A new calculus for two dimensional vortex dynamics

Darren Crowdy

Department of Mathematics
Imperial College London
180 Queen's Gate
London, SW7 2AZ
United Kingdom

d.crowdy@imperial.ac.uk

Prepared for the Proceedings volume of the IUTAM conference
"150 Years of Vortex Dynamics" , Copenhagen, October 2008.

FIRST DRAFT

In remembrance of Philip Geoffrey Saffman (1931–2008).

Abstract

This article provides a user's guide to a new calculus for finding the complex potentials associated with point vortex motion in geometrically complicated planar domains, with multiple boundaries, in the presence of background flows. The key to the generality of the approach is the use of conformal mapping theory together with a special transcendental function called the Schottky-Klein prime function. Illlustrative examples are given.

1 Introduction

A first course in theoretical fluid dynamics usually includes, quite early on, a study of ideal fluids where viscosity is neglected and, by an appeal to Kelvin's circulation theorem which guarantees the persistence of an initially irrotational flow in an ideal fluid, goes on to introduce the notion of a complex potential – an analytic function of a complex variable $z = x + iy$ – from which the student can build up various flows of interest. These include uniform flows, irrotational straining flows, flows with fluid sources and sinks and those involving a distribution of point vortices where their dynamical evolution can be computed and studied using a system of ordinary differential equations. A canonical solution that every student sees, not least because of its importance in basic aerofoil theory, is for steady uniform potential flow past a circular obstacle. Often, conformal mapping is then introduced to resolve the flow around more realistic aerofoil shapes.

All this has arguably found its place in the canon of theoretical fluid dynamics. Treatments of it can be found in texts such as Acheson [1], Batchelor [2] and Milne-Thomson [22]. But there is one obvious extension of this basic theory that is not discussed in any of the aforementioned textbooks and which is clearly of fundamental importance: how does the theory change when the flow domain contains more than one cylinder (or obstacle)? What is the complex potential for uniform flow past *two* cylinders? Or *three*? These are natural questions and the student will struggle to find the answers in the extant fluid dynamics literature.

It turns out that it is possible to present a rather complete mathematical theory – which we go so far as to call a "calculus" – which fills this lacuna in the classic fluid dynamical canon. It is a natural extension of what is already well-known (when just one obstacle is present) and, in this author's view, should be known to anybody interested in theoretical fluid dynamics. Flows in geometrically complex domains are ubiquitous in fluid dynamics: aerodynamicists are interested in multi-aerofoil configurations, civil engineers compute the forces on an array of bridge supports in a laminar flow, geophysical fluid dynamicists study the motion of oceanic eddies around island clusters and even in the resurgent field of biofluid dynamics there is much interest in modelling, for example, the motion and behaviour of schools of fish interacting both with each other and with their wake structures. All the basic mechanisms in such problems can be studied using the calculus to be presented here.

The standard fluid dynamical literature is, at best, patchy when it comes to planar flows in multiply connected domains. Isolated special results have been reported, mostly for the doubly connected case when just two obstacles are present (in early aeronautics, this was the famous biplane problem). Indeed, as evidence of the incoherence of the literature on this topic, the results for doubly connected domains appear to have been rediscovered more than once throughout the decades (e.g. [19] [15] [18]). Treatments of the doubly connected case invariably use the theory of elliptic functions. For triply and higher connected regions, few analytical results exist (prior to the work of the present author and his collaborators). Our strategy is to build the special calculus we need using a single special function called the *Schottky-Klein prime function*. The doubly connected situation is contained in this calculus as a special case and is thus capable of capturing all existing results.

We refer to the framework described here as a "calculus" because, just as in standard calculus, the introduction of just a few basic special functions, together with their defining properties, provides the means to construct a rich variety of more complicated functions and solutions to more difficult problems. The same is true of the new calculus here: actually, the *only* special function we need to introduce is the Schottky-Klein prime function. Since this paper is intended to constitute a "user's guide", we adopt a practical "how to" approach and omit all unnecessary (albeit important) subsidiary mathematical details and proofs. The reader should refer to the author's original papers for those supporting facts. The principal objective of this article is to encourage constructive use of this calculus in the field by showcasing its simplicity. This is akin to the student of calculus making liberal and unhindered use of the fact that $\sin\theta$ is a 2π-periodic function without worrying too much about why that is true, or indeed, how to show it.

Standard calculus is best appreciated by the execution of specific examples. We therefore include illustrative examples showing the scope and versatility of the methods. It will be possible to systematically derive analytical expressions for the complex potentials for general two dimensional invisicid flows in general multiply connected fluid domains. The presence of point vortices and solid boundaries will be incorporated. There will be no restrictions on the number of such solid boundaries, nor indeed, on their shapes. Conformal mapping theory takes care of that.

2 Riemann mapping theorem

First, we describe the setting in which the new calculus is couched. It is founded on one of the most important achievements of 19th century mathematics: the Riemann mapping theorem and its extensions. The Riemann mapping theorem, in its original form, states that any simply connected region of the plane is conformally equivalent to a unit disc. Expressed another way, the theorem guarantees the existence of a conformal mapping from the interior of the unit disc in, say, a parametric ζ-plane, to any given simply connected fluid region in the z-plane. Let this conformal mapping be $z(\zeta)$. If the fluid region is unbounded, there must be some point $\zeta = \beta$ inside or on the boundary of the unit ζ-disc which maps to the point at infinity. Indeed, $z(\zeta)$ must have a simple pole at $\zeta = \beta$. There are three real degrees of freedom in the specification of this conformal map. One way to pin these down is to *choose* the value of β arbitrarily. Then, the map is determined up to a single rotational degree of freedom. Often this is set by insisting that, near $\zeta = \beta$, the map has the local behaviour

$$z(\zeta) = \frac{a}{\zeta - \beta} + \text{analytic} \tag{1}$$

where a is taken to be real.

The aim is to build a calculus for finding complex potentials associated with flows that are irrotational apart from a set of isolated singularities. The first step is to acknowledge that, owing to the conformal invariance of the boundary value problems to be solved, we can equally well find the required complex potentials as functions of the parametric variable ζ (rather than as functions of z). For geometrically complicated domains, this observation provides a major simplification.

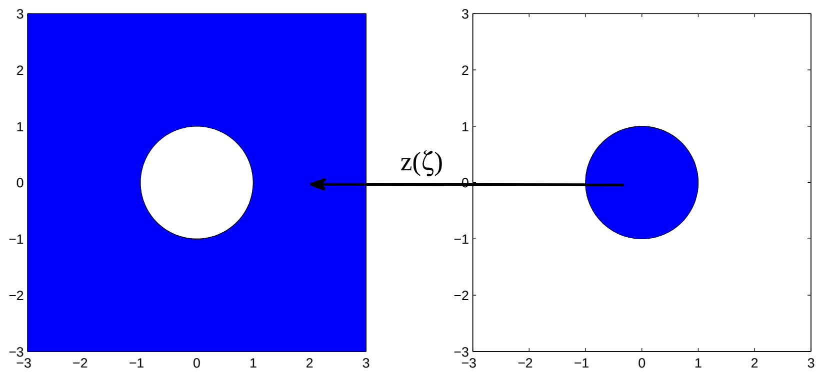

Figure 1: Conformal mapping $z(\zeta) = 1/\zeta$ from the interior to the exterior of the unit disc.

3 A motivating example

To illustrate the ideas, consider the unbounded fluid region exterior to a unit radius cylinder centred at $z = 0$. This region is the conformal image of a unit ζ-disc, in a parametric ζ-plane, under the simple conformal mapping

$$z(\zeta) = \frac{1}{\zeta}. \tag{2}$$

Clearly, $\zeta = 0$ maps to $z = \infty$.

