

# Point vortex motion on the surface of a sphere with impenetrable boundaries

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A conformal mapping approach to the problem of the motion of a single point vortex in a simply connected region bounded by impenetrable walls on the surface of a sphere is presented. Several illustrative examples are given, including those considered by previous authors using arguments based on the method of images. A new example of the motion of a vortex around a straight barrier along a great circle on the spherical surface is studied in detail. Finally, a theoretical connection with a boundary value problem for a generalized Liouville-type quasilinear partial differential equation is also made. © 2006 American Institute of Physics. [DOI: 10.1063/1.2183627]

## I. INTRODUCTION

The study of point vortex motion is an important paradigm in theoretical vortex dynamics. By concentrating vorticity in a distribution of “points” with no spatial extent and analyzing their interaction, important deductions on the dynamical behavior of the system can be made with, arguably, a minimum of mathematical complexity. The recent monograph by Newton<sup>1</sup> provides a valuable overview of the  $N$ -vortex problem with emphasis on the case of the point vortex model.

In comparison with the extensive literature on point vortex motion in unbounded domains (see Ref. 2 for a comprehensive review), the study of point vortex motion in the presence of walls is modest. Newton<sup>1</sup> and Saffman<sup>3</sup> include discussions of vortex motion in the presence of walls. Classic approaches include the celebrated “method of images”—a rather special technique limited to cases where the domain of interest has certain geometrical symmetries so that an appropriate distribution of image vorticity can be ascertained, essentially by inspection. This image vorticity is placed in nonphysical regions of the plane in order to satisfy the boundary conditions that the walls act as impenetrable barriers for the flow. The most important mathematical tool, in the case of planar flow regions, is the Hamiltonian approach associated with the names of Kirchhoff and Routh (see, e.g., Lamb<sup>4</sup> for a summary of this work). They established the result that the problem of  $N$ -vortex motion in any bounded simply connected domain is a Hamiltonian dynamical system. Moreover, it turns out that the Hamiltonian has simple transformation properties when a given flow domain of interest is mapped conformally to another. Much later, Lin<sup>5,6</sup> established that the same Hamiltonian structure survives in the case of a multiply connected planar region, an important observation that has recently been brought to implementational fruition by Crowdy and Marshall.<sup>7,8</sup> The latter authors have produced explicit formulas, up to

conformal mapping, for the Hamiltonians in multiply connected fluid regions of arbitrary finite connectivity. In combination, all these results provide a versatile array of mathematical techniques for studying the motion of  $N$  point vortices in bounded regions of the plane.

What about the motion of point vortices, in bounded domains with impenetrable barriers, on the surface of a sphere? Such paradigms are clearly relevant to planetary-scale oceanographic flows in which oceanic eddies interact with topography such as ridges and land masses or evolve in closed basins. With such motivation in mind, Kidambi and Newton<sup>9</sup> have recently considered precisely this general problem (this article also contains a detailed list of references to the literature concerning point vortex motion on a sphere without boundaries). The method in Ref. 9 relies on the use of the classical method of images and, as in the planar case, requires the flow domain to have certain symmetry properties. As just mentioned in the planar case, techniques of conformal mapping can be profitably employed in cases where the method of images fails. Indeed, at the end of their article, Kidambi and Newton<sup>9</sup> point out the desirability of extending the conformal mapping approach to the case of a spherical shell. This article presents such a generalization. For simplicity, we restrict attention to the case of single-vortex motion in a simply connected region on the spherical shell.

Our modified formulation means that the simply connected domains under consideration are no longer restricted to have special symmetries. By way of illustration, we retrieve (and, in fact, slightly generalize) all the examples involving simply connected fluid regions tackled in Ref. 9 using the method of images approach. In addition, we present a new example consisting of the generalization to the surface of a sphere of the recent investigation by Johnson and McDonald<sup>10</sup> of the motion of a vortex near a gap in a straight wall. On both the plane and a spherical shell, the solution of this problem is not amenable to the standard method of images.

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## II. POINT VORTEX MOTION ON A SPHERE

Consider the motion of a single point vortex in some bounded simply connected region  $D_\Sigma$  on the surface of the sphere  $\Sigma$ . Without loss of generality, let the sphere have unit radius. Let  $(\theta, \phi)$  denote the usual polar and meridional angles in spherical polar coordinates.

