Computable and incomputable boundary extensions

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The BIG question of computability theory

Question

What can be computed?

Computability theory/ theory of computation

- Limitations of computers (*i.e.* discrete computing devices)
- When a problem can be solved by a computer, how difficult is it to solve? (in terms of time/memory) **Complexity Theory**
- When a problem can not be solved by any computer, how impossible is it (relative to other impossible problems)?

Fundamental notion: algorithm

- *i.e.* a procedure that can be carried out without thinking.
- Viewpoint: computers are just devices for implementing algorithms, so algorithms are the real focus of study.

Fundamental definition

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Definition

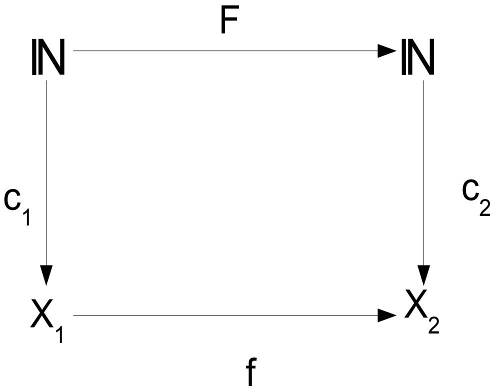
A function $f :\subseteq \mathbb{N}^m \to \mathbb{N}$ is *computable* if there is an algorithm that given any $n_1, \ldots, n_m \in \mathbb{N}$ as input, halts with output $f(n_1, \ldots, n_m)$ if $(n_1, \ldots, n_m) \in \mathrm{dom}(f)$ and does not halt if $(n_1, \ldots, n_m) \notin \mathrm{dom}(f)$.

Examples:

- Addition, multiplication, division
- gcd's
- probably any function you can think of.

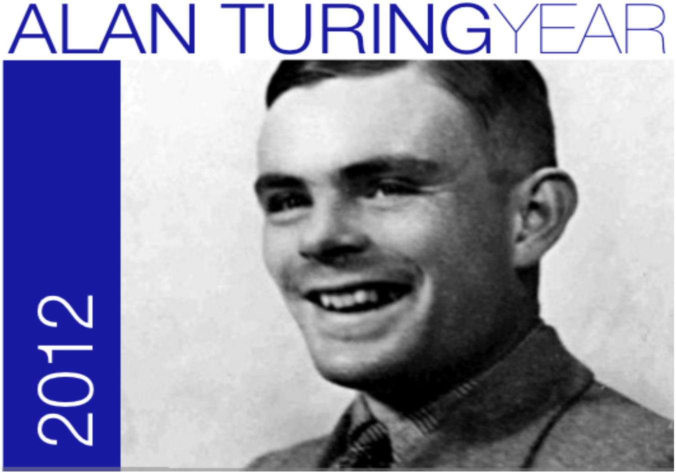
Definition

A *coding* of a set X is a function $c :\subseteq \mathbb{N} \rightarrow X$ that is onto.



By means of codings, can extend computability to

- $\mathbb{Z}$
- $\mathbb{Q}$
- set of all finite graphs, *etc.*


ALAN TURING YEAR
2012

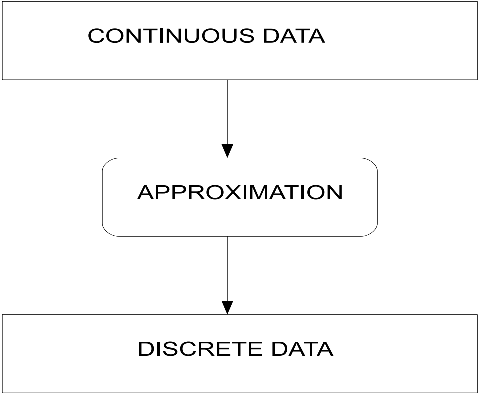
Formalizing ‘algorithm’

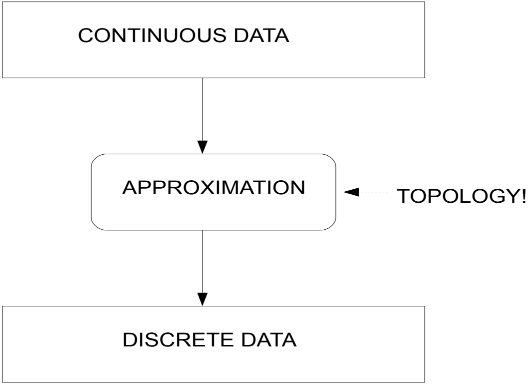
- Turing machines (A. Turing)
- Partial recursive functions (A. Church)
- Unlimited register machine (J. Von Neumann)
- Flowchart computability (Wang)

These are all equivalent!

Computable analysis

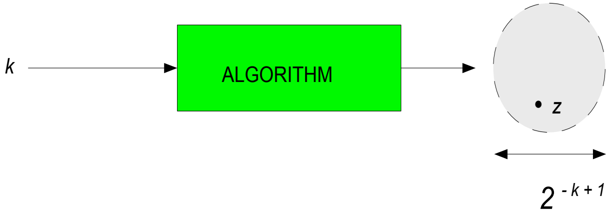
Theory of computation with *continuous* data


CONTINUOUS DATA
APPROXIMATION
DISCRETE DATA


CONTINUOUS DATA
APPROXIMATION
TOPOLOGY!
DISCRETE DATA

Definition

A point $z \in \mathbb{C}$ is *computable* if there is an algorithm that, given any $k \in \mathbb{N}$ as input, produces a rational point q such that $|z - q| < 2^{-k}$.



Examples:

- Any rational point.
- $e, \pi, \sqrt{2}$
- Almost any point you can think of.

