Constrained Surface Registration using Extremal Teichmüller maps (T-Map)

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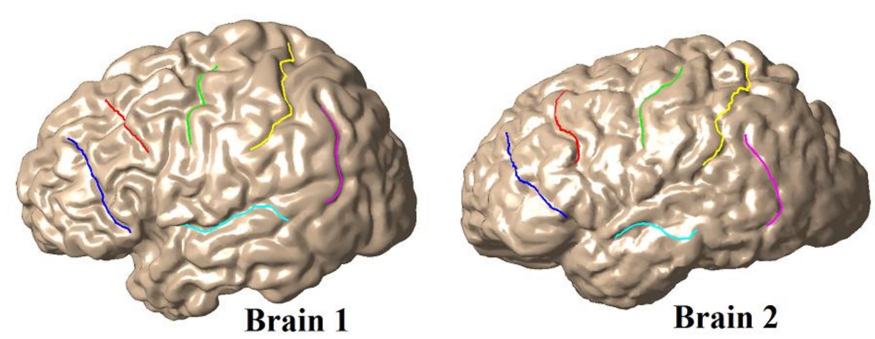
Workshop on Conformal Geometry in Mapping, Imaging and Sensing
June 20-21, 2013

Outline Of The Talk

- Motivation
- Mathematical Background
- Computational Algorithms
- Applications
- Conclusion



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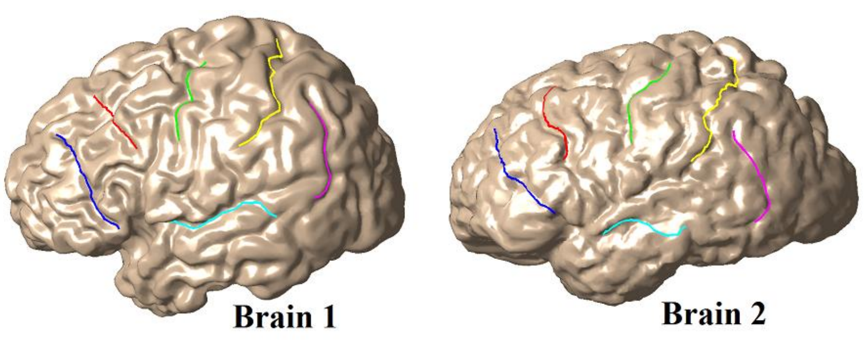
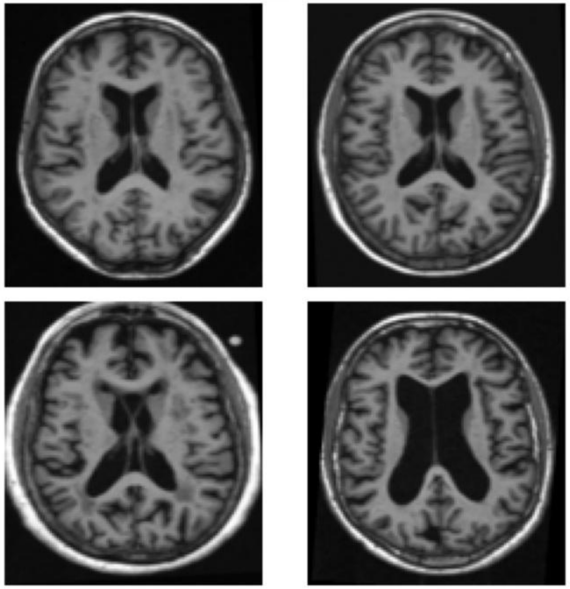
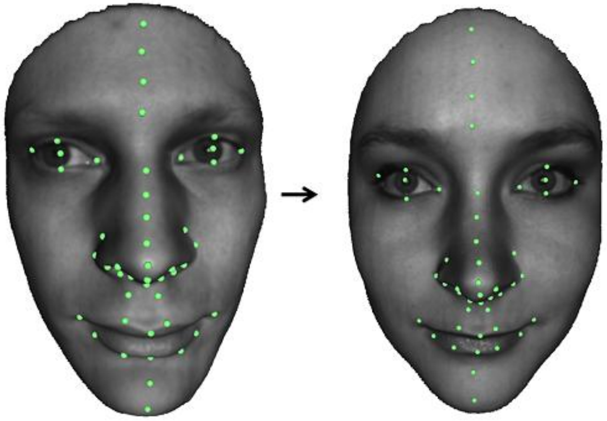


Motivation: Registration

- **Main Goal: Registration**
  - **Image registration:** medical imaging, image super-resolution, video compression…
  - **Surface registration:** face recognition, texture mapping, medical shape analysis…

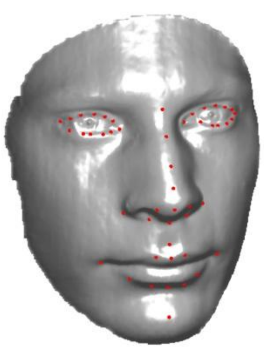


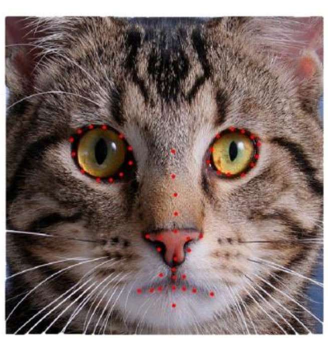


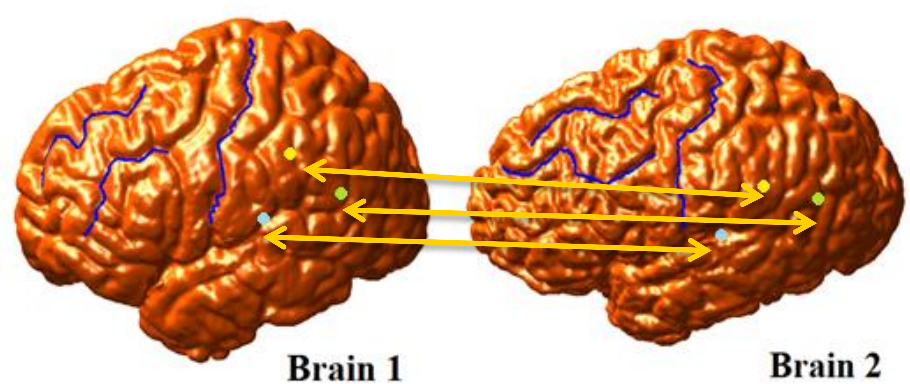
Motivation: Registration

- Categories of Registration:
  - Intensity-based registration: based on image intensity for image registration or geometric quantities (curvatures) for surface registration.
  - Landmark-based registration: based on salient features or landmarks (e.g. sulcal/gyral landmarks on brains)

Goal: Look for a registration with minimum geometric distortion!







Our Goal

- **Develop algorithm to compute constrained registration:**
  - **Preserve bijectivity (many landmarks/large deformations)**
  - **Preserve local geometry**
  - **Match landmark consistently**
  - **Uniqueness (won't jump into local minimum)**
  - **Efficiency**
  - **Independence of the mesh structure**

Consider a special class of bijective map, called T-Map:

- Minimizes the local geometric distortion
- Uniform local geometric distortion
- Always bijective

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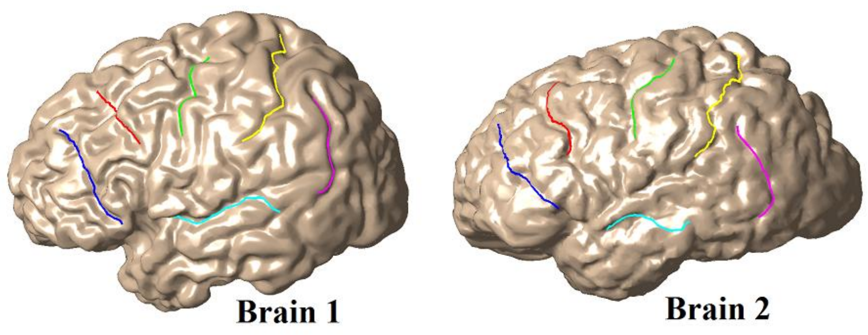


What is Quasi-conformal map?

- Generalization of conformal maps (angle-preserving);
- Orientation preserving homeomorphism between Riemann surfaces;
- Bounded conformality distortion;
- Intuitively, map infinitesimal circle to ellipse;
- Mathematically, it satisfies: $\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z}$

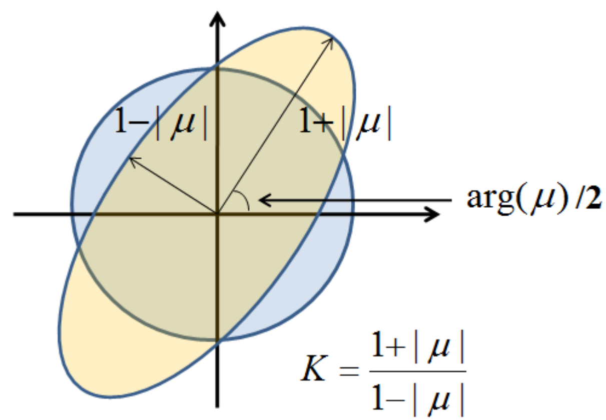
Beltrami coefficient

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right);$$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

Conformal $\Leftrightarrow \mu = 0 \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0$

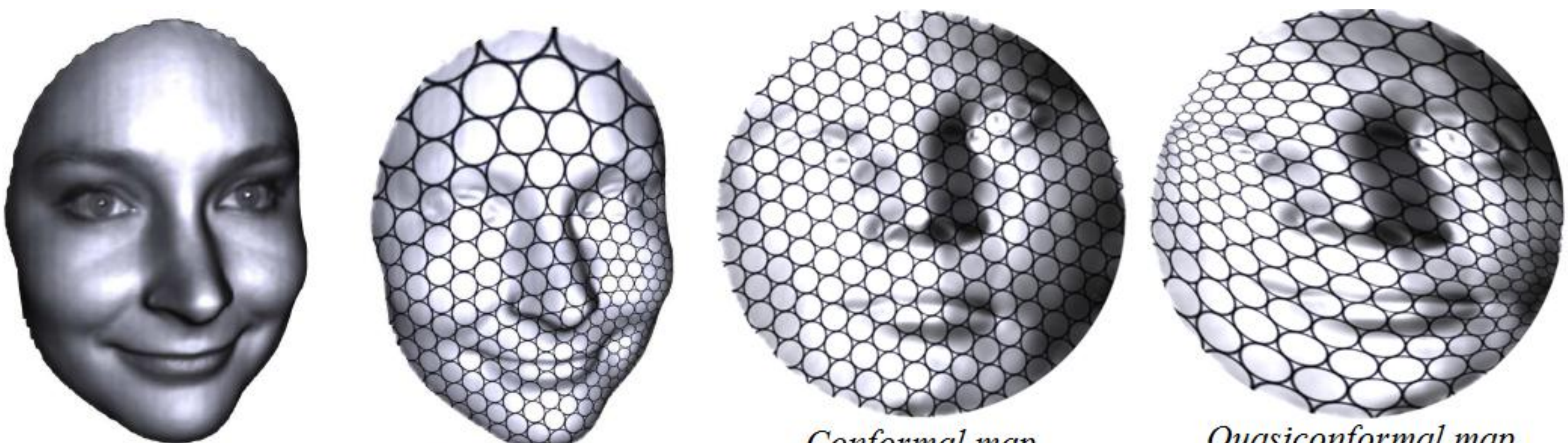
- Beltrami coefficient:
  - Measure conformality distortion;
  - Invariant under conformal



Examples of QC maps

In term of Riemannian metric,

Conformal	Quasi-conformal
$f^*(ds_E^2) = \lambda \lvert dz \rvert^2$	$f^*(ds_E^2) = \lvert \frac{\partial f}{\partial z} \rvert^2 \lvert dz + \mu(z) d\overline{z} \rvert^2$

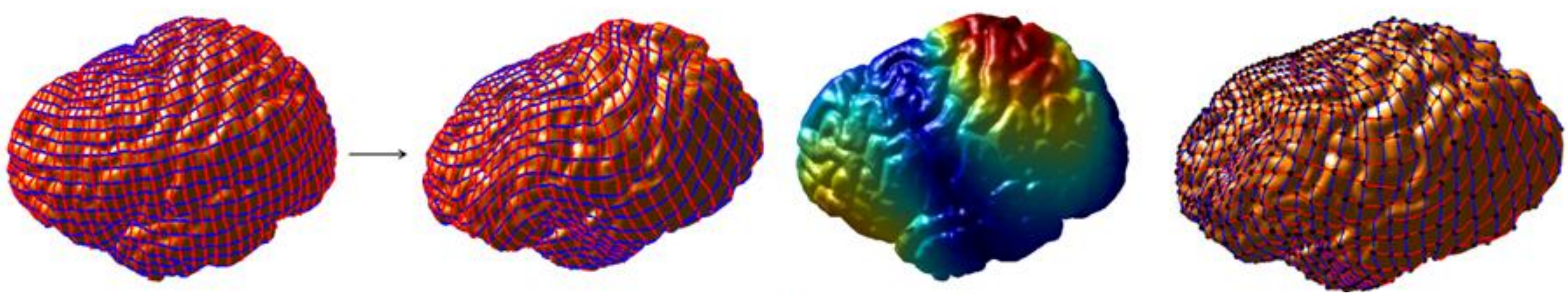


Original face — *Circle packing* — *Conformal map (Circles to circles)* — *Quasiconformal map (Circles to ellipses)*

Discrete QC Maps

Discrete Measurable Riemann Mapping Theorem

Theorem (Discrete measurable Riemann mapping). *Suppose K_1 and K_2 are genus 0 (simply-connected) closed (open) surface meshes. Fixing three (two) points correspondence, there is a 1-1 correspondence between the set of discrete BC defined on K_1 and the set of discrete QC maps between K_1 and K_2.*



Brain 1 | Brain 2 | Beltrami Coefficient | Reconstructed

Extremal Map

- Extremal map: minimizes conformality distortion.