It is assumed that the fluid motion in  $D_\Sigma$  is irrotational except for the vorticity associated with the single point vortex of strength  $\kappa$ . At first sight, this assumption may seem unusual in respect of point vortex motion on the surface of a sphere since it is normally assumed that any point vortex on a sphere is embedded in a sea of uniform vorticity covering the whole spherical surface. But this is because it is also normally assumed that the vortical motion is taking place on the entire spherical surface; the sea of uniform vorticity is added to ensure that the Gauss constraint (which says that the global integral of the vorticity over the spherical surface is zero) is always satisfied whenever a new delta-function distribution of vorticity is added (or subtracted) at any point on the surface. If the motion is confined to a bounded sub-region of the spherical surface of a sphere, however, it is no longer necessary to embed a point vortex in uniform vorticity (unless this happens to be relevant to the flow problem being considered).

The flow is incompressible so there exists a stream function  $\psi(\theta, \phi)$ . The instantaneous boundary value problem satisfied by  $\psi(\theta, \phi)$  is that it satisfies the partial differential equation

$$\nabla_\Sigma^2 \psi = 0 \tag{1}$$

everywhere in  $D_\Sigma$  except for a  $\delta$ -function singularity at some point corresponding to the point vortex.  $\nabla_\Sigma^2$  denotes the Laplace-Beltrami operator on the sphere given explicitly, in terms of  $\theta$  and  $\phi$ , as

$$\nabla_\Sigma^2 \equiv \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}. \tag{2}$$

It also satisfies the boundary condition that it is constant (without loss of generality, in the simply connected case, taken equal to zero) on the boundary of  $D_\Sigma$ , that is,

$$\psi = 0 \quad \text{on} \quad \partial D_\Sigma, \tag{3}$$

where  $\partial D_\Sigma$  denotes the boundary of  $D_\Sigma$ . This ensures  $\partial D_\Sigma$  is a streamline. Now introduce the stereographic projection of  $D_\Sigma$  onto a region  $D$  in a complex  $z$  plane of projection so that

$$z = \cot(\theta/2)e^{i\phi}. \tag{4}$$

A schematic of this stereographic projection is given in Fig. 1 (see Crowdy and Cloke<sup>11</sup> for more details). Note that as  $D_\Sigma$  is always a bounded region,  $D$  can be unbounded if  $D_\Sigma$  includes the north pole. Let  $z_\alpha$  be the projection of the point vortex. The stream function  $\psi(\theta, \phi)$  can now be rewritten as a function of the new independent variables  $(z, \bar{z})$ . Let this function be  $\psi^{(z)}(z, \bar{z}; z_\alpha, \bar{z}_\alpha)$ . It can be shown<sup>11</sup> that, in terms of these variables, (1) assumes the form

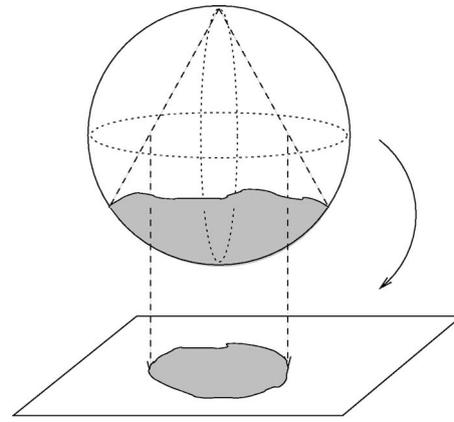


FIG. 1. Schematic of the stereographic projection of a region  $D_\Sigma$  on a spherical surface to a region  $D$  in a complex  $z$  plane through the equator.

$$(1 + z\bar{z})^2 \frac{\partial^2 \psi^{(z)}}{\partial z \partial \bar{z}} = 0. \tag{5}$$

The key advantage of assuming that the point vortex is embedded in an otherwise irrotational flow now comes into play. It means that, when considered in the  $z$  plane of projection, the boundary value problem satisfied by the stream function is conformally invariant under conformal transformations of the region  $D$  of the  $z$  plane. That is, if the stream function can be found in some region of the  $z$  plane that is conformally equivalent to  $D$ , then the solution in  $D$  has effectively also been found. This conformal invariance property of the boundary value problem (1) and (3) is easily verified once (5) is used.

To exploit this, the conformal mapping to  $D$  from some simple region in a parametric  $\zeta$  plane is introduced. Since the details are different, two separate cases will be considered: conformal mappings to the upper-half  $\zeta$  plane and to the unit  $\zeta$  disk.