3.1 A point vortex outside a cylinder

Suppose a point vortex of unit circulation is situated at some point z_α outside the unit-radius cylinder. Let $\zeta = \alpha$ be the preimage of this point under the mapping $z(\zeta)$ so that

$$\alpha = \frac{1}{z_\alpha}. \tag{3}$$

The complex potential for a point vortex of unit circulation at $\zeta = \alpha$ existing in free space is well-known to be given by

$$w(\zeta) = -\frac{\mathrm{i}}{2\pi} \log(\zeta - \alpha). \tag{4}$$

This function does not, however, satisfy the boundary condition that the unit circle $|\zeta| = 1$ is a streamline. Since the streamfunction is the imaginary part of the complex potential, we must find a complex potential which is, say, purely real on $|z| = 1$ so that it has constant imaginary part equal to zero there. One function, built from (4), which is certainly real on $|\zeta| = 1$ is

$$w(\zeta) + \overline{w(\zeta)}. \tag{5}$$

But this function is not an analytic function of ζ so it is not a candidate for the required complex potential. It is also real *everywhere* in the complex ζ-plane while we only need it to be real on $|\zeta| = 1$. To fix all this, we exploit the fact that on $|\zeta| = 1$ it is true that $\zeta = 1/\overline{\zeta}$ so we can replace ζ in the final term of (5) to give

$$w(\zeta) + \overline{w(1/\overline{\zeta})}. \tag{6}$$

Two things then happen: first, *this* function is now an analytic function of ζ (it is no longer a function of $\overline{\zeta}$); second, it is still purely real on $|\zeta| = 1$ (but not necessarily off this boundary circle). Also, owing to the presence of the term $w(\zeta)$, it has the required logarithmic singularity at $\zeta = \alpha$ reflecting the presence of a point vortex there. In this way, we have constructed a solution to our problem. Substitution of (4) into (6) produces

$$-\frac{\mathrm{i}}{2\pi} \log\left(\frac{\zeta - \alpha}{1/\zeta - \overline{\alpha}}\right), \tag{7}$$

or, on rearrangement, it can be written as

$$-\frac{\mathrm{i}}{2\pi} \log\left(\frac{\zeta - \alpha}{|\overline{\alpha}|(\zeta - 1/\overline{\alpha})}\right) - \frac{\mathrm{i}}{2\pi} \log\zeta + \text{ constant}. \tag{8}$$

Incidentally, the construction we have just presented is basically the content of the so-called *Milne-Thomson circle theorem* (see, for example, Acheson [1] or Milne-Thomson [22]).

3.2 Circulation around the obstacle or island

(8) happens to be only one of many possible solutions to the problem as stated. It is easily checked that other possible complex potentials for the same problem are

$$-\frac{\mathrm{i}}{2\pi}\log\left(\frac{\zeta-\alpha}{|\overline{\alpha}|(\zeta-1/\overline{\alpha})}\right)-\frac{\mathrm{i}\gamma}{2\pi}\log\zeta+\text{ real constant,} \tag{9}$$

where γ is an arbitrary real constant: (9) is also an analytic function of ζ having the required logarithmic singularity at α as well as the property that it is purely real on $|\zeta|=1$. Clearly, to pin down a unique solution to the problem it is necessary to additionally specify the required value of the circulation around the object in the original problem statement.

Consider the first function in (9), let's call it $G_0(\zeta,\alpha)$, so that

$$G_0(\zeta,\alpha)\equiv-\frac{\mathrm{i}}{2\pi}\log\left(\frac{\zeta-\alpha}{|\overline{\alpha}|(\zeta-1/\overline{\alpha})}\right). \tag{10}$$

It is purely real on the boundary $|\zeta|=1$ of the circular object; it also has logarithmic singularities at $\zeta=\alpha$ and $\zeta=1/\overline{\alpha}$ corresponding to point vortices at these points. It is easy to check that this complex potential leads to a circulation equal to -1 around the cylinder. Actually, by the conformal invariance of the problem, (10) is the complex potential for a point vortex exterior to an object of *arbitrary* shape: to complete the solution, it is only necessary to know the form of the function $z(\zeta)$ mapping the unit ζ-disc to the region exterior to the object.

On the other hand, the term

$$-\frac{\mathrm{i}\gamma}{2\pi}\log\zeta \tag{11}$$

is also purely real on $|\zeta|=1$ and it has logarithmic singularities at $\zeta=0$ and $\zeta=\infty$ (since $\zeta=0$ maps to $z=\infty$ this corresponds to a point vortex at infinity in the physical plane). It gives a circulation $-\gamma$ around the cylinder. Combining the two contributions (10) and (11) together to form (9) it is clear that if we want, say, the total circulation around the cylinder to be zero then we need $-1-\gamma=0$. Expressed differently, if we require the circulation around the cylinder to be zero we must add to (10) the complex potential associated with a point vortex of circulation $\gamma = -1$ at physical infinity.

The idea behind the calculus is to find the appropriate generalizations of expression (10) for a point vortex at some position α outside a collection of obstacles. We will also need to find a way to independently control the circulations around the various obstacles.

4 Multiply connected conformal mapping

The calculus we are developing applies to fluid domains involving *any finite number* of obstacles, or islands, in the flow. An unbounded fluid domain containing just one obstacle is simply connected, if it contains two obstacles it is doubly connected, and so on. It is therefore important to understand something about the conformal mapping of multiply connected domains.

The generalization of the Riemann mapping theorem to multiply connected regions was given in the early 20th century by Koebe [16]: any multiply connected domain (with finite connectivity) is conformally equivalent to some multiply connected *circular domain*. Such a domain consists of the unit disc in a ζ-plane with M smaller circular discs excised. Let this domain be D_ζ and let C_j, for $j = 1, ..., M$, denote the circular boundary of the j-th excised circular disc. Also, let us denote the unit circle $|\zeta| = 1$ by C_0. The only geometrical parameters needed to uniquely specify such a domain are the centres $\{\delta_j | j = 1, ..., M\}$ and the radii $\{q_j | j = 1, ..., M\}$ of the circles $\{C_j | j = 1, ..., M\}$. We shall refer to the data $\{\delta_j, q_j | j = 1, ..., M\}$ as the *conformal moduli* of D_ζ. It is important to note that the values of the conformal moduli cannot be picked arbitrarily; rather, they are determined (up to the usual three real degrees of freedom of the mapping theorem mentioned earlier) by the target domain in the z-plane.

4.1 A three cylinder example

This is best illustrated by example. Consider the unbounded fluid region exterior to three equal circular obstacles as shown in Figure 2. The obstacles all have radius s and are centred at $-d, 0$ and d. From the geometrical

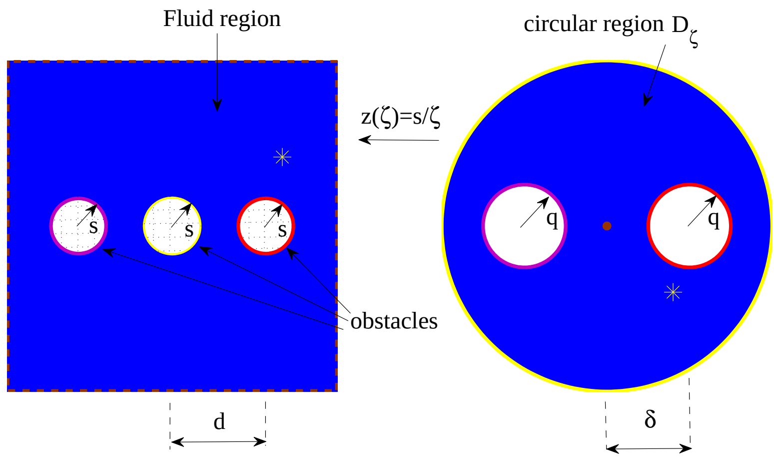

Figure 2: An example conformal map from a circular domain D_ζ to the exterior of three circular obstacles.

