When complexifying simplifies the problem

Alexander Antonov **talks to Dr Darren Crowdy, one of the two recent recipients of the Leverhulme prize in the Mathematics department, and finds out about complex analysis and its real applications.**

What attracted you to mathematics?

A few things, I guess. It may sound strange but, contrary to the common perception that mathematicians are 'geeks', as a teenager I remember thinking it was really cool that people could be fluent in, and communicate ideas in, some weird esoteric language of symbols. When I was fourteen, I remember we had a maths class at school straight after the upper sixth formers so, when we walked in the room, the board was always covered in all kinds of calculus stuff that, at that time, I had no idea about. I remember wanting to learn it.

Was there anyone in particular who inspired you to pursue a maths career?

Now that you ask, I don't recall making any conscious decision at any time to 'pursue' a maths career. I guess I subconsciously floated into it, urged on by people around me who recognized my talent. I've just turned 34, and sometimes I think I should stop messing around with mathematics and get a proper job. I'm only just realising that this is a proper job: it's just a really great one. I'm paid to think about whatever I think is interesting and give a few lectures.

What motivates you to do the research that you do?

In my last two years as an undergraduate I focused on applied mathematics – I wanted to do something that was useful to the real world. To be honest, I felt disappointed by what I learned. The real world is messy, the governing equations are difficult, and few have nice closed-form solutions. Science was losing its attraction for me. I first learned about complex analysis in my second year at Cambridge, from a lecturer called Alan Beardon – one of the truly great expositors of the subject. Then, to my delight, I realized that a small group of 'applied mathematicians' use those beautiful ideas from complex analysis to find remarkable solutions to problems arising in the physical sciences. Forget Fermat's Last Theorem: you can also use ideas from algebraic geometry to solve problems that physical scientists actually care about! My love of applying complex analysis to problems arising in physics has never deserted me, and it guides all my research.

Why is it important to study complex analysis?

When I'm in the pub and people ask me what I do, after telling them (and after they've then told me how crap at maths they were), one of the most frequent questions is "what the hell are complex numbers all about?" I was even asked this by an immigration officer when I re-entered the US on the way back to Caltech after one Christmas break! People think: you can't just make up new "imaginary" numbers to solve your mathematical problems and inconsistencies. Well, that's true. Just making up new numbers to patch things up would be a sham. But the beauty of it all is that you only have to invent one new number. Invent the square root of -1, make it obey some natural mathematical rules, and everything fits into place.

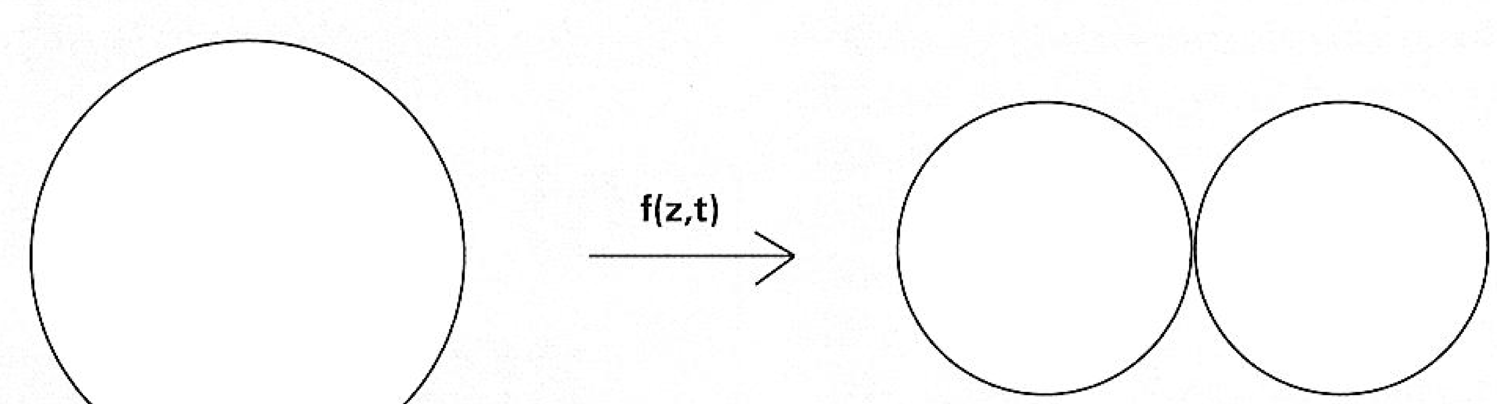
Complex analysis is one of the most beautiful, rationally coherent, powerful and widely-used areas of mathematics that exists. Have you ever wondered why every decent scientific university everywhere in the world teaches its undergraduates, not just mathematicians but physical scientists and engineers, to understand complex numbers? It's because it's so important and powerful. Unfortunately, a typical undergraduate never gets to fully experience its true power. In the fifties and sixties theories of complex analysis were studied really hard. Computers weren't anything like they are now, and scientists spent their lives trying to find nice analytical results: complex analysis was really powerful. In this day and age people have veered away from it, and they just do things numerically now. It always disappoints me that most engineers see of complex analysis is a few boring inverse Fourier transforms.