### A. The upper-half plane

First, introduce a one-to-one conformal mapping  $\zeta = \zeta(z)$  from the simply connected region  $D$  to the upper-half plane in a parametric  $\zeta$  plane. Let  $\zeta = \alpha$  correspond to the point vortex of strength  $\kappa$  at  $z = z_\alpha$  so that  $\alpha = \zeta(z_\alpha)$ . It is easy to write the required stream function in terms of  $\zeta(z)$ , indeed,

$$\psi^{(z)}(z, \bar{z}; z_\alpha, \bar{z}_\alpha) = -\kappa \log \left| \frac{\zeta - \alpha}{\zeta - \bar{\alpha}} \right| = -\kappa \log \left| \frac{\zeta(z) - \zeta(z_\alpha)}{\zeta(z) - \zeta(z_\alpha)} \right|. \tag{6}$$

This function has a logarithmic singularity at  $\zeta = \alpha$  corresponding to the point vortex at  $z = z_\alpha$ . It is also equal to zero on the real  $\zeta$  axis, as is easily checked. Locally, near  $z = z_\alpha$ ,

$$\begin{aligned} \psi^{(z)}(z, \bar{z}; z_\alpha, \bar{z}_\alpha) &= -\kappa \log |(z - z_\alpha)\zeta'(z_\alpha)| + \kappa \log |\alpha - \bar{\alpha}| \\ &\quad + \mathcal{O}(z - z_\alpha, \bar{z} - \bar{z}_\alpha) \\ &= -\kappa \log |z - z_\alpha| - \kappa \log \left| \frac{\zeta'(z_\alpha)}{(\alpha - \bar{\alpha})} \right| \\ &\quad + \mathcal{O}(z - z_\alpha, \bar{z} - \bar{z}_\alpha). \end{aligned} \tag{7}$$

It should also be noted that  $\psi^{(z)}$  given in (6) satisfies the reciprocity condition

$$\psi^{(z)}(z, \bar{z}; z_\alpha, \bar{z}_\alpha) = \psi^{(z)}(z_\alpha, \bar{z}_\alpha; z, \bar{z}). \tag{8}$$

Now, the stream function  $\psi_{pv}$  for an isolated point vortex of strength  $\kappa$  at the projected point  $z_\alpha$  on a sphere without boundaries is<sup>1,11</sup>

$$\psi_{pv}(z, \bar{z}; z_\alpha, \bar{z}_\alpha) = -\frac{\kappa}{2} \log\left(\frac{(z - z_\alpha)(\bar{z} - \bar{z}_\alpha)}{(1 + z\bar{z})(1 + z_\alpha\bar{z}_\alpha)}\right), \tag{9}$$

where the normalization of this stream function has been chosen so that, again, it satisfies a reciprocity condition of the form

$$\psi_{pv}(z, \bar{z}; z_\alpha, \bar{z}_\alpha) = \psi_{pv}(z_\alpha, \bar{z}_\alpha; z, \bar{z}). \tag{10}$$

Locally, near  $z = z_\alpha$ , this has the form

$$\begin{aligned} \psi_{pv}(z, \bar{z}; z_\alpha, \bar{z}_\alpha) &= -\kappa \log|z - z_\alpha| + \frac{\kappa}{2} \log(1 + z_\alpha\bar{z}_\alpha)^2 \\ &+ \mathcal{O}(z - z_\alpha, \bar{z} - \bar{z}_\alpha). \end{aligned} \tag{11}$$

To determine the motion of the point vortex, it is required to ascertain the function, which we shall call  $\hat{\psi}$ , satisfying the condition that

$$\psi^{(z)}(z, \bar{z}; z_\alpha, \bar{z}_\alpha) = \psi_{pv}(z, \bar{z}; z_\alpha, \bar{z}_\alpha) + \hat{\psi}(z, \bar{z}; z_\alpha, \bar{z}_\alpha). \tag{12}$$

Then, given  $\hat{\psi}$ , the evolution equation for  $z_\alpha$  is

$$\frac{d\bar{z}_\alpha}{dt} = -\frac{i}{2}(1 + z_\alpha\bar{z}_\alpha)^2 \left. \frac{\partial \hat{\psi}}{\partial z_\alpha} \right|_{(z_\alpha, \bar{z}_\alpha)}, \tag{13}$$

where we have used the relation (see, e.g., Crowdy and Cloke<sup>11</sup>)

$$U - iV = -\frac{i}{2}(1 + z\bar{z})^2 \partial \hat{\psi} / \partial z, \tag{14}$$

