COMPLEX AND REAL VARIABLE HELE-SHAW MOVING BOUNDARY PROBLEM

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The aims:
- to report on the method of reduction of the Hele-Shaw model to an abstract Cauchy-Kovalevsky problem;
- to present results for unbounded and bounded Hele-Shaw cell;
- to give a formulation of the real-variable Hele-Shaw model;
- to describe the way of application of certain asymptotic methods.

Plan:

$\diamond$ Complex-variable Hele-Shaw model

$\diamond$ Mathematical model for Hele-Shaw problem

$\diamond$ Nirenberg-Nishida-Ovsjannikov theorem

$\diamond$ Case 1. Complex-variable Hele-Shaw model with kinetic undercooling regularization

$\diamond$ Case 2. Complex-variable Hele-Shaw flow in a bounded cell.

$\diamond$ Real-variable Hele-Shaw model

Hele-Shaw model. *Hele-Shaw flow* is the flow of a viscous fluid between two closely related parallel plate. The Navier-Stokes equations neglecting the gravity

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = \frac{1}{\rho}\left(-\nabla p + \mu \triangle \mathbf{V}\right), \quad \nabla \cdot \mathbf{V} = 0. \tag{1}$$

Assumptions

$$\frac{\partial \mathbf{V}}{\partial t} = 0, \quad V_3 = 0.$$

Under these the equations (1) become

$$\begin{aligned} \left(V_1 \frac{\partial}{\partial x_1} + V_2 \frac{\partial}{\partial x_2}\right) V_1 &= -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \frac{\mu}{\rho}\triangle V_1, \\ \left(V_1 \frac{\partial}{\partial x_1} + V_2 \frac{\partial}{\partial x_2}\right) V_2 &= -\frac{1}{\rho}\frac{\partial p}{\partial x_2} + \frac{\mu}{\rho}\triangle V_2, \\ 0 &= -\frac{1}{\rho}\frac{\partial p}{\partial x_3}, \end{aligned}$$

with boundary conditions

$$V_1\Big|_{x_3=0,h} = V_2\Big|_{x_3=0,h} = 0.$$

If h is sufficiently small and the flow is slow, then

$$\frac{\partial p}{\partial x_1} = \mu \frac{\partial^2 V_1}{\partial x_3^2}, \quad \frac{\partial p}{\partial x_2} = \mu \frac{\partial^2 V_2}{\partial x_3^2}, \quad 0 = \frac{\partial p}{\partial x_3}.$$

The boundary conditions then imply

$$V_1 = \frac{1}{2} \frac{\partial p}{\partial x_1} \left(\frac{x_3^2}{\mu} - \frac{h x_3}{\mu} \right), \;\; V_2 = \frac{1}{2} \frac{\partial p}{\partial x_2} \left(\frac{x_3^2}{\mu} - \frac{h x_3}{\mu} \right).$$

The integral mean $\widetilde{\mathbf{V}}$ of $\mathbf{V}$ satisfies *Hele-Shaw equation*

$$\widetilde{\mathbf{V}} = -\frac{h^2}{12\mu} \nabla p. \qquad (2)$$

A point sink/source (x_1^0, x_2^0) of constant strength. Then

$$\iint\limits_{U_\varepsilon} \frac{h^2 \rho}{12\mu} \triangle p \, dx_1 dx_2 = const.$$

On the fluid boundary the balance of forces gives

$$p = \text{exterior air pressure} + \text{surface tension}.$$

Let $\Omega(t)$ be the bounded simply connected domain and consider suction/injection through a single sink/source at the origin. $\Gamma(t) = \partial\,\Omega(t)$ $(\Omega(0) =: \Omega_0, \Gamma(0) =: \Gamma_0)$.

$$\triangle p = Q\delta_0(z), \quad z = x + iy \in \Omega(t). \tag{3}$$

The zero surface tension *dynamic* boundary condition is given by

$$p(z,t) = 0, \quad \forall z \in \Gamma(t). \tag{4}$$

$$v_n = \mathbf{V}\Big|_{\Gamma(t)} \cdot \mathbf{n}(t), \tag{5}$$

The *kinematic* boundary condition

$$\frac{\partial\, p}{\partial\, \mathbf{n}} = -v_n. \tag{6}$$

The complex potential $\chi(z,t), \mathrm{Re}\,\chi = p$.

$$\frac{\partial \chi}{\partial z} = \frac{\partial p}{\partial x_1} - i\frac{\partial p}{\partial x_2}.$$

$$\chi(z,t) = \frac{Q}{2\pi}\log z + \chi_0(z,t). \tag{7}$$

The Polubarinova-Galin equation: find a family of conformal mappings $f(z,t) : G_1 = \{z \in \mathbb{C} : |z| < 1\} \to \Omega(t)$, s.t.

$$\mathrm{Re}\left\{\dot{f}(\zeta,t)\overline{\zeta f'(\zeta,t)}\right\} = -\frac{Q}{2\pi}, \;\; \zeta = e^{i\phi}. \tag{8}$$

Mathematical model for the complex-variable Hele-Shaw problem.