symmetries of this domain, it is reasonable to seek a circular preimage region D_ζ which shares these symmetries. We therefore pick that $\beta = 0$ maps to infinity and consider the unit ζ disc with two smaller circular discs, each of radius q, centred at $\pm\delta$ (see the rightmost diagram in Figure 2). Let us introduce the conformal mapping

$$z(\zeta) = \frac{s}{\zeta}. \tag{12}$$

This takes $|\zeta| = 1$ to the circle $|z| = s$ and it also maps $\zeta = 0$ to $z = \infty$. It is also easy to verify that if we pick q and δ such that

$$q = \frac{s^2}{d^2 - s^2}, \quad \delta = \frac{sd}{d^2 - s^2} \tag{13}$$

then the circle $|\zeta - \delta| = q$ will map to $|z - d| = s$ while $|\zeta + \delta| = q$ maps to $|z + d| = s$. Clearly, the conformal moduli depend on the geometry of the domain in the physical plane.

Once the conformal moduli are known, we can also define a set of M

Möbius maps by means of the relations

$$\theta_j(\zeta) \equiv \delta_j + \frac{q_j^2 \zeta}{1 - \overline{\delta_j} \zeta}. \tag{14}$$

Knowledge of the domain D_ζ, its conformal moduli $\{\delta_j, q_j | j = 1, ..., M\}$ and the maps $\{\theta_j(\zeta) | j = 1, ..., M\}$, together with the conformal mapping $z(\zeta)$ are all that is needed to devise the new calculus.

5 A fact from function theory

We want to generalize the expression (10) to the multi-obstacle case. We will now do something which, at first, seems pedantic: we introduce the notation

$$\omega(\zeta, \alpha) \equiv (\zeta - \alpha) \tag{15}$$

for the simple monomial function $(\zeta - \alpha)$. With this definition, (10) takes the form

$$G_0(\zeta, \alpha) = -\frac{\mathrm{i}}{2\pi} \log \left(\frac{\omega(\zeta, \alpha)}{|\overline{\alpha}| \omega(\zeta, 1/\overline{\alpha})} \right). \tag{16}$$

Recall that this is just the complex potential for a point vortex of circulation $+1$ around a single obstacle of arbitrary shape. It also gives circulation -1 around the obstacle itself.

Suppose now that we have a point vortex of unit circulation exterior to a collection of $M+1$ obstacles of arbitrary shape. For $M > 0$ the fluid region, which we will call D_z, is multiply connected. By the multiply connected Riemann mapping theorem, we know that there is a conformal mapping $z(\zeta)$ from a conformally equivalent circular domain D_ζ, with some choices of the centres and radii $\{q_j, \delta_j | j = 1, ..., M\}$ (the conformal moduli) and with some point β in D_ζ mapping to $z = \infty$.

We can now ask the natural question: what is the (generalized) complex potential associated with a point vortex of circulation $+1$ at a point α in D_z and having circulation -1 around the island whose boundary is the image of C_0?

We now state the remarkable fact that lies at the heart of the new calculus: the answer is again given by formula (16)!

5.1 The Schottky-Klein prime function

What do we mean by the last statement? What is meant is that there exists a special function which we will continue to denote by $\omega(\zeta, \alpha)$ – even though it is only for $M = 0$ that it is given by the simple formula (15) – such that (16) remains the formula for the complex potential for the flow generated by a point vortex of circulation $+1$ at position α in the multiply connected region D_ζ. This complex potential produces circulation -1 around the obstacle whose boundary is the image of C_0 and happens to produce zero circulation around all other obstacles. It is remarkable that the required fluid dynamical formula remains the same; all that changes is what we *mean* by the function $\omega(\zeta, \alpha)$.

The function $\omega(\zeta, \alpha)$ is called the *Schottky-Klein prime function* and it plays a fundamental role in complex function theory that extends far beyond the realm of fluid dynamics. When $M = 0$ (the simply connected case) $\omega(\zeta, \alpha)$ is defined by (15); for $M > 0$ it is a more complicated function. We will say much more about this function, and how to compute it, later. Since it is a function of two complex variables, henceforth we will refer to it as $\omega(.,.)$.

Here are the only two facts that are important:

(1) $\omega(\zeta, \alpha)$ has a simple zero at $\zeta = \alpha$.

(2) The properties of $\omega(.,.)$ are such that $G_0(\zeta, \alpha)$ has constant imaginary part on all the boundary circles of D_ζ (this means that all the obstacle boundaries are streamlines).

Let us assume for now that $\omega(.,.)$ is a known (and computable) special function associated with a circular domain D_ζ that is conformally equivalent to some multiply connected fluid domain D_z of interest. Let $z(\zeta)$ be the conformal map taking D_ζ to D_z.

6 Building the new calculus

We have already claimed that

$$G_0(\zeta, \alpha) \equiv -\frac{\mathrm{i}}{2\pi} \log \left(\frac{\omega(\zeta, \alpha)}{|\alpha|\omega(\zeta, 1/\overline{\alpha})} \right)$$

is the complex potential for a unit circulation point vortex outside any number of objects. It produces a circulation -1 around the particular obstacle which is the image of C_0 and circulation 0 around the other obstacles. It generalizes expression (10) to the multi-obstacle situation.

6.1 N point vortices exterior to multiple objects

Suppose there are N point vortices in a multiply connected fluid region D_z with $M+1$ obstacles. Let the vortex at the point z_k have circulation Γ_k where $k = 1, ..., N$. Suppose too that we require the circulations around all $M+1$ obstacles to be zero. The complex potential associated with this flow can be constructed as follows. Consider

$$\sum_{k=1}^{N} \Gamma_k G_0(\zeta, \alpha_k) \tag{17}$$

where α_k is the pre-image of z_k under the conformal mapping, i.e.,

$$z_k = z(\alpha_k), \quad k = 1, ..., N. \tag{18}$$

Owing to its various logarithmic singularities, the complex potential (17) produces the required point vortex distribution. But it also produces a total circulation of $-\Gamma_T$, where $\Gamma_T = \sum_{k=1}^{N} \Gamma_k$, around the obstacle that is the image of C_0 and has zero circulation around all the other obstacles. To ensure zero circulation around *all* the obstacles we must add in an extra term:

$$-\Gamma_T G_0(\zeta, \beta) + \sum_{k=1}^{N} \Gamma_k G_0(\zeta, \alpha_k) \tag{19}$$

where β maps to $z = \infty$. The extra term corresponds to adding a point vortex of circulation $-\Gamma_T$ at infinity. This is exactly what we did in the simply connected case ($M = 0$) considered earlier. Indeed, (19) is the complex potential we seek.

This analytical result for the complex potentials associated with point vortices around multiple obstacles (with zero circulation around the islands) was first described in Crowdy & Marshall [4] in the context of Kirchhoff-Routh theory (which is further elucidated in §9.1).

6.2 Adding circulation around the obstacles

(19) results in zero circulation around *all* the obstacles. But suppose we want to be able to specify that there is a circulation $\gamma_j \neq 0$ around the j-th obstacle. How do we construct the relevant complex potential?