where  $(U, V)$  are the Cartesian components of the velocity field in the plane of projection. In the case of unbounded motion on the sphere (i.e., when no boundaries are present), the local condition (13) is the one ensuring that the point vortex is in force-free motion (i.e., that there are no external forces on the vortex) so, since it is a local condition, it must also be the correct one to impose when boundaries are present somewhere on the spherical shell. It is a simple matter to show that, provided  $\hat{\psi}$  satisfies a reciprocity condition of the form

$$\hat{\psi}(z, \bar{z}; z_\alpha, \bar{z}_\alpha) = \hat{\psi}(z_\alpha, \bar{z}_\alpha; z, \bar{z}), \tag{15}$$

then the contours

$$\hat{\psi}(z_\alpha, \bar{z}_\alpha; z_\alpha, \bar{z}_\alpha) = \text{const} \tag{16}$$

are the solutions of (13). But, by (8) and (10) and the definition (12) of  $\hat{\psi}$ , it follows that the latter function satisfies (15). Direct calculation, on use of (6), (9), and (12), yields

$$\begin{aligned} \hat{\psi}(z_\alpha, \bar{z}_\alpha; z_\alpha, \bar{z}_\alpha) &= -\kappa \log \left| \frac{(1 + z_\alpha\bar{z}_\alpha)\zeta'(z_\alpha)}{\alpha - \bar{\alpha}} \right| \\ &= -\kappa \log \left| \frac{(1 + z_\alpha\bar{z}_\alpha)\zeta'(z_\alpha)}{\zeta(z_\alpha) - \bar{\zeta}(z_\alpha)} \right|, \end{aligned} \tag{17}$$

where  $\zeta'(z) \equiv d\zeta(z)/dz$ . In fact, it can be shown that the function on the left-hand side of (16) plays the role of the vortex Hamiltonian for this problem.

It follows that the trajectories for single-vortex motion in the bounded region of the stereographic  $z$  plane which maps conformally to the upper-half  $\zeta$  plane via the conformal mapping  $\zeta(z)$  are given by the set of curves

$$\left| \frac{(1 + z_\alpha\bar{z}_\alpha)\zeta'(z_\alpha)}{\zeta(z_\alpha) - \bar{\zeta}(z_\alpha)} \right| = \text{const}. \tag{18}$$

### B. The unit $\zeta$ disk

The case where the conformal mapping is from  $D$  to the unit  $\zeta$  disk is only slightly different. The required stream function in terms of  $\zeta(z)$  is

$$\psi^{(z)}(z, \bar{z}; z_\alpha, \bar{z}_\alpha) = -\kappa \log \left| \frac{1}{\alpha} \frac{(\zeta(z) - \alpha)}{(\zeta(z) - \bar{\alpha}^{-1})} \right|. \tag{19}$$

This function has a singularity at  $\zeta = \alpha$  corresponding to the point vortex at  $z = z_\alpha$  and satisfies the condition that it vanishes on  $|\zeta| = 1$ , as is easily verified.

To determine the motion of the point vortex, we must ascertain the function  $\hat{\psi}$  defined by the condition

$$\psi^{(z)}(z, \bar{z}; z_\alpha, \bar{z}_\alpha) = \psi_{pv}(z, \bar{z}; z_\alpha, \bar{z}_\alpha) + \hat{\psi}(z, \bar{z}; z_\alpha, \bar{z}_\alpha). \tag{20}$$

Direct calculation, on use of (19), (9), and (20), yields

$$\hat{\psi}(z_\alpha, \bar{z}_\alpha; z_\alpha, \bar{z}_\alpha) = -\frac{\kappa}{2} \log\left(\frac{(1 + z_\alpha\bar{z}_\alpha)^2 \zeta'(z_\alpha) \bar{\zeta}'(z_\alpha)}{(\zeta(z_\alpha) \bar{\zeta}(z_\alpha) - 1)^2}\right), \tag{21}$$

where  $\zeta'(z) = d\zeta(z)/dz$ . It follows that the trajectories of the point vortex in this case are given by

$$\left| \frac{(1 + z_\alpha\bar{z}_\alpha)\zeta'(z_\alpha)}{\zeta(z_\alpha)\bar{\zeta}(z_\alpha) - 1} \right| = \text{const}. \tag{22}$$

## III. EXAMPLES

As a check on our approach, in this section we first retrieve the three examples considered in Kidambi and Newton.<sup>9</sup> Then we present the new example of point vortex motion around a barrier, along a great circle, on the spherical surface.