$$\begin{aligned} \mathrm{Re}\left(\frac{1}{z}\frac{\partial f}{\partial t}(z,t)\overline{\frac{\partial f}{\partial z}(z,t)}\right) &= -\frac{Q(t)}{2\pi}, \; (z,t) \in \partial G_1 \times [0,T); && (9) \\ f(z,0) &= f_0(z), \;\; z \in G_1; && (10) \\ f(0,t) &= 0, \;\; t \in [0,T). && (11) \end{aligned}$$

Nirenberg-Nishida-Ovsjannikov theorem. *Let us consider the problem*

$$d_t w = F(t,w), \;\; w(0) = 0, \tag{12}$$

in a scale of Banach spaces $\{B_s, \|\cdot\|_s\}_{0<s\leq 1}$*, and let*

- *for each fixed* $s, 0 < s \leq 1$*, the mapping* $F(t,w)$ *of* $[0,T] \times \{w \in B_s : \|w\|_s < R\}$ *to* $B_{s'}$*,* $0 < s' < s$*, is continuous with respect to* t*;*
- *for all* $0 < s' < 1$ *the continuous function* $F(t,0)$ *satisfies*

$$\|F(t,0)\|_{s'} \leq \frac{K}{1-s'};$$

- *for all* $0 < s' < s \leq 1$, $t \in [0,T]$, $w_1, w_2 \in \{\|w\|_s < R\}$

$$\|F(t,w_1) - F(t,w_2)\|_{s'} \leq \frac{C}{s-s'} \|w_1 - w_2\|_s .$$

Then the problem (12) has a unique solution

$$w \in \mathcal{C}^1 \left([0, a_0(1-s)), B_s\right)_{0<s<1}, \; \|w(t)\|_s < R,$$

where a_0 *is a suitable positive constant.*

Case 1. Complex-variable Hele-Shaw model with kinetic undercooling regularization. Problem (P_α):

$$\mathrm{Re}\left(\frac{1}{z}\frac{\partial f}{\partial t}(z,t)\overline{\frac{\partial f}{\partial z}(z,t)}\right) = -\frac{Q(t)}{2\pi} + \mathrm{Re}\,(z w_{reg}(z,t)), \tag{13}$$

$$f(z,0) = f_0(z), \tag{14}$$

$$f(0,t) = 0, \tag{15}$$

$$\mathrm{Im}\,(z w_{reg}(z,t)) = \alpha \partial_\theta \left(\left|\frac{\partial f}{\partial z}\right|^{-1} \Big(Q(t) + \mathrm{Re}\,(z w_{reg}(z,t))\Big)\right), \tag{16}$$

where $\alpha > 0, z = e^{i\theta}$.

For complex potential $\chi = \chi(f(z,t),t)$ we have

$$\mathrm{Re}\left(\frac{1}{z}\frac{\partial f}{\partial t}\overline{\frac{\partial f}{\partial z}}\right) = \mathrm{Re}\left(z\frac{\partial \chi}{\partial z}\right),$$

$$\mathrm{Re}\chi = -\alpha\left|\frac{\partial f}{\partial z}\right|^{-1}\mathrm{Re}\left(\frac{1}{z}\frac{\partial f}{\partial t}\overline{\frac{\partial f}{\partial z}}\right), \alpha > 0.$$

The ansatz for $\chi = Q(t)\log z + \chi_{reg}$, and for $\overline{w} := \partial_z \chi = Q(t)/z + w_{reg}$.

On structure of the problem (P_α). Under the additional assumption

$$\operatorname{Im}\left(\frac{1}{z}\frac{\partial f}{\partial t}(z,t)\left(\frac{\partial f}{\partial z}\right)^{-1}(z,t)\right)(0,t) = 0, \tag{17}$$

one can rewrite (13) by using Schwarz's integral formula as follows:

$$\frac{\partial f}{\partial t}(z,t) - z\frac{\partial f}{\partial z}(z,t)\frac{1}{2\pi i}\int_{|\zeta|=1}\left|\frac{\partial f}{\partial \zeta}\right|^{-2}(Q(t)+\operatorname{Re}(\zeta w_{reg}))\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} = 0.$$

Differentiating and setting $\omega(z,t) := z w_{reg}(z,t)$, $\phi(z,t) := \left(\frac{\partial f}{\partial z}\right)^{-1}(z,t)$ gives

Problem (Q_α):

- abstract Cauchy-Kovalevsky problem

$$\frac{\partial \phi}{\partial t}(z,t) - z\frac{\partial \phi}{\partial z}(z,t)\mathbf{T}_t(\phi,\omega) + \phi(z,t)\frac{\partial}{\partial z}\left(z\mathbf{T}_t(\phi,\omega)\right) = 0, \tag{18}$$

$$\phi_0(z) := \phi(z,0) = \left(\frac{\partial f_0}{\partial z}\right)^{-1}, \mathbf{T}_t(\phi,\omega) := \frac{1}{2\pi i}\int_{|\zeta|=1} |\phi|^2\left(Q(t) + \mathrm{Re}\,\omega\right)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}. \tag{19}$$