Definition *Let $f : S_1 \to S_2$ be a quasi-conformal mapping between S_1 and S_2. f is said to be an extremal mapping if for any quasi-conformal mapping $h : S_1 \to S_2$ isotopic to ϕ relative to the boundary,*

$$||\mu(f)||_\infty \leq ||\mu(h)||_\infty$$

It is uniquely extremal if the inequality is strict.

Properties of extremal map:

- Minimizes the conformality distortion
- Extremal map always exists but may not unique
- Under suitable condition on the boundary/landmark constraints, extermal map is unique.

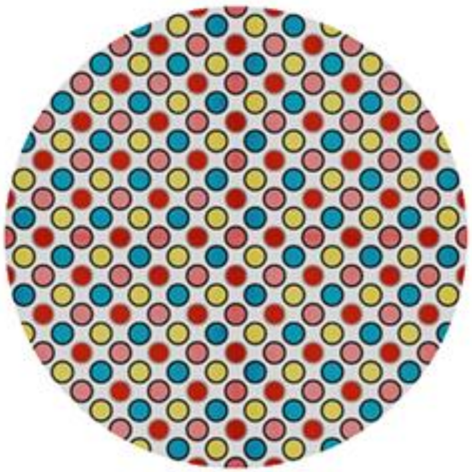
What is T-Map?

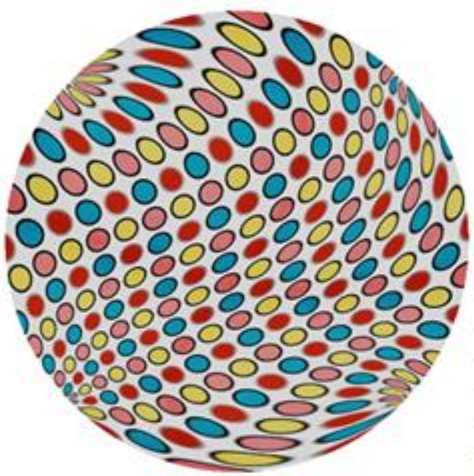
- Quasi-conformal mapping with uniform conformality distortion.

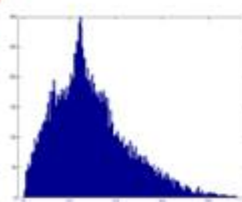
Definition *Let $f : S_1 \to S_2$ be a quasi-conformal mapping. f is said to be a Teichmüller mapping associated to the integrable holomorphic function $\varphi : S_1 \to \mathbb{C}$ if its associated Beltrami differential is of the form:*

$$\mu(f) = k\frac{\overline{\varphi}}{|\varphi|}$$

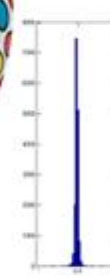
for some constant $k < 1$ and holomorphic function $\varphi \neq 0$ with $||\varphi||_1 = \int_{S_1} |\varphi| < \infty$.







General Q.C.



T-Map

T-Map v.s. Extremal Map

- Huge relationship between T-Map and Extremal map!

Definition (Boundary dilation). *Suppose S_1 and S_2 are open Riemann surfaces with the same topology. The boundary dilation $K_1[f]$ of f is defined as:*

$$K_1[f] = \inf_C \{K(h|_{S_1 \setminus C}) : h \in \mathfrak{F}, C \subseteq S_1, C \text{ is compact.}\}$$

where $\mathfrak{F}$ is the family of quasi-conformal homeomorphisms of S_1 onto S_2 which are homotopic to f modulo the boundary.

- Under suitable condition, T-Map = Extremal map!

Theorem (Strebel's theorem). *Let f be an extremal quasi-conformal mapping with $K(f) > 1$. If $K_1[f] < K(f)$, then f is a Teichmüller map associated with an integrable holomorphic function on S_1.*

Hence, f is also an unique extremal mapping.

T-Map v.s. Extremal Map

- Under suitable boundary condition, T-Map = Extremal map on disk!

Theorem *Let $g : \partial\mathbb{D} \to \partial\mathbb{D}$ be an orientation-preserving homeomorphism of $\partial\mathbb{D}$. Suppose further that $h'(e^{i\theta}) \neq 0$ and $h''(e^{i\theta})$ is bounded. Then there is a Teichmüller mapping f that is the unique extremal extension of g to $\mathbb{D}$. That is, $f : \mathbb{D} \to \mathbb{D}$ is an extremal mapping with $f|_{\partial\mathbb{D}} = g$.*

Main idea:

- For open surfaces with disk topology, if the boundary correspondence satisfies "good" conditions for their derivatives,

EXTREMAL MAP = T-MAP!

Landmark matching T-Map

- T-Map exists and unique even with interior landmark constraints enforced!

Theorem (Landmark-matching Teichmüller mapping). *Let S_1 and S_2 be open Riemann surfaces with the same topology. Let $\{p_i\}_{i=1}^n \in S_1$ and $\{q_i\}_{i=1}^n \in S_2$ be the corresponding interior landmark constraints. Let $f : S_1 \setminus \{p_i\}_{i=1}^n \to S_2 \setminus \{q_i\}_{i=1}^n$ be the extremal quasi-conformal mapping, such that p_i corresponds to q_i for all $1 \leq i \leq n$. If $K_1[f] < K(f)$, then f is a Teichmüller map associated with an integrable holomorphic function on $S_1 \setminus \{p_i\}_{i=1}^n$. Hence, f is an unique extremal mapping.*

Main idea:

- If boundary dilation is under certain condition, there **EXISTS** landmark matching T-Map;
- T-Map is unique extremal map. Hence, given a prescribed set of landmark constraints, the associated T-Map is **UNIQUE**!
- T-Map has BC with norm k < 1. Hence, T-Map is **BIJECTIVE**!

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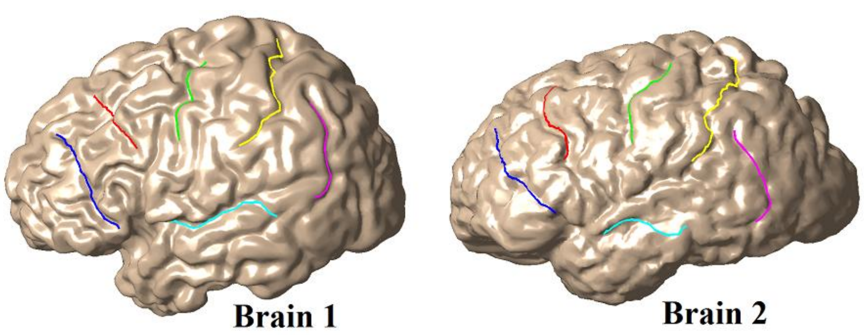


Our problem

Problem:

- Find a T-Map which satisfies some boundary condition and interior landmark constraints.
- Find a mapping which has **LEAST** and **UNIFORM** (everywhere the same) conformality distortion.

Mathematically:

- Find a T-Map such that:

$$\frac{\partial f}{\partial \bar{z}} = k \frac{\overline{\varphi}}{|\varphi|} \frac{\partial f}{\partial z} \quad \text{and} \quad f|_{\partial D_1} = g \quad \text{on } \partial D_1$$

Beltrami equation

$$f(a_i) = b_i; \;\; f(p_j) = q_j; \;\; \text{for } i = 1, ..., n; \;\; j = 1, ..., m$$

Interior landmark points/curves constraints

Variational formulation

Main idea:

- Solving the above problem is difficult!
- Propose a variational formulation of the problem.
- Iterative method, called the QC iterations, will be developed.

Variational formulation:

- T-Map *f* is extremal in the sense that:

$$||\mu(f)||_\infty \leq ||\mu(h)||_\infty \quad \text{for any } h : D_1 \to D_2 \text{ satisfying } h|_{\partial D_1} = g$$

- Our problem can be formulated as:

$$f = \mathbf{argmin}_{f:D_1\to D_2} E_1(f) \ := \mathbf{argmin}_{f:D_1\to D_2}\{||\mu(f)||_\infty\}$$

subject to:

- $f|_{\partial D_1} = g$ (boundary condition); $f(p_j) = q_j$; for $i = 1, ..., n$; $j = 1, ..., m$
- $\mu(f) = k\frac{\overline{\varphi}}{|\varphi|}$ for some constant $0 \leq k < 1$ and holomorphic function $\varphi : D_1 \to \mathbb{C}$.

Variational formulation

Difficulty:

- Solving the above variational problem over f is difficult!

$$f = \mathbf{argmin}_f \{||\mu(f)||_\infty\} = \mathbf{argmin}_f || \frac{\partial f / \partial \bar{z}}{\partial f / \partial z} ||_\infty$$

- Propose to minimize it over the Beltrami coefficients!

Variational formulation:

- We formulate the variational problem over Beltrami coefficients:

$$(\nu, f) = \mathbf{argmin}_{\nu : D_1 \to \mathbb{C}} E_2(\nu) := \mathbf{argmin}_{\nu : D_1 \to \mathbb{C}} \{||\nu||_\infty\}$$

subject to:

- $\nu = \mu(f)$ and $||\nu||_\infty < 1$;
- $\nu = k\frac{\overline{\varphi}}{|\varphi|}$ for some constant $0 \leq k < 1$ and holomorphic function $\varphi : D_1 \to \mathbb{C}$;
- $f|_{\partial D_1} = g$ (boundary condition). $f(p_j) = q_j$; for $i = 1, ..., n$; $j = 1, ..., m$

Computational Algorithm

Main idea:

- In each iterations, smooth and average the BC;
- Find the “best” associated qc map that fixes landmark and boundary constraints.

Tools that we need:

- **Linear Beltrami Solver (LBS):** Provides a way to go between Beltrami coefficient and its associated QC map.

  (should be efficient so that fast computation in each iterations)
- **Quasi-conformal (QC) Iterations:** Provides a way to minimizes the variational model for computing the T-Map.