### A. A spherical cap

In example 1 of Ref. 9 the motion of a vortex in a spherical cap is considered. Suppose the cap corresponds, in the  $z$  plane of projection, to the circular disk  $|z| \leq r_0$ . The corresponding conformal mapping to the upper-half  $\zeta$  plane is then

$$\zeta(z) = i \left( \frac{r_0 - z}{r_0 + z} \right). \tag{23}$$

Let  $r=|z_\alpha|$ . On substitution into (18), and on use of the facts that

$$\zeta(z_\alpha) - \overline{\zeta(z_\alpha)} = \frac{2i(r_0^2 - r^2)}{(r_0 + z_\alpha)(r_0 + \bar{z}_\alpha)}, \quad \zeta'(z_\alpha) = -\frac{2ir_0}{(z_\alpha + r_0)^2}, \tag{24}$$

the trajectories are found to be given by

$$\frac{(1 + r^2)^2}{(r^2 - r^2)^2} = \text{const}, \tag{25}$$

which is equivalent to the set of circles  $r=\text{const}$ .

Alternatively, the conformal mapping from the unit  $\zeta$  disk to the same domain is

$$z(\zeta) = r_0 \zeta \tag{26}$$

so that  $\zeta(z)=z/r_0$ . On use of this in (22), the same condition (25) is obtained.

### B. A longitudinal wedge

In example 2 of Ref. 9 point vortex motion in a longitudinal wedge is considered. There, the motion is assumed to take place in a sector of the spherical surface bounded by the longitudes 0 and  $\pi/m$ , where  $m$  is a positive integer. The projection onto the stereographic plane then consists of the infinite planar wedge between two infinite rays emanating from the origin. The conformal mapping from the  $z$  projection of such a wedge to the upper-half  $\zeta$  plane is

$$\zeta(z) = z^m. \tag{27}$$

Let  $r=|z_\alpha|$ . Substitution into (18), and use of the fact that

$$\begin{aligned} |\zeta(z_\alpha) - \overline{\zeta(z_\alpha)}|^2 &= (r^m e^{im\phi} - r^m e^{-im\phi})(r^m e^{-im\phi} - r^m e^{im\phi}) \\ &= 2r^{2m}(1 - \cos(2m\phi)) \\ &= 4r^{2m} \sin^2(m\phi), \end{aligned} \tag{28}$$

yields the trajectories

$$\frac{(1 + r^2)^2}{r^2 \sin^2(m\phi)} = \text{const}, \tag{29}$$

which agree with those obtained by Kidambi and Newton.<sup>9</sup> In fact, we have already generalized the case considered by Kidambi and Newton<sup>9</sup> as the analysis here does not require the wedge angle to be  $\pi/m$  for some positive integer  $m$ . Here, the wedge angle can be arbitrary and  $m$  can be an arbitrary positive real number. The trajectories are still described by (29).

### C. A half-longitudinal wedge

In example 3 of Ref. 9, point vortex motion in a half-longitudinal wedge is considered. There, the motion now takes place in a *half*-sector (or spherical triangle) bounded by the longitudes 0 and  $\pi/m$  (where, again,  $m$  is restricted to be a positive integer) and the equator. The conformal mapping

from the  $z$  projection of such a wedge to the upper-half  $\zeta$  plane is

$$\zeta(z) = \left( \frac{1 + z^m}{1 - z^m} \right)^2. \tag{30}$$

Let  $r=|z_\alpha|$ . On substitution into (18) and on use of the facts that

$$\zeta'(z) = -\frac{4mz^{m-1}(z^m + 1)}{(z^m - 1)^3}, \tag{31}$$

$$\zeta(z) - \overline{\zeta(z)} = \frac{4(|z|^{2m} - 1)(\bar{z}^m - z^m)}{(z^m - 1)^2(\bar{z}^m - 1)^2},$$

shows the trajectories to be given by

$$\frac{(1 + r^2)^2(r^{4m} + 1 - 2r^{2m} \cos(2m\phi))}{(r^{2m} - 1)^2 r^2 \sin^2(m\phi)} = \text{const}. \tag{32}$$

This agrees with the result of Kidambi and Newton.<sup>9</sup> Again, (32) generalizes example 3 of Ref. 9 in that the derivation here does not require  $m$  to be an integer.

### D. Motion through a gap in a wall

Now that the theory has been divorced from a reliance on the method of images, a wider range of examples can be analyzed. A new example is now presented that extends, to the surface of a sphere, the recent study of Johnson and McDonald<sup>10</sup> on the motion of a single point vortex, in the plane, approaching a single gap in an infinite straight wall. They find that if the vortex starts off far from the gap at a distance less than half the gap width from the wall then it will penetrate the gap and travel back, along the other side of the wall, in the direction it came. Otherwise, the vortex does not penetrate the gap.