- Riemann-Hilbert-Poincaré problem for $\omega = \omega(z,t)$:

$$\mathrm{Im}\left(\omega(z,t)\right) = \alpha\partial_\theta\Big(|\phi(z,t)|\Big(Q(t) + \mathrm{Re}\left(\omega(z,t)\right)\Big)\Big) \textit{ on } \partial G_1 \times [0,T). \tag{20}$$

Mathematical treatment of problem (Q_α). Let us fix constants r_0, r_1, $1 < r_0 < r_1$, a positive constant b, and a parameter $s \in (0,1)$. By $\mathcal{H}(G_{(s)})$ we denote the space of functions which are holomorphic in $G_{(s)}$, where $G_{(s)} := \{z \in \mathbb{C} : |z| < r_0 + s(r_1 - r_0)\}$.

$$\mathbf{B} := \Big\{ g = g(z,t) \in \bigcup_{0<s<1} \mathcal{C}\left([0, b(1-s)), \mathcal{H}(G_{(s)}) \cap \mathcal{C}^{1,\lambda}(\overline{G}_{(s)})\right) :$$

$$\|g\|_{\mathbf{B}} = \max \left\{ \sup_{s\in(0,1), h<b(1-s)} \max_{t\in[0,h]} \|g(\cdot,t)\|_{\mathcal{C}^\lambda(\overline{G}_{(s)})}; \right.$$

$$\left. \sup_{s\in(0,1), t<b(1-s)} \left\| \frac{\partial g}{\partial z}(\cdot,t) \right\|_{\mathcal{C}^\lambda(\overline{G}_{(s)})} \left(1 - \frac{t}{b(1-s)}\right)^{\frac{1}{2}} \right\} < \infty \Big\}.$$

The problem (18) can be rewritten as:

$$\begin{aligned} \mathbf{I}(\phi) = \mathbf{I}(\phi_0) \;\; &+ \;\; \mathbf{A} \circ \mathbf{M}\Big(\phi, \mathbf{M}\Big(z, \mathbf{T}_t(\phi,\omega)\Big)\Big) \\ &- \;\; 2\mathbf{J} \circ \mathbf{M}\Big(\phi, \partial_z \circ \mathbf{M}\Big(z, \mathbf{T}_t(\phi,\omega)\Big)\Big), \end{aligned} \tag{21}$$

$\mathbf{M}(\phi,\omega) := \phi \cdot \omega,\ \mathbf{J}\psi := \int\limits_0^t \psi(\cdot,\tau)d\tau.$

$$\mathbf{B}_a := \Big\{ g = g(z,t) \in \bigcup_{0<s<1} \mathcal{C}\left([0, b(1-s)), \mathcal{H}(A_{(s)}) \cap \mathcal{C}^{1,\lambda}(\overline{A}_{(s)})\right) :$$

$$\|g\|_{\mathbf{B}_a} = \max \left\{ \sup_{s\in(0,1), h<b(1-s)} \max_{t\in[0,h]} \|g(\cdot,t)\|_{\mathcal{C}^\lambda(\overline{A}_{(s)})}; \right.$$

$$\left. \sup_{s\in(0,1), t<b(1-s)} \left\| \frac{\partial g}{\partial z}(\cdot,t) \right\|_{\mathcal{C}^\lambda(\overline{A}_{(s)})} \left(1 - \frac{t}{b(1-s)}\right)^{\frac{1}{2}} \right\} < \infty \Big\},$$

$$A_{(s)} = \left\{ z \in \mathbb{C} : \frac{1}{r_0 + (r_1 - r_0)s} < |z| < r_0 + (r_1 - r_0)s \right\},\ 0 < s < 1.$$

Existence result. For each $\omega \in \mathbf{B}$ there exists a constant $b = b(\omega)$ such that the Cauchy-Kovalevsky problem (18) has a unique solution $\phi \in \mathbf{B}$, holomorphic and non-vanishing function in G_d. We prove that ω connected with ϕ by

$$\mathrm{Im}\,(\omega(z)) = \alpha\partial_\theta\Big(|\phi(z)|\Big(Q + \mathrm{Re}\,(\omega(z))\Big)\Big) \text{ on } \partial G_1, \tag{22}$$

$$\omega(0) = 0, \tag{23}$$

is also holomorphic in G_d. Let $\phi \in \mathcal{H}(G_1) \cap \mathcal{C}^{1,\lambda}(\overline{G}_1)$, $\phi(z) \neq 0$. Then problem (22)–(23) possesses a unique solution $\omega(z) \in \mathcal{H}(G_1) \cap \mathcal{C}^{1,\lambda}(\overline{G}_1)$.
Main Theorem. *There exists an interval of time $[0, b)$ such that the problem (Q_α) has a unique solution (ϕ, ω). $\phi = \phi(z, t)$ has no zeros on $\overline{G}_{r_0} \times [0, b)$ and belongs to the space $\mathcal{C}^1\Big([0, b), \mathcal{H}(G_{r_0}) \cap \mathcal{C}^{1,\lambda}(\overline{G}_{r_0})\Big)$. $\omega = \omega(z, t)$ belongs to the space $\mathcal{C}\Big([0, b), \mathcal{H}(G_{r_0}) \cap \mathcal{C}^{1,\lambda}(\overline{G}_{r_0})\Big)$. The constant r_0 is taken as in the definition of $\mathbf{B}$.*