  (should converge fast, so that only few iterations are needed)

Linear Beltrami Solver

Main idea:

- Build a discrete analogue of the generalized version of Beltrami equation:

$$\text{Let } f = u + \sqrt{-1}v. \quad \text{Let } \mu(f) = \rho + \sqrt{-1}\,\tau.$$

$$\begin{aligned} -v_y &= \alpha_1 u_x + \alpha_2 u_y; \\ v_x &= \alpha_2 u_x + \alpha_3 u_y. \end{aligned}$$

$$\text{where } \alpha_1 = \frac{(\rho-1)^2+\tau^2}{1-\rho^2-\tau^2};\ \alpha_2 = -\frac{2\tau}{1-\rho^2-\tau^2};\ \alpha_3 = \frac{1+2\rho+\rho^2+\tau^2}{1-\rho^2-\tau^2}.$$

- Take divergence on both sides:

$$\nabla \cdot \left(A \begin{pmatrix} u_x \\ u_y \end{pmatrix} \right) = 0 \text{ and } \nabla \cdot \left(A \begin{pmatrix} v_x \\ v_y \end{pmatrix} \right) = 0$$

$$\text{where } A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix}.$$

Linear Beltrami Solver

Discrete analogue:

- In the discrete setting, assume mapping to be piecewise linear.
- On triangulation mesh, let BC be defined on each faces.
- Then on each face, we have:

$$-d_T = \alpha_1(T)a_T + \alpha_2(T)b_T$$
$$c_T = \alpha_2(T)a_T + \alpha_3(T)b_T$$

where

$$D_x u(T) = a_T,\ D_y u(T) = b_T,\ D_x v(T) = c_T \text{ and } D_y v(T) = d_T.$$

- Discrete divergence can be defined by (using divergence theorem on mesh):

$$Div(\vec{V})(v_i) = \sum_{T \in N_i} A_i^T V_1(T) + B_i^T V_2(T)$$

where:

$$A_i^T = (h_j - h_k)/Area(T),\ A_j^T = (h_k - h_i)/Area(T),\ A_k^T = (h_i - h_j)/Area(T);$$
$$B_i^T = (g_k - g_j)/Area(T),\ B_j^T = (g_i - g_k)/Area(T),\ B_k^T = (g_j - g_i)/Area(T);$$

Linear Beltrami Solver

Linear system to get the associated QC map:

- A sparse symmetric positive definite linear system can be obtained.

$$Div\left(A\begin{pmatrix} D_x u \\ D_y u \end{pmatrix}\right) = 0 \text{ and } Div\left(A\begin{pmatrix} D_x v \\ D_y v \end{pmatrix}\right) = 0 \text{ where } A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix}.$$

- This is equivalent to the following linear system:

$$\sum_{T \in N_i} A_i^T [\alpha_1(T) a_T + \alpha_2(T) b_T] + B_i^T [\alpha_2(T) a_T + \alpha_3(T) b_T] = 0$$

$$\sum_{T \in N_i} A_i^T [\alpha_1(T) c_T + \alpha_2(T) d_T] + B_i^T [\alpha_2(T) c_T + \alpha_3(T) d_T] = 0$$

Quasi-conformal (QC) Iterations

Main idea:

- Iteratively minimizes the variational model for computing T-Map
- Recall: $$(\nu, f) = \mathbf{argmin}_{\nu: D_1 \to \mathbb{C}} \{ ||\nu||_\infty \}$$

  subject to: (1) $\nu = \mu(f)$ with $||\nu||_\infty < 1$; (2) $\nu = k\frac{\overline{\varphi}}{|\varphi|}$

  (3) f satisfies certain boundary condition and/or landmark constraints.

- Initially, we consider an initial map:

$$f_0 = \mathbf{LBS}_{LM}(\mu_0 := 0)$$

- Compute the BC of the initial map: $\nu_0 = \mu(f_0)$

  Hence, obtain the initial pair: (ν_0, f_0)

- Laplace smooth and Averaging:

$$\mathfrak{L}(\nu_0)(T) := \sum_{T_i \in \mathrm{Nbhd}(T)} \nu_0(T) \; / |\mathrm{Nbhd}(T)|$$ (Laplace smooth)

$$\mathcal{A}(\tilde{\mu_1})(T) := \left(\frac{\sum_{T \in \text{ all faces of } K_1} |\tilde{\mu_1}|(T)}{\text{No. of faces of } K_1} \right) \frac{\tilde{\mu_1}(T)}{|\tilde{\mu_1}(T)|}$$ (Averaging)

Quasi-conformal (QC) Iterations

Detailed algorithm:

$$\mu_{n+1} := \mathcal{A}(\mathfrak{L}(\nu_n));$$
$$f_{n+1} := \mathbf{LBS}_{LM}(\mu_{n+1});$$
$$\nu_{n+1} := \mu(f_{n+1}).$$

Algorithm: *(QC iteration)*
Input : *Triangular meshes:* K_1 *and* K_2*; the desired landmark constraints and/or boundary condition.*
Output : *Optimal Beltrami coefficient* ν *and the T-Map* f

1. *Obtain the initial mapping* $f_0 = \mathbf{LBS}_{LM}(\mu_0 := 0)$. *Set* $\nu_0 = \mu(f_0)$;
2. *Given* ν_n, *compute* $\mu_{n+1} := \mathcal{A}(\mathfrak{L}(\nu_n))$; *Compute* $f_{n+1} := \mathbf{LBS}_{LM}(\mu_{n+1})$ *and set* $\nu_{n+1} := \mu(f_{n+1})$;
3. *If* $||\nu_{n+1} - \nu_n|| \geq \epsilon$, *continue. Otherwise, stop the iteration.*

Convergence Analysis

Summary of QC iterations:

- Laplace smooth BC;
- Projection of BC into the space of BCs of Teichmuller type

Why it works:

- QC iterations = Minimization of harmonic energy under the distorted metric given by BC.

Theorem (T-Map and harmonic energy). *Let*

$$T(S_1) := \{\mu : S_1 \to \mathbb{C} : \mu = k\frac{\overline{\varphi}}{|\varphi|}\}.$$

Define the harmonic energy with respect to $\mu \in T(S_1)$ *by:*

$$E(\mu) = \min_{f:S_1 \to S_2}\{\int_{S_1} ||\nabla_\mu f||^2_{S_2}\} = \min_{f:S_1 \to S_2}\{\int_{S_1} |\frac{\partial f}{\partial \bar{z}} - \mu\frac{\partial f}{\partial z}|^2 dS_1\}$$

Then: $E : T(S_1) \to \mathbb{R}^+$ *is convex and its global minimizer is the Teichmüller extremal map.*

L.M. Lui, X.F. Gu, S.T. Yau, *Convergence of an Iterative Algorithm for Teichmüller Maps via Harmonic Energy Minimization*, UCLA CAM report 13-36, June 2013

Numerical experiments 1

Analytic example:

For a ring domain with inner and outer radii r_0 and r_1, we map it to a ring with inner and outer radii $r_0{}'$ and $r_1{}'$.
The extremal map has the polar formal

$$(r,\theta) \to \left(r_0' \left(\frac{r}{r_0} \right)^K, \theta \right)$$

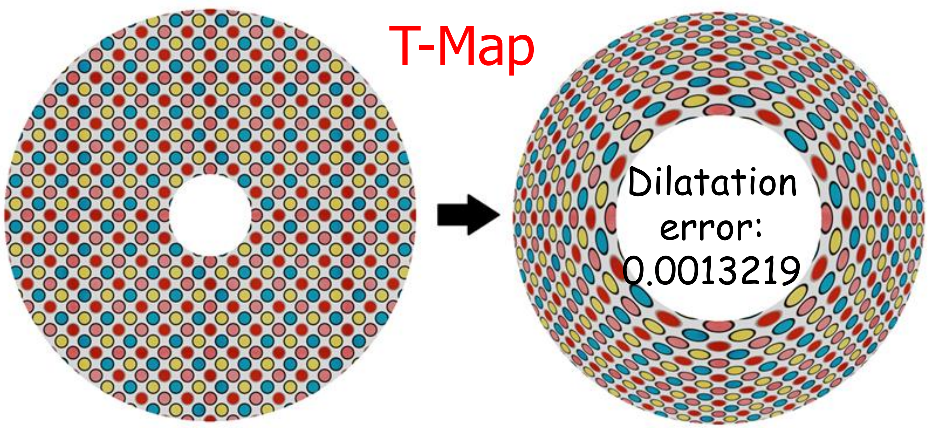
where

$$K = \frac{\ln\left(\frac{r_1'}{r_0'}\right)}{\ln\left(\frac{r_1}{r_0}\right)}$$

In the following, experiments with different situation is illustrated. Most of the program codes are still in Matlab version. C++ version or even GPU calculating can be applied to speed up the calculation.

Numerical experiments 1

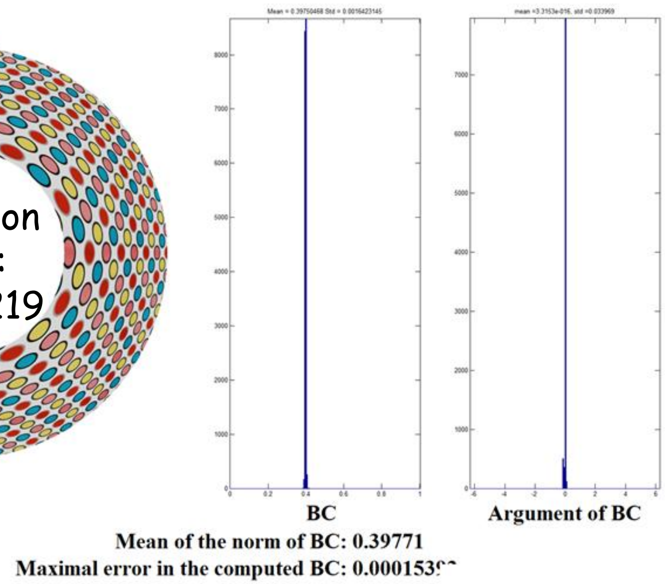
Analytic example:



Time: ~2s
Vertices:8257
Faces: 16256

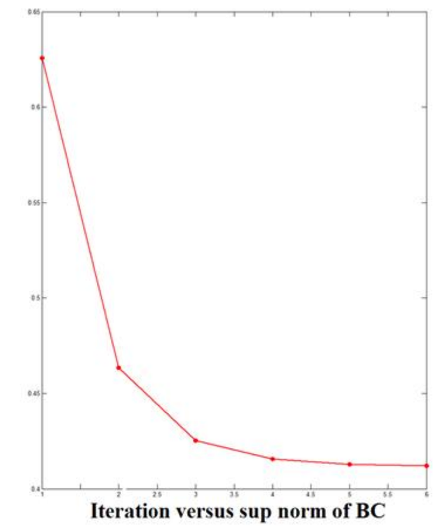
(A)

Histogram of BC norm



(B) (C)

BC norm in each iterations



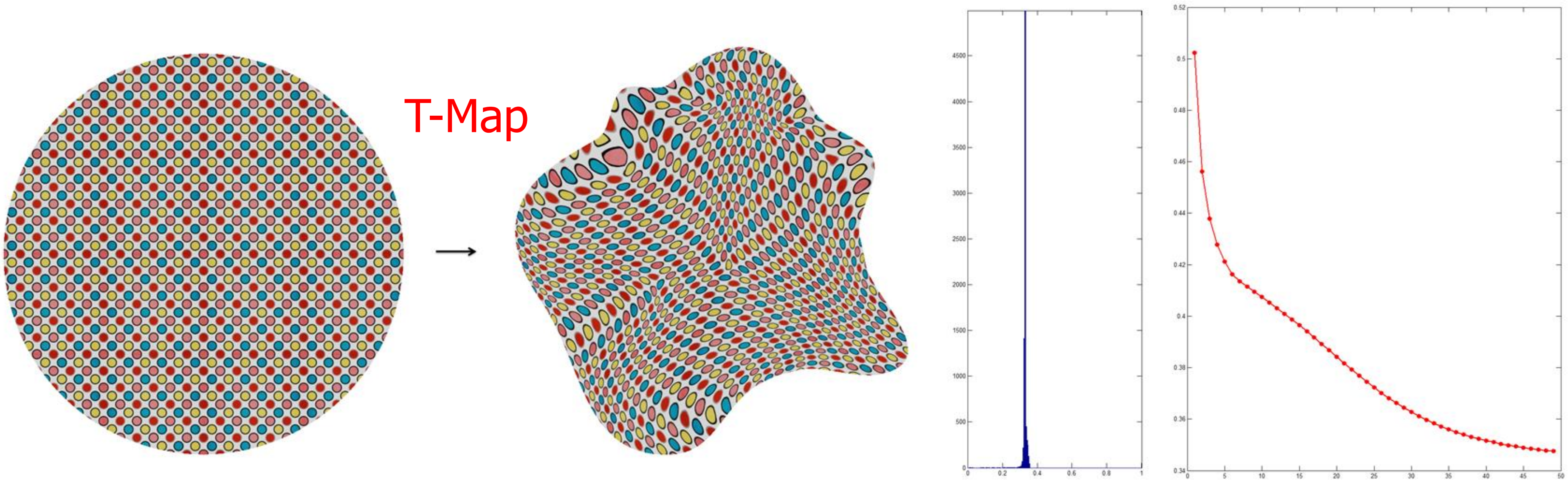
(D)