To answer this question involves constructing more functions in our calculus but, importantly, they are built from the *same* special functions already introduced. It should not be surprising that we actually need M additional functions since we have M other obstacles which may possibly have circulation around them. Therefore, let us define the M functions

$$\boxed{G_j(\zeta,\alpha) = -\frac{\mathrm{i}}{2\pi}\log\left(\frac{\omega(\zeta,\alpha)}{|\alpha|\omega(\zeta,\theta_j(1/\overline{\alpha}))}\right), \quad j = 1,...,M.}$$

Note the appearance of $\theta_j(\zeta)$ which, for each j, is one of the M Möbius maps (14) introduced earlier. $G_j(\zeta,\alpha)$ has the following significance: it is the complex potential corresponding to a point vortex of unit circulation at the point α but *now* with circulation -1 around the obstacle corresponding to the image of C_j and circulation 0 around all other obstacles. Our choice of notation is helpful: the subscripts reflect *which* obstacle has non-zero circulation (in fact, a circulation of -1) around it.

Now, adding circulations around the obstacles is easy: since, for a given $j = 0, 1, ..., M$, the function $G_j(\zeta,\beta)$ gives a circulation of -1 around the obstacle whose boundary is the image of the circle C_j under the conformal mapping, the required complex potential is

$$-\sum_{j=0}^{M}\gamma_j G_j(\zeta,\beta). \tag{20}$$

This analytical result for the complex potentials associated with specifying the circulations around multiple obstacles was first described by Crowdy

[6] in the context of adding circulations around stacks of aerofoils in ideal flow.

6.3 Uniform flow past multiple objects

Remarkably, the complex potential for uniform flow past multiple obstacles can also be constructed from the function $G_0(\zeta,\alpha)$ (hereafter denoted simply by G_0). We seek a complex potential which looks, as $z \to \infty$, like $Ue^{-\mathrm{i}\chi}z$ and has constant imaginary part on all the boundaries of D_ζ. This corresponds to uniform potential flow with speed U at angle χ to the x axis.

Let $\alpha = \alpha_x + \mathrm{i}\alpha_y$. Consider the two functions

$$\Phi_1(\zeta,\alpha) = \frac{\partial G_0}{\partial \alpha_x}, \quad \Psi_1(\zeta,\alpha) = \frac{\partial G_0}{\mathrm{i}\partial \alpha_y}. \tag{21}$$

Owing to the identities

$$\frac{\partial G_0}{\partial \alpha_x} = \frac{1}{2}\left(\frac{\partial G_0}{\partial \alpha} + \frac{\partial G_0}{\partial \overline{\alpha}}\right), \quad \frac{1}{\mathrm{i}}\frac{\partial G_0}{\partial \alpha_y} = \frac{1}{2}\left(\frac{\partial G_0}{\partial \alpha} - \frac{\partial G_0}{\partial \overline{\alpha}}\right), \tag{22}$$

it is simple to check that both $\Phi_1(\zeta,\alpha)$ and $\Psi_1(\zeta,\alpha)$ have a simple pole, each with residue $\mathrm{i}/(4\pi)$, at α. Since the imaginary part of G_0 is constant on all boundaries of D_ζ, so are the imaginary parts of $\Phi_1(\zeta,\alpha)$ and $\mathrm{i}\Psi_1(\zeta,\alpha)$ (this is because we take *parametric derivatives* with respect to the real and imaginary parts of α, not derivatives with respect to ζ). We can now take *real* linear combinations of the two functions $\Phi_1(\zeta,\alpha)$ and $\mathrm{i}\Psi_1(\zeta,\alpha)$ in order to give us the required singularity at infinity. Indeed the analytic function

$$-4\pi U\cos\chi[\mathrm{i}\Psi_1(\zeta,\alpha)] - 4\pi U\sin\chi[\Phi_1] \tag{23}$$

where U and χ are real constants has a simple pole, with residue $Ue^{-i\chi}$ at α. It also has constant imaginary part on all the boundaries of D_ζ. On use of (22), (23) can be written

$$2\pi U\mathrm{i}\left(e^{\mathrm{i}\chi}\frac{\partial G_0}{\partial \overline{\alpha}} - e^{-\mathrm{i}\chi}\frac{\partial G_0}{\partial \alpha}\right). \tag{24}$$

If, as $\zeta \to \beta$, we have

$$z(\zeta) = \frac{a}{\zeta - \beta} + \text{analytic} \tag{25}$$

for some constant a, then the required complex potential for the uniform flow is given by

$$\boxed{2\pi U a \mathrm{i} \left(e^{\mathrm{i}\chi} \frac{\partial G_0}{\partial \overline{\alpha}} - e^{-\mathrm{i}\chi} \frac{\partial G_0}{\partial \alpha} \right) \bigg|_{\alpha=\beta}}$$

It is important, in this formula, to take derivatives with respect to α and $\overline{\alpha}$ *before* letting $\alpha = \beta$.

This derivation of the complex potentials associated with uniform flow past multiple obstacles was first described in [7]. It has been employed in Crowdy [6] for computing the lift and interference forces on stacks of aerofoils when they are in a uniform flow and have non-zero circulations around them.

6.4 Straining flows around multiple objects

We can even build the complex potentials for higher order flows using nothing more than the functions we have already introduced. It should be clear that we need to take higher order parametric derivatives of $G_0(\zeta, \alpha)$ (again, denoted hereafter by G_0). Let us seek a complex potential which looks, as $z \to \infty$, like $\Omega e^{\mathrm{i}\lambda} z^2$ (where Ω and λ are some real constants) and which has constant imaginary part on all the boundaries of D_ζ.

Therefore, consider the three second parametric derivatives given by

$$\Phi_2(\zeta, \alpha) = \frac{\partial^2 G_0}{\partial \alpha_x^2}, \quad \Psi_2(\zeta, \alpha) = \frac{\partial^2 G_0}{\partial \alpha_x \partial(\mathrm{i}\alpha_y)}, \quad \Pi_2(\zeta, \alpha) = \frac{\partial^2 G_0}{\partial \alpha_y^2}. \tag{26}$$

The functions $\Phi_2(\zeta, \alpha), \mathrm{i}\Psi_2(\zeta, \alpha)$ and $\Pi_2(\zeta, \alpha)$ have constant imaginary part on all the boundaries of ∂D_ζ. What about their singularities? By virtue of (22), we have

$$\begin{aligned}
\Phi_2(\zeta, \alpha) &= \frac{1}{4} \left(\frac{\partial^2 G_0}{\partial \alpha^2} + \frac{\partial^2 G_0}{\partial \overline{\alpha}^2} + \frac{\partial^2 G_0}{\partial \alpha \partial \overline{\alpha}} \right), \\
\Psi_2(\zeta, \alpha) &= \frac{1}{4} \left(\frac{\partial^2 G_0}{\partial \alpha^2} - \frac{\partial^2 G_0}{\partial \overline{\alpha}^2} \right), \\
\Pi_2(\zeta, \alpha) &= -\frac{1}{4} \left(\frac{\partial^2 G_0}{\partial \alpha^2} + \frac{\partial^2 G_0}{\partial \overline{\alpha}^2} - \frac{\partial^2 G_0}{\partial \alpha \partial \overline{\alpha}} \right).
\end{aligned} \tag{27}$$

Now the quantity

$$\frac{\partial^2 G_0}{\partial \alpha \partial \overline{\alpha}} \tag{28}$$

has a δ-function singularity at $\zeta = \alpha$ that we must eliminate. It is therefore natural to consider the combination

$$\hat{\Phi}_2(\zeta, \alpha) \equiv \frac{1}{2} \left(\Phi_2(\zeta, \alpha) - \Pi_2(\zeta, \alpha) \right) = \frac{1}{4} \left(\frac{\partial^2 G_0}{\partial \alpha^2} + \frac{\partial^2 G_0}{\partial \overline{\alpha}^2} \right). \tag{29}$$