I see that one of the problems you have applied complex analysis to is the study of bubbles. Can you tell me more about your research in this area?

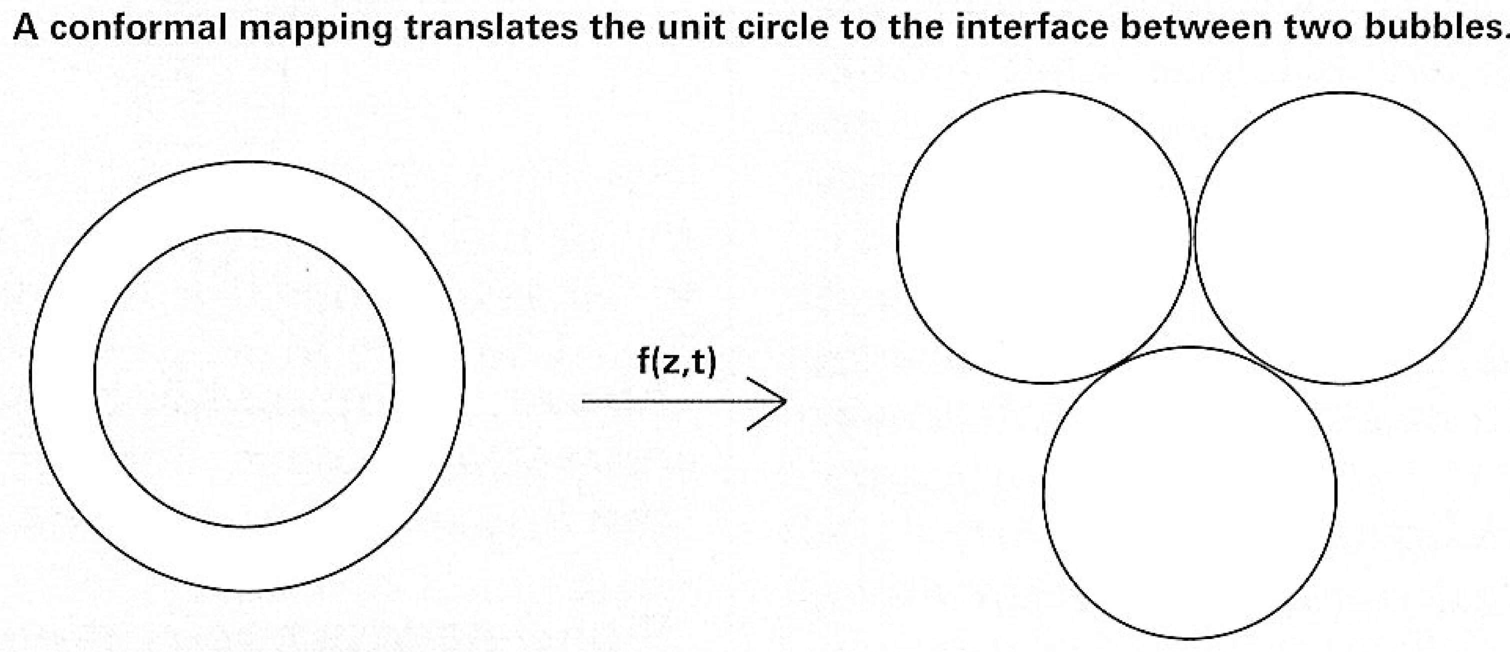
One of the cases where bubbles arise is in very viscous flows. For example, think of something like glass heated up to a very high temperature so that it becomes molten. If you want objects with rather complicated shapes it's often not efficient to machine them, so people usually get glass or metal powder, put it in a kiln and then heat it up (this process is called sintering). What happens is that the connecting region between the spherical powder particles becomes a region of fluid and, since there are surface tension forces on the boundaries, this opens it all up. Gradually, the regions or pores in between the particles will close up, and eventually you'll get a contiguous object with no holes and in the right shape.