Suppose there exists an impenetrable wall around the great circle corresponding to  $\phi=0, \pi$  except for a single gap, symmetrical about the south pole, spanning the latitudes  $[\pi - \theta_0, \pi]$  where  $\theta_0$  is some angle between 0 and  $\pi$  (of course, if the gap is too large this problem is more properly thought of as point vortex motion around a straight wall, perhaps modelling a long extended island on the spherical surface). In the  $z$  plane of projection the wall corresponds to the segments of the real axis

$$(-\infty, -L] \cup [L, \infty), \tag{33}$$

where

$$L = \cot(\theta_0/2). \tag{34}$$

It is necessary to find a conformal map from the region exterior to this wall to the upper-half  $\zeta$  plane.

In this problem, it is more convenient to find the conformal map from the unit  $\zeta$  disk to the region exterior to the wall (33). To do so, consider the sequence of maps given by

$$\zeta_1(\zeta) = \frac{1}{2}(\zeta + \zeta^{-1}), \quad z(\zeta_1) = \frac{L}{\zeta_1}. \tag{35}$$

The first of these is a degenerate Joukowski map and takes the interior of the unit  $\zeta$  disk to the entire  $\zeta_1$  plane exterior to

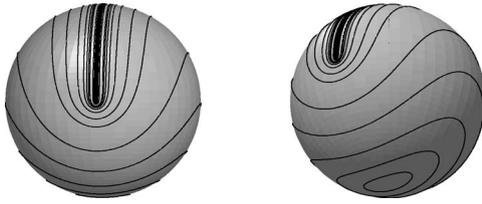


FIG. 2. Vortex trajectories around a wall when  $\theta_0=\pi/2$ ,  $L=1$ . The wall spans half a great circle.

a slit, of length 2, between  $[-1, 1]$ . The second map takes the latter region to the region exterior to the slit region given in (33). The second map is needed in order to have the gap centered on the south pole at  $\zeta=0$  and the barrier centered at the north pole. A composition of the maps yields

$$z(\zeta) = \frac{2L\zeta}{\zeta^2 + 1}. \tag{36}$$

It is convenient to parametrize the trajectories using the pre-image variable  $\alpha$  in favor of  $z_\alpha$ . Substitution of (36) into (22) yields, after some algebra, the trajectories to be given by

$$\left| \frac{(1 + \alpha^2)(1 + \bar{\alpha}^2) + 4L^2\alpha\bar{\alpha}}{(1 - \alpha^2)(\alpha\bar{\alpha} - 1)} \right| = \text{const.} \tag{37}$$

Figures 2–5 show the vortex trajectories for four walls of different lengths corresponding to  $L=1, 1/\sqrt{2}=0.7071, 0.6682$ , and  $0.4142$ . For clarity, two different views of the same sphere are given to the right and left of each figure.  $L=1$  corresponds to a wall spanning exactly half a great circle. In this case, all trajectories are qualitatively the same and are closed contours encircling the wall. The south pole is an elliptic stationary point. As  $L$  decreases so that the length of the wall increases, it is found that there is a critical value at which the distribution of trajectories changes qualitatively. The south pole becomes a hyperbolic stationary point and spawns two elliptic stationary points that emerge symmetrically to either side along the great circle at right angles to the one corresponding to the wall, i.e., along latitudes  $\phi = \pm \pi/2$ . Thus, two regions of closed trajectories form on either side of the wall. These regions of enclosed streamlines grow in size as the length of the wall increases (so that the gap width decreases). As the gap width gets small (compared to the radius of the sphere), locally, in the region near the gap where the effect of the global curvature of the sphere is only slight, the trajectories resemble those computed by Johnson and McDonald<sup>10</sup> for point vortex motion near a gap in a wall on the plane (see the right-most diagram in Fig. 5).

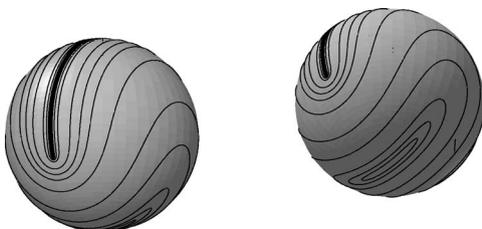


FIG. 3. Vortex trajectories in the critical case  $L=1/\sqrt{2}=0.7071$ .