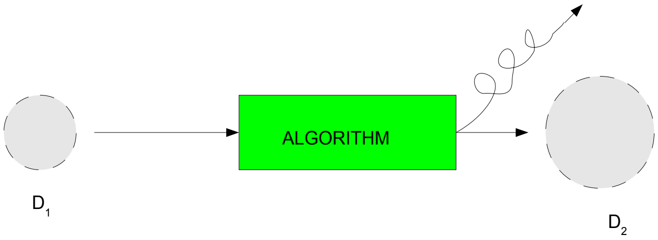
Definition

A sequence of complex numbers $\{z_n\}_{n \in \mathbb{N}}$ is *computable* if there is an algorithm that, given any $k, n \in \mathbb{N}$ as input, produces a rational point q such that $|z_n - q| < 2^{-k}$.

Definition

Let $f :\subseteq \mathbb{C} \rightarrow \mathbb{C}$. f is *computable* if there is an algorithm P that has the following three properties:

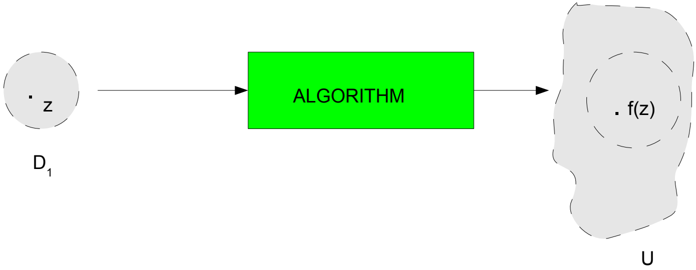
Property 1: approximation



Property 2: correctness

$$f[D_1] \subseteq D_2$$

Property 3: convergence



Proposition

If $f :\subseteq \mathbb{C} \to \mathbb{C}$ is computable, and if $z \in \mathrm{dom}(f)$ is computable, then $f(z)$ is computable.

Theorem (Pommerenke 1991(?))

Suppose f is a bounded conformal map of $\mathbb{D}$ onto D. Then, f has a boundary extension if and only if the boundary of D is locally connected.

- Recall: X is locally connected if and only if for every $p \in X$, every neighborhood of p includes a connected neighborhood of p

Theorem (Hahn-Mazurkiewicz Theorem)

If X is a Hausdorff topological space, then there is a continuous map of $[0, 1]$ onto X if and only if X is metrizable, compact, connected, and locally connected.

Definition

A conformal map $\phi : \mathbb{D} \to \mathbb{C}$ is *computably extendible* if its boundary extension is computable.

Definition

A set $X \subseteq \mathbb{R}^n$ is *computably traceable* if there is a computable surjection of $[0, 1]$ onto X.

Theorem (M., 2012)

There is a computable conformal map of $\mathbb{D}$ onto a Jordan domain D that is not computably extendible even though the boundary of D is computably traceable.

Definition (really a theorem (sort of))

$X \subseteq \mathbb{C}$ is *effectively locally connected* if there is a computable map $g : \mathbb{N} \to \mathbb{N}$ such that whenever $k \in \mathbb{N}$ and $p, q \in X$ are such that $0 < |p - q| < 2^{-g(k)}$, X includes an arc A from p to q whose diameter is at most 2^{-k}.

Theorem (M., 2011)

Suppose ϕ is a bounded computable conformal map of $\mathbb{D}$ onto D. If the boundary of D is effectively locally connected, then ϕ is computably extendible.

- *i.e.* effective local connectivity provides *sufficient* information for computing boundary extensions.
- Proof based on following lemma:

Lemma (M., 2011)

Suppose $0 < 1 - s_0 < r_0 < 1$*. Suppose* C *is an arc from a point* $p \in A_{s_0}$ *to a point* $q \in \partial D$ *such that* $C \cap \partial D = \{q\}$ *and such that* $|\phi(t) - z| \geq m(s_0, N_0, r_0)$ *whenever* $t \in [1 - s_0, r_0]$ *and* $z \in C$*. Then,* $\phi(t) \notin C$ *for all* $t \in [1 - s_0, 1]$*.*

Theorem (M., 2012)

There is a computably extendible conformal map of $\mathbb{D}$ onto a domain D whose boundary is not effectively locally connected.

- Proof based on following estimate:

Theorem (M., 2012)

Suppose ϕ *is a conformal map on the unit disk that has a boundary extension and let D denote its range. Suppose* r_0, R_0 *are positive numbers such that* $r_0 < \frac{R_0}{\exp(16\pi^2)}$. *and such that* R_0 *is smaller than the minimum of* $|\phi(z) - \zeta|$ *as z ranges over all points whose modulus is at most* $1/2$ *and* ζ *ranges over all boundary points of D. Let* ζ_0 *be a boundary point of D, and choose a point* $z_0 \in D$ *so that* ζ_0 *is a boundary point of the connected component of* z_0 *in* $D_{r_0}(\zeta_0) \cap D$. *Then,* $\zeta_0 = \phi(\zeta)$ *for some* ζ *such that*

$$\left|\text{Arg}\left(\frac{\zeta}{\phi^{-1}(z_0)}\right)\right| < \arcsin\left(\frac{2\pi}{\sqrt{\ln(R_0/r_0)}}\right).$$

T.H. McNicholl, *Computing boundary extensions of conformal maps*, Submitted. Preprint available at http://arxiv.org/abs/1110.5271v1.

______, *Conformal maps and jagged boundaries*, Submitted. Preprint available at http://arxiv.org/abs/1304.1915.

______, *An effective Carathéodory theorem*, Theory of Computing Systems **50** (2012), no. 4, 579 – 588.

Ch. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 299, Springer-Verlag, Berlin, 1992.