ_1 and χ_2 are not equal, and is the direct sum of $\chi_1 \circ \det$ and an irreducible representation of dimension q otherwise. Compute the characters of these irreducible representations.

1f. Show that the restriction of any irreducible cuspidal representation of $\mathrm{GL}_2(\mathbb{F}_q)$ to $\overline{M}$ is a direct sum of copies of $\mathrm{Ind}_{\overline{N}}^{\overline{M}} \overline{\psi}$. Use character theory, and your character computations above, to show that there are $\frac{q^2-q}{2}$ irreducible cuspidal representations of $\mathrm{GL}_2(\mathbb{F}_q)$, each of which is of dimension $q - 1$.

1g. Finish off the character table of $\overline{G}$. In particular show that for each unordered pair (θ, θ^q) of characters of $\mathbb{F}_{q^2}^\times$ such that $\theta \neq \theta^q$, there is a unique irreducible cuspidal representation π_θ of $\mathrm{GL}_2(\mathbb{F}_q)$ such that the trace of $\pi_\theta(e)$ is $-\theta(e) - \theta^q(e)$ for all e in $\mathbb{F}_{q^2}^\times$. (Here we consider $\mathbb{F}_{q^2}^\times$ as a subgroup of $\mathrm{GL}_2(\mathbb{F}_q)$ by choosing an $\mathbb{F}_q$-basis for $\mathbb{F}_{q^2}$ and sending e in $\mathbb{F}_{q^2}^\times$ to the matrix of multiplication by e.)

2. Now fix a prime ℓ such that ℓ divides $q + 1$, and consider representations of G and $\overline{G}$ over $\overline{\mathbb{F}}_\ell$.

2a. Show that $\mathrm{Ind}_{\overline{B}}^{\overline{G}} 1$ has length three, has a unique irreducible subreprentation isomorphic to the trivial representation, a unique irreducible quotient which is also trivial, and an irreducible cuspidal subquotient $\overline{\pi}$.

2b. Show that $\overline{\pi}$ is the reduction modulo ℓ of π_θ, where $\theta : \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$ is a character such that $\theta \neq \theta^q$ and the reduction modulo ℓ of θ is the trivial character. (Note that such characters only exist when ℓ divides $q+1$.

2c. Show that $\operatorname{Ind}_B^G 1$ also has length three over $\overline{\mathbb{F}}_\ell$, and that its unique irreducible quotient is $(\chi \circ \det)$, where χ is the unique unramified character of $F^\times$ of order two. (In particular, modulo ℓ cuspidal representations can occur as subquotients of parabolic inductions- this fact has profound implications ofr the modular representation theory of reductive groups.)

2d. Denote the cuspidal subquotient of $\operatorname{Ind}_B^G 1$ by π. Show that the $\overline{G}$-representation π^{K_1} is isomorphic to $\overline{\pi}$, where K_1 denotes the kernel of the map $\mathrm{GL}_2(\mathcal{O}_F) \to \overline{G}$.

3a. Let K be the subgroup $\mathrm{GL}_2(\mathcal{O}_F)$ of $\mathrm{GL}_2(F)$. Show (as claimed in the lectures) that $G = KB$ (the Iwasawa decomposition) and that $G = KTK$ (the Cartan decomposition).

3b. Let W be the module c-$\operatorname{Ind}_K^G 1$, where 1 is the trivial character of K, and let $H(G,K)$ denote the endomorphism algebra of W, regarded as the space of left and right K-invariant functions on G. Show that $H(G,K)$ is isomorphic to $\mathbb{C}[T_{\varpi,1}, T_{\varpi,\varpi}^{\pm 1}]$, where $T_{\varpi,1}$ is the characteristic function of $K \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} K$ and $T_{\varpi,\varpi}$ is the characteristic function of $K \begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix} K$.

3c. Show that one has an isomorphism:

$$\operatorname{Hom}_G(W, \operatorname{Ind}_B^G \chi_1 \otimes \chi_2) \cong (\operatorname{Ind}_B^G \chi_1 \otimes \chi_2)^K,$$

and that the latter is one-dimensional if χ_1 and χ_2 are unramified, and zero-dimensional otherwise. Compute the action of $T_{\varpi,1}$ and $T_{\varpi,\varpi}$ on this space (when it is nonzero).

4. Fix a nontrivial character ψ of N of level one, and a tame admissible pair (E,χ). Compute the epsilon factors: $\epsilon(\operatorname{Ind}_{W_E}^{W_F} \chi, \psi, X)$ and $\epsilon(\pi_{E,\chi}, \psi, X)$ and show that they do not coincide.