This function also has constant imaginary part on all the boundaries of D_ζ. $\hat{\Phi}_2(\zeta, \alpha)$ and $\Psi_2(\zeta, \alpha)$ each have a second order pole, with strength $\mathrm{i}/(8\pi)$, at α. Now we find *real* linear combinations of $\hat{\Phi}_2(\zeta, \alpha)$ and $\mathrm{i}\Psi_2(\zeta, \alpha)$ giving the required singularity at infinity. The required function is

$$-8\pi\Omega \cos\lambda[\mathrm{i}\Psi_2(\zeta, \alpha)] + 8\pi\Omega \sin\lambda \hat{\Phi}_2(\zeta, \alpha). \tag{30}$$

This function has a second order pole of strength $\Omega e^{i\lambda}$ at $\zeta = \alpha$. On making use of (22) it can be written as

$$2\pi\Omega\mathrm{i} \left(e^{-\mathrm{i}\lambda} \frac{\partial^2 G_0}{\partial \overline{\alpha}^2} - e^{\mathrm{i}\lambda} \frac{\partial^2 G_0}{\partial \alpha^2} \right). \tag{31}$$

Then if $z(\zeta)$ has the behaviour given in (25) at $\zeta = \beta$ then the required complex potential for the quadratic straining flow is

$$\boxed{2\pi\Omega a^2\mathrm{i} \left(e^{-\mathrm{i}\lambda} \frac{\partial^2 G_0}{\partial \overline{\alpha}^2} - e^{\mathrm{i}\lambda} \frac{\partial^2 G_0}{\partial \alpha^2} \right) \Bigg|_{\alpha=\beta}}$$

In fact, this appears to be the first time that the complex potential for a general irrotational straining flow past multiple objects has been explicitly written down (although the possibility of doing so was advertised in Crowdy [7]).

6.5 Moving objects

What if the obstacles in the flow are moving, perhaps with some prescribed velocity? (so far we have assumed that the solid objects in the flow are stationary). Once again, the calculus of the functions we have already introduced can be adapted to this circumstance too. The complex potentials associated with such motion can be written down using the prime function $\omega(.,.)$. This time, there is a slight difference that the expressions involve integrals. It is established in Crowdy [10] that the complex potential in which the jth obstacle is moving with complex velocity U_j for $j = 0, 1, ..., M$ is given by

$$
\begin{aligned}
W_{\mathbf{U}}(\zeta) = &\frac{1}{2\pi}\oint_{C_0}\left[\mathrm{Re}[-\mathrm{i}\overline{U_0}z(\zeta')]\right]\left(d\log\omega(\zeta',\zeta) - d\log\overline{\omega}(\zeta'^{-1},\zeta^{-1})\right) \\
&-\sum_{j=1}^{M}\frac{1}{2\pi}\oint_{C_j}\left[\mathrm{Re}[-\mathrm{i}\overline{U_j}z(\zeta')] + d_j\right]\left(d\log\omega(\zeta',\zeta) - d\log\overline{\omega}(\overline{\theta_j}(\zeta'^{-1}),\zeta^{-1})\right),
\end{aligned}
\tag{32}
$$

where the constants $\{d_j | j = 1, ..., M\}$ solve a linear system that is recorded in [10] (we refer the reader there for full details). The subscript on $W_{\mathbf{U}}(\zeta)$ is a vector $\mathbf{U} \equiv (U_0, U_1, ..., U_M)$ of the complex velocities of the $M + 1$ obstacles.

7 Examples

To illustrate the flexibility of the new calculus, and how to use it, we will consider some illustrative examples. There are just a few keys steps in analysing any given problem. They are as follows:

(1) Analyse the geometry and determine the conformal moduli $\{q_j, \delta_j | j = 1, ..., M\}$ and the mapping $z(\zeta)$. Note that well-known numerical methods of multiply connected conformal mapping may be necessary here. Note that for *circular* objects the required conformal map is a simple Möbius map and everything is explicit.

(2) Construct the Möbius maps $\{\theta_j(\zeta) | j = 1, ..., M\}$ and compute the Schottky-Klein prime function $\omega(.,.)$.

(3) Do calculus with the functions $\{G_k(\zeta, \alpha) | j = 0, 1, ..., M\}$ to solve the given fluid problem.

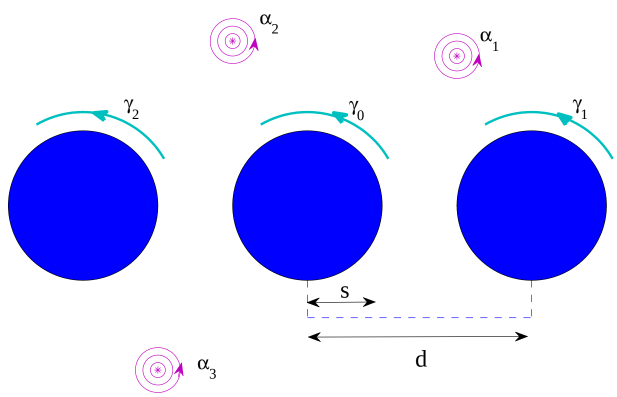

Figure 3: Three point vortices moving around three circular islands. There are non-zero circulations around each of the islands.

7.1 Three point vortices near three circular islands

Problem 1: Suppose that there are three circular islands, all of radius s, centred at $(-d, 0), (0, 0)$ and $(d, 0)$ and each having a circulation γ around it. The fluid also contains 3 point vortices at positions α_1, α_2 and α_3 each of which have circulation Γ. What is the instantaneous complex potential?

1. The geometry: This fluid region has already been considered in an earlier example (Figure 2). The circular region D_ζ is the unit ζ-disc with two smaller discs excised, each of radius q and centred at $\pm\delta$. The point $\beta = 0$ maps to infinity.

2. Möbius maps: There are two Möbius maps in this case given by

$$\theta_1(\zeta) = \delta + \frac{q^2}{1 - \bar{\delta}\zeta}, \quad \theta_2(\zeta) = -\delta + \frac{q^2}{1 + \bar{\delta}\zeta}. \tag{33}$$

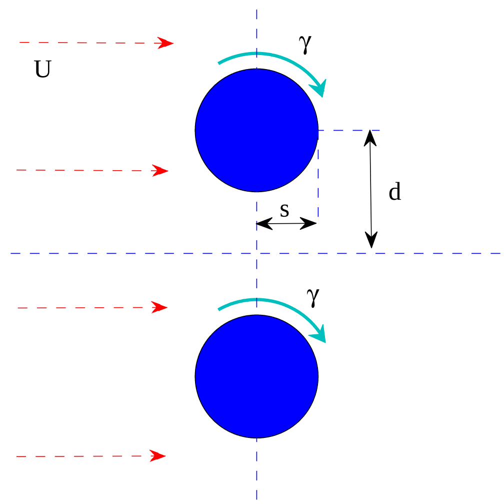

Figure 4: Uniform flow past two equal cylinders.

3. Do calculus: The complex potential in this case is

$$\begin{aligned} w_1(\zeta) = &\sum_{k=1}^{3} \Gamma G_0(\zeta, \alpha_k) \quad \leftarrow \text{point vortices} \\ &- 3\Gamma G_0(\zeta, 0) \quad \leftarrow \text{make all round-obstacle circulations zero} \\ &- \sum_{j=0}^{2} \gamma_j G_j(\zeta, 0) \quad \leftarrow \text{add in required round-obstacle circulations.} \end{aligned} \tag{34}$$

7.2 What is the lift on a biplane?

Problem 2: Consider two circular aerofoils stacked vertically both of radius s and centred at $\pm id$. Far away, the flow is uniform with speed U parallel to x-axis. Suppose there is a circulation γ around each of them. If, as shown in Figure 4, this circulation is negative, one can generally expect there to be a lift force on each aerofoil in the vertical direction. We must find the relevant complex potential.