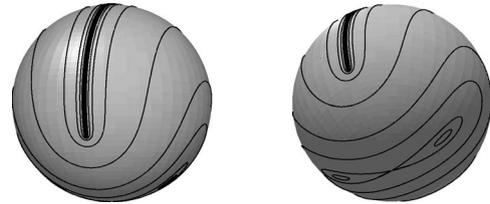


FIG. 4. Vortex trajectories when  $\theta_0=5\pi/8$ ,  $L=0.6682$ . The south pole is now a hyperbolic critical point.

The critical value of  $L$  at which the transition between the two qualitatively different types of vortex behavior occurs can be found analytically. By symmetry, the critical separatrix trajectory goes through the south pole at  $\zeta=0$ . Thus, the critical trajectory is the algebraic curve in the pre-image  $\zeta$  plane given by

$$P(\alpha, \bar{\alpha}) \equiv (1 + \alpha^2)^2(1 + \bar{\alpha}^2)^2 + 8L^2\alpha\bar{\alpha}(1 + \alpha^2)(1 + \bar{\alpha}^2) + 16L^4\alpha^2\bar{\alpha}^2 - (1 - \alpha^2)(1 - \bar{\alpha}^2)(\alpha\bar{\alpha} - 1)^2 = 0. \tag{38}$$

By letting  $\alpha=x+iy$ , and examining when the determinant of the Hessian matrix

$$\begin{pmatrix} \partial^2 P/\partial x^2 & \partial^2 P/\partial x\partial y \\ \partial^2 P/\partial x\partial y & \partial^2 P/\partial y^2 \end{pmatrix} \tag{39}$$

is zero, it is found that

$$L = \frac{1}{\sqrt{2}} \tag{40}$$

is the critical value at which the two qualitatively different types of behavior cross over. Typical trajectories in this case are shown in Fig. 3. For values of  $L$  just below this critical value, two elliptic stationary points emerge to either side of the south pole as shown in Fig. 4.

#### IV. CONNECTION WITH A LIOUVILLE-TYPE PARTIAL DIFFERENTIAL EQUATION

The partial differential equation

$$\nabla^2\psi = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi = ce^{d\psi}, \tag{41}$$

where  $c$  and  $d$  are real constants, is known as the elliptic Liouville equation.<sup>12</sup> In planar vortex dynamics, it arises in two quite unrelated contexts. First, the Hamiltonian governing the motion of a single vortex in a simply connected region of the plane bounded by impenetrable walls satisfies this equation, together with the condition that it is infinite

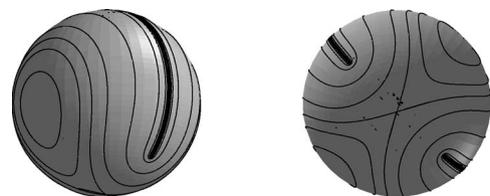


FIG. 5. Vortex trajectories when  $\theta_0=3\pi/4$ ,  $L=0.4142$ . Near the gap, these resemble the trajectories in the planar case (Ref. 10).

everywhere on the boundary of the flow region. Gustafsson<sup>13</sup> (see also Ref. 14) and Richardson<sup>15</sup> independently discovered this fact. Second, it also arises if one seeks steady solutions of the incompressible Euler equations consisting of an everywhere-smooth vorticity distribution where the vorticity and stream function are exponentially related. This leads to exact solutions of the steady Euler equations known as Stuart vortices.<sup>16</sup> Stuart<sup>16</sup> used these solutions as a model of the mixing layer. In this section we demonstrate the interesting result that, when considering the analogues of both of these vortex dynamics problems generalized to a spherical surface, precisely the same circumstance arises: both problems again lead to the same partial differential equation. Instead of Liouville’s equation, however, the relevant partial differential equation that arises is

$$\nabla_{\Sigma}^2 \psi = ce^{d\psi} + 2/d, \tag{42}$$

where  $\nabla_{\Sigma}^2$  is the Laplace-Beltrami operator on the spherical surface.

To see this, let

$$z_{\alpha} = \cot(\theta_{\alpha}/2)e^{i\phi_{\alpha}} \tag{43}$$

and define the operator

$$\hat{\nabla}_{\Sigma}^2 \equiv \frac{1}{\sin \theta_{\alpha}} \frac{\partial}{\partial \theta_{\alpha}} \left( \sin \theta_{\alpha} \frac{\partial}{\partial \theta_{\alpha}} \right) + \frac{1}{\sin^2 \theta_{\alpha}} \frac{\partial^2}{\partial \phi_{\alpha}^2}. \tag{44}$$