Case 2. Hele-Shaw flow in a bounded domain. The polymer is injected through the gate AD under a constant pressure p_0 and occupied a domain Ω_t at the time t. The flow reaches the back wall along of arcs $\{(\overset{\smile}{\alpha_k, \beta_k}), k = 1, 2, \ldots, n\}$. The arcs $\{(\overset{\smile}{\gamma_k, \delta_k}), k = 1, 2, \ldots, n+1\}$, where $\gamma_1 \in (\overset{\smile}{A, B})$, $\gamma_k = \beta_{k-1}, k = 2, \ldots, n+1$, $\delta_k = \alpha_k, k = 1, \ldots, n$, $\delta_{n+1} \in (\overset{\smile}{C, D})$ is a free part $\Gamma_{f,t}$ of $\partial\Omega_t$, $\Gamma_{w,t}$ is the part of $\partial\Omega_t$ which reached by the polymer (*i.e.*, $(\overset{\smile}{A, \gamma_1}) \bigcup_{k=1}^{n} (\overset{\smile}{\alpha_k, \beta_k}) \bigcup (\overset{\smile}{\delta_{n+1}, D})$), and by Γ_* the gate $(\overset{\smile}{A, D})$.

The flow satisfying Darcy law is described in terms $\omega = \varphi + i\psi$, $\varphi(z, t) \equiv \varphi(x, y, t)$ is a velocity potential, $\psi(z, t) \equiv \psi(x, y, t)$ is a stream function. It is assumed that the (impermeable) walls $(\overset{\smile}{A, B}), (\overset{\smile}{D, C})$, as well as the arcs $(\overset{\smile}{\alpha_k, \beta_k})$ are stream lines, but the free arcs are equipotential lines. More precisely, $\psi_{|(\overset{\smile}{A,B})} = \nu_1$, $\psi_{|(\overset{\smile}{D,C})} = \nu_2$, $\psi_{|(\overset{\smile}{\alpha_k,\beta_k})} = \nu, k = 1, 2, \ldots, n$.

It leads

$$\mathrm{Re}\left[i\frac{\partial\omega}{\partial t}\right]=0,\ \text{ on the wall.} \tag{24}$$

On the gate $(\overset{\smile}{A,D})$ we have a constant pressure $p=p_0$, wlog:

$$\varphi=0,\ \text{ on the free boundary.} \tag{25}$$

Besides,

$$\frac{\partial\varphi}{\partial n}=\frac{\partial n}{\partial t},\ \text{ on the free boundary.} \tag{26}$$

Combining (25) and (26) one can get

$$\frac{\partial\varphi}{\partial t}=-\left(\frac{\partial n}{\partial t}\right)^2,\ \text{ on the free boundary,}$$

or in terms of the complex potential:

$$\mathrm{Re}\omega=0,\ \mathrm{Re}\left[\frac{\partial\omega}{\partial t}\right]=-\left|\frac{\partial\omega}{\partial z}\right|^2,\ \text{ on the free boundary.} \tag{27}$$

Hele-Shaw flow in a bounded cell. Mathematical model. Let $f(\zeta,t)$ be a conformal mapping of the unit disc G_1 onto domain Ω_t, $W(\zeta,t)$ be analytic in the unit disc G_1, s.t.:

$$\mathrm{Re}\left\{\overline{\Lambda(\tau)}W(\tau,t)\right\} = c(\tau) = \begin{cases} 0, & \tau \in \tilde{\Gamma}_{f,t}, \\ \nu, & \tau \in \bigcup\limits_{k=1}^{n} (\tilde{\alpha}_k, \tilde{\beta}_k), \\ \nu_1, & \tau \in (\tilde{A}, \tilde{\gamma}_1), \\ \nu_2, & \tau \in (\tilde{\delta}_{n+1}, \tilde{D}), \\ p_0, & \tau \in (\tilde{A}, \tilde{D}), \end{cases} \tag{28}$$

$$\Lambda(\tau) = \begin{cases} 1, & \tau \in \tilde{\Gamma}_{f,t}, \\ -i, & \tau \in \tilde{\Gamma}_{w,t}, \\ 1, & \tau \in \tilde{\Gamma}_{*}. \end{cases} \tag{29}$$

$$\omega(z,t) = W\left[f^{-1}(z,t), t\right]. \tag{30}$$

Then the problem is equivalent to the Riemann-Hilbert problem

$$\text{Re}\left\{\overline{\Lambda(\tau)}\Phi(\tau,t)\right\} = d(\tau,t) = \begin{cases} 0, & \tau \in \tilde{\Gamma}_{w,t} \bigcup \tilde{\Gamma}_{*}, \\ \left|\frac{W'(\tau,t)}{f'(\tau,t)}\right|^2, & \tau \in \tilde{\Gamma}_{f,t}, \end{cases} \tag{31}$$

$$\Phi(\zeta,t) := W'(\zeta,t)\frac{\dot{f}(\zeta,t)}{f'(\zeta,t)}. \tag{32}$$

$$\text{Re}\left\{\overline{\Lambda(\tau)}W(\tau)\right\} = c(\tau), \;\; \tau \in \mathbb{T} \setminus \{\tau_1^t, \dots, \tau_{2n+4}^t\}, \tag{33}$$