Numerical experiments 3

Arbitrary shapes:

Histogram of the BC norm

Norm of BC in each iteration

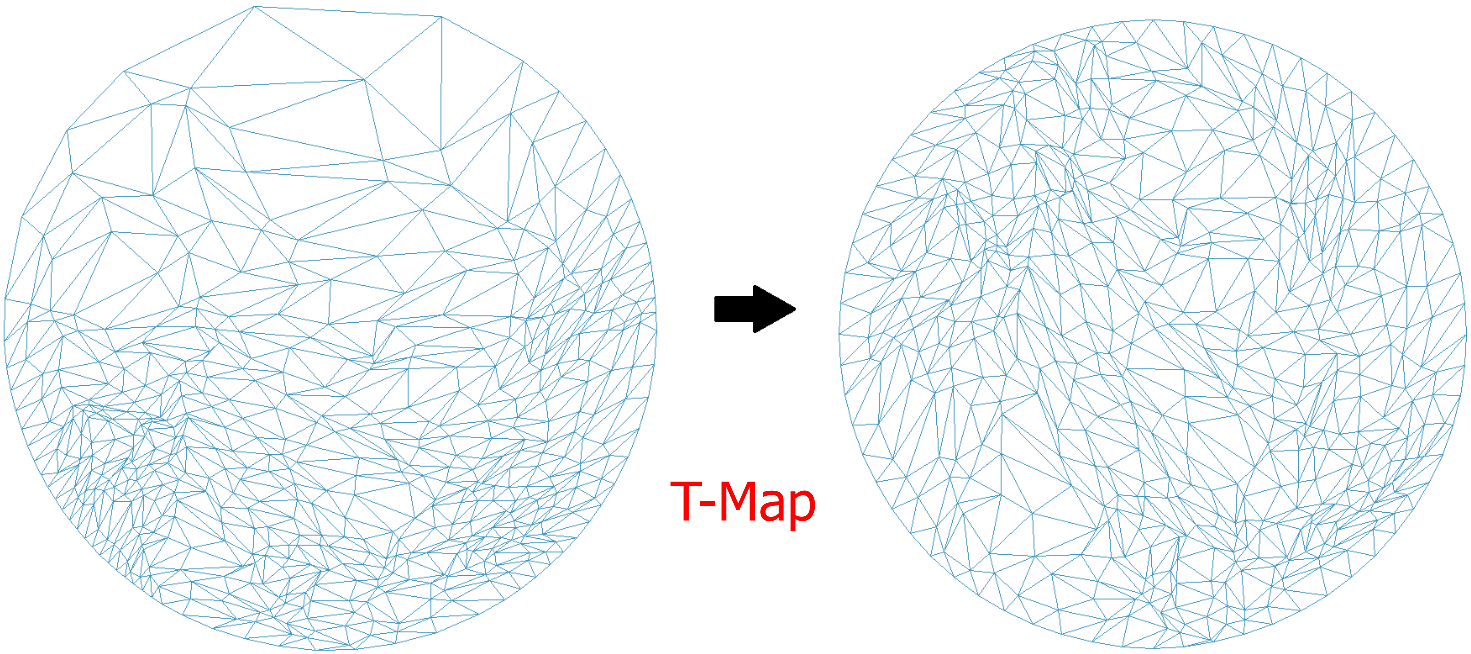


(A) (B) (C)

Numerical experiments 4

Irregular triangulation:

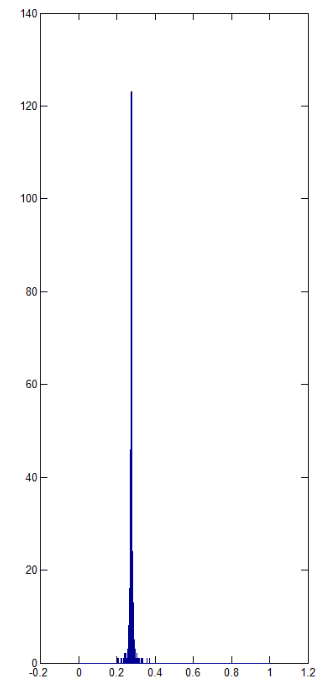
Irregular mesh



T-Map

(A)

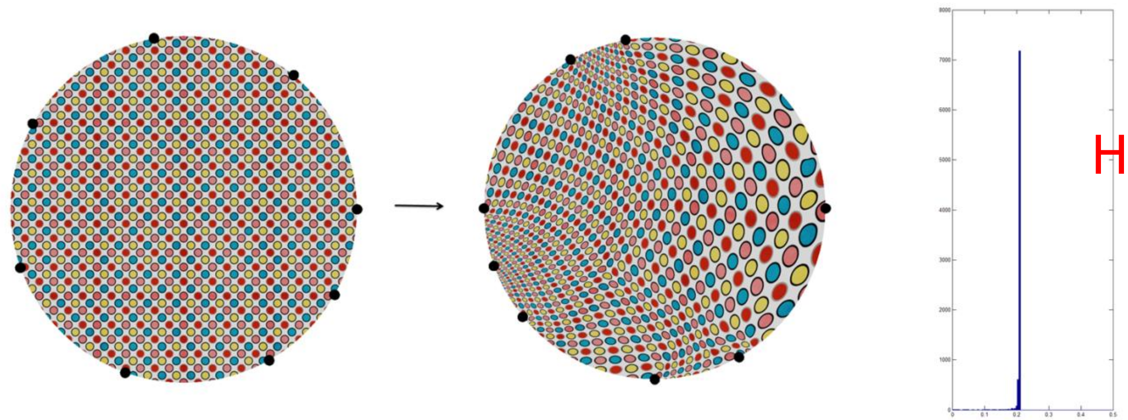
Histogram of the BC norm



(B)

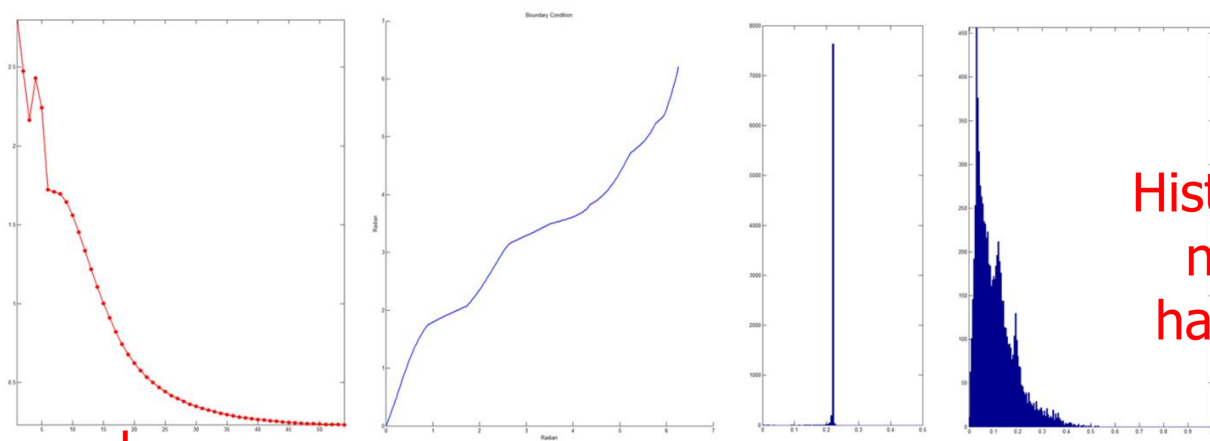
Numerical experiments 5

Without fixing the whole boundary:



(A) (B)

Histogram of BC norm under T-Map



(C) (D) (E) (F)

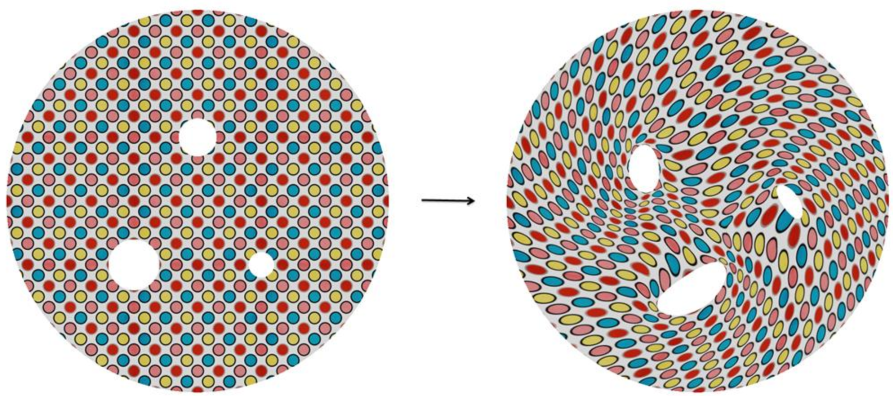
Histogram of BC norm under harmonic map

Norm of BC in each iteration

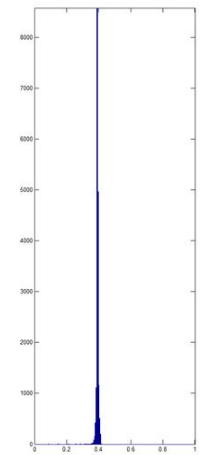
Automatic detected optimal boundary correspondence

Numerical experiments 6

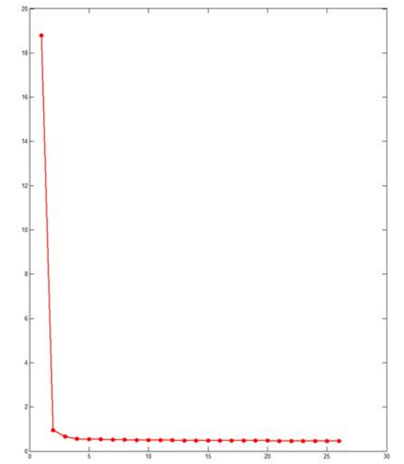
Multiply-connected domains:



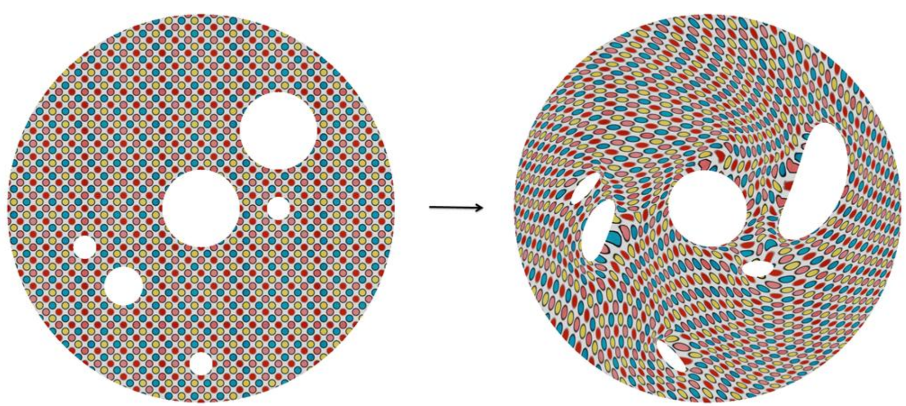
(A)



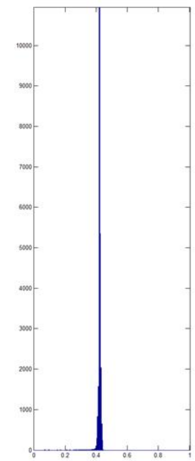
(B)



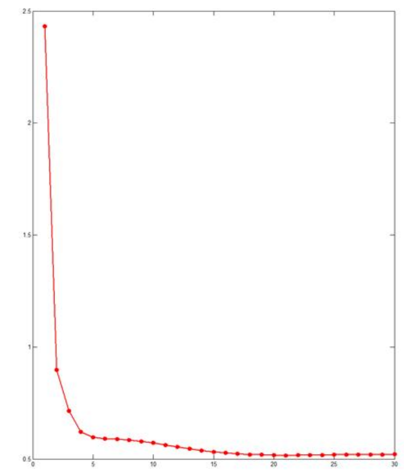
(C)



(A)