1. The geometry: The fluid domain in this case is doubly connected so there is a conformal mapping to it from a concentric annulus $\rho < |\zeta| < 1$ (which is the domain D_ζ for this case). The required conformal mapping is a Möbius map with the form

$$z(\zeta) = iA\left(\frac{\zeta - \sqrt{\rho}}{\zeta + \sqrt{\rho}}\right), \tag{35}$$

where

$$\rho = \frac{1-(1-(s/d)^2)^{1/2}}{1+(1-(s/d)^2)^{1/2}}, \quad A = d\left(\frac{1-\rho}{1+\rho}\right) = \sqrt{d^2 - s^2}. \tag{36}$$

Note that $\zeta = -\sqrt{\rho}$ maps to infinity with local behaviour

$$z(\zeta) = \frac{a}{\zeta + \sqrt{\rho}} + \text{analytic} \tag{37}$$

where

$$a = -2\mathrm{i}A\sqrt{\rho}. \tag{38}$$

2. Möbius maps: There is only a single Möbius map in this case given by

$$\theta_1(\zeta) = \rho^2\zeta. \tag{39}$$

3. Do calculus: The complex potential is then given by

$$\begin{aligned} w_2(\zeta) = 2\pi Ua\mathrm{i}\left(\frac{\partial G_0}{\partial\overline{\alpha}} - \frac{\partial G_0}{\partial\alpha}\right)\Bigg|_{\alpha=-\sqrt{\rho}} \quad &\leftarrow \text{uniform flow } (\chi = 0) \\ -\sum_{j=0}^{1}\gamma G_j(\zeta, -\sqrt{\rho}) \quad &\leftarrow \text{round obstacle circulations} \end{aligned} \tag{40}$$

To compute the lift distribution, $w_2(\zeta)$ can be inserted into the usual Blasius integral formula for the force on an object [1]. The forces (and torques) on any number of aerofoils can be computed similarly and various calculations of this kind can be found in [6].

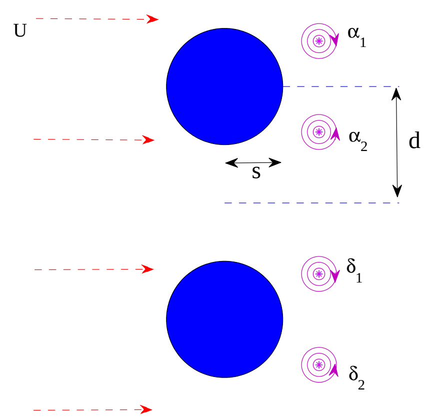

Figure 5: Generalized Föppl flows with two cylinders.

7.3 Generalized Föppl flows with two cylinders

Problem 3: Consider the same two cylinders as in Example 2, but we suppose there is now zero circulation around the cylinders and two point vortices in the wake of each cylinder. Each pair of point vortices are supposed to be of equal and opposite sign, say $\pm\Gamma$. This is a generalization of the classic Föppl steady vortex pair behind a cylinder [26] to the situation where two cylinders are present.

The complex potential in this case is

$$\begin{aligned} w_3(\zeta) = 2\pi U a \mathrm{i} \left(\frac{\partial G_0}{\partial \overline{\alpha}} - \frac{\partial G_0}{\partial \alpha} \right) \Bigg|_{\alpha=-\sqrt{\rho}} &\quad \leftarrow \text{uniform flow } (\chi = 0) \\ + \Gamma G_0(\zeta, \alpha_1) - \Gamma G_0(\zeta, \alpha_2) + \Gamma G_0(\zeta, \delta_1) - \Gamma G_0(\zeta, \delta_2) &\quad \leftarrow \text{point vortices} \end{aligned} \tag{41}$$

Although we have placed 4 point vortices in the flow (which, as we have seen from Example 1, can be expected to require an additional point vortex at infinity) note that their total circulation is zero. This means that there is no need for any additional point vortex at infinity.

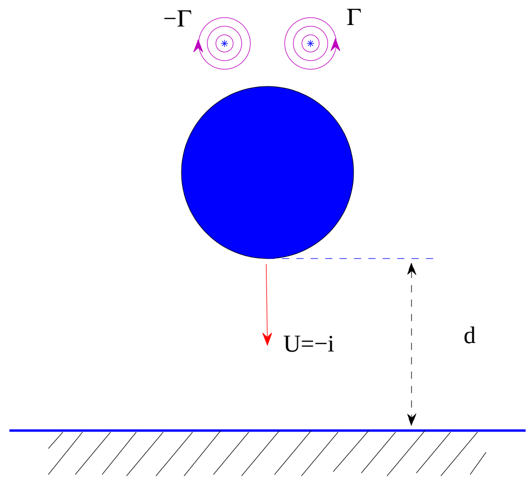

Figure 6: A moving cylinder, with point vortex wake, approaching a stationary wall at constant speed.

This finite parameter space can, in principle, be investigated to find generalized Föppl equilibrium configurations.

7.4 Cylinder with wake approaching a wall

Problem 4: Suppose a cylinder, of unit radius, is positioned such that its lowest point is at a height of d units above an infinite straight wall along the real axis. Suppose the wall is stationary but that the cylinder it is moving at complex speed $U = -\mathrm{i}$ towards the wall. Furthermore, to model the wake behind the cylinder, we place two point vortices at symmetrical positions behind the moving cylinder: a point vortex of circulation Γ is at position $z = \alpha$ while another, of circulation $-\Gamma$, is at $-\overline{\alpha}$. What is the instantaneous complex potential?

1. The geometry: The domain is again doubly connected so we take D_ζ as $\rho < |\zeta| < 1$. The conformal mapping from this annulus to the fluid domain is again a Möbius map of the form

$$z(\zeta) = \frac{\mathrm{i}(1-\rho^2)}{2\rho}\left(\frac{\zeta+\rho}{\zeta-\rho}\right) \tag{42}$$

where

$$d = \frac{(1-\rho)^2}{2\rho}. \tag{43}$$

The circle $|\zeta| = 1$ maps to the boundary of the cylinder while $|\zeta| = \rho$ maps to the infinite plane wall.

2. Möbius maps: There is only a single Möbius map in this case given by

$$\theta_1(\zeta) = \rho^2\zeta. \tag{44}$$

3. Do calculus:

$$\begin{aligned} w_4(\zeta) = \Gamma G_0(\zeta,\alpha) - \Gamma G_0(\zeta,-\overline{\alpha}) \quad &\leftarrow \text{point vortices} \\ + W_{\mathbf{U}}(\zeta) \quad &\leftarrow \text{flow due to moving cylinder} \end{aligned} \tag{45}$$

where $\mathbf{U} = (-\mathrm{i}, 0)$.

The dynamics of the vortices in this problem was studied by Crowdy, Surana and Yick [11] using Kirchhoff-Routh theory.

7.5 Model of school of swimming fish

Suppose we have three objects, with ellipse-like shapes, each travelling at some complex speed U_j for $j = 0, 1, 2$ in a uniform ambient flow of speed U parallel to the x-axis. Assume there is no circulation around any of the objects. In the fluid around them, at some instant, there are six point vortices having circulations Γ_k (for $k = 1, ..., 6$) and situated at positions α_k (for $k = 1, ..., 6$). This configuration might, for example, model a school of three fish swimming in a uniform ambient flow with the point vortices modelling their instantaneous wakes. What is the instantaneous complex potential for this flow?

