A new calculus for two dimensional vortex dynamics

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In remembrance

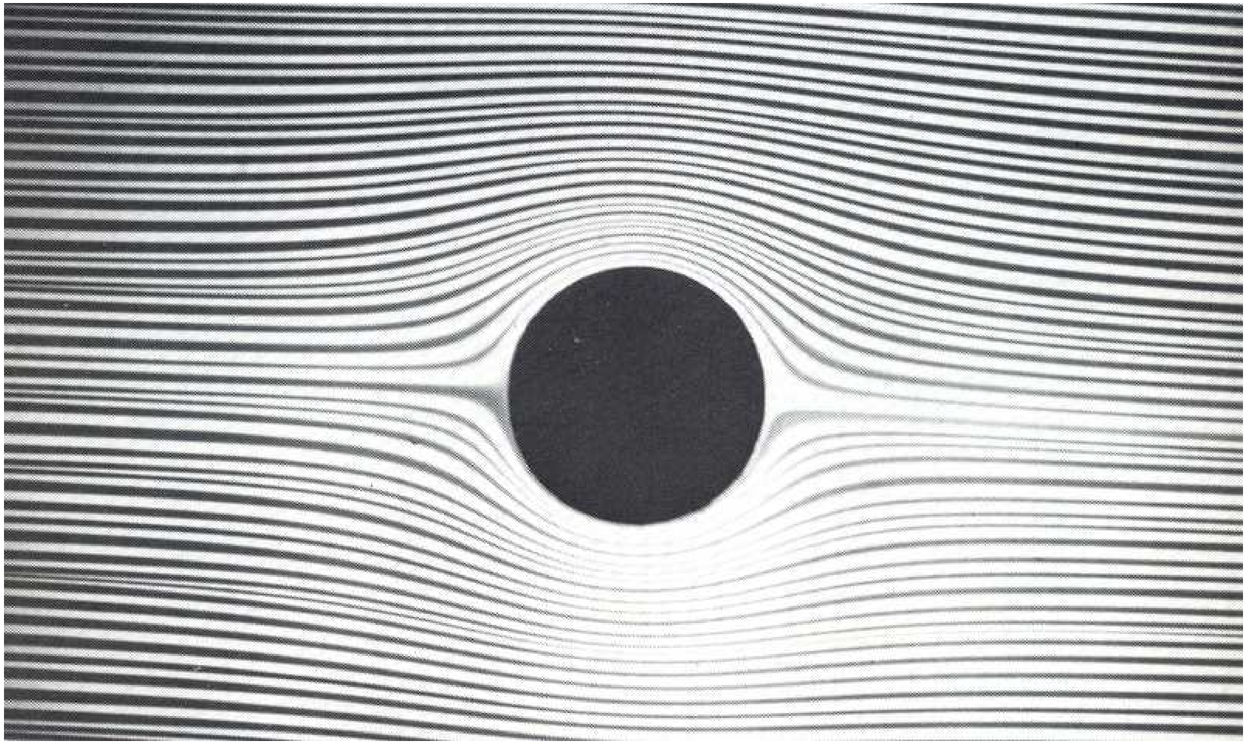
Philip Geoffrey Saffman, FRS (1931–2008)

In remembrance

Philip Saffman, FRS   Derek Moore, FRS

(1931–2008)

Uniform flow past a cylinder



The first photograph in Van Dyke's *Album of Fluid Motion*

Complex potential

$$w(z) = U\left(z + \frac{1}{z}\right)$$

Simple. But what about flow past *multiple* objects?

The biplane problem

THE FRICTIONLESS FLOW IN THE REGION AROUND TWO CIRCLES

By M. Lagally

Zeitschrift für angewandte Mathematik und Mechanik
Vol. 9, No. 4, August, 1929

The complex potential will be

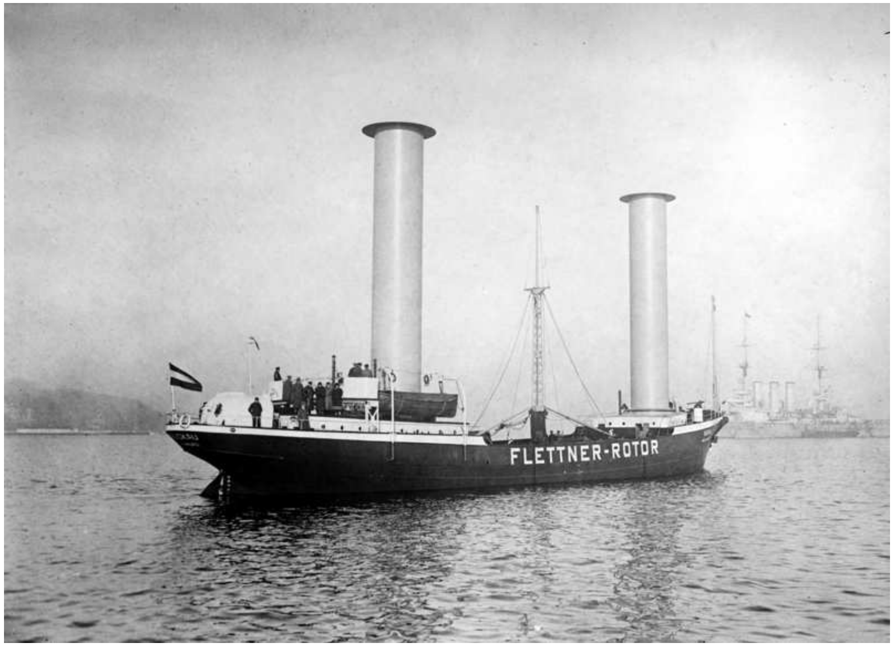
$$\Omega = -i\,\frac{\Gamma}{2\pi}\log_e\frac{\sigma(Z)}{\sigma(Z+2\beta)} + 2c\,[w_\infty\,\zeta(Z) - \bar{w}_\infty\,\zeta(Z+2\beta)] + i\kappa Z + \kappa' . \qquad (15)$$

Weierstrass σ-function ↗ and ζ-function ↗

Here κ' is a new additive constant which can be chosen as we please. For simplicity we shall put $\kappa' = 0$. However, it should be noted that Ω is also an infinitely multiple-valued function; if $\log_e \frac{\sigma(Z)}{\sigma(Z+2\beta)}$ changes by $\pm 2\pi i$ by a revolution around a pole, then Ω itself changes by $\pm\Gamma$.

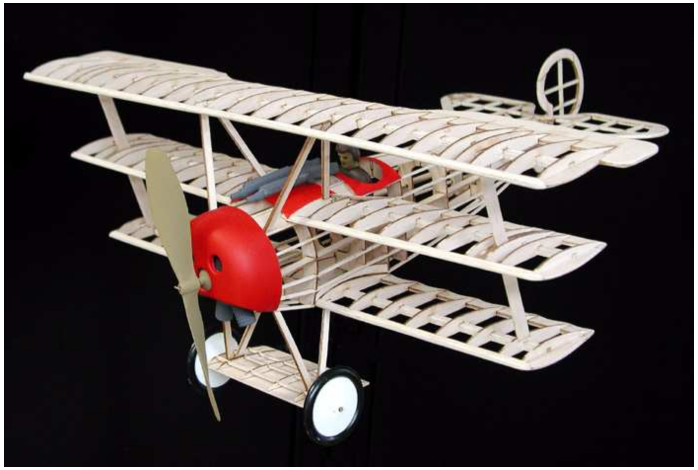
*Burkhardt-Faber, "Elliptische Funktionen," section 30.

The Flettner rotor ship



- Flettner rotors are rotating cylinders which exploit the Magnus effect for propulsion
- This mechanism was explored in the 1920's and 1930's

Triply connected analogues



The triplane

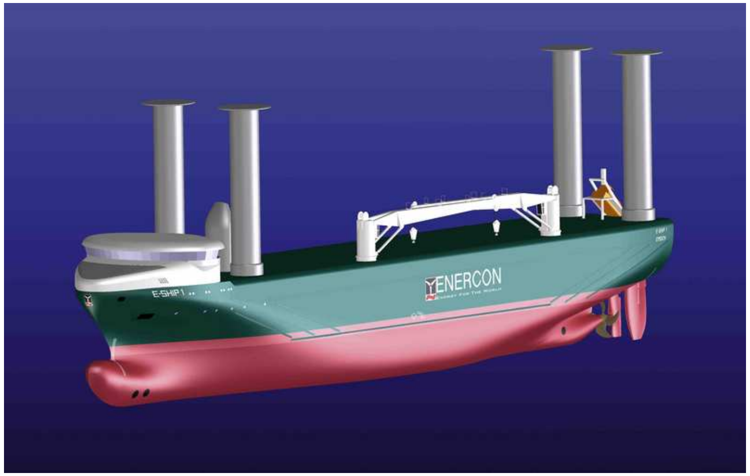
3-rotor Flettner yacht

These are examples where flows past three obstacles are relevant

Quadruply connected rotor vessels

(source: BBC website)

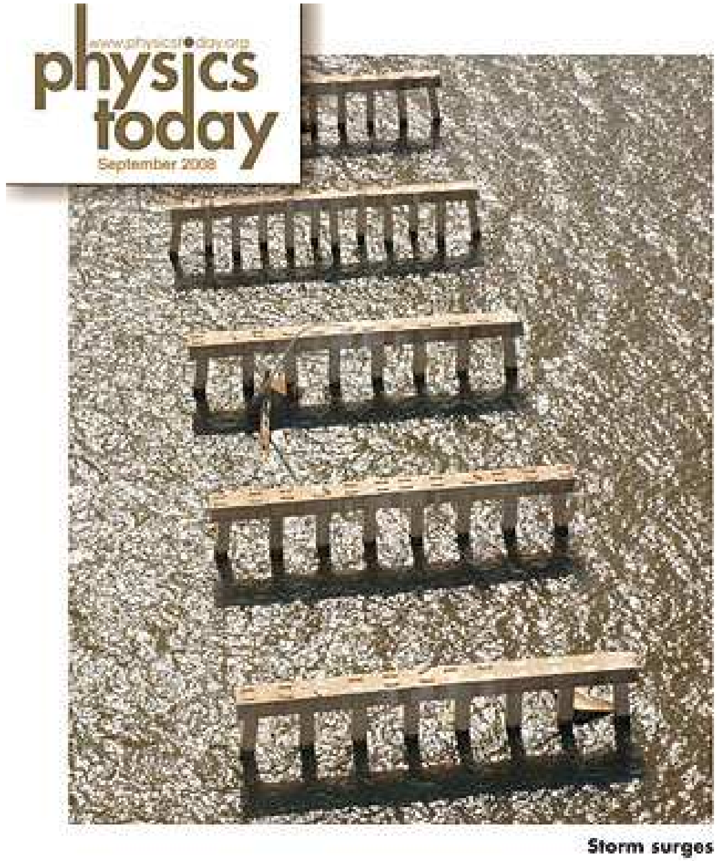
Futuristic “cloudseeder yachts” – wind-powered, unmanned vessels