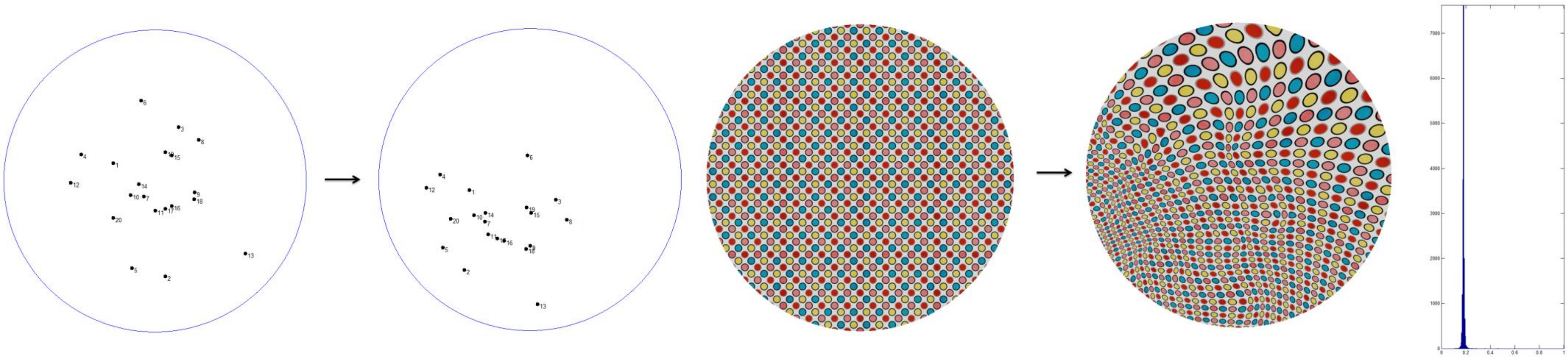
(B)



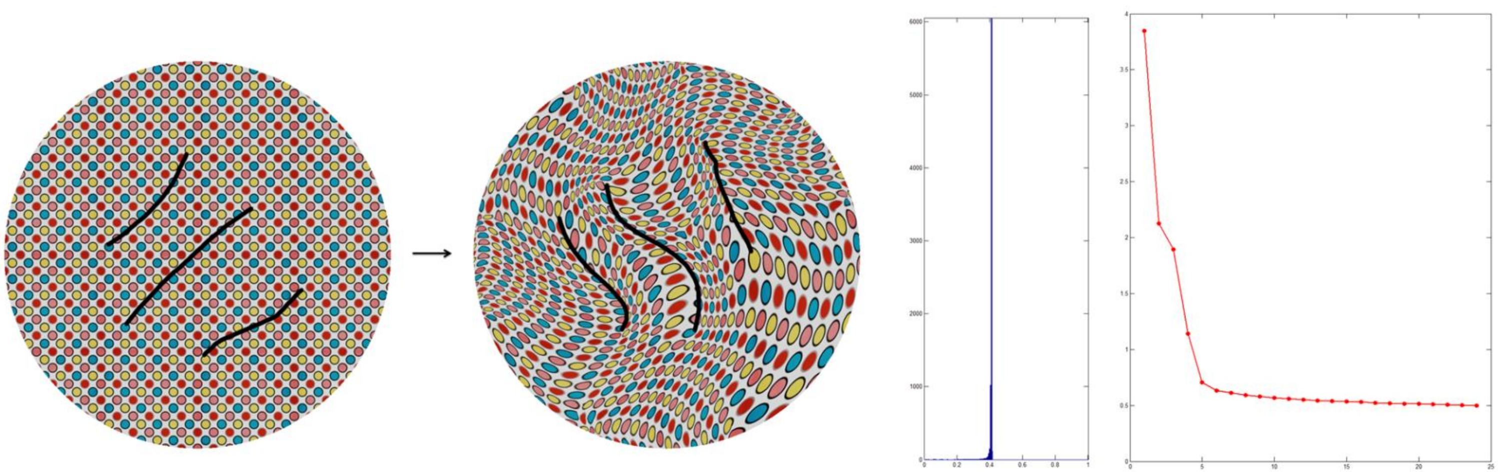
(C)

Numerical experiments 7

Interior landmark constraints:



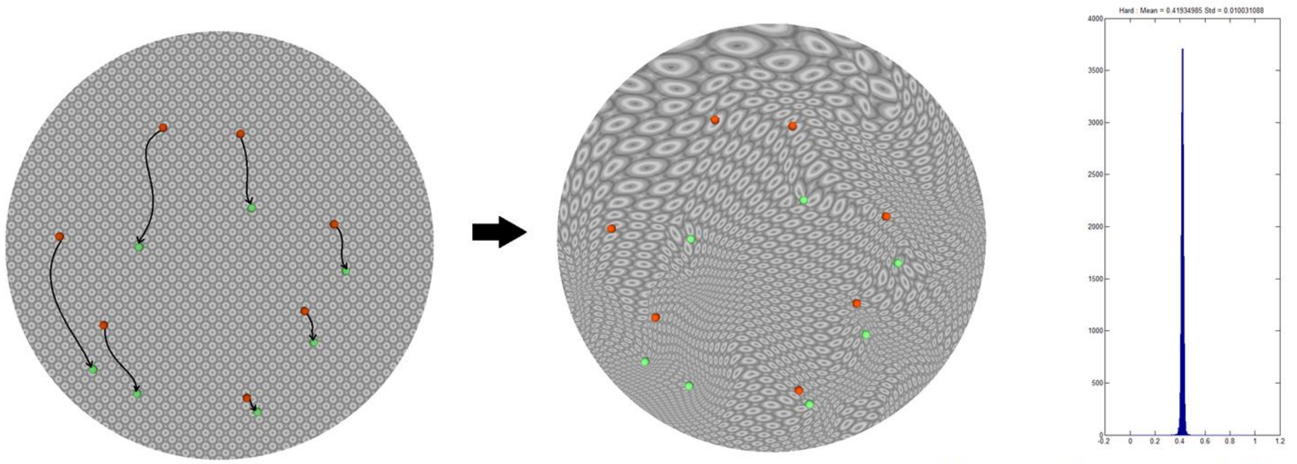
(A) (B) (C)



(A) (B) (C)

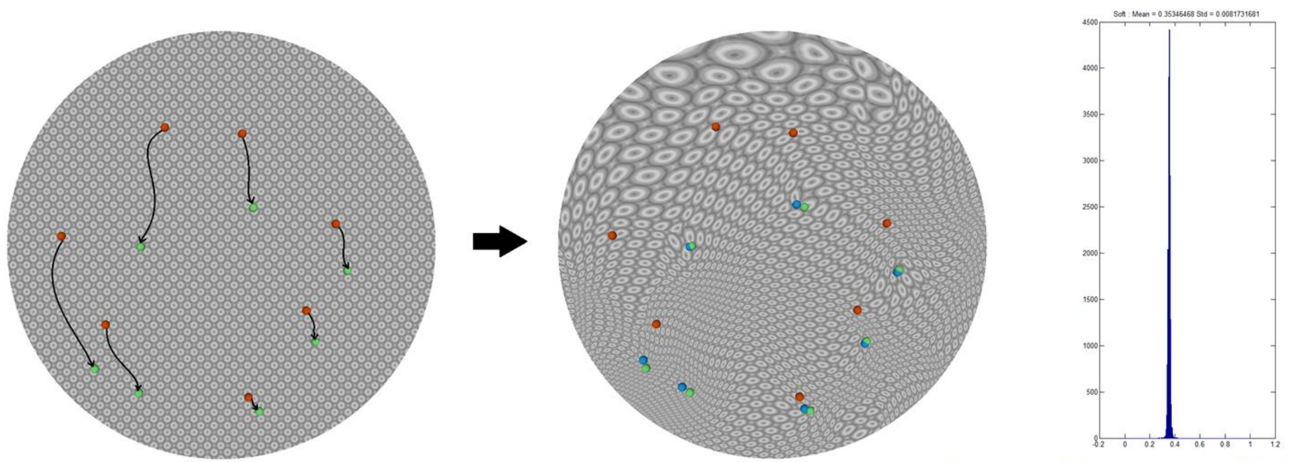
Numerical experiments 8

Soft landmark constraints:



