M4A32: Summer 2005 (Solutions)

1.(a) The complex potential is

$$w(z,t) = -\frac{i\Gamma}{2\pi}\log(z - z_1(t)) - \frac{i\Gamma}{\pi}\log(z - z_2(t)).$$

2 marks

(b) By imposing the conditions that each vortex moves with the local non-self-induced velocity, the dynamical equations are

$$\frac{d\bar{z}_1}{dt} = -\frac{i\Gamma}{\pi}\frac{1}{z_1 - z_2} \qquad [1]$$

and

$$\frac{d\bar{z}_2}{dt} = -\frac{i\Gamma}{2\pi}\frac{1}{z_2 - z_1} = \frac{i\Gamma}{2\pi}\frac{1}{z_1 - z_2}. \qquad [2]$$

3 marks

(c) Let $\mathcal{Z} = z_1 - z_2$ then, from [1] and [2],

$$\frac{d\overline{\mathcal{Z}}}{dt} = -\frac{3i\Gamma}{2\pi\mathcal{Z}}$$

which implies

$$\mathcal{Z}\frac{d\overline{\mathcal{Z}}}{dt} = -\frac{3i\Gamma}{2\pi} \qquad [3]$$

and, by taking complex conjugate,

$$\overline{\mathcal{Z}}\frac{d\mathcal{Z}}{dt} = \frac{3i\Gamma}{2\pi} \qquad [4]$$

Adding [3] and [4] implies

$$\mathcal{Z}\frac{d\overline{\mathcal{Z}}}{dt} + \overline{\mathcal{Z}}\frac{d\mathcal{Z}}{dt} = \frac{d|\mathcal{Z}|^2}{dt} = 0$$

which shows that the separation $|z_1 - z_2|$ of the vortices is a constant (and equal to 1, by the initial conditions).

4 marks

(d) Adding 1/2*[1] to [2]:

$$\frac{d}{dt}\left(\frac{1}{2}\bar{z}_1 + \bar{z}_2\right) = 0$$

which implies

$$z_0 = \frac{1}{2} z_1 + z_2$$

is a constant of the motion.

4 marks

(e) Defining

$$\mathcal{Z}_1 = z_1(t) - z_0 = \frac{z_1}{2} - z_2,$$
$$\mathcal{Z}_2 = z_2(t) - z_0 = -\frac{z_1}{2},$$

then note that

$$\mathcal{Z}_1 - \mathcal{Z}_2 = z_1 - z_2.$$

Also, on use of [1] and [2],

$$\frac{d\overline{\mathcal{Z}}_1}{dt} = -\frac{i\Gamma}{\pi} \frac{1}{\overline{\mathcal{Z}_1 - \mathcal{Z}_2}},$$
$$\frac{d\overline{\mathcal{Z}}_2}{dt} = \frac{i\Gamma}{2\pi} \frac{1}{\overline{\mathcal{Z}_1 - \mathcal{Z}_2}}.$$

Now seek solutions of form $\mathcal{Z}_1(t) = r_1 e^{i\theta(t)}$ and $\mathcal{Z}_2(t) = -r_2 e^{i\theta(t)}$ where r_1 and r_2 are constants. But $r_1 + r_2 = 1$ (using fact that separation is fixed to be unity). Also, pick $z_0 = 0$ then

$$\frac{z_1}{2} + z_2 = \left(\frac{r_1}{2} - r_2\right) e^{i\theta} = 0$$

then

$$r_1 = \frac{2}{3}, \quad r_2 = \frac{1}{3}.$$

2 marks

Substituting into the equations of motion leads to

$$-ir_1 \frac{d\theta}{dt} e^{-i\theta} = -\frac{i\Gamma}{\pi(r_1 + r_2)} e^{-i\theta}$$

which implies that the angular velocity is

$$\frac{d\theta}{dt} = \frac{\Gamma}{\pi r_1 (r_1 + r_2)} = \frac{3\Gamma}{2\pi}.$$