1. The geometry: The domain is triply connected. For the particular geometry shown in Figure 7, D_ζ is the unit disk with two circular discs excised,

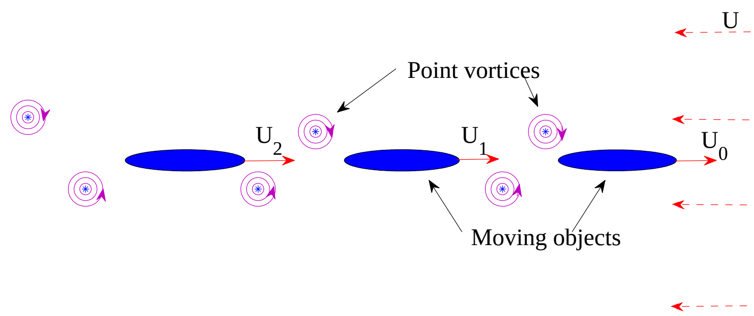

Figure 7: Three moving ellipse-like bodies in an ambient uniform flow together with an array of point vortices.

each of radius q and centred at $\pm\delta$. The functional form of the mapping happens to be given by

$$z(\zeta) = \left[-a \left.\frac{\partial}{\partial \alpha}\right|_{\alpha=0} + b \left.\frac{\partial}{\partial \overline{\alpha}}\right|_{\alpha=0} \right] G_0(\zeta, \alpha) + c, \tag{46}$$

where a, b and c are real constants. We will not explain this formula but will simply mention that such shapes happen to be exact solutions of a free boundary problem involving bubbles in Hele-Shaw flows [13]. It is interesting to note, though, that it is built from the same function G_0 that we have used to construct the calculus (it turns out that G_0 has intimate connections with conformal slit maps, a fact used to study the motion of point vortices through gaps in walls by Crowdy & Marshall [8]). The key point is that the vast literature of conformal mapping can be imported into our calculus to deal with multiply connected geometries of (in principle) any shape. In the map (46), the point $\beta = 0$ maps to infinity and, locally,

$$z = \frac{a}{\zeta} + \text{analytic}. \tag{47}$$

2. Möbius maps: There are two Möbius maps in this case given by

$$\theta_1(\zeta) = \delta + \frac{q^2}{1 - \bar{\delta}\zeta}, \quad \theta_2(\zeta) = -\delta + \frac{q^2}{1 + \bar{\delta}\zeta}. \tag{48}$$

3. Do calculus: Again we construct the complex potential by adding together the appropriate components:

$$\begin{aligned} w_5(\zeta) = & \sum_{k=1}^{6} \Gamma_k G_0(\zeta, \alpha_k) \quad \leftarrow \text{point vortices} \\ & - \left(\sum_{k=1}^{6} \Gamma_k \right) G_0(\zeta, 0) \ \leftarrow \text{make round-obstacle circulations zero} \\ & + 2\pi U a \mathrm{i} \left(\frac{\partial G_0}{\partial \overline{\alpha}} - \frac{\partial G_0}{\partial \alpha} \right) \Bigg|_{\alpha=0} \quad \leftarrow \text{uniform flow } (\chi = 0) \\ & + W_{\mathbf{U}}(\zeta) \quad \leftarrow \text{flow generated by moving bodies} \end{aligned} \tag{49}$$

where $\mathbf{U} = (U_0, U_1, U_2)$.

8 How to compute the SK prime function?

With this powerful calculus at hand, just one question remains: how does one evaluate the SK prime function? One possibility is to use a classical infinite product formula for it as recorded, for example, in Baker [3]. It is given by

$$\omega(\zeta, \alpha) = (\zeta - \alpha) \prod_{\theta_k} \frac{(\theta_k(\zeta) - \alpha)(\theta_k(\alpha) - \zeta)}{(\theta_k(\zeta) - \zeta)(\theta_k(\alpha) - \alpha)} \tag{50}$$

where the product is over all compositions of the basic maps $\{\theta_j, \theta_j^{-1} | j = 1, ..., M\}$ excluding the identity and all inverse maps.

In the doubly connected case, if we take D_ζ to be the concentric annulus $\rho < |\zeta| < 1$ with $0 < \rho < 1$ then there is just a single Möbius map given by $\theta_1(\zeta) = \rho^2 \zeta$. The infinite product (50) is then over all mappings of the form

$$\{\theta_j(\zeta) = \rho^{2j}\zeta | j \geq 1\}. \tag{51}$$

Using this in (50) leads to the expression

$$\omega(\zeta,\alpha) = -\frac{\alpha}{C}P(\zeta/\alpha,\rho) \tag{52}$$

where $C = \prod_{k=1}^{\infty}(1-\rho^{2k})$ and the function

$$P(\zeta,\rho) \equiv (1-\zeta)\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta)(1-\rho^{2k}\zeta^{-1}). \tag{53}$$

Since $P(\zeta,\rho)$ is analytic in the annulus $\rho < |\zeta| < 1$ it also has a convergent Laurent series there. It is given by the (rapidly convergent) series

$$P(\zeta,\rho) = A\sum_{n=-\infty}^{\infty}(-1)^n\rho^{n(n-1)}\zeta^n, \tag{54}$$

where

$$A = \prod_{n=1}^{\infty}(1+\rho^{2n})^2 \Big/ \sum_{n=0}^{\infty}\rho^{n(n-1)}. \tag{55}$$

This Laurent series converges everywhere in the fundamental annulus $\rho < |\zeta| < \rho^{-1}$ and proves a much faster way of evaluating $P(\zeta,\rho)$ than a method based on (53).

It is interesting to point out that the function $P(\zeta,\rho)$ is closely connected to the first Jacobi theta function Θ_1 [28] which is one way to see the connection between the approach here and the approach to doubly connected problems using elliptic function theory that usually appears in the literature [19] [15] [18].

For the case of general multiply connected domains, it is not known if the product (50) converges for all choices of the parameters $\{q_j,\delta_j|j=1,...,M\}$ and, even if it does, its rate of convergence can be so slow as to make use of (50) impractical in many circumstances. It can, however, be safely used in some cases, especially of small connectivity. It is then natural to ask: is there is a Laurent series representation, analogous to (54) in the $M=1$ case, for cases where $M>1$?. Such a representation will obviate the need to use the infinite product (50). Recently, Crowdy and Marshall [14] have devised a novel numerical algorithm based on precisely such Laurent series representations. It can be used to evaluate $\omega(\zeta,\alpha)$, with great speed and accuracy, for broad classes of domains and without resorting to use of the infinite product (50). The algorithm works by writing

$$X(\zeta,\alpha) = (\zeta-\alpha)^2 \hat{X}(\zeta,\alpha) \tag{56}$$

where $X(\zeta,\alpha) = \omega^2(\zeta,\alpha)$ and then computing the coefficients in the following Laurent expansion of $\hat{X}(\zeta,\alpha)$:

$$\hat{X}(\zeta,\alpha) = A\left(1 + \sum_{k=1}^{M}\sum_{m=1}^{\infty} \frac{c_m^{(k)} q_k^m}{(\zeta-\delta_k)^m} + \sum_{k=1}^{M}\sum_{m=1}^{\infty} \frac{d_m^{(k)} Q_k^m}{(\zeta-\delta_k')^m}\right). \tag{57}$$

It is important to note that the algorithm in [3] does <u>not</u> depend on a sum or product over a Schottky group. This renders this method of evaluating the prime function much faster in practice than making use of (50). Full details of this numerical algorithm can be found in [14] and freely downloadable `MATLAB` M-files will soon be available at the website:
`www.ma.ic.ac.uk/~dgcrowdy/SKPrime`.

9 Other considerations

The purpose of this user's guide has been to show how to write down analytic expressions for the instantaneous complex potentials for any given ideal flow in two dimensions. These results can be applied in various circumstances and extended in different directions. For completeness, we will give a brief overview of these additional matters.

9.1 Kirchhoff-Routh theory

Here, we have been solely concerned with finding the instantaneous complex potential associated with distributions of point vortices in geometrically complicated domains, possibly with additional background flows (such as uniform or straining flows) and with possible circulations around any obstacles. But what about the *dynamics* of the point vortices?