(from Enercon press release)

“Enercon’s E-ship uses “sailing rotors” to cut fuel costs by 30%”

Civil engineering



Civil engineers are interested in forces on multiple objects (e.g. bridge supports) in laminar flows

T. Yamamoto, “Hydrodynamic forces on multiple circular cylinders”, *J. Hydraulics Division*, ASCE, **102**, (1976)

Topology of laminar mixing

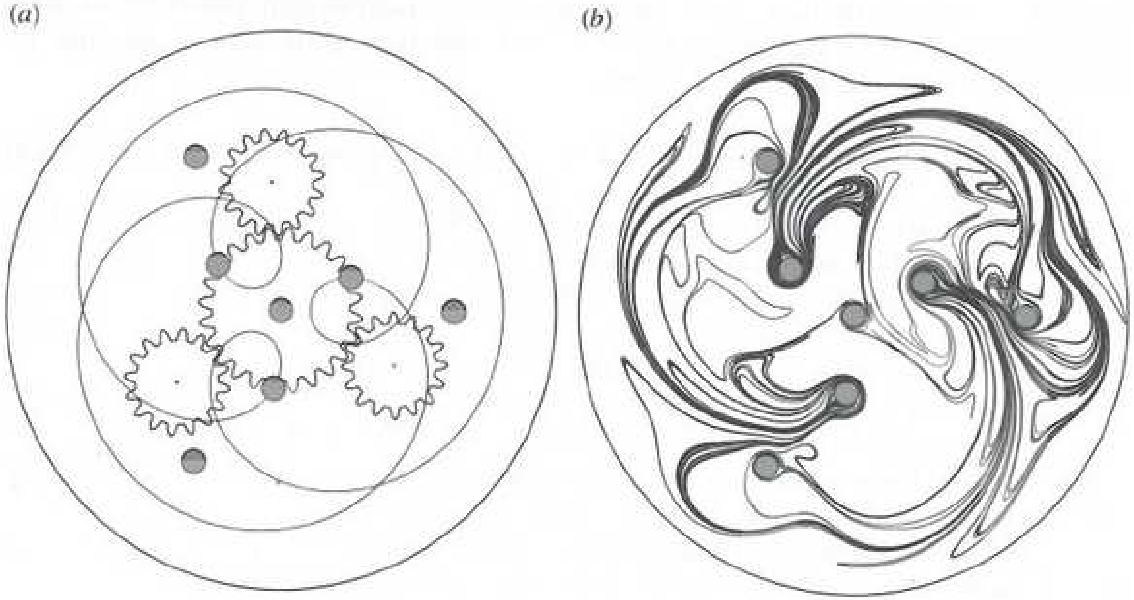


Figure 3. (a) An arrangement of gears with attached stirring rods (grey circles) that intertwine in a way that is topologically rich enough to produce (b) pseudo-Anosov chaotic advection of a stirred marker according to Thurston–Nielsen theory. (Courtesy of M. D. Finn and J.-L. Thiffeault.)

Mixing in an octuply connected flow domain

Biolocomotion

Mining Smartness from Nature

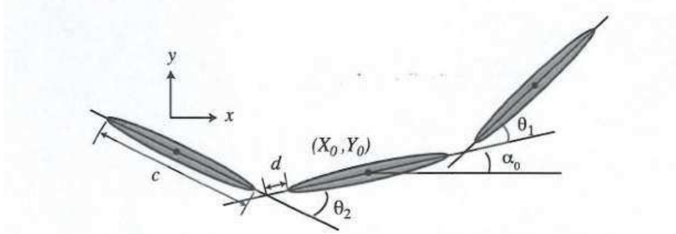


Figure 1: Diagram of articulated three-link fish, with labeled hinges

An Exploration of Passive and Active Flexibility in Biolocomotion through Analysis of Canonical Problems

Jeff D. Eldredge[1,a], Megan Wilson[1,b] and Daniel Hector[1,c]

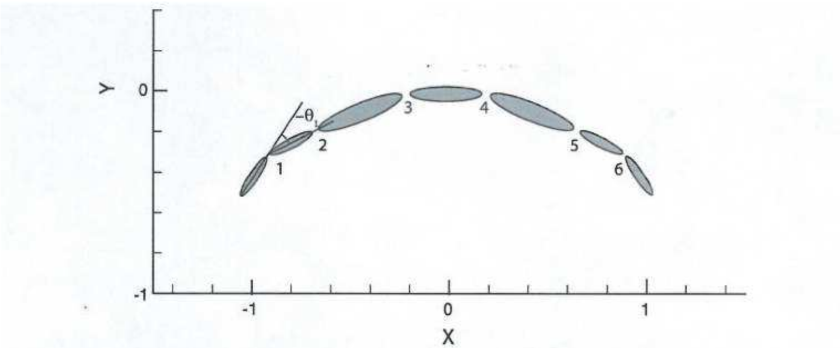
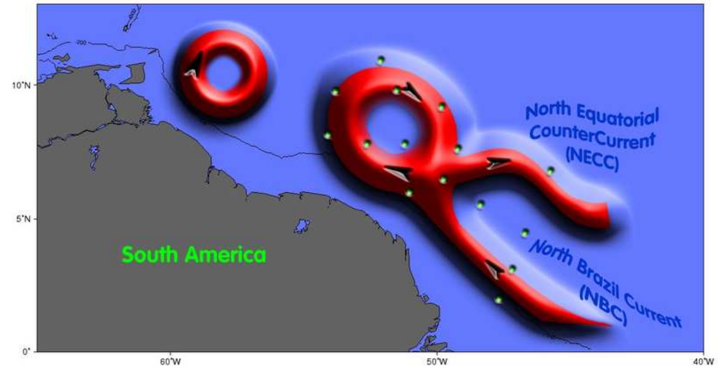
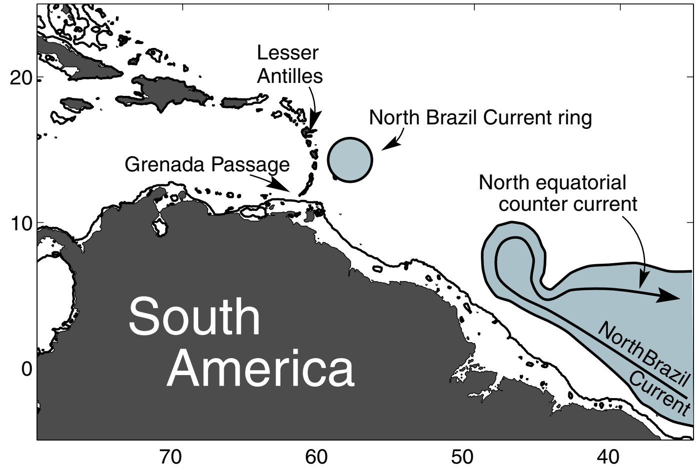


Figure 4: Diagram of hinge-body setup for jellyfish-like locomotion simulations.

Recent interest in biolocomotion has led to resurgence in flow modelling techniques originally pioneered in aeronautics

Oceanic eddies





Geophysical fluid dynamicists want to model motion of oceanic eddies in complicated island topographies

Other engineering challenges

Other engineering challenges

“The World”

History of the two-obstacle problem

W.M. Hicks, On the motion of two cylinders in a fluid, *Q. J. Pure Appl. Math.*, (1879)

A. G. Greenhill, Functional images in Cartesians, *Q. J. Pure Appl. Math.*, (1882)

M. Lagally, The frictionless flow in the region around 2 circles, *ZAMM*, (1929).

C. Ferrari, Sulla trasformazione conforme di due cerchi in due profili alari, *Mem. Real. Accad. Sci. Torino, (1930)*

T. Yamamoto, Hydrodynamic forces on multiple circular cylinders, *J. Hydr. Div, ASCE*, (1976).

E.R. Johnson & N. Robb McDonald, The motion of a vortex near two circular cylinders, *Proc. Roy. Soc. A*, (2004)

Burton, D.A., Gratus, J. & Tucker, R.W., Hydrodynamic forces on two moving discs, *Theor. Appl. Mech.*, (2004)

No prior analytical results for more than two aerofoils

Standard fluids literature contains almost nothing on multi-obstacle flows

This talk seeks to fill this gap with an analytical treatment

Riemann mapping theorem

- Any simply connected domain D_z (bounded or unbounded) in the plane can be conformally mapped to the unit disc (and vice versa)
- Let the unit disc in a complex ζ-plane be denoted D_ζ
- Let the conformal mapping from D_ζ to D_z be $z(\zeta)$

If the domain is unbounded then a point $\beta \in D_\zeta$ maps to infinity and, locally

$$z(\zeta) = \frac{a}{\zeta - \beta} + \text{analytic}$$

There are three degrees of freedom in the mapping theorem
This means, for example, that we can <u>pick β arbitrarily</u>

A point vortex outside a cylinder

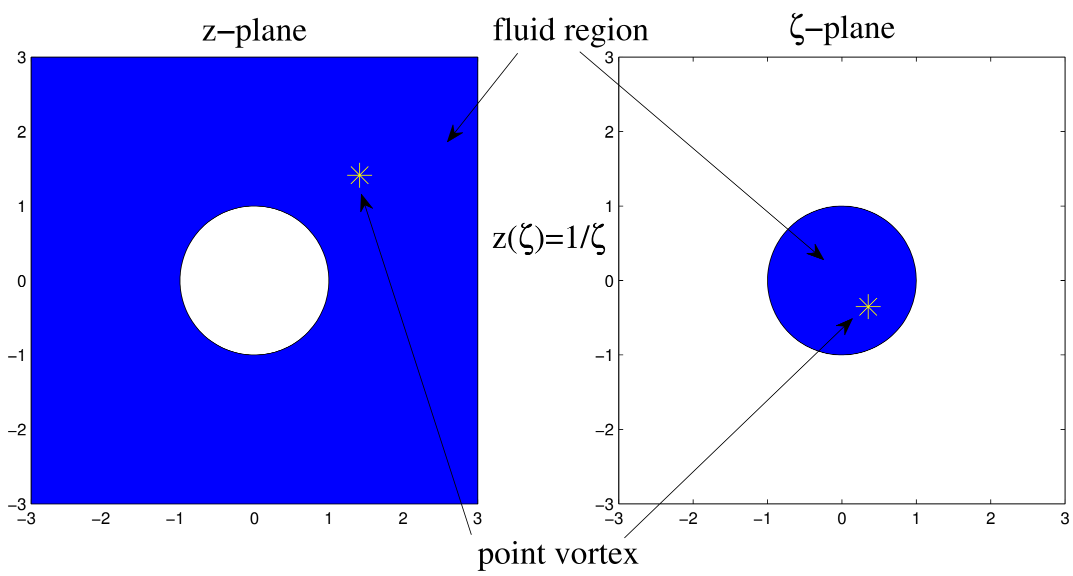
Consider a single point vortex outside a unit-radius cylinder

Conformal map from the interior to the exterior of unit disc:

$$z(\zeta) = \frac{1}{\zeta}$$

We have chosen $\beta = 0$ to map to $z = \infty$.