$$\text{Re}\left\{\overline{\Lambda(\tau)}W'(\tau)\frac{\dot{f}(\tau,t)}{f'(\tau,t)}\right\} = d(\tau,t), \;\; \tau \in \partial G_1 \setminus \{\tau_1^t, \dots, \tau_{2n+4}^t\}, \tag{34}$$

$$f(\zeta,0) = f_0(\zeta), \; \zeta \in \mathbb{U}. \tag{35}$$

The problem (33) has the form

$$\text{Re}\left\{\overline{\Lambda(\tau)}g(\tau)\right\} = h(\tau), \;\; \tau \in \partial G_1 \setminus \{\tau_1^t, \dots, \tau_{2n+4}^t\}, \tag{36}$$

$$\varkappa = n + 2;$$

$$W(\zeta) = W_p(\zeta) + X(\zeta)P_\varkappa(\zeta),$$

$$X(\zeta) = C\left(\prod_{j=1}^{2n+4}(\zeta - \tau_j^t)\right)^{-1/2},$$

$$P_\varkappa(\zeta) = c_0\zeta^\varkappa + c_1\zeta^{\varkappa-1} + \dots + c_\varkappa, \;\; c_k = \overline{c_{\varkappa-k}}, \; k = 0, 1, \dots, \varkappa,$$

$$W_p(\zeta) = \frac{X(\zeta)}{2\pi i}\int_{\mathbb{T}} \frac{\tau^{\varkappa+1} + \zeta^{\varkappa+1}}{X^+(\tau)\tau^{\varkappa+1}}\frac{h(\tau)d\tau}{\tau - \zeta}.$$

By using the solution operator $\mathbf{T}_{\Lambda}^{t}$ to (36) we rewrite the boundary condition (34) in the following equivalent form

$$\dot{f}\frac{W'}{f'} = \mathbf{T}_{\Lambda}^{t}(d), \;\; \zeta \in \overline{G_1} \setminus \{\tau_1^t, \ldots, \tau_{2n+4}^t\}. \tag{37}$$

By using new unknown function

$$\phi(\zeta, t) := \frac{W'(\zeta, t)}{f'(\zeta, t)}, \;\; \zeta \in \overline{G_1} \setminus \{\tau_1, \ldots, \tau_{2n+4}\}, \tag{38}$$

we rewrite the relations (34), (35) in the form of an abstract Cauchy-Kovalevsky problem

$$\begin{cases} \frac{\partial \phi}{\partial t}(\zeta, t) = \frac{1}{W'(\zeta,t)}\frac{\partial \phi}{\partial \zeta}(\zeta, t)\mathbf{T}_{\Lambda}^{t}(d)(\zeta, t) \\ \qquad\qquad -\frac{1}{W'(\zeta,t)}(\zeta)\phi(\zeta, t)\frac{\partial}{\partial \zeta}\mathbf{T}_{\Lambda}^{t}(d)(\zeta, t), \;\; (\zeta, t) \in \partial G_1 \times [0, T], \\ \phi(\zeta, 0) = \frac{W'(\zeta,0)}{(f_0)'(\zeta)}, \qquad\qquad \zeta \in G_1. \end{cases} \tag{39}$$

Let us fix numbers $\alpha_k \in \mathbb{C}$ and define the weight functions

$$\rho^{(j)}(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k + j}, \quad j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{40}$$

$$\varphi \in \mathbb{A}^r_{\mu;n}(\Gamma; t_1, \ldots, t_m) \Leftrightarrow \begin{cases} \varphi\Big|_{\overset{\smile}{\tau_k, \tau_{k+1}}} \in \mathbb{B}^r_\mu(t_k, t_{k+1}), & k = 1, 3, \ldots, \\ \\ \varphi\Big|_{\overset{\smile}{\tau_k, \tau_{k+1}}} \in \mathbb{KZ}^0_{n+\mu}(t_k, t_{k+1}), & k = 2, 4, \ldots. \end{cases} \tag{41}$$

The conditions on the arcs with even indexes ($k = 2, 4, \ldots$):

$$\begin{aligned} \varphi \in \mathbb{KZ}^0_{n+\mu}(t_k, t_{k+1}) \ &\Leftrightarrow \ \varphi_j := \rho^{(j)} \partial^j \varphi \in \mathbb{Z}^0_\mu(t_k, t_{k+1}) \ \ (j = 0, 1, \ldots, n) \\ &\Leftrightarrow \ \varphi_j \in \mathbb{Z}_\mu(t_k, t_{k+1});\ \varphi_j(t_k) = \varphi_j(t_{k+1}) = 0. \end{aligned} \tag{42}$$

For each α_k the expression $\rho^{(j)}\partial^j\varphi$ is defined by the equality

$$\rho^{(j)}\partial^j\varphi := \rho^{(j)}(\omega(\tau))\frac{\partial^j\varphi(\omega(\tau))}{\partial\tau^j}. \tag{43}$$