The new calculus can help us with the dynamical problem too. To see how, we invoke an important result from 1941 due to C.C. Lin [20]. He showed that the problem of point vortex motion in general multiply connected domains is a Hamiltonian system and wrote down a formula for the governing Hamiltonians in terms of a "special Green's function" associated with the domain in which the vortices were moving. Lin did not provide any explicit way to construct this Green's function but derived his results based purely on its existence. Actually, the special Green's function discussed by Lin has the explicit expression, in multiply connected circular domains D_ζ, given by the explicit formula for $G_0(\zeta, \alpha)$ introduced in (17); indeed, $G_0(\zeta, \alpha)$ *is* precisely Lin's special Green's function. This crucial fact was first pointed out by Crowdy & Marshall [4]. Furthermore, in a second 1941 paper, Lin [21] went on to show how the Hamiltonian transforms under conformal mapping: if $H^{(\zeta)}$ is the Hamiltonian for N-vortex motion in D_ζ, then the Hamiltonian $H^{(z)}$ in any domain D_z that is the conformal image of D_ζ under the map $z(\zeta)$ is just

$$H^{(z)}(\{z_k\}) = H^{(\zeta)}(\{\alpha_k\}) + \sum_{k=1}^{N} \frac{\Gamma_k^2}{4\pi} \log\left|\frac{dz}{d\zeta}\right|_{\alpha_k}, \tag{58}$$

where $z_k = z(\alpha_k)$. This second result completes our theory: it means that it is *enough* to know the functional form of the special Green's function in multiply connected circular domains (and we already do) since (58) then gives the Hamiltonians in any other conformally equivalent domain. Crowdy & Marshall have explored this theory in the context of single vortex motion around circular islands [5] and through gaps in walls [8].

9.2 Contour dynamics

The calculus built around the Schottky-Klein prime function can help in more sophisticated situations too. The most popular model of vorticity, after the point vortex model, is the vortex patch model [26] where vortex structures are modelled as finite-area regions of uniform vorticity. Owing to the fact that, in two dimensions, vorticity is convected with the flow, it is enough to follow just the boundaries of any such uniform vortex regions thereby reducing the dynamical model to the tracking of a set of contours in the fluid. Such numerical methods are collectively known as contour dynamics methods [25]. In recent work, Crowdy & Surana [12] have shown how to generalize such methods to finding the evolution of vortex patches in geometrically complicated, multiply connected domains. Again, these methods rely on combined use of conformal mapping methods with the properties of the Schottky-Klein prime function. The analysis of such methods is in its infancy but they appear to be highly effective.

9.3 Vortex motion on a sphere

It turns out that our theoretical development generalizes quite naturally to the flows on the surface of a sphere once it is endowed with a complex analytic structure by means of a stereographic projection. Surana & Crowdy [27] have shown how to generalize the calculus both to the case of point vortex motion and vortex patch motion in complicated domains on a spherical surface.

Acknowledgments: DGC acknowledges an EPSRC Advanced Research Fellowship. He is grateful to J.S. Marshall for helpful discussions.

The author's interest in vortex dynamics was inspired by his PhD advisor, Philip Saffman. This article is dedicated to him.

References

[1] D.J. Acheson, Elementary fluid dynamics, Oxford University Press, Oxford, (1990).

[2] G.K. Batchelor, An introduction to fluid dynamics, Cambridge University Press, Cambridge, (1967).

[3] H.F. Baker, Abelian functions: Abel's theorem and the allied theory of theta functions, Cambridge University Press, Cambridge, (1897).

[4] D.G. Crowdy and J.S. Marshall, Analytical formulae for the Kirchhoff Routh path function in multiply connected domains, *Proc. Roy. Soc. A.*, **461**, 2477–2501, (2005).

[5] D.G. Crowdy & J.S. Marshall, The motion of a point vortex around multiple circular islands, *Phys. Fluids*, **17**, 056602, (2005).

[6] D.G. Crowdy, Calculating the lift on a finite stack of cylindrical aerofoils, *Proc. Roy. Soc. A.*, **462**, 1387–1407, (2006).

[7] D.G. Crowdy, Analytical solutions for uniform potential flow past multiple cylinders, *Eur. J. Mech. B/Fluids* , **25**(4), 459-470, (2006).

[8] D.G. Crowdy and J.S. Marshall, The motion of a point vortex through gaps in walls, *J. Fluid Mech.*, **551**, 31–48, (2006).

[9] D.G. Crowdy & J.S. Marshall, Conformal mappings between canonical multiply connected domains, *Comput. Methods Funct. Theory* **6(1)**, 59–76, (2006).

[10] D.G. Crowdy, Explicit solution for the potential flow due to an assembly of stirrers in an inviscid fluid, *J. Eng. Math.*, (in press).

[11] D.G. Crowdy, A. Surana & K-Y Yick, The irrotational flow generated by two planar stirrers in inviscid fluid, *Phys. Fluids*, **19**, 018103, (2007).

[12] D.G. Crowdy & A. Surana, Contour dynamics in complex domains, *J. Fluid Mech.*, **593**, 235-254, (2007).

[13] D. Crowdy, Multiple steady bubbles in a Hele-Shaw cell, *Proc. Roy. Soc. A.*, (to appear).

[14] D.G. Crowdy and J.S. Marshall, Computing the Schottky-Klein prime function on the Schottky double of planar domains, *Comput. Methods Funct. Theory*, **7**(1), 293-308, (2007).

[15] C. Ferrari, Sulla trasformazione conforme di due cerchi in due profili alari, *Memoire della Reale Accad. della Scienze di Torino*, Ser. II 67 (1930).

[16] G.M. Goluzin, Geometric theory of functions of a complex variable, Amer. Math. Soc., Providence, RI, (1969).

[17] D.A. Hejhal, Theta functions, kernel functions and Abelian integrals, Mem. Amer. Math. Soc., **129**, Amer. Math. Soc., Providence, R.I., (1972).

[18] E.R. Johnson, N.R. McDonald, The motion of a vortex near two circular cylinders, *Proc. Roy. Soc. London Ser. A*, **460**, 939954, (2004).

[19] M. Lagally, Die reibungslose Strmung im Aussengebiet zweier Kreise,*Z. Angew. Math. Mech.,* **9**, 299305, (1929). English translation: The frictionless flow in the region around two circles, N.A.C.A., Technical Memorandum No 626, (1931).

[20] C.C. Lin, On the motion of vortices in two dimensions I: existence of the Kirchhoff-Routh function, *Proc. Natl. Acad. Sci.*, **27**, 570–575, (1941).

[21] C.C. Lin, On the motion of vortices in two dimensions II: some further investigations on the Kirchhoff-Routh function, *Proc. Natl. Acad. Sci.*, **27**, 576–577, (1941).

[22] L.M. Milne-Thomson, Theoretical hydrodynamics, Dover, New York, (1996).

[23] Z. Nehari, Conformal mapping, Dover, New York, (1952).

[24] W.J. Prosnak, Computation of fluid motions in multiply connected domains, Wissenschaft & Technik, (1987).

[25] D.I. Pullin, Contour dynamics methods, *Ann. Rev. Fluid Mech.*, **24**, 89–115, (1992).

[26] P.G. Saffman, Vortex dynamics, Cambridge University Press, Cambridge, (1992).

[27] A. Surana & D.G. Crowdy, Vortex dynamics in complex domains on a spherical surface, *J. Comp. Phys.*, **227**(12), 6058-6070, (2008).

[28] E.T. Whittaker & G.N. Watson, A course of modern analysis, Cambridge University Press, Cambridge, (1927).