Let the unit circle $|\zeta| = 1$ be denoted C_0



A point vortex outside a cylinder

Complex potential for isolated point vortex at $\zeta = \alpha$:

$$w(\zeta) = -\frac{\mathrm{i}}{2\pi}\log(\zeta - \alpha)$$

But, we need it to be real on $|\zeta| = 1$ (so it is a streamline)

A function, built from $w(\zeta)$, that is real:

$$w(\zeta) + \overline{w(\zeta)}$$

But this is not analytic. On $|\zeta| = 1$, $\zeta = 1/\overline{\zeta}$, so consider

$$w(\zeta) + \overline{w(1/\overline{\zeta})}$$

$$= -\frac{\mathrm{i}}{2\pi}\log\left(\frac{\zeta - \alpha}{1/\zeta - \overline{\alpha}}\right) = -\frac{\mathrm{i}}{2\pi}\log\left(\frac{\zeta - \alpha}{|\alpha|(\zeta - 1/\overline{\alpha})}\right) - \frac{\mathrm{i}}{2\pi}\log\zeta + \mathrm{c}$$

Circulation around the cylinder?

Another possible solution:

$$-\frac{\mathrm{i}}{2\pi}\log\left(\frac{\zeta-\alpha}{|\alpha|(\zeta-1/\overline{\alpha})}\right)-\frac{\mathrm{i}\gamma}{2\pi}\log\zeta$$

where γ is any real number

Consider the two terms separately:

$$-\frac{\mathrm{i}}{2\pi}\log\left(\frac{\zeta-\alpha}{|\alpha|(\zeta-1/\overline{\alpha})}\right)$$ $\longleftarrow$ circulation -1 around cylinder, vortex at α

and

$$-\frac{\mathrm{i}\gamma}{2\pi}\log\zeta$$ $\longleftarrow$ circulation $-\gamma$ around cylinder, vortex at $\zeta=0$ ($z=\infty$)

Note: we are free to choose the round-obstacle circulation

Pick $-1-\gamma=0$ if want zero circulation around cylinder

The function $G_0(\zeta, \alpha)$

It seems pedantic, but introduce the notation

$$\omega(\zeta, \alpha) \equiv (\zeta - \alpha)$$

Also introduce notation $G_0(\zeta, \alpha)$:

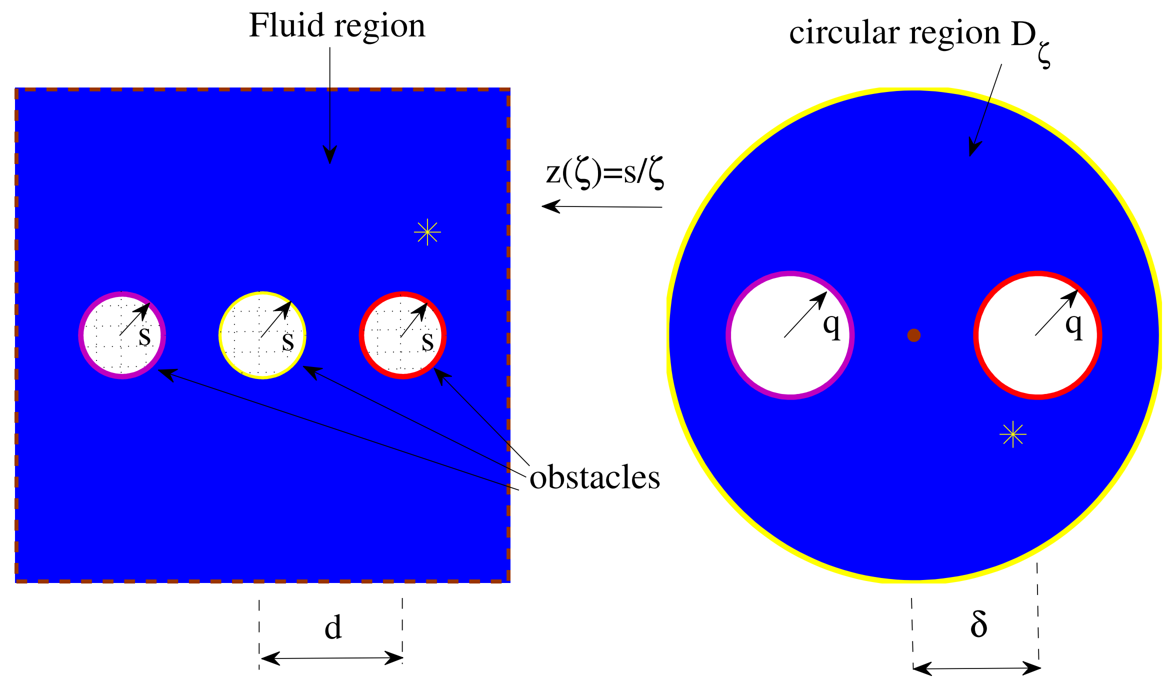
$$G_0(\zeta, \alpha) = -\frac{\mathrm{i}}{2\pi} \log \left(\frac{\zeta - \alpha}{|\alpha|(\zeta - 1/\overline{\alpha})} \right) = -\frac{\mathrm{i}}{2\pi} \log \left(\frac{\omega(\zeta, \alpha)}{|\alpha|\omega(\zeta, 1/\overline{\alpha})} \right)$$

Recall, this is complex potential for:

- a point vortex, circulation $+1$, at α
- it has circulation -1 around obstacle whose boundary is image of C_0 (hence subscript)
- it has constant imaginary part on C_0

What about three cylinders?

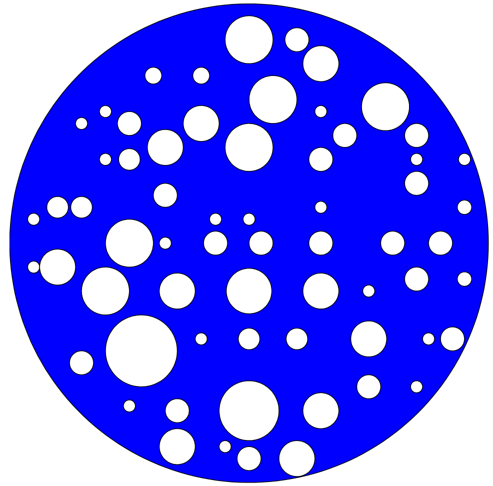
Now consider fluid region exterior of three circular cylinders



$$z(\zeta) = \frac{s}{\zeta} \quad \text{with} \quad q = \frac{s^2}{d^2 - s^2}, \quad \delta = \frac{sd}{d^2 - s^2}$$

Geometry of D_ζ depends on geometry of given domain

Generalized Riemann Theorem



D_ζ

Any multiply $(M+1)$-connnected domain can be conformally mapped to from a circular domain D_ζ consisting of the unit disc with M smaller circular discs excised

The radii of the discs will be $\{q_j | j = 1, ..., M\}$

The centres of the discs will be $\{\delta_j | j = 1, ..., M\}$

Let unit circle be C_0; all other circular boundaries $\{C_j | j = 1, .., M\}$

Back to $G_0(\zeta, \alpha)$

Let's go back to $G_0(\zeta, \alpha)$ for the single cylinder example:

$$G_0(\zeta, \alpha) = -\frac{\mathrm{i}}{2\pi} \log \left(\frac{\omega(\zeta, \alpha)}{|\alpha| \omega(\zeta, 1/\overline{\alpha})} \right)$$

Recall, this is complex potential for:

- a point vortex, circulation $+1$, at α
- it has circulation -1 around obstacle whose boundary is image of C_0 (hence subscript)
- it has constant imaginary part on C_0

What is analogous complex potential for the three cylinder example?

Higher connected generalization?

Remarkable fact

$$G_0(\zeta,\alpha) \equiv -\frac{\mathrm{i}}{2\pi}\log\left(\frac{\omega(\zeta,\alpha)}{|\alpha|\omega(\zeta,1/\overline{\alpha})}\right)$$

is the required complex potential!

It has exactly the same functional form!!

It has

- a point vortex, circulation $+1$, at α
- a circulation -1 around object whose boundary is image of C_0 (hence subscript)
- circulation 0 around all other objects
- it has constant imaginary part on C_j $(j = 0, 1, \ldots .M)$

A fact from function theory

What can we possibly mean by this?

Fact: there exists a special transcendental function of two variables $\omega(.,.)$ – it is just a function of the data $\{q_j, \delta_j | j = 1, .., M\}$ – such that:

(1) $\omega(\zeta, \alpha)$ has a simple zero at $\zeta = \alpha$

(2) $G_0(\zeta, \alpha)$ has constant imaginary part on all the boundary circles of D_ζ (so that all the obstacle boundaries are streamlines)

The function $\omega(\zeta, \alpha)$ is called the *Schottky-Klein prime function*

It plays a fundamental role in complex function theory that extends far beyond the realm of fluid dynamics.

Consider it just a computable special function (cf: $\sin(x), J_k(x)$)

Adding circulation around the other obstacles

What if we want non-zero circulations around the other M obstacles? Then we need M additional complex potentials:

$$G_j(\zeta,\alpha) = -\frac{i}{2\pi}\log\left(\frac{\omega(\zeta,\alpha)}{|\alpha|\omega(\zeta,\theta_j(1/\overline{\alpha}))}\right), \quad j = 1,..,M$$

The complex potential $G_j(\zeta,\alpha)$ has

- a point vortex, circulation $+1$, at α
- a circulation -1 around cylinder whose boundary is image of C_j (hence the subscript)
- circulation 0 around all other cylinders
- it has constant imaginary part on C_k $(k = 0, 1, .., M)$

The functions $\{\theta_j(\zeta) | j = 1, .., M\}$

The functions $\{\theta_j(\zeta) | j = 1, .., M\}$ are simple functions of the data $\{q_j, \delta_j | j = 1, .., M\}$:

$$\theta_j(\zeta) \equiv \delta_j + \frac{q_j^2 \zeta}{1 - \overline{\delta_j} \zeta}, \quad j = 1, .., M$$

These functions are fully determined by $\{q_j, \delta_j | j = 1, .., M\}$

Example: Suppose D_ζ is the annulus $\rho < |\zeta| < 1$ then there is just one Möbius map ($M = 1$): $\delta_1 = 0, q_1 = \rho$

$$\theta_1(\zeta) = \rho^2 \zeta$$

Uniform flow past multiple objects

What about adding background flows?