On use of the fact<sup>11</sup> that

$$\hat{\nabla}_{\Sigma}^2 = (1 + z_{\alpha}\bar{z}_{\alpha})^2 \frac{\partial^2}{\partial z_{\alpha} \partial \bar{z}_{\alpha}} \tag{45}$$

it is now possible to check directly that  $\hat{\psi}$ , as given in (17), satisfies the quasilinear partial differential equation

$$\hat{\nabla}_{\Sigma}^2 \hat{\psi} = -\kappa e^{-2\hat{\psi}/\kappa} - \kappa. \tag{46}$$

Equation (46) is not the usual Liouville equation since the operator on the left-hand side is not the usual planar Laplacian whereas the right-hand side is not just an exponential term. However, (46) is precisely the generalized quasilinear partial differential equations posed by Crowdy<sup>17</sup> as being relevant for generalizing the planar Stuart vortex solutions to the surface of a sphere. There, (46) was referred to as a *modified Liouville equation* and it was considered in the general form

$$\nabla_{\Sigma}^2 \psi = ce^{d\psi} + g, \tag{47}$$

where  $c$ ,  $d$ , and  $g$  are arbitrary real constants. It was shown in Ref. 17 that the general solution to this partial differential equation can be found in the special case when  $c$  is arbitrary and when  $g$  and  $d$  satisfy the relation

$$g = \frac{2}{d}. \tag{48}$$

The general solution given in Ref. 17 when  $cd < 0$  is then

$$\psi(z_{\alpha}, \bar{z}_{\alpha}) = \frac{1}{d} \log \left( \frac{2f'(z_{\alpha})\bar{f}'(\bar{z}_{\alpha})(1 + z_{\alpha}\bar{z}_{\alpha})^2}{-cd(f(z_{\alpha})\bar{f}(\bar{z}_{\alpha}) + 1)^2} \right), \tag{49}$$

where  $f(z)$  is an arbitrary analytic function in  $D$  except possibly for a finite number of simple pole singularities. It is easy to show that another general solution in the case  $cd > 0$ , and for an arbitrary  $f(z)$  with the same properties as previously given, is

$$\psi(z_{\alpha}, \bar{z}_{\alpha}) = \frac{1}{d} \log \left( \frac{2f'(z_{\alpha})\bar{f}'(\bar{z}_{\alpha})(1 + z_{\alpha}\bar{z}_{\alpha})^2}{cd(f(z_{\alpha})\bar{f}(\bar{z}_{\alpha}) - 1)^2} \right). \tag{50}$$

In respect of (47) and (46) corresponds to the choice

$$c = -\kappa, \quad d = -\frac{2}{\kappa}, \quad g = -\kappa \tag{51}$$

which, significantly, satisfies (48). Indeed, (17) corresponds to the particular choice

$$f(z_{\alpha}) = \zeta(z_{\alpha}) \tag{52}$$

in (50). As  $\partial D$  corresponds to  $|\zeta(z)| = 1$ , it follows that (17) satisfies the boundary condition that  $\hat{\psi}$  is infinite on the boundary  $\partial D$ . This is precisely the same boundary condition relevant in the planar case.<sup>13-15</sup>

## V. DISCUSSION

Formulas for the trajectories of a single vortex in simply connected bounded regions on the surface of a sphere, where the stereographic projections of the regions are obtained by conformal mappings from both the upper-half plane and the unit disk in a parametric  $\zeta$  plane, have been presented. Formulas (18) and (22) embody the key new results of this article.

The theory should be amenable to development in a number of directions. It is of interest to rephrase our results in terms of a generalization of the classical Kirchhoff-Routh theory<sup>1,3</sup> for point vortex motion in bounded regions on the surface of a sphere. In such a formulation, a Hamiltonian (or *Kirchhoff-Routh path function*) is considered together with its transformation properties under the effects of a conformal mapping of the fluid domain (indeed, the function  $\hat{\psi}$  used in this article is essentially the Hamiltonian in the single vortex case). From this perspective, the recent work of Crowdy and Marshall<sup>7</sup> implementing the Kirchhoff-Routh theory in planar multiply connected fluid domains of arbitrary finite connectivity should be generalizable to the surface of a sphere. Note that Kidambi and Newton<sup>9</sup> have presented an example of point vortex motion in a doubly connected domain on the surface of a sphere using their method-of-images approach. These various matters are under investigation.

This work also points to the importance of quasilinear partial differential equation (47) with  $g = 2/d$ . The evidence herein suggests that it is a natural analogue of the planar Liouville equation when working on the surface of a sphere. It would be interesting to see if it arises in any other applied mathematical contexts.

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