(A) Hard constraints

Mean of the norm of BC: 0.4193

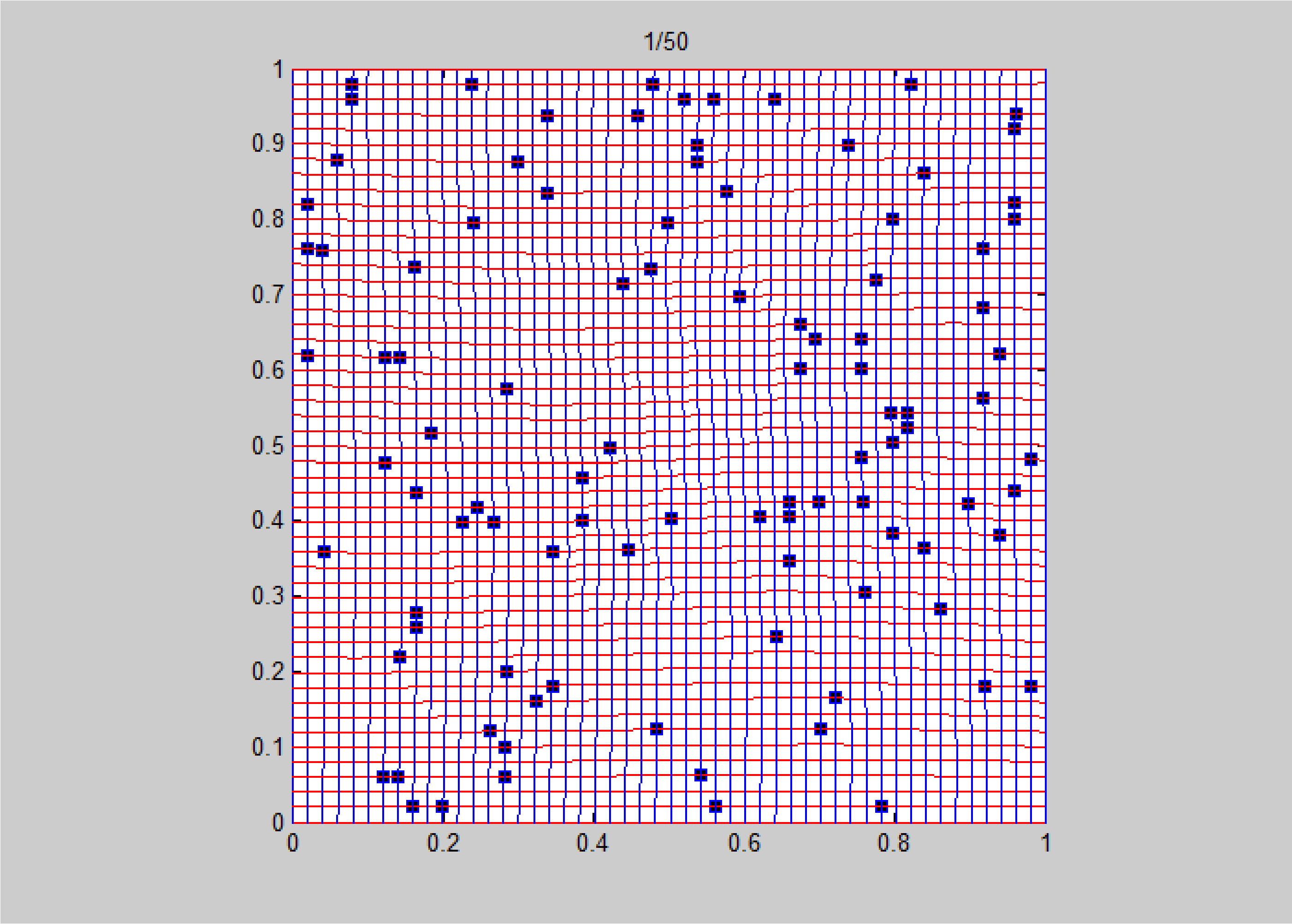


(B) Soft constraints

Mean of the norm of BC: 0.3535

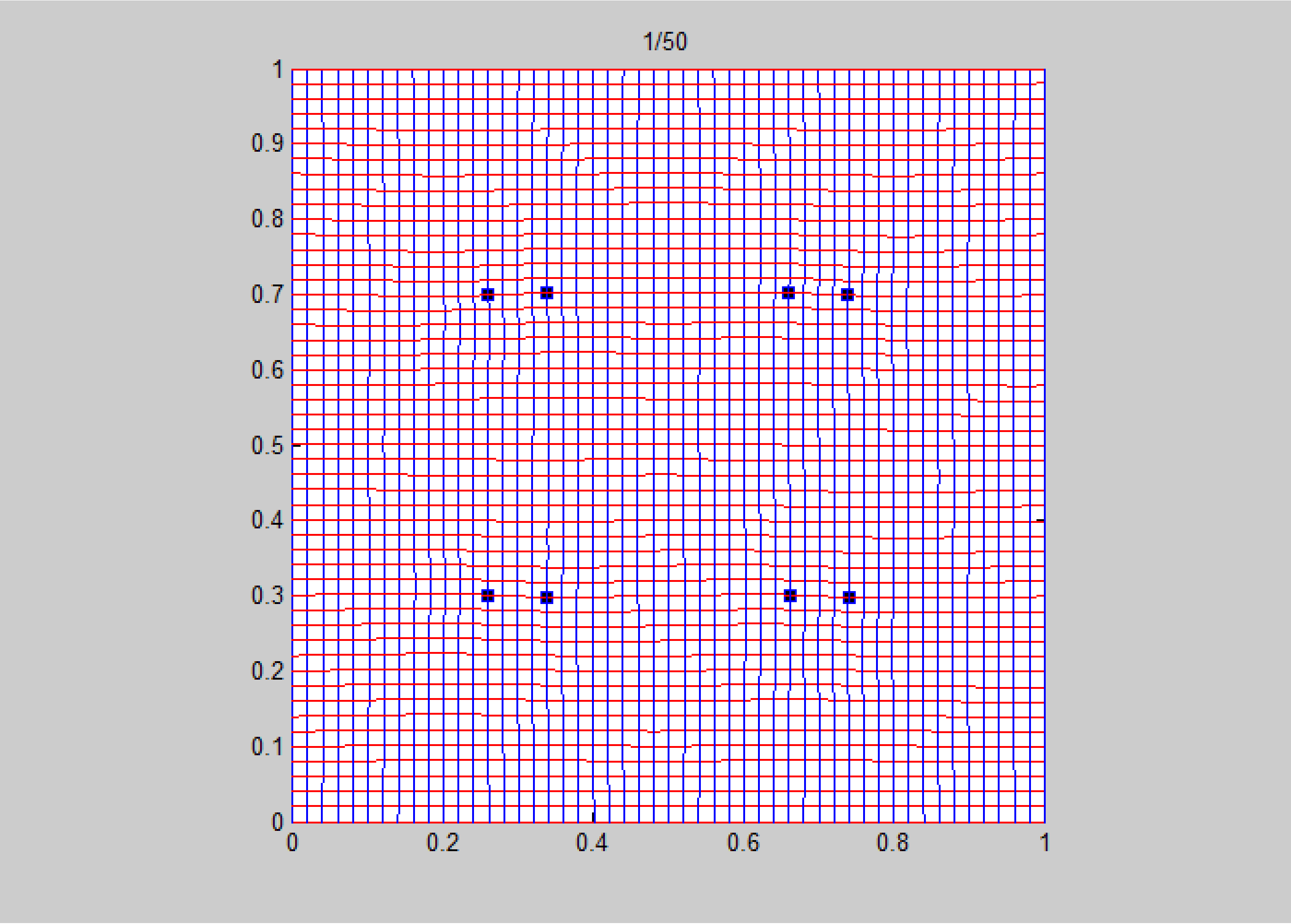
QC Iterations for large deformation

QC iterations obtain bijective map even with large deformation:



QC Iterations for large deformation

QC iterations obtain bijective map even with large deformation:



Computational time for QC iterations

Laptop machine: Intel Core i7 2.70 GHz CPU; 8 GB RAM (implemented using MATLAB)

Computational time for QC iterations

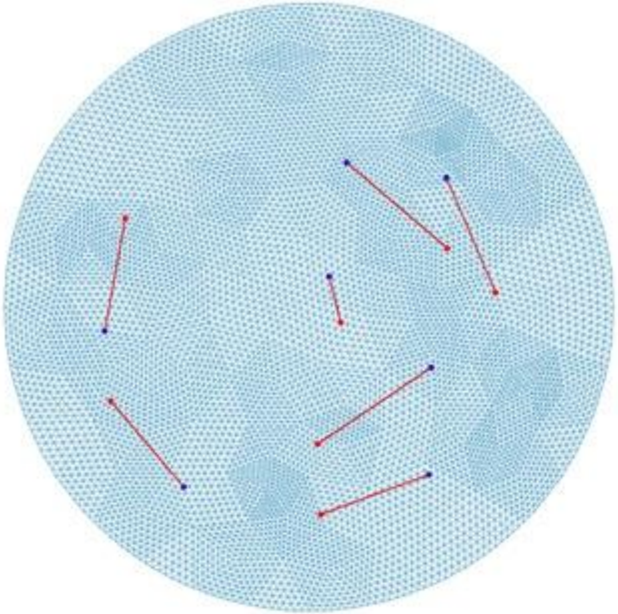
	Vertex number	Time	$\|\|\mu\|\|_\infty$
Analytic example	8936	1.297 s	0.4122
4 points on boundary + 3 landmark curves	8257	4.46 s	0.4154
Disk (Dirichlet boundary)	8257	5.420 s	0.2295
8 points on boundary	8257	6.645 s	0.2120
4 points on boundary + 20 landmarks	8257	8.467 s	0.2843
Arbitrary shape	8257	10.056 s	0.3475
Disk free boundary + 20 landmarks	8257	18.579 s	0.1855
Three holes disk	17746	15.030 s	0.4088
Six holes disk	22979	17.680 s	0.4433
Sphere	10242	34.679 s	0.3086

Comparison with other methods

Comparison with: 1. Harmonic Map 2. TPS 3. LDDMM

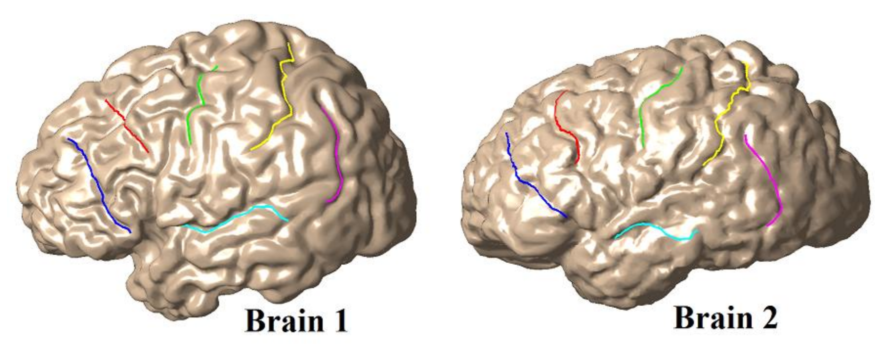
Comparison with other methods

	Teichmüller map	Harmonic map	TPS	LDDMM
Time	6.276 s	1.697s	0.117 s	416.247 s
Overlap faces	0	110	13	0
$\|\|\mu\|\|_\infty$	0.594	5.389	1	0.805



Outline Of The Talk

- Motivation
- Mathematical Background
- Computational Algorithms
- Applications
- Conclusion



Brain landmark matching registration

Goal:

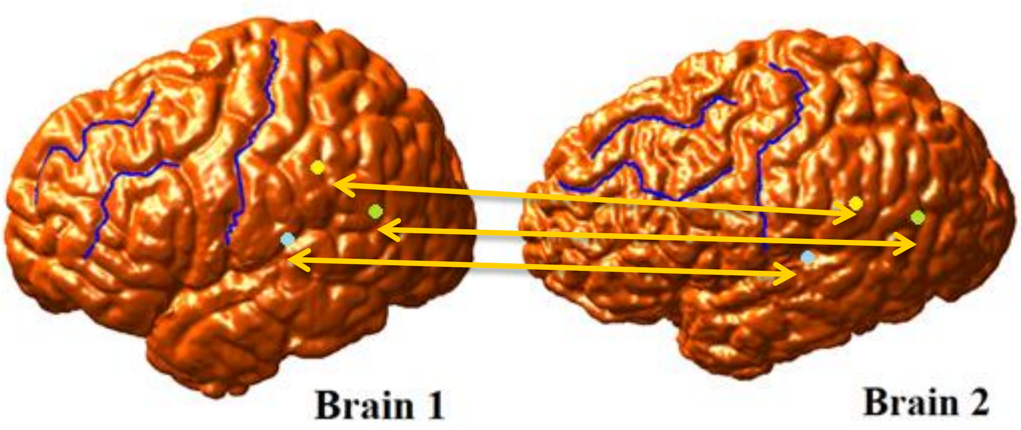
Find meaning 1-1 correspondence between brain surface matches sulci consistently.

Applications:

Statistical analysis; morphometry; processing,...

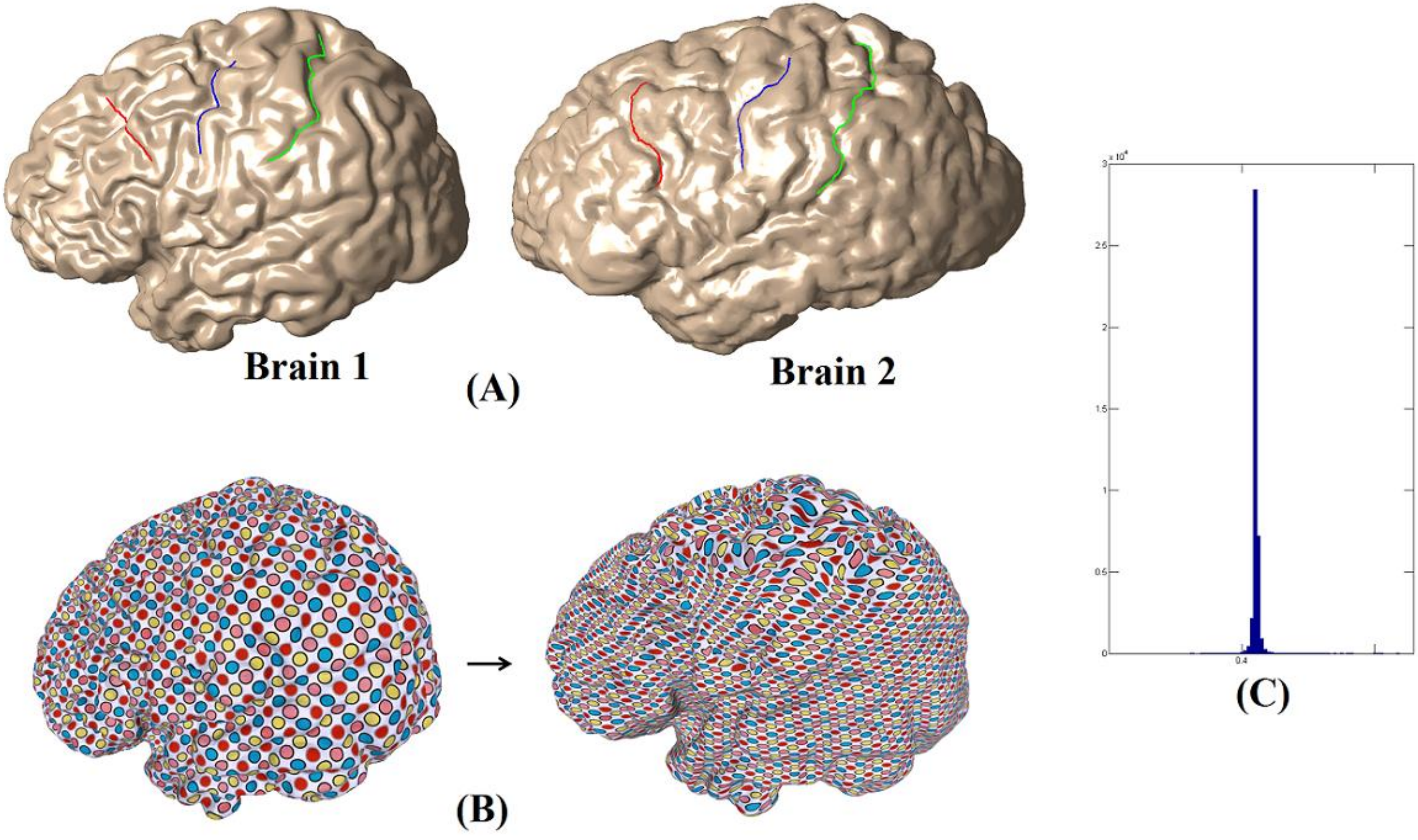
Difficulties:

Overlaps or flips occurs when there are large number of landmarks or with large deformation.