Suppose, as $\zeta \to \beta$, we have

$$z(\zeta) = \frac{a}{\zeta - \beta} + \text{analytic}$$

for some constant a, and we want the complex potential for uniform flow of speed U at angle χ to the x-axis

The required complex potential is

$$2\pi U a \mathrm{i} \left(e^{\mathrm{i}\chi} \frac{\partial G_0}{\partial \overline{\alpha}} - e^{-\mathrm{i}\chi} \frac{\partial G_0}{\partial \alpha} \right) \Bigg|_{\alpha=\beta}$$

It is just a function of $G_0(\zeta, \alpha)$

If we let $\omega(\zeta,\alpha) = (\zeta - \alpha)$ so that

$$G_0(\zeta,\alpha) = -\frac{\mathrm{i}}{2\pi}\log\left(\frac{\zeta-\alpha}{|\alpha|(\zeta - 1/\overline{\alpha})}\right)$$

then the formula

$$2\pi U a \mathrm{i}\left(e^{\mathrm{i}\chi}\frac{\partial G_0}{\partial \overline{\alpha}} - e^{-\mathrm{i}\chi}\frac{\partial G_0}{\partial \alpha}\right)\Bigg|_{\alpha=\beta}$$

with

$$z(\zeta) = \frac{1}{\zeta} \qquad \leftarrow \text{flow past circular cylinder } (\beta = 0)$$

reduces to

$$U\left(\frac{1}{\zeta} + \zeta\right) = U\left(z + \frac{1}{z}\right) \qquad \text{(choosing } \chi = 0\text{)}$$

Straining flows around multiple objects

What about higher-order flows?

Suppose, as $\zeta \to \beta$, we have

$$z(\zeta) = \frac{a}{\zeta - \beta} + \text{analytic}$$

for some constant a, and we want the complex potential for straining flow tending to $\Omega e^{i\lambda} z^2$ as $z \to \infty$

The required complex potential is

$$2\pi\Omega a^2 \mathrm{i} \left(e^{-\mathrm{i}\lambda} \frac{\partial^2 G_0}{\partial \overline{\alpha}^2} - e^{\mathrm{i}\lambda} \frac{\partial^2 G_0}{\partial \alpha^2} \right) \Big|_{\alpha=\beta}$$

Again, it is just a function of $G_0(\zeta, \alpha)$

What if the objects move?

Complex potential when jth obstacle moves with complex velocity U_j:

$$W_{\mathbf{U}}(\zeta) = \frac{1}{2\pi}\oint_{C_0} \left[\mathrm{Re}[-\mathrm{i}\overline{U_0} z(\zeta')]\right] \left[d \log\left(\frac{\omega(\zeta',\zeta)}{\overline{\omega}(\zeta'^{-1},\zeta^{-1})}\right)\right]$$

$$- \sum_{j=1}^{M} \frac{1}{2\pi}\oint_{C_j} \left[\mathrm{Re}[-\mathrm{i}\overline{U_j} z(\zeta')] + d_j\right] \left[d \log\left(\frac{\omega(\zeta',\zeta)}{\overline{\omega}(\overline{\theta_j}(\zeta'^{-1}),\zeta^{-1})}\right)\right]$$

$$\mathbf{U} \equiv (U_0, U_1, ..., U_M)$$

The constants $\{d_j | j = 1, ..., M\}$ solve a linear system
This time, expression is an integral depending on $\omega(.,.)$

(Useful for modelling biological organisms, vortex control problems)

Strategy for problem solving

Step 1: Analyse the geometry and determine D_ζ, the data $\{q_j, \delta_j | j = 1, ..., M\}$ and map $z(\zeta)$ (use numerical conformal mapping if necessary)

$\Downarrow$

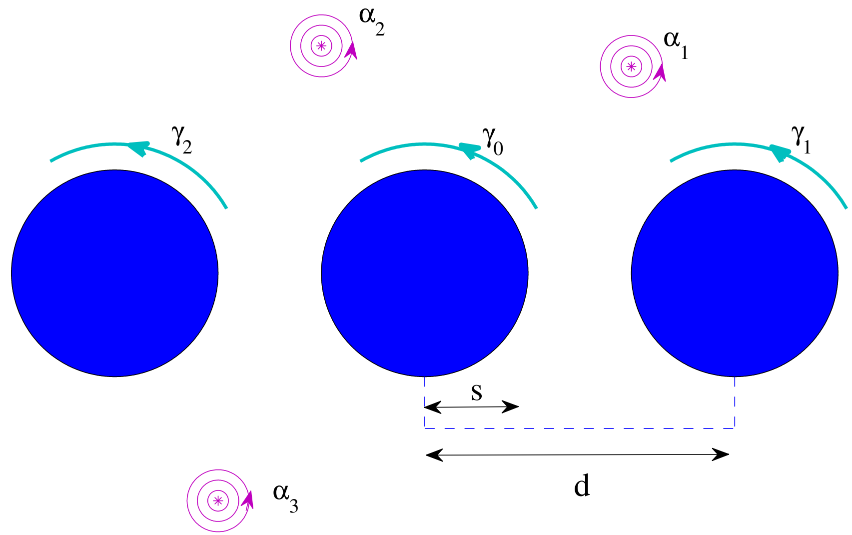
Step 2: Construct the Möbius maps $\{\theta_j(\zeta) | j = 1, ..., M\}$ and compute $\omega(.,.)$ (it all depends on this function!)

$\Downarrow$

Step 3: Do calculus with the functions $\{G_j(\zeta, \alpha) | j = 0, 1, .., M\}$ to solve the fluid problem

Let's do some examples!

Three point vortices near three circular islands

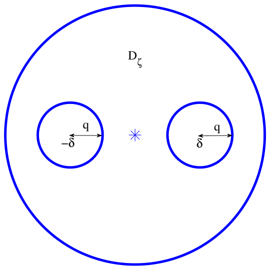


D_ζ is the unit ζ-disc with two smaller discs excised, each of radius q and centred at $\pm\delta$.

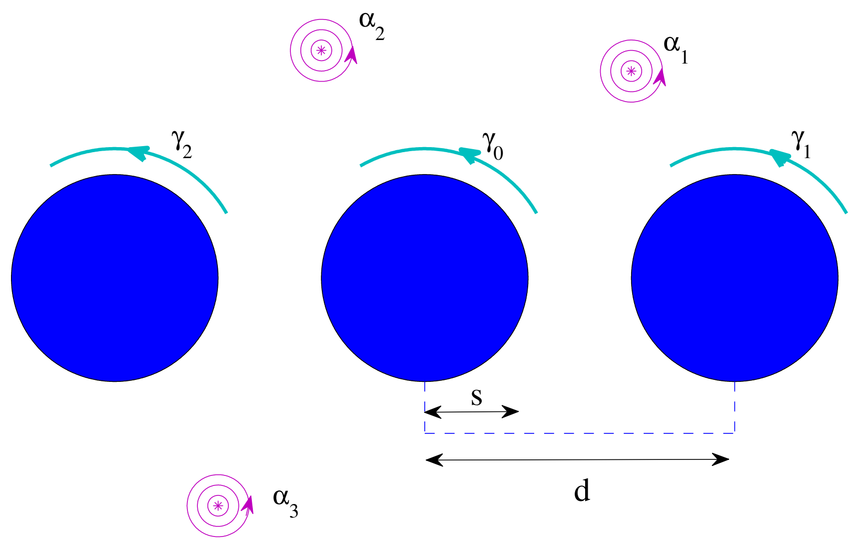
The point $\beta = 0$ maps to infinity.

Assume point vortices all have circulation Γ

Assume circulation γ_j around island D_j



Three point vortices near three circular islands

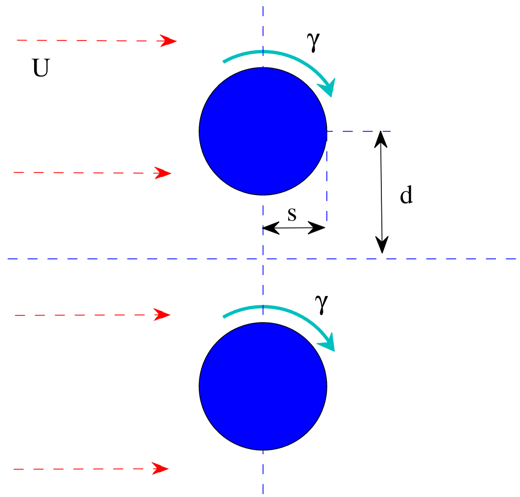


$$w_1(\zeta) = \sum_{k=1}^{3} \Gamma G_0(\zeta, \alpha_k)$$ ← point vortices

$$- 3\Gamma G_0(\zeta, 0)$$ ← round-obstacle circulations zero

$$- \sum_{j=0}^{2} \gamma_j G_j(\zeta, 0)$$ ← add in round-obstacle circulations

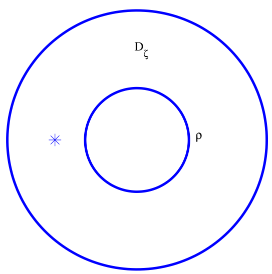
What is the lift on a biplane?



Take D_ζ as the annulus $\rho < |\zeta| < 1$:

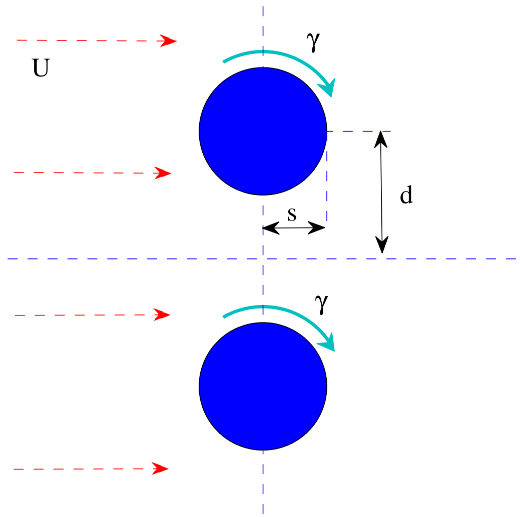
$$z(\zeta) = i\sqrt{d^2 - s^2}\left(\frac{\zeta - \sqrt{\rho}}{\zeta + \sqrt{\rho}}\right),$$

$$\rho = \frac{1 - (1 - (s/d)^2)^{1/2}}{1 + (1 - (s/d)^2)^{1/2}},$$



We have $\quad z(\zeta) = \dfrac{a}{\zeta + \sqrt{\rho}} +$ analytic, $\quad a = -2\mathrm{i}\sqrt{\rho(d^2 - s^2)}$

What is the lift on a biplane?