The spaces $\mathbb{Z}_\mu(t_k, t_{k+1})$, $\mathbb{Z}^0_\mu(t_k, t_{k+1})$:

$$\left\|\varphi\left|\mathbb{Z}^0_\mu(t_k, t_{k+1})\right\| = \left\|\rho\varphi\left|\mathbb{Z}_\mu(t_k, t_{k+1})\right\|, \tag{44}$$

$$\begin{aligned}\left\|\psi\left|\mathbb{Z}_\mu(t_k, t_{k+1})\right\| := \sup_{\tau\in(\tau_k,\tau_{k+1})} |\psi(\omega(\tau))| + \\ + \sup_{\tau\in(\tau_k,\tau_{k+1});h>0} h^{-\mu}\left|\psi(\omega(\tau+h)) - 2\psi(\omega(\tau)) + \psi(\omega(\tau-h))\right|.\end{aligned} \tag{45}$$

The norm in the space $\mathbb{KZ}^0_{n+\mu}(t_k, t_{k+1})$ is defined by the standard way:

$$\left\|\varphi \left| \mathbb{KZ}^0_{n+\mu}(t_k, t_{k+1})\right.\right\| := \sum_{j=0}^{n-1} \sup_{\tau \in (\tau_k, \tau_{k+1})} |\varphi_j(\omega(\tau))| + \left\|\varphi_n \left| \mathbb{Z}^0_\mu(t_k, t_{k+1})\right.\right\|. \tag{46}$$

$$\partial^j \varphi = o\left(\rho^{(j)}\right), \; t \to t_k \text{ or } t \to t_{k+1}. \tag{47}$$

It will be said that $\varphi \in \mathbb{B}^r_\mu(t_k, t_{k+1})$, $k = 1, 3, \ldots$, if $\exists r > 0$ such that

$$\varphi \in \mathcal{C}^\infty(t_k, t_{k+1}) : \sup_{j \in \mathbb{N}_0} \left\|\varphi_j \left| \mathbb{Z}_\mu(t_k, t_{k+1})\right.\right\| \cdot \frac{r^j}{\Gamma_1(j)} < +\infty. \tag{48}$$

The last supremum determines the norm in the space $\mathbb{B}^r_\mu(t_k, t_{k+1})$:

$$\left\|\varphi_j \left| \mathbb{B}^r_\mu(t_k, t_{k+1})\right.\right\| := \sup_{j \in \mathbb{N}_0} \left\|\rho^{(j)} \partial^{(j)} \varphi \left| \mathbb{Z}_\mu(t_k, t_{k+1})\right.\right\| \cdot \frac{r^j}{\Gamma_1(j)}. \tag{49}$$

Scale of Banach spaces. We choose $\mathbb{A}^r_{\mu;n}(cl\,D; t_1, \ldots, t_m)$ as a subspace of n-times piece-wise continuously differentiable functions.

$$\left\|\varphi\middle|\mathbb{A}^r_{\mu;n}(\Gamma; t_1, \ldots, t_m)\right\| := \max\left\{\max_{k=1,3,\ldots}\left\|\varphi\Big|_{\overset{\smile}{t_k, t_{k+1}}}\middle|\mathbb{B}^r_\mu(t_k, t_{k+1})\right\|, \max_{k=2,4,\ldots}\left\|\varphi\Big|_{\overset{\smile}{t_k, t_{k+1}}}\middle|\mathbb{KZ}^0_{n+\mu}(t_k, t_{k+1})\right\|\right\}. \tag{50}$$

$$\mathbb{A}^r_{\mu;n}(cl\,D; t_1, \ldots, t_m) := \mathcal{H}(D) \cap \mathbb{A}^r_{\mu;n}(\Gamma; t_1, \ldots, t_m). \tag{51}$$

$$\mathbb{A}_s\left(\Gamma_{(s)}\right) := \mathbb{A}^r_{\mu;n}\left(\Gamma_{(s)}; t_1(s), \ldots, t_m(s)\right). \tag{52}$$

Introduce also the space of functions defined in the domains:

$$\mathbb{A}_s\left(\mathrm{cl}\, D_{(s)}\right) := \mathcal{H}\left(D_{(s)}\right) \cap \mathbb{A}_s\left(\Gamma_{(s)}\right). \tag{53}$$

Let us fix a positive number r and numbers $1 < r_0 < r_1$.

$$D_{\{s\}} := D_{(r_0+s(r_1-r_0))}, \Gamma_{\{s\}} := \partial\, D_{\{s\}}. \tag{54}$$