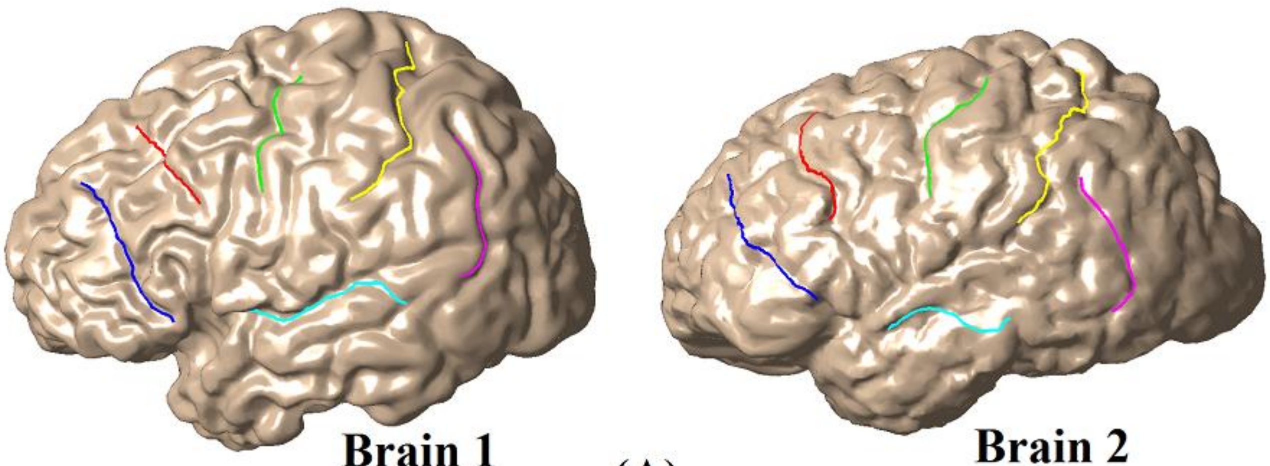
Brain landmark matching registration

T-Map for Brain registration with 3 sulcal landmarks

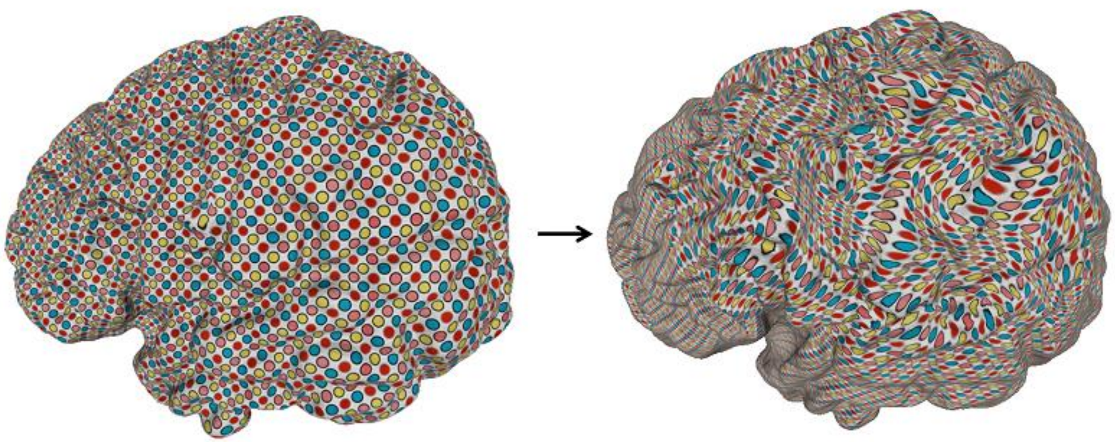


Brain landmark matching registration

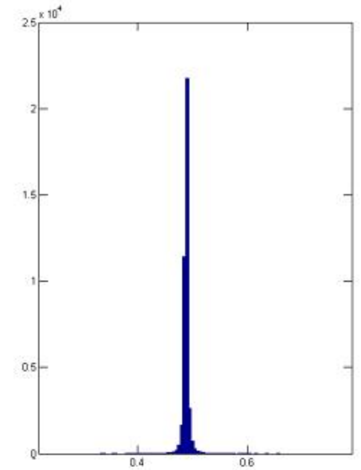
T-Map for Brain registration with 6 sulcal landmarks



(A)



(B)



(C)

Brainstem registration

Brainstem:

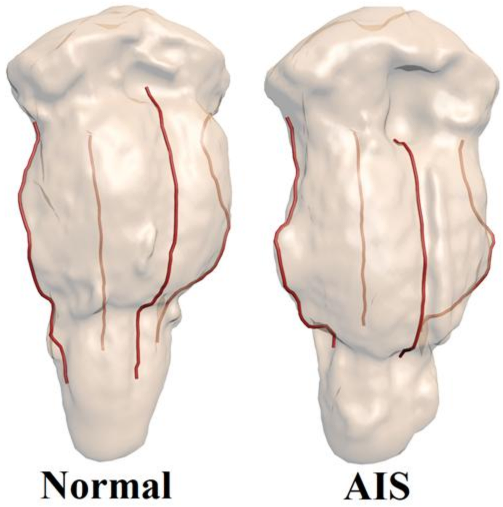
Anatomical brain structures which govern the balance control; regulate cardiac/respiratory function…

Goal:

Study Adolescent Idiopathic Scholiosis = 3D structural deformity of the spine

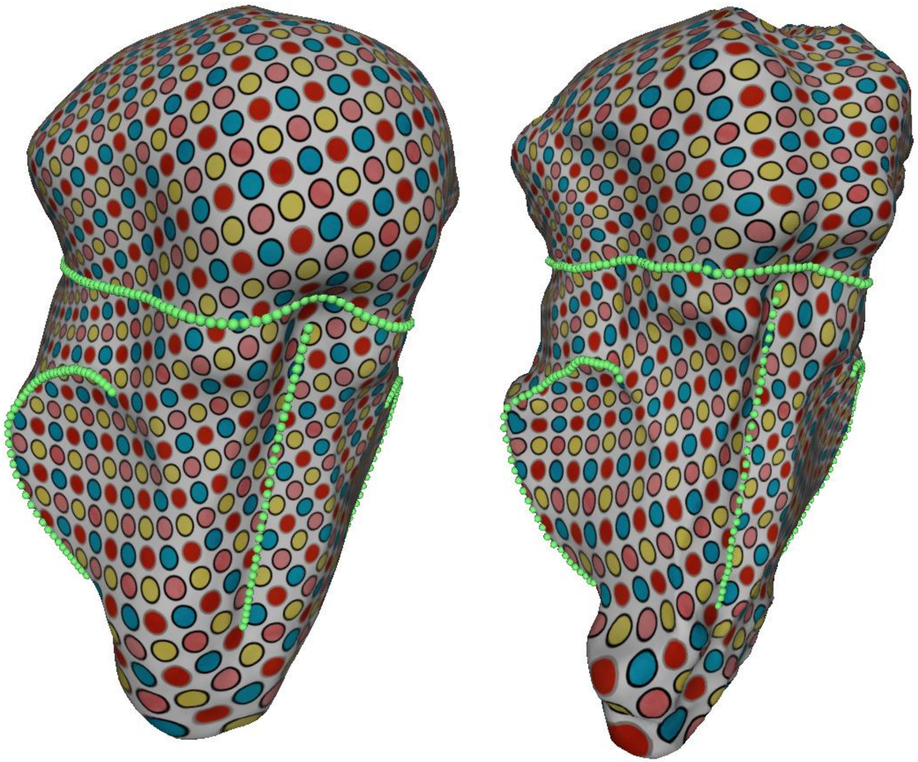
Method:

1. *Find meaningful surface registration;*
2. *Statistical shape analysis.*



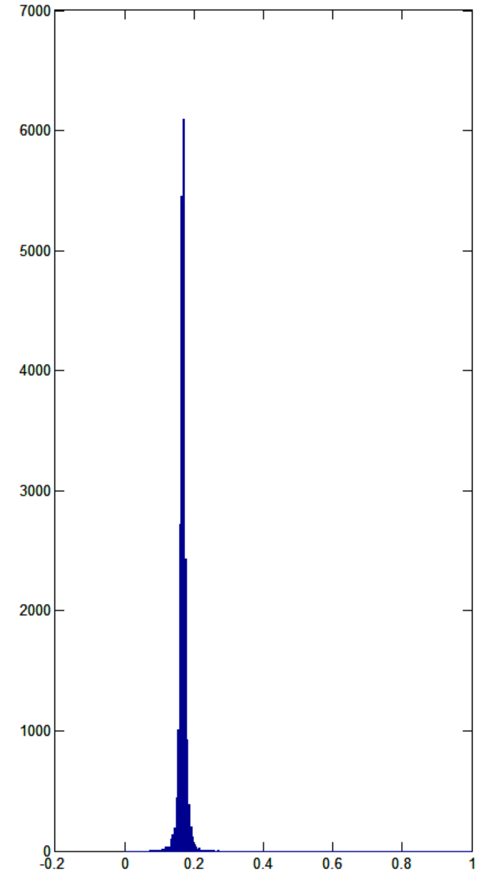
Brainstem registration

T-Map



Normal

AIS

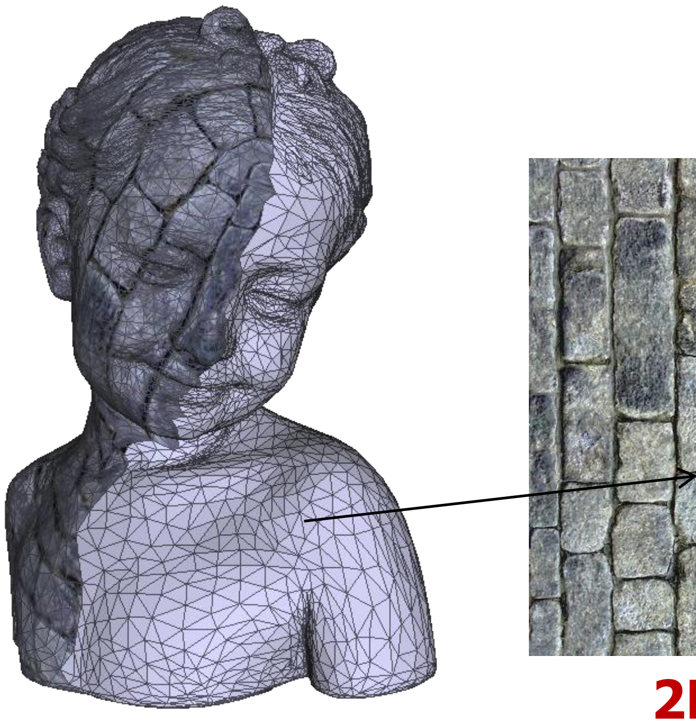


Histogram of the BC norm

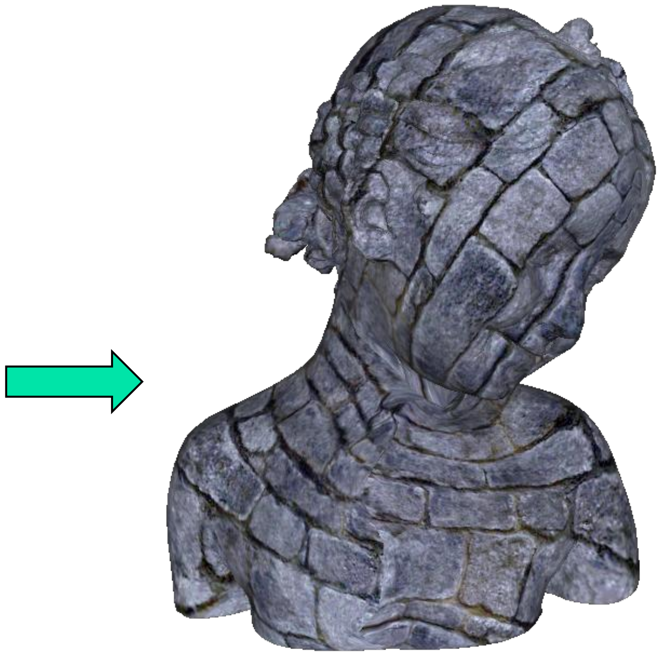
Constrained Texture Mapping

Texture mapping = map image onto a surface
(for surface decoration etc)

Idea: 1. Map vertices to 2D positions of an image;
(Correspondence guided by landmark features)
2. Color value is assigned for each vertex;
3. Color value inside the face by linear interpolation.