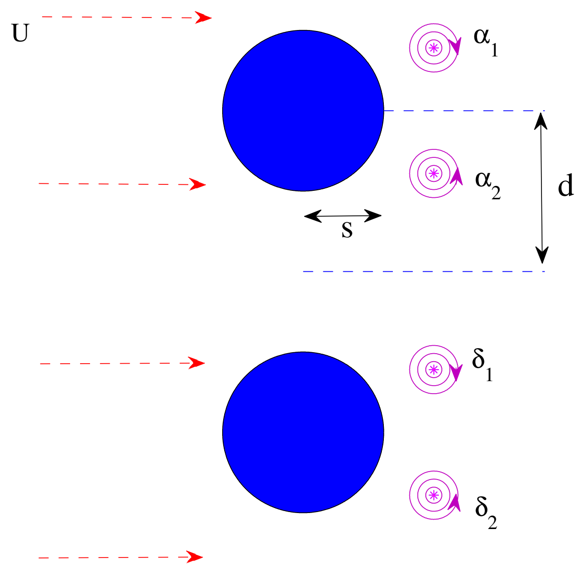


$$w_2(\zeta) = 2\pi U a \mathrm{i} \left(\frac{\partial G_0}{\partial \overline{\alpha}} - \frac{\partial G_0}{\partial \alpha} \right) \Big|_{\alpha = -\sqrt{\rho}} \quad \leftarrow \text{uniform flow } (\chi = 0)$$

$$- \sum_{j=0}^{1} \gamma G_j(\zeta, -\sqrt{\rho}) \quad \leftarrow \text{add round obstacle circulations}$$

(now use Blasius integral formula for $F_x - iF_y$)

Generalized Föppl flows with two cylinders



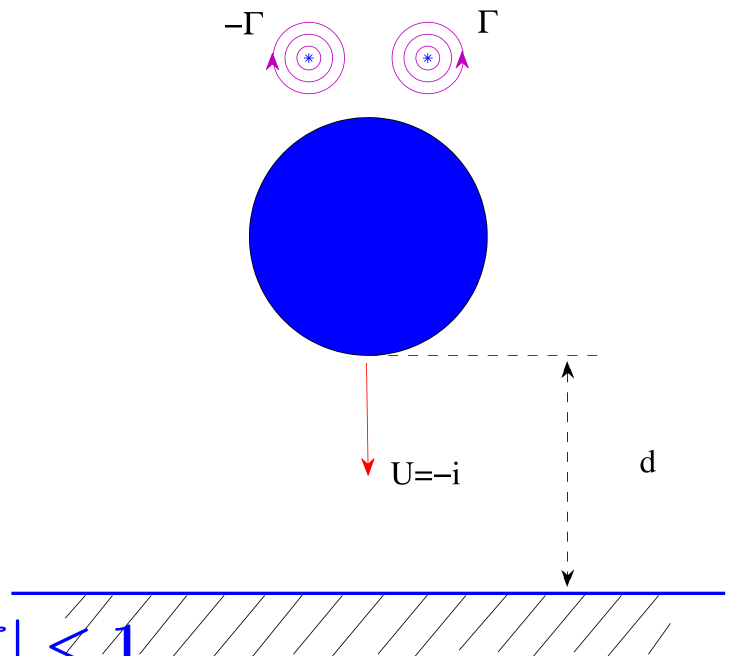
$$w_3(\zeta) = 2\pi U a \mathrm{i} \left(\frac{\partial G_0}{\partial \overline{\alpha}} - \frac{\partial G_0}{\partial \alpha} \right) \bigg|_{\alpha = -\sqrt{\rho}} \leftarrow \text{uniform flow } (\chi = 0)$$

$$+ \Gamma G_0(\zeta, \alpha_1) - \Gamma G_0(\zeta, \alpha_2)$$

$$+ \Gamma G_0(\zeta, \delta_1) - \Gamma G_0(\zeta, \delta_2) \leftarrow \text{point vortices}$$

(now search finite dimensional parameter space for equilibria)

Cylinder with wake approaching a wall



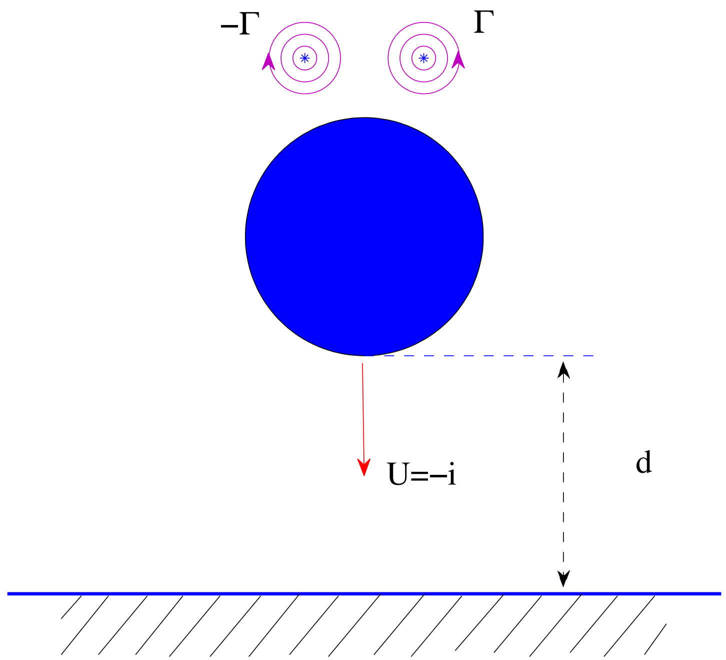
Take D_ζ as $\rho < |\zeta| < 1$

$$z(\zeta) = \frac{\mathrm{i}(1-\rho^2)}{2\rho}\left(\frac{\zeta+\rho}{\zeta-\rho}\right), \quad d = \frac{(1-\rho)^2}{2\rho}.$$

$|\zeta| = 1$ maps to the boundary of the cylinder

$|\zeta| = \rho$ maps to wall.

Cylinder with wake approaching a wall

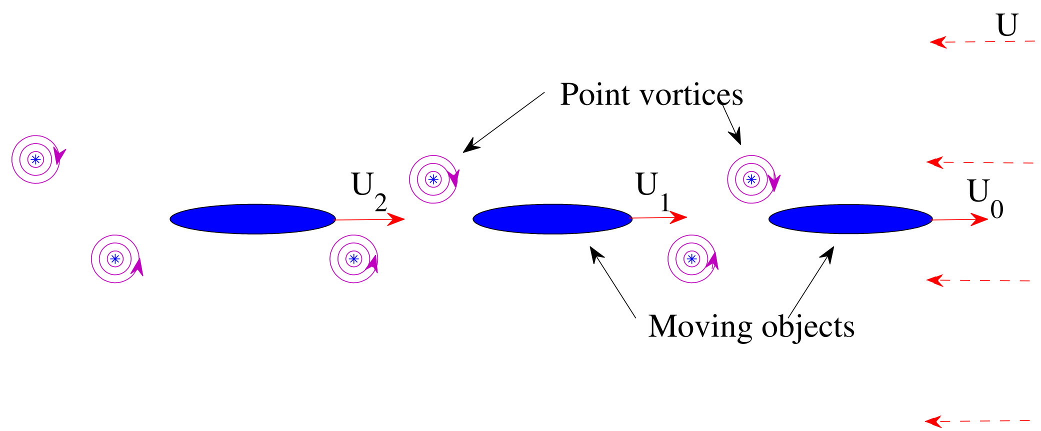


$$w_4(\zeta) = \Gamma G_0(\zeta, \alpha) - \Gamma G_0(\zeta, -\overline{\alpha}) \quad \leftarrow \text{point vortices}$$

$$+ W_U(\zeta) \quad \leftarrow \text{flow due to moving cylinder}$$

where $\mathbf{U} = (-\mathbf{i}, \mathbf{0})$

Model of school of swimming fish

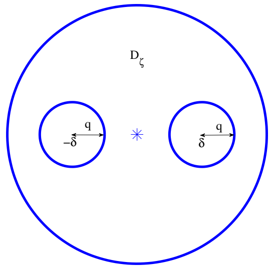


Conformal mapping non-trivial in this case. It happens to be

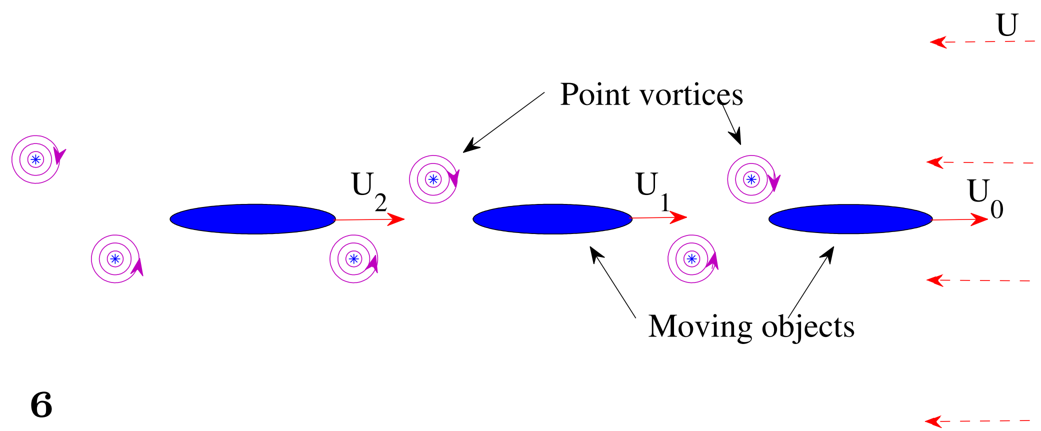
$$z(\zeta) = \left[-a \left. \frac{\partial}{\partial \alpha} \right|_{\alpha=0} + b \left. \frac{\partial}{\partial \overline{\alpha}} \right|_{\alpha=0} \right] G_0(\zeta, \alpha) + c$$

Near $\zeta = 0$ (so $\beta = 0$):

$$z = \frac{a}{\zeta} + \text{analytic}$$



Model of school of swimming fish



$$w_5(\zeta) = \sum_{k=1}^{6} \Gamma_k G_0(\zeta, \alpha_k) \quad \leftarrow \text{point vortices}$$

$$- \left(\sum_{k=1}^{6} \Gamma_k \right) G_0(\zeta, 0) \quad \leftarrow \text{round-obstacle circulations zero}$$

$$+ 2\pi U a i \left(\frac{\partial G_0}{\partial \overline{\alpha}} - \frac{\partial G_0}{\partial \alpha} \right) \Bigg|_{\alpha=0} \quad \leftarrow \text{uniform flow } (\chi = 0)$$

$$+ W_{\mathbf{U}}(\zeta) \quad \leftarrow \text{flow due to moving bodies } \mathbf{U} = (U_0, U_1, U_2)$$

How to compute $\omega(.,.)$?