5 marks

2. (a) Use the usual stereographic projection. Then

$$\zeta = \cot(\theta/2)\, e^{i\phi}.$$

Assume, at some instant, the two vortices are projected onto points $\zeta = \pm a$ where a is real so that

$$a = \cot(\theta_0/2).$$

The expression for the streamfunction is

$$\psi = -\frac{\kappa}{2}\log\left(\frac{(\zeta-a)(\bar{\zeta}-a)}{1+\zeta\bar{\zeta}}\right) - \frac{\kappa}{2}\log\left(\frac{(\zeta+a)(\bar{\zeta}+a)}{1+\zeta\bar{\zeta}}\right).$$

The complex velocity of the vortex at $\zeta = a$ is

$$\frac{2\zeta}{\sin\theta_0}\left(-\frac{\kappa}{2}\right)\left[\frac{1}{\zeta+a} - \frac{\bar{\zeta}}{1+\zeta\bar{\zeta}}\right]_{\zeta=a} = -\frac{\kappa a}{\sin\theta_0}\left[\frac{1}{2a} - \frac{a}{1+a^2}\right].$$

Simplifying, this becomes

$$-\frac{\kappa}{2\sin\theta_0}\frac{1-a^2}{1+a^2}.$$

This is purely real so the velocity is purely zonal. If angular velocity is Ω then we must have

$$\begin{aligned}\Omega\sin\theta_0 &= -\frac{\kappa}{2\sin\theta_0}\frac{1-a^2}{1+a^2}\\ &= \frac{\kappa\cos\theta_0}{2\sin\theta_0}\end{aligned}$$

This means

$$\Omega = \frac{\kappa\cos\theta_0}{2\sin^2\theta_0}.$$

6 marks

(b) Now add a point vortex of strength κ_s at south pole (i.e., at $\zeta = 0$). Streamfunction is now

$$\psi = -\frac{\kappa}{2}\log\left(\frac{(\zeta-a)(\bar{\zeta}-a)}{1+\zeta\bar{\zeta}}\right) - \frac{\kappa}{2}\log\left(\frac{(\zeta+a)(\bar{\zeta}+a)}{1+\zeta\bar{\zeta}}\right) - \frac{\kappa_s}{2}\log\left(\frac{\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}}\right).$$

It is clear that, in order for the angular velocity to double, we need

$$-\frac{\kappa}{2}\left(\frac{1}{\zeta+a}-\frac{\bar\zeta}{1+\zeta\bar\zeta}\right)\Big|_{\zeta=a}=-\frac{\kappa_s}{2}\left(\frac{1}{\zeta}-\frac{\bar\zeta}{1+\zeta\bar\zeta}\right)\Big|_{\zeta=a}.$$

On rearrangement,

$$\kappa_s=\kappa\left(\frac{1-a^2}{2}\right)$$

or

$$\kappa_s=\kappa\left(\frac{1-\cot^2(\theta_0/2)}{2}\right)=\frac{\kappa\cos\theta_0}{\cos\theta_0-1} \qquad [1]$$

7 marks

(c) Now add a point vortex of strength κ_n at north pole. Then the streamfunction is now

$$\psi=-\frac{\kappa}{2}\log\left(\frac{(\zeta-a)(\bar\zeta-a)}{1+\zeta\bar\zeta}\right)-\frac{\kappa}{2}\log\left(\frac{(\zeta+a)(\bar\zeta+a)}{1+\zeta\bar\zeta}\right)-\frac{\kappa_s}{2}\log\left(\frac{\zeta\bar\zeta}{1+\zeta\bar\zeta}\right)-\frac{\kappa_n}{2}\log\left(\frac{1}{1+\zeta\bar\zeta}\right)$$

To render entire configuration stationary, it is clear on grounds of symmetry, that it is enough to require that

$$-\frac{\kappa}{2}\left(\frac{1}{\zeta+a}-\frac{\bar\zeta}{1+\zeta\bar\zeta}\right)\Big|_{\zeta=a}-\frac{\kappa_s}{2}\left(\frac{1}{\zeta}-\frac{\bar\zeta}{1+\zeta\bar\zeta}\right)\Big|_{\zeta=a}-\frac{\kappa_n}{2}\left(-\frac{\bar\zeta}{1+\zeta\bar\zeta}\right)\Big|_{\zeta=a}=0.$$

On rearrangement, and use of [1], we get

$$\kappa_n=\kappa\left(\frac{1-a^2}{a^2}\right)=-\frac{2\kappa\cos\theta_0}{1+\cos\theta_0}.$$

7 marks

3.(a) The Green's function $G(\zeta;\alpha,\bar\alpha)$ is the function which is harmonic everywhere in $|\zeta|<1$ except that, near $\zeta=\alpha$,

$$G(\zeta;\alpha,\bar\alpha)=-\frac{1}{2\pi}\log|\zeta-\alpha|+\textit{regular}$$

and is such that $G = 0$ everywhere on the boundary $|\zeta| = 1$.