$t_1^{(s)}, t_2^{(s)}, \ldots, t_m^{(s)}$ - knots. $\mathbb{A}_{\{s\}}(\mathrm{cl}\, D_{\{s\}}) := \mathbb{A}_{r_0+s(r_1-r_0)}(\mathrm{cl}\, D_{(r_0+s(r_1-r_0))})$, $\mathbb{A}_{\{s\}}(\Gamma_{\{s\}}) := \mathbb{A}_{r_0+s(r_1-r_0)}(\Gamma_{(r_0+s(r_1-r_0))})$. The scale of Banach spaces

$$\left\|\varphi\middle|\mathbb{B}_{s,p}\left(D_{\{s\}};\Gamma_1;\rho\right)\right\| := \max\left\{\sup_{z\in D_{\{s\}}}\left|\varphi(z)\right|, \max_{k=1,3,\ldots}\left\|\varphi\Big|_{\overset{\smile}{t_k^{(s)},t_{k+1}^{(s)}}}\middle|\mathbb{B}_\mu^{pr}(t_k^{(s)},t_{k+1}^{(s)})\right\|,\right.$$

$$\left.\max_{k=2,4,\ldots}\left\|\varphi\Big|_{\overset{\smile}{t_k^{(s)},t_{k+1}^{(s)}}}\middle|\mathbb{KZ}_{n+\mu}^0(t_k^{(s)},t_{k+1}^{(s)})\right\|\right\} < +\infty. \tag{55}$$

$$\mathbf{B}_{pw,n} := \Bigg\{ g(\zeta,t) \in \bigcup_{0<s<1, 0<p<1} \mathcal{C}\left([0,\delta(1-s)), \mathbb{B}_{s,p}\left(D_{\{s\}};\Gamma_1;\rho\right)\right) : \quad (56)$$

$$\|g\|_{\mathbf{B}_{pw,n}} := \max \Bigg\{ \sup_{s\in(0,1), d<\delta(1-s)} \max_{t\in[0,d]} \sup_{z\in\mathbb{D}_{\{s\}}} \|g(z,t)\|,$$

$$\sup_{s\in(0,1), p\in(0,1), d<\delta(1-s)} \max_{k=1,3,\ldots,2n+3} \max_{t\in[0,d]} \left\| g(\cdot,t)\Big|_{\overset{\smile}{\tau_k^{(s)},\tau_{k+1}^{(s)}}} \Big| \mathbb{B}_{\mu}^{pr}(\tau_k^{(s)},\tau_{k+1}^{(s)}) \right\|,$$

$$\sup_{s\in(0,1), d<\delta(1-s)} \max_{k=2,4,\ldots,2n+2} \max_{t\in[0,d]} \left\| g(\cdot,t)\Big|_{\overset{\smile}{\tau_k^{(s)},\tau_{k+1}^{(s)}}} \Big| \mathbb{KZ}_{1+\mu}^{0}(\tau_k^{(s)},\tau_{k+1}^{(s)}) \right\|,$$

$$\sup_{s\in(0,1), p\in(0,1), d<\delta(1-s)} \max_{t\in[0,d]} \left\| \rho^{(1)}(\cdot)\frac{\partial g}{\partial \zeta}(\cdot,t) \right\|_{\mathbb{Z}_{\{\mu\}}\left(\overline{D}_{\{s\}}\right)} \left(1 - \frac{t}{\delta(1-s)}\right)^{1/2} \Bigg\} < \infty \Bigg\}$$

Main Theorem. *Let* $\nu, \nu_1, \nu_2, p_0 > 0$ *be positive constants. Let* $f_0 = f_0(\zeta)$ *maps* G_1 *onto* Ω_0*, and can be continued for the function analytic and univalent in a neighborhood* $G_1 \setminus \left\{\tau_1^0, \ldots, \tau_{2n+4}^0\right\} \equiv G_1 \bigcup_{k=1}^{n+1} U(\tau_{2k}^0, \smile \tau_{2k+1}^0)$,

$$U(\tau_{2k}^0, \smile \tau_{2k+1}^0) \bigcap \mathrm{cl}\, [\tau_{2k}^0, \smile \tau_{2k+1}^0] = (\tau_{2k}^0, \smile \tau_{2k+1}^0).$$

$$t_k^0 \equiv f_0(\tau_k^0),\, k = 1, \ldots, 2n+4,$$

be the images of τ_k^0 *under the mapping* f_0.

Then for each $t \in [0, \delta)$ *with* δ *sufficiently small there exists a unique solution* $\omega(z,t) \equiv W\left[f^{-1}(z,t), t\right]$ *to (33), (34), (35) in the space*

$$\mathcal{C}^1\left([0,\delta), \mathcal{H}\Big(D_{\{0\}}\Big) \bigcap \Big(\bigcup_{k=1}^{n+1} \mathbb{KZ}^0_{2+\mu}(\tau_{2k}^{(0)}, \tau_{2k+1}^{(0)})\Big) \bigcap \Big(\bigcup_{k=1}^{n+2} \mathbb{B}_\mu^{pr}(\tau_{2k-1}^{(0)}, \tau_{2k}^{(0)})\Big)\right).$$

Real-variable Hele-Shaw model.
Viscous incompressible fluid occupies a doubly connected domain $D_1(t)$ at a time instant $t \geq 0$. Internal domain F is a fixed small obstacle (hole). The simply connected domain without hole will be denoted $D(t)$, $\delta := diam\, F > 0$, $D(0)$ is open bounded set with a smooth boundary

$$dist\ \{\partial\, F, \partial\, D(0)\} = 2d > 0. \tag{57}$$

$$\partial\, F \in \mathcal{C}^{2,\alpha}, \quad \partial\, D(0) \in \mathcal{C}^{1,\alpha},\ 0 < \alpha < 1. \tag{58}$$