Option 1: There is a classical infinite product formula for it:

$$\omega(\zeta,\alpha) = (\zeta - \alpha)\prod_{\theta_k} \frac{(\theta_k(\zeta) - \alpha)(\theta_k(\alpha) - \zeta)}{(\theta_k(\zeta) - \zeta)(\theta_k(\alpha) - \alpha)}$$

Example: In the doubly connected case, D_ζ to be $\rho < |\zeta| < 1$

There is just a single Möbius map given by $\theta_1(\zeta) = \rho^2\zeta$

The infinite product is then

$$\omega(\zeta,\alpha) \propto P(\zeta/\alpha,\rho)$$

where

$$P(\zeta,\rho) \equiv (1-\zeta)\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta)(1-\rho^{2k}\zeta^{-1}).$$

Infinite sum representations

$P(\zeta,\rho)$ is analytic in $\rho < |\zeta| < 1$, so also has Laurent series

$$P(\zeta,\rho) = A \sum_{n=-\infty}^{\infty} (-1)^n \rho^{n(n-1)} \zeta^n,$$

where A is a constant. This converges faster than product!

Crowdy & Marshall have extended this idea to produce a fast numerical algorithm for higher connectivity

It is based on Fourier-Laurent representations (not infinite products)

MATLAB M-files will be freely available soon at
`www.ma.ic.ac.uk/~dgcrowdy/SKPrime`.

Crowdy & Marshall, "Computing the Schottky-Klein prime function on the Schottky double of planar domains", *Comput. Methods Func. Th.*, **7**, (2007)

Relation to Lagally?

It can be shown that

$$P(\zeta, \rho) = -\frac{\mathrm{i}Ce^{-\tau/2}}{\rho^{1/4}}\Theta_1(\mathrm{i}\tau/2, \rho)$$

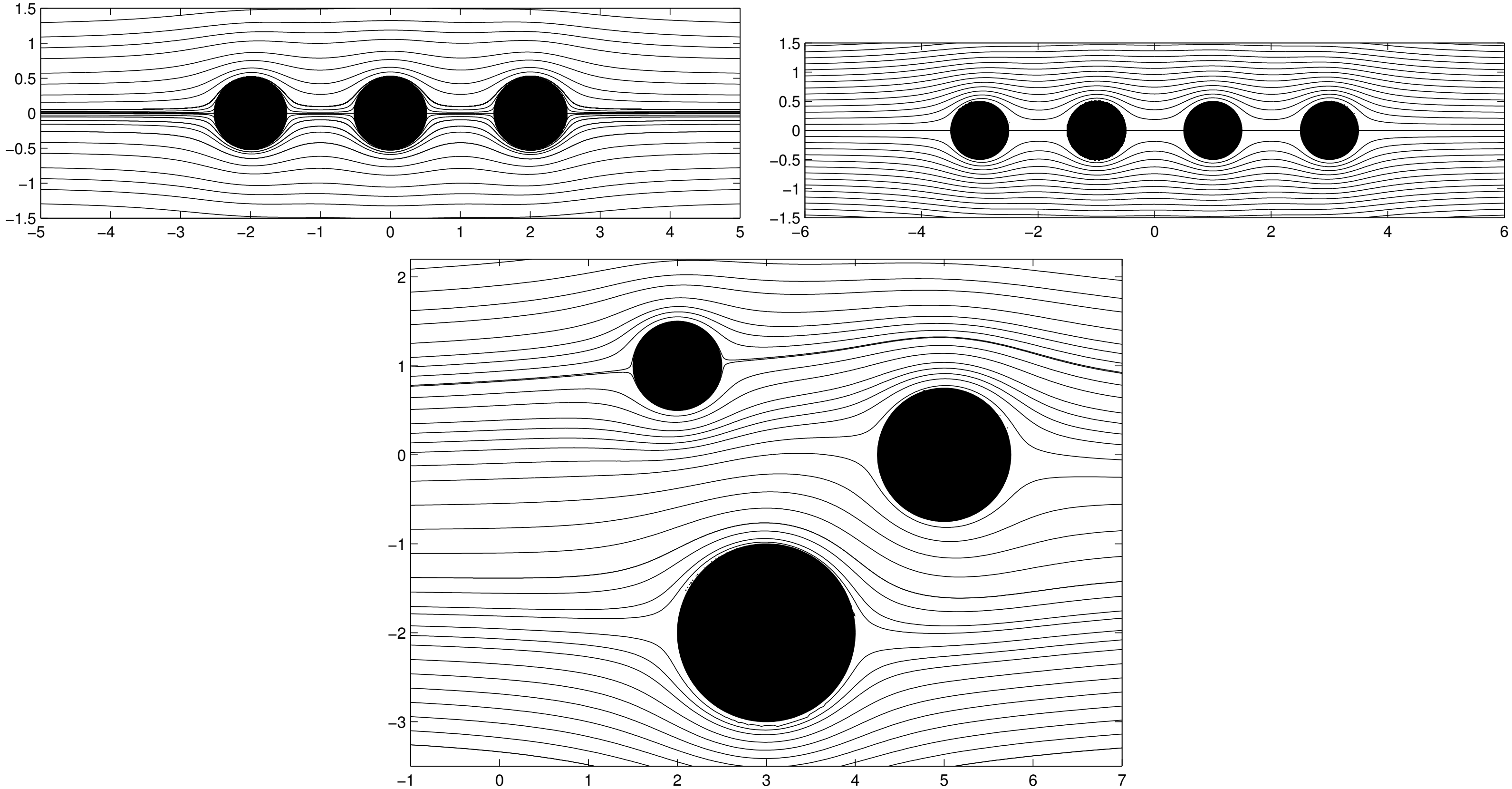
where $\tau = -\log\zeta$ and Θ_1 is first Jacobi theta function

The Jacobi theta function can be related to the Weierstrass σ and ζ function

Recall: Lagally (1929) used the latter functions in his solution to the biplane problem

Our calculus simplies and extends this two-obstacle result

Streamlines for uniform flow

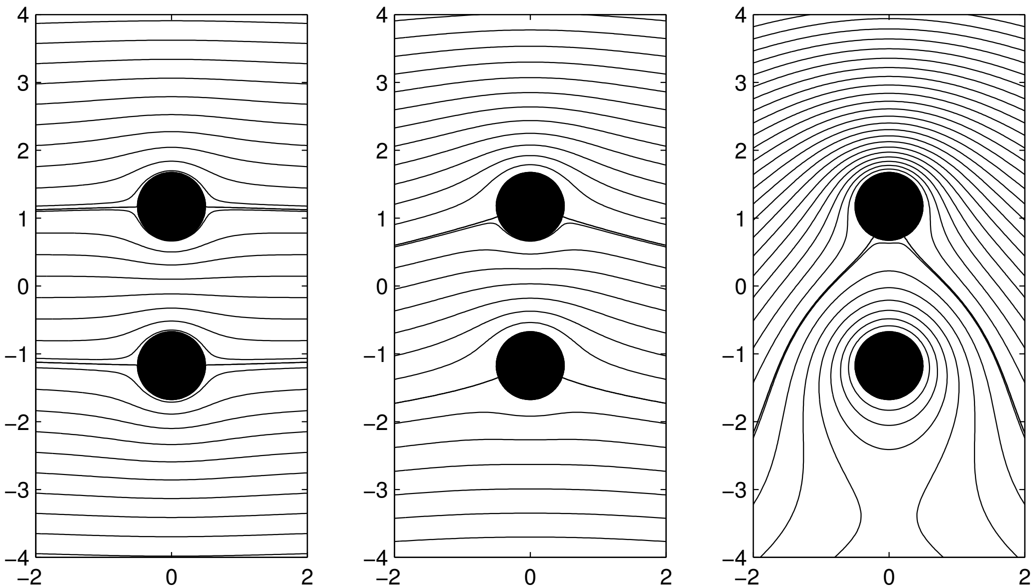


Very easy to plot using analytical formulae for complex potential

Conformal maps from circular domains D_ζ are just Möbius maps

This answers the question prompted by Van Dyke's first photograph!

Two aerofoils in unstaggered stack



Two aerofoils with gradually increasing circulation

Kirchhoff-Routh theory

In 1941, C.C. Lin wrote two papers in which he established that N-vortex motion in multiply connected domains is Hamiltonian

He relied on the existence of a "*special Green's function*"

This special Green's function is precisely $G_0(\zeta, \alpha)$!

He also showed the following transformation property of Hamiltonians:

$$H^{(z)}(\{z_k\}) = H^{(\zeta)}(\{\zeta_k\}) + \sum_{k=1}^{N} \frac{\Gamma_k^2}{4\pi} \log \left| \frac{dz}{d\zeta} \right|_{\zeta_k}$$

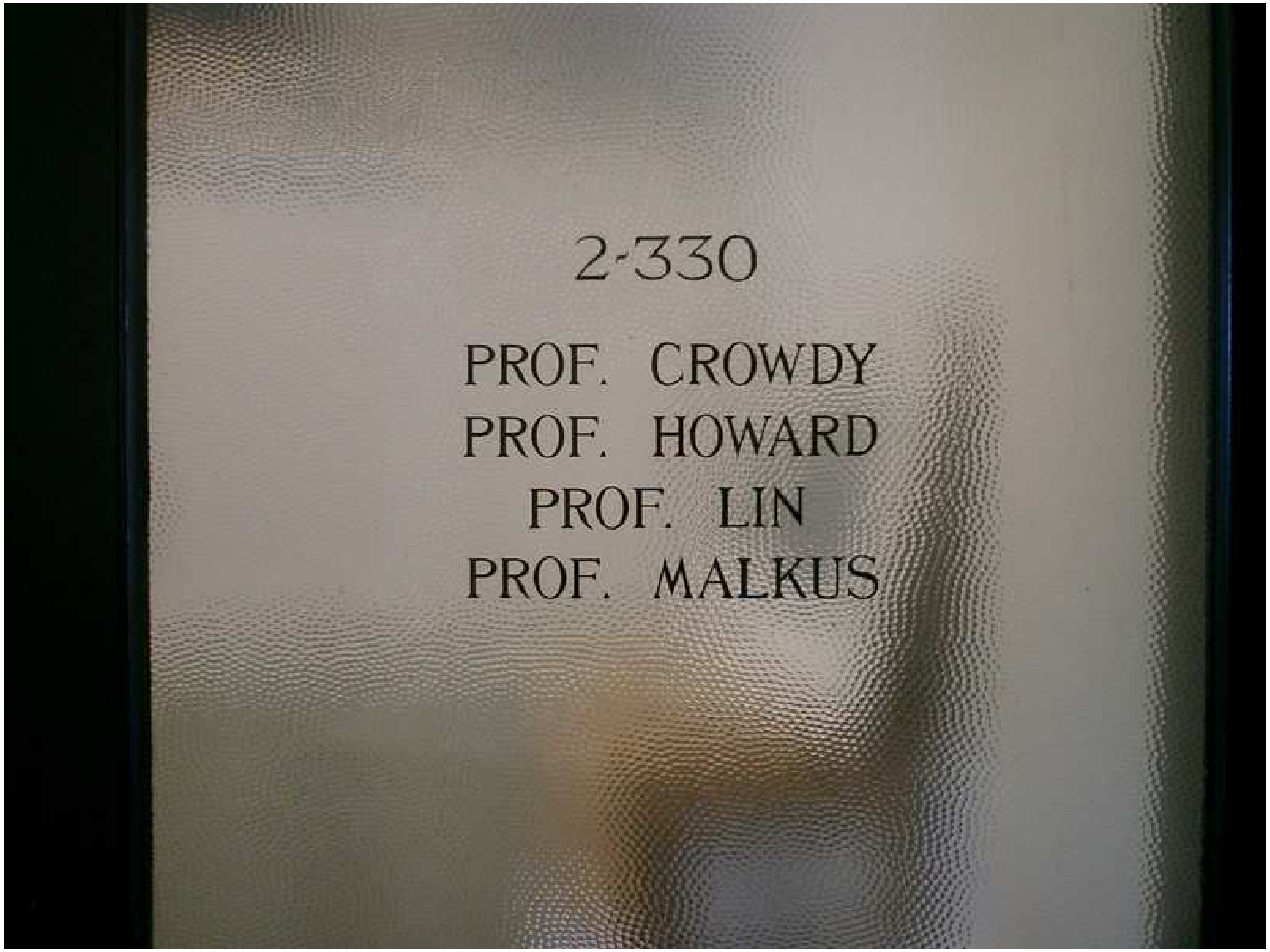
where $z_k = z(\zeta_k)$

This fact completes the theory!