This process can take quite a long time, so manufacturers want to minimise the time it's in the kiln and optimise the time to full densification (when all the pores have closed up). They don't want any holes inside the material after the process has finished because it will compromise its strength properties and, if it's a metal, its conductivity. So people need to understand how regions of very viscous fluid move around when the driving force is surface tension on the boundaries.

In 1990 this guy showed that the problem, if you start with two cylindrical particles just touching, can be reduced, using complex analysis, to two easily solvable ordinary



A conformal mapping translates the unit circle to the interface between two bubbles.



A conformal mapping translates the unit annulus maps to the interface between three bubbles.

Complex analysis techniques facilitate the study of involute time-evolving fluid-air interfaces.

differential equations (ODEs) using conformal maps. These maps take a simple geometric region like the unit circle and translate it to the more complicated interface between the coalescing particles. The maps therefore encode all the information about your changing fluid-air interface. You can then study the dynamics of these maps, which is a lot more analytically tractable because you can apply the known techniques of complex analysis. Previously, people have been using things like the finite element method, 256x256 meshes, and you can now do it by integrating two ODEs! Do you see the power of these things?

The question that my adviser put me onto was whether any results that are as impressive as these exist for the more general problem. The problem with the 1990 solution is that it only applies to two particles, so there are no pores between them, and you need at least three to have a pore. Topologically, in the two-particle problem you have a simply-connected region, which means it hasn't got any holes, but in a three-particle case it's doubly-connected, which means it has one hole. From a mathematical point of view you need completely different technologies to cope with these.

Does it not matter that you're modelling this in 2D?

In the viscous sintering problem, where you have spherical particles, the three-dimensionality is of course crucial. But many of the insights that you have from how the pores evolve in two dimensions can be carried over in a qualitative way in three dimensions and because studying the 2D problem becomes so easy now, you can do lots of simulations really quickly. Most complex analysis problems are in 2D, but that's OK. For example, in vortex dynamics, if you're modelling the atmosphere, its stratification means that actually storm systems are quasi-2D. So a lot of vortex dynamics that you model for meteorological purposes is done using 2D models with extra effects added in to account for the 3-dimensionality.

How do you attack a new maths problem?

There's only one way to solve a problem. Think of a theory or idea and find whatever way you can to test it, starting with the simplest cases first. That's how most mathematicians, pure or applied, operate. The most important thing you learn as a researcher is that the path to discovery, even of the most elegant end-results, often involves hopelessly clumsy, circuitous and long-winded routes. You piece together clues, experiment, and see where it leads. The most important thing a PhD student must learn, for example, is to trust his or her own instincts and never to be afraid to invest time testing their own ideas.

Have you been tempted to try any of the Millennium problems?

No. I'm aware of them, as any professional mathematician should be, but they're too difficult for me. In any case, it's my natural instinct to be turned off by something that's in the public consciousness. It's the same reason I haven't read any of the Harry Potter books.

Gauss famously declared that maths is the queen of all the sciences. Do you think that's true and why?

Undoubtedly. Physicists, in particular, are famous for coming up with wacky new theories often based on hand-waving arguments and a smattering of empirical evidence. But it's not usually considered an established and accepted theory unless it is underpinned by a mathematical formulation, which can make verifiable predictions. In that respect, mathematics rules.