3 marks

(b) First, it is clear that $G(\zeta;\alpha,\bar{\alpha})$ is the imaginary part of the analytic function of ζ given by

$$-\frac{i}{2\pi}\log R_0(\zeta;\alpha,\bar{\alpha})$$

where

$$R_0(\zeta;\alpha,\bar{\alpha}) \equiv \frac{(\zeta-\alpha)}{|\alpha|(\zeta-\bar{\alpha}^{-1})}$$

so it is harmonic everywhere in $|\zeta| < 1$ except at $\zeta = \alpha$ where, clearly,

$$\begin{aligned} G(\zeta;\alpha,\bar{\alpha}) &= -\frac{1}{2\pi}\log|\zeta-\alpha| + \frac{1}{2\pi}\log|\alpha(\zeta-\bar{\alpha}^{-1})| \\ &= -\frac{1}{2\pi}\log|\zeta-\alpha| + \mathit{regular}. \qquad [1] \end{aligned}$$

Note that $\bar{\alpha}^{-1}$ is outside the unit ζ-disc.

2 marks

Note also that, for ζ on the unit circle where $\bar{\zeta} = \zeta^{-1}$,

$$\overline{R_0(\zeta;\alpha,\bar{\alpha})} = \frac{\zeta^{-1}-\bar{\alpha}}{|\alpha|(\zeta^{-1}-\alpha^{-1})} = \frac{1}{R_0(\zeta;\alpha,\bar{\alpha})}$$

so that $G = 0$ on the unit circle, as required.

3 marks

(c) From [1], it is clear that

$$g(\zeta;\alpha,\bar{\alpha}) = \frac{1}{2\pi}\log|\alpha(\zeta-\bar{\alpha}^{-1})|$$

so that

$$H(\alpha,\bar{\alpha}) = \frac{1}{4\pi}\log|\alpha\bar{\alpha}-1| = \frac{1}{4\pi}\log(1-\alpha\bar{\alpha}). \qquad [2]$$

3 marks

(d) The trajectories are given by contours of H which, in this case, are $|\alpha|^2 - 1 = \text{cst}$ – i.e., circles.

1 mark

(e) First, note that if $\alpha = x + iy$ then

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial\alpha\partial\bar{\alpha}}$$

Then

$$\begin{aligned}\nabla^2 H = 4\frac{\partial^2 H}{\partial\alpha\partial\bar{\alpha}} &= -\frac{1}{\pi}\left(\frac{1}{1-\alpha\bar{\alpha}} - \frac{\alpha\bar{\alpha}}{(1-\alpha\bar{\alpha})^2}\right)\\ &= -\frac{1}{\pi}\frac{1}{(1-\alpha\bar{\alpha})^2}\\ &= -\frac{1}{\pi}e^{-8\pi H}\end{aligned}$$

Therefore

$$\nabla^2 H = -\frac{1}{\pi}e^{-8\pi H}$$

as required.

8 marks

4.(a) The point vortex trajectories in the single-vortex case correspond to the contours of $H^{(\zeta)}$ so they are

$$\alpha - \bar{\alpha} = \text{constant}$$

i.e., they are lines with constant imaginary part or lines parallel to the real axis.

2 marks

(b) The conformal mapping $z = \sqrt{\zeta}$ or $\zeta = z^2$ maps the upper half-plane in the ζ-plane to the first quadrant of the z-plane. Therefore, if

$$z_\alpha = z(\alpha) \quad \text{or} \quad \alpha = z_\alpha^2$$

then

$$H^{(z)}(z_\alpha, \bar{z}_\alpha) = \frac{\Gamma^2}{4\pi}\log\left|(\alpha-\bar{\alpha})z_\zeta(\alpha)\right| = \frac{\Gamma^2}{4\pi}\log\left|\frac{(z_\alpha^2 - \bar{z}_\alpha^2)}{2z_\alpha}\right|$$