A general analytical framework now exists for N-vortex motion

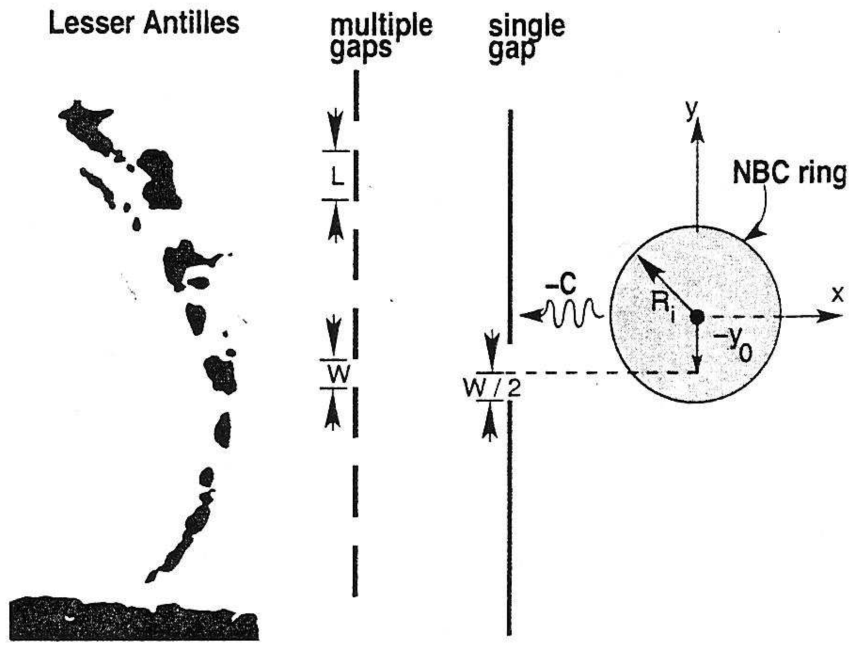
Crowdy & Marshall, Analytical formulae for the Kirchhoff-Routh path function in multiply connected domains", *Proc. Roy. Soc. A*, **461**, (2005)

Life's little ironies



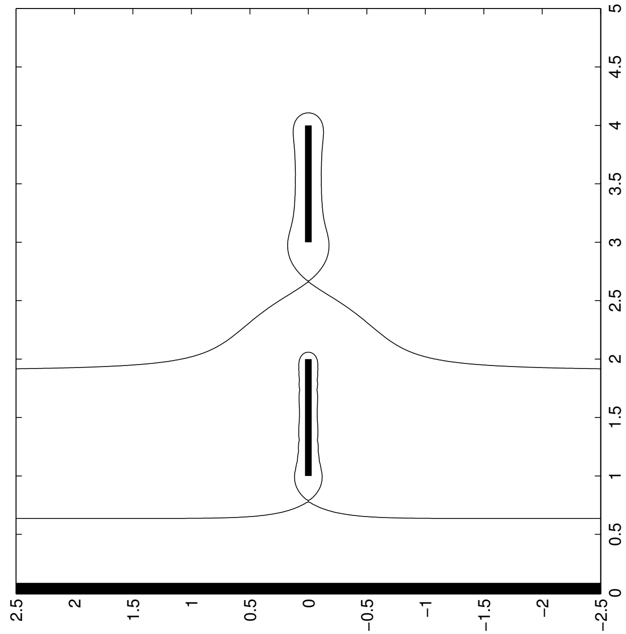
My office door at MIT

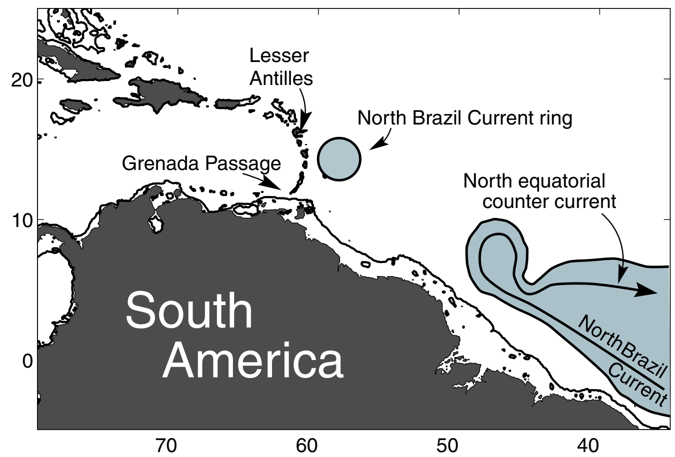
Modelling geophysical flows



Simmons & Nof, “The squeezing of eddies through gaps”, J. Phys. Ocean., (2002).

Vortex motion through gaps in walls

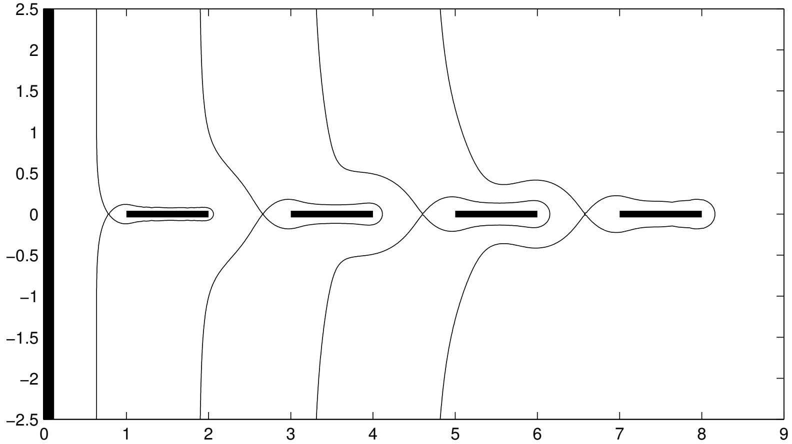




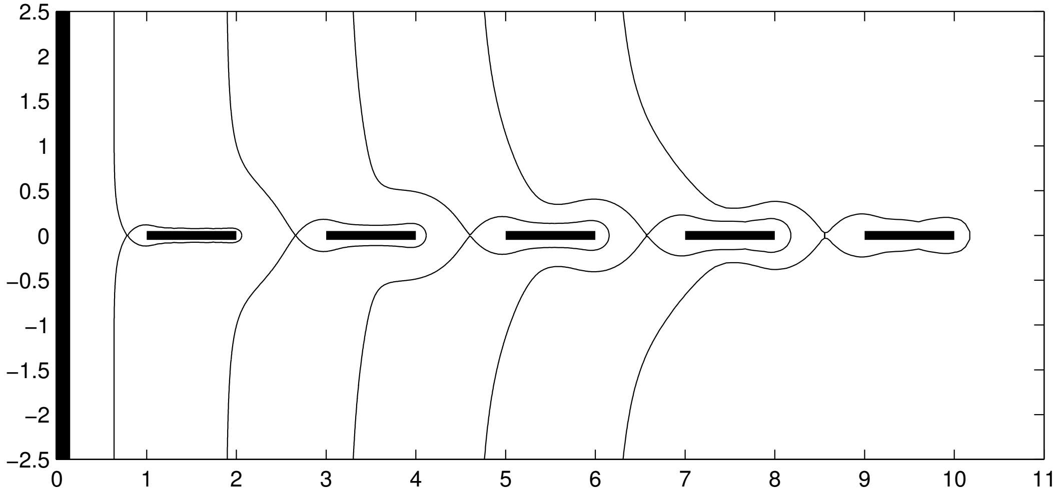
Critical vortex trajectories for two offshore islands

Crowdy & Marshall, “The motion of a point vortex through gaps in walls” *J. Fluid Mech.*, **551**, (2006)

Critical vortex trajectories



Four offshore islands



Five offshore islands

Note: Even the conformal slit maps are obtained analytically!

Other applications of the calculus

The calculus has many other applications:

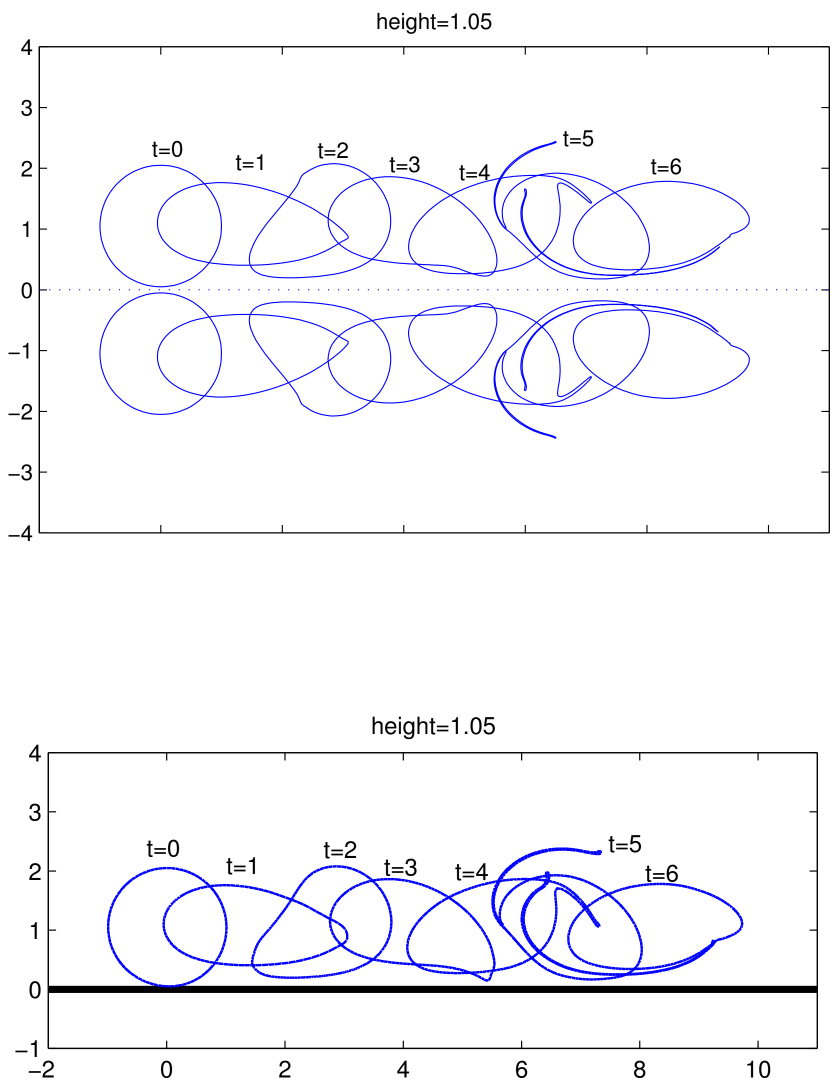
Contour dynamics: Facilitates numerical determination of vortex patch dynamics
(kernels in contour integrals expressed using $\omega(.,.)$)