2D potential flow of incompressible fluid in the Hele-Shaw cell:

$$\frac{\partial\, \mathbf{V}}{\partial\, t} = 0,\ V_3 = 0, \tag{59}$$

$$\mathbf{V} = -\frac{h^2}{12\mu}\nabla\, p. \tag{60}$$

$$\triangle\, p = 0. \tag{61}$$

Unique sourse/sink at a fixed point $z_0 = (x_0, y_0)$

$$z_0 \in D_1(0) = D(0) \setminus cl\, F, \quad O \in int\, F.$$

$$p(z,t) \sim -\frac{Q(t)}{2\pi} \log |z - z_0|, \quad |z| \to z_0. \tag{62}$$

$$\frac{\partial p}{\partial n} = 0,\ z \in \partial F. \tag{63}$$

$$p(z) = 0, \quad z \in \Gamma(t), \tag{64}$$

$$\frac{d\Gamma}{dt} = \mathbf{V}, \quad z \in \Gamma(t). \tag{65}$$

$$\frac{d\Gamma}{dt} = -\frac{h^2}{12\mu} \nabla p. \tag{66}$$

Let us now introduce new unknown function, a one-parametric family of $\mathcal{C}^2$-diffeomorphisms

$$w(s,t) = (u(s,t), v(s,t)) : \partial G_1 \times I \to \Gamma(t), \quad G_1 = \{s = (s_1, s_2) \in \mathbb{R}^2 : |s| < 1\}. \tag{67}$$

The function $w(s,t)$ in (67) determines an unknown parametrization of the free boundary $\Gamma(t)$, namely,

(i) $w(s,t) \in \Gamma(t)$ for all $(s,t) \in \partial G_1 \times I$,

(ii) $w(\cdot,t) : \partial G_1 \to \Gamma(t)$ is a $\mathcal{C}^2$-diffeomorphism for each fixed $t \in I$,

(iii) $w(\cdot,\cdot) \in \mathcal{C}^2\left(\partial G_1 \times I; \mathbb{R}^2\right)$.

$$p = Q \cdot \mathcal{G}_{D_1(t)}, \tag{68}$$

and $\mathcal{G}_{D_1(t)}$ is the solution to

$$\triangle \mathcal{G}_{D_1(t)}(z, z_0) + \delta_0(z - z_0) = 0, \quad z \in D_1(t), \tag{69}$$

$$\mathcal{G}_{D_1(t)}(z, z_0) = 0, \quad z \in \Gamma(t), \tag{70}$$

$$\frac{\partial \mathcal{G}_{D_1(t)}}{\partial n}(z, z_0) = 0, \quad z \in \partial F. \tag{71}$$

Problem $(\mathbf{HS_0})$. *Find a pair* $\{w(s,t); \mathcal{G}(z,z_0;t)\}$, *such that* $w(s,t) : \partial G_1 \times I \to \mathbb{R}^2$ *is a* $\mathcal{C}^2$*-diffeomorphism satisfying*

(i) $w(s,t) \in \Gamma(t)$ *for all* $(s,t) \in \partial G_1 \times I$*;*

(ii) $w(\cdot,t) : \partial G_1 \to \Gamma(t)$ *is a* $\mathcal{C}^2$*-diffeomorphism for each fixed* $t \in I$*;*

(iii) $w^{(0)}(s) = w(s,0)$ *is a given* $\mathcal{C}^2$*-diffeomorphism of the unit circle* ∂G_1*, which describes the boundary* $\Gamma(0)$ *of initial domain* $D_1(0)$*;*

(iv) $\mathcal{G}(z,z_0;t)$ *is Green's function of the operator* $-\triangle$ *in the doubly connected domain* $D_1(t)$ *with the homogeneous Neumann data on the fixed boundary* ∂F *and the homogeneous Dirichlet data on the free boundary* $\Gamma(t)$*, i.e. satisfies conditions (69)–(71) for each fixed* $t \in I$*;*

(v) $\partial_t w(s,t) = -\frac{Qh^2}{12\mu} \cdot \nabla \mathcal{G}(w(s,t), z_0; t)$ *for all* $(s,t) \in \partial G_1 \times I$*.*

We consider four essentially different situations:

(a) the source z_0 and ALL points $z = w(s,t)$ of the boundary $\partial\Gamma(0)$ of the initial domain $D_1(0)$ are distant from the boundary $\partial\omega_\varepsilon$ of the obstacle;

(b) the source z_0 is close to the boundary $\partial\omega_\varepsilon$ of the obstacle, but ALL points $z = w(s,t)$ of $\partial\Gamma(0)$ are distant from $\partial\omega_\varepsilon$;

(c) the source z_0 and SOME points $z = w(s,t)$ of $\partial\Gamma(0)$ are close to the boundary $\partial\omega_\varepsilon$ of the obstacle; we consider those points $z = w(s,t)$ which are distant from $\partial\omega_\varepsilon$;

(d) the source z_0 and SOME points $z = w(s,t)$ of $\partial\Gamma(0)$ are close to the boundary $\partial\omega_\varepsilon$ of the obstacle; we consider those points $z = w(s,t)$ which are close to $\partial\omega_\varepsilon$.

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