6 marks

(c) The point vortex trajectories are the contours of $H^{(z)}$. So, letting $z_\alpha = re^{i\theta}$ they are

$$(z_\alpha^2 - \bar{z}_\alpha^2)(\bar{z}_\alpha^2 - z_\alpha^2) = Ar^2$$

where A is some constant. On rearrangement, this becomes

$$r^2\left(1 - e^{-4i\theta} - e^{4i\theta} + 1\right) = A$$

or

$$2r^2(1 - \cos 4\theta) = A = 4r^2 \sin^2 2\theta$$

so trajectories are

$$r \sin 2\theta = \text{constant}.$$

5 marks

(d) By the method of images, image system consists of two point vortices of circulation $-\Gamma$ at $z = 1 - i$ and $z = -1 + i$ and a point vortex of circulation Γ at $z = -1 - i$. Thus, instantaneous complex potential is

$$w(z) = -\frac{i\Gamma}{2\pi}\log(z - (1+i)) - \frac{i\Gamma}{2\pi}\log(z + (1+i)) + \frac{i\Gamma}{2\pi}\log(z - (1-i)) + \frac{i\Gamma}{2\pi}\log(z - (-1+i)).$$

This simplifies to

$$w(z) = -\frac{i\Gamma}{2\pi}\log\left(\frac{z^2 - 2i}{z^2 + 2i}\right).$$

Complex velocity is

$$u - iv = \frac{dw}{dz} = -\frac{i\Gamma}{2\pi}\left(\frac{2z}{z^2 - 2i} - \frac{2z}{z^2 + 2i}\right)$$

Evaluating at $z = 1$ gives

$$u - iv = \frac{4\Gamma}{5\pi}.$$

7 marks

5.(a) The conformal mapping from unit circle to exterior of patch is

$$z = \frac{1}{\zeta} + b\zeta^2 \qquad [1]$$

where b is real. The complex velocity field associated with a patch of uniform vorticity is known to be

$$u - iv = \begin{cases} -\frac{i\omega_0}{2}\left(\bar{z} - C_i(z)\right) & z \text{ in patch} \\ \frac{i\omega_0}{2}C_o(z) & z \text{ outside patch} \end{cases}$$

2 marks

But, on patch boundary, we must have continuous velocity field so that

$$\bar{z} = C_i(z) - C_0(z), \quad \text{on patch boundary.}$$

2 marks

But patch boundary corresponds to $|\zeta| = 1$ where

$$\bar{z} = \zeta + \frac{b}{\zeta^2}.$$

But, from [1],

$$\frac{1}{\zeta} = z - b\zeta^2$$

so

$$\bar{z} = \zeta + b(z - b\zeta^2)^2 = \zeta + b(z^2 - 2zb\zeta^2 + b^2\zeta^4).$$

Substituting once again for z from [1] in the third term on the right hand side gives

$$\bar{z} = bz^2 - 2b^2(\zeta + b\zeta^4) + \zeta + b^3\zeta^4$$

4 marks

Note that bz^2 is analytic inside the patch so

$$C_i(z) = bz^2.$$

It follows that the velocity field inside the patch is

$$u - iv = -\frac{i\omega_0}{2}\left(\bar{z} - bz^2\right).$$

2 marks

(b) Since $-2b^2(\zeta + b\zeta^4) + \zeta + b^3\zeta^4$ is analytic inside $|\zeta| < 1$ (and tends to zero as $\zeta \to 0$) which corresponds to the region exterior to the patch it follows that

$$C_o(z) = 2b^2(\zeta + b\zeta^4) - \zeta - b^3\zeta^4.$$

4 marks

Since, from [1], $\zeta \sim z^{-1}$ as $z \to \infty$, then

$$u - iv \sim \frac{i\omega_0}{2}(2b^2 - 1)\zeta = \frac{i\omega_0(2b^2 - 1)}{2z}$$

2 marks

(c) Complex potential for a point vortex at $z = 0$ with circulation Γ is

$$w(z) = -\frac{i\Gamma}{2\pi} \log z$$

so corresponding velocity is

$$u - iv = \frac{dw}{dz} = -\frac{i\Gamma}{2\pi z}.$$

This has same form as last part of part (b) provided we identify

$$\Gamma = \pi\omega_0(1 - 2b^2).$$

4 marks