Crowdy & Surana, Contour dynamics in complex domains, *J. Fluid Mech.*, **593**, (2007)

Surface of a sphere
(need to endow spherical surface with complex analytic structure by means of stereographic projection)

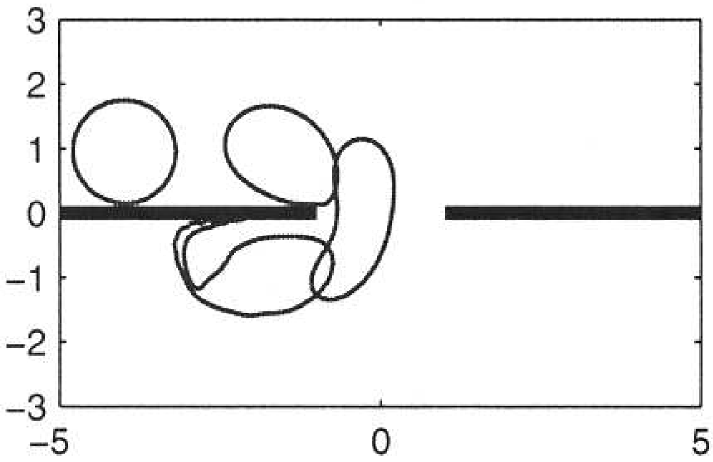
Surana & Crowdy, Vortex dynamics in complex domains on a spherical surface, *J. Comp. Phys.*, **227**, (2008)

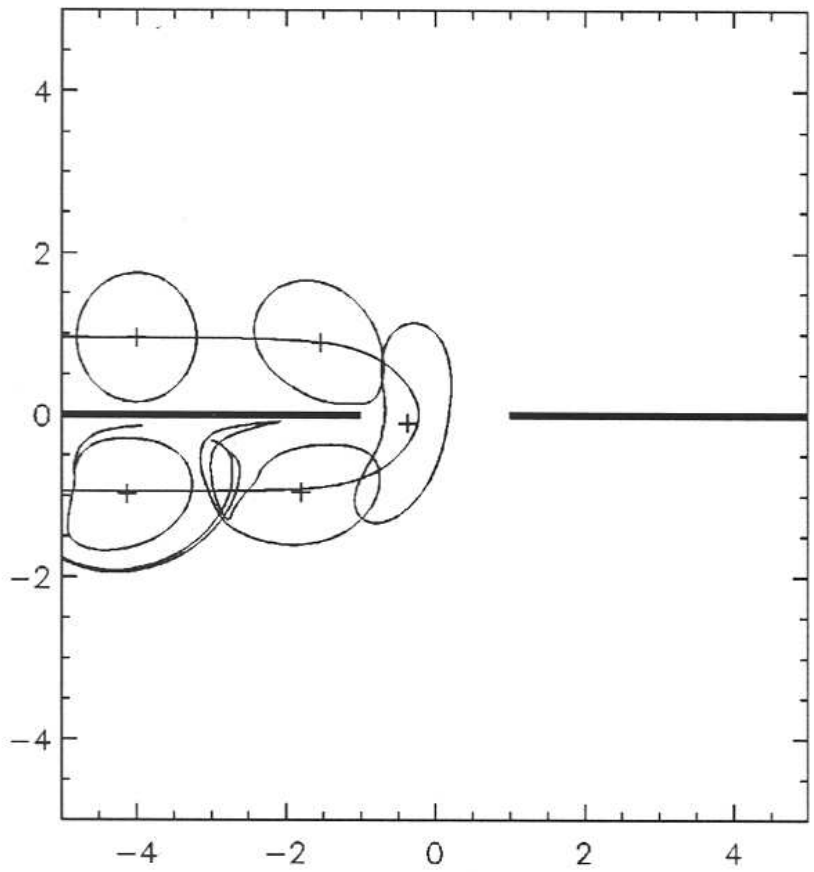
Comparison with “free space” code [Dritschel (1989)]:



Patch motion through a gap in a wall

Compares well with Johnson & MacDonald, *Phys. Fluids*, (2004):

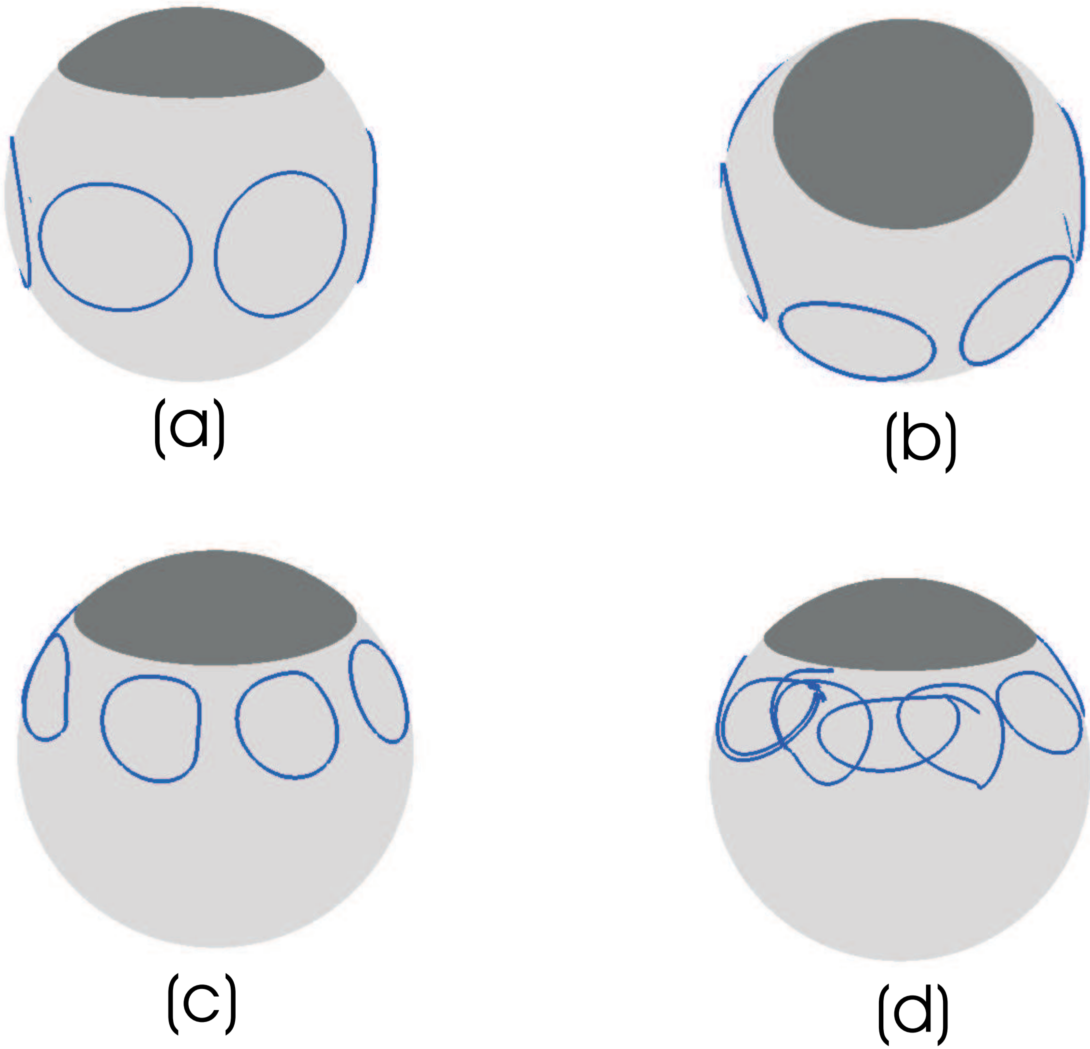




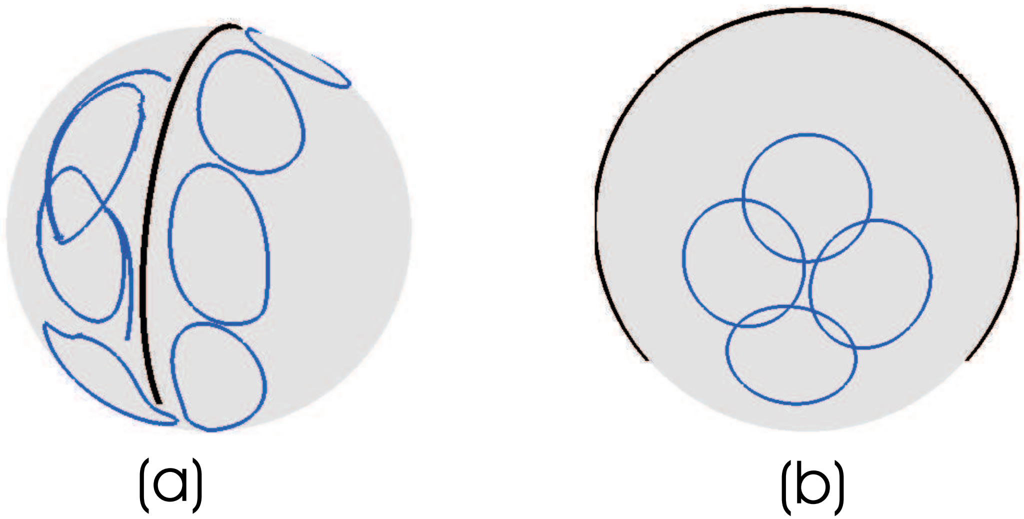
Crowdy/Surana

Johnson/McDonald

Patch motion near a spherical cap



Patch motion near a barrier on a sphere



Simulation of vortex patch penetrating a barrier on a spherical surface

References and resources

For

- Published papers
- A PDF copy of this talk
- A preprint of the paper: "A new calculus for two dimensional vortex dynamics"
- Downloadable MATLAB M-files for computing $\omega(.,.)$ (soon)

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