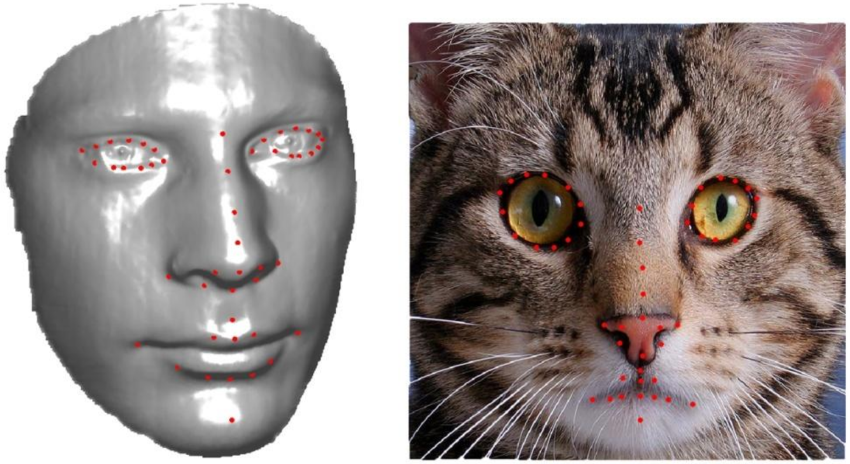
2D Image



Textured surface mesh

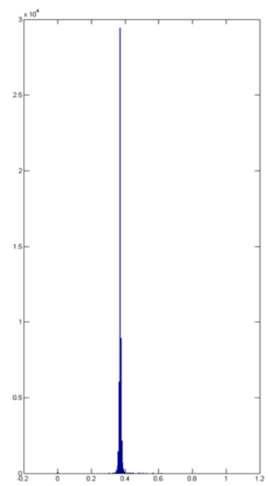
Constrained Texture Mapping

T-Map for constrained texture mapping



(A)

(B)

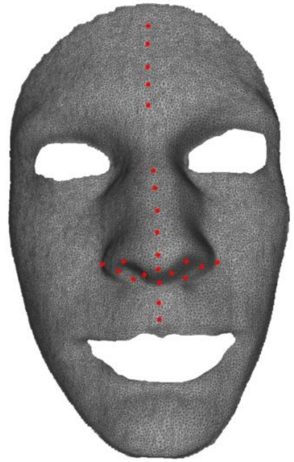


(C)

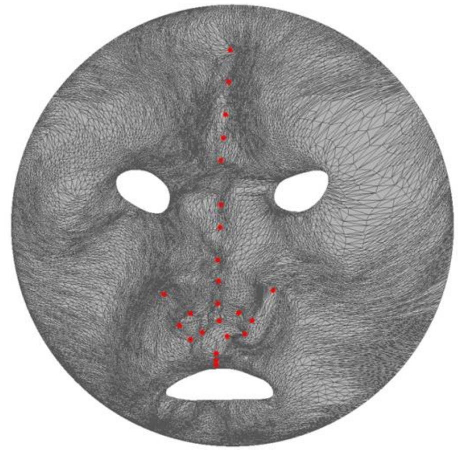
Constrained Texture Mapping

T-Map for constrained texture mapping

(A)

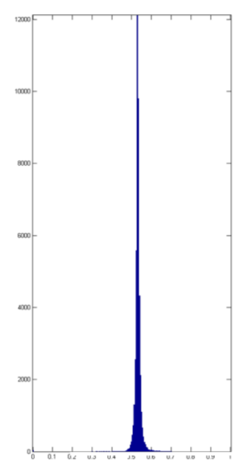


(B)



(C)

(D)

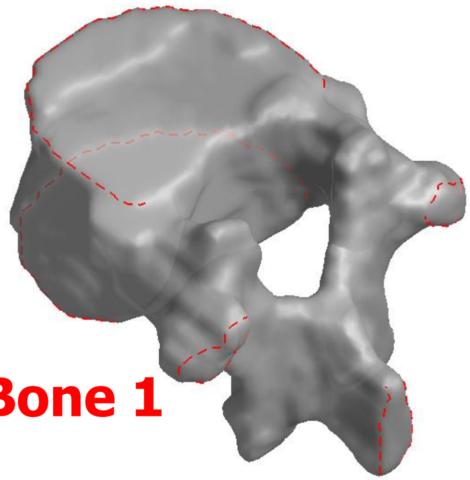


(E)

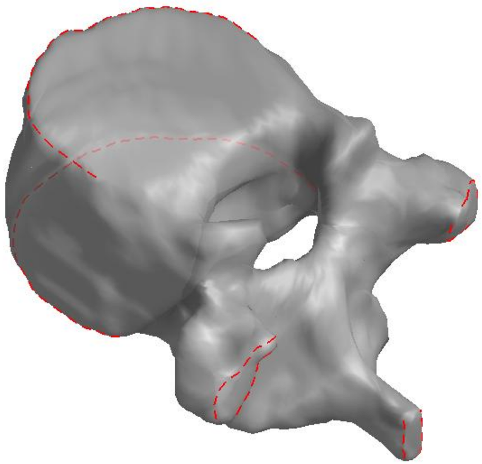
T-Map for high-genus surfaces

Vertebrae bone (genus-1) registration

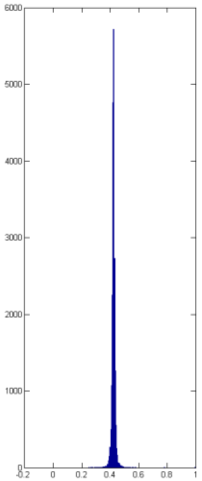
Shape morphometry: analysis of bone cancer, AIS etc…



Bone 1

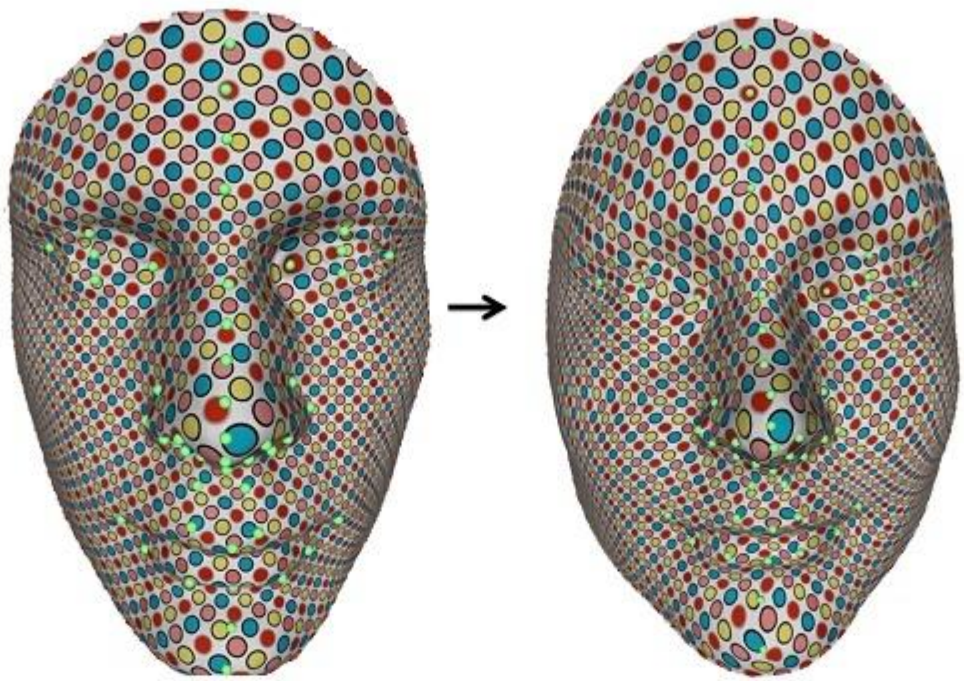


Bone 2

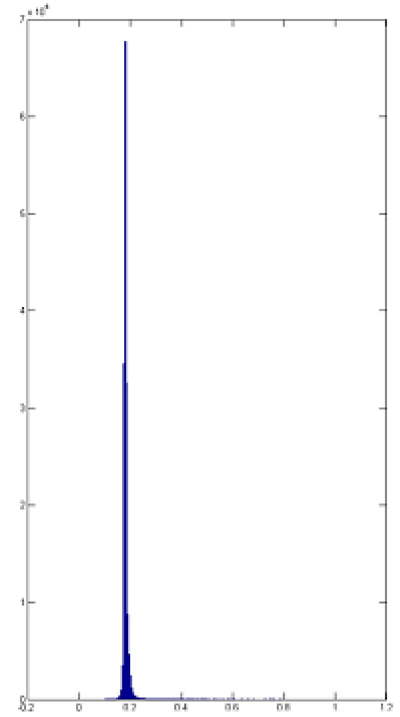


Human Face registration

T-Map for face registration



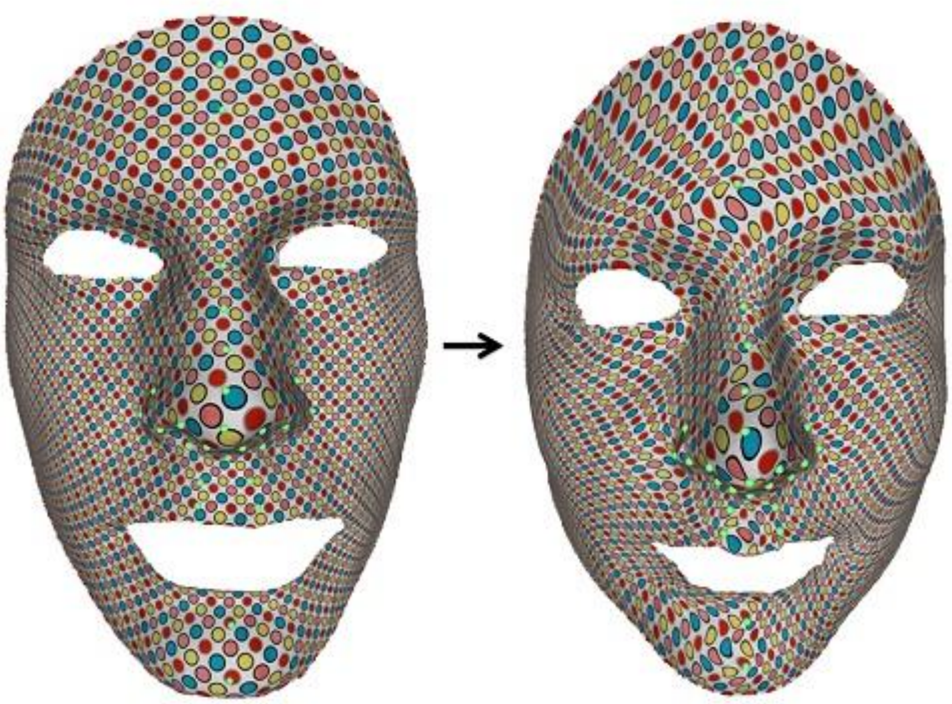
(B)



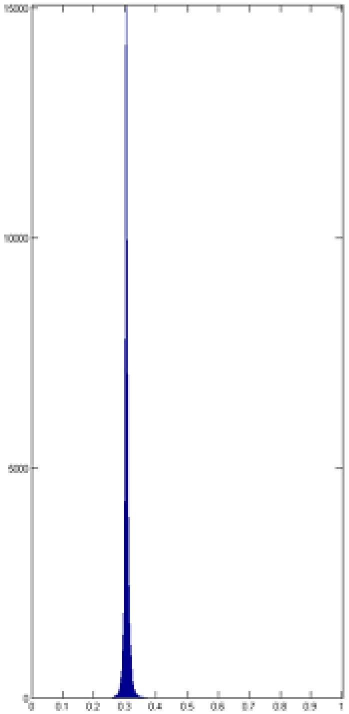
(C)

Human Face registration

T-Map for face registration



(B)



(C)

QC iterations with intensity matching

Extension of QC iterations:

Goal:
1. **Intensity matching;**
2. **Landmark matching;**
3. **Allow non-uniform conformality distortion.**

Key idea:

$$\mathcal{I}_{matching}(\nu) := \mathbf{argmin}_{\mu}\{\int_{S_1}(I_1 - I_2(f^{\mu})^2 + \int_{S_1}|\mu - \nu|^p + \int_{S_1}|\nabla\mu|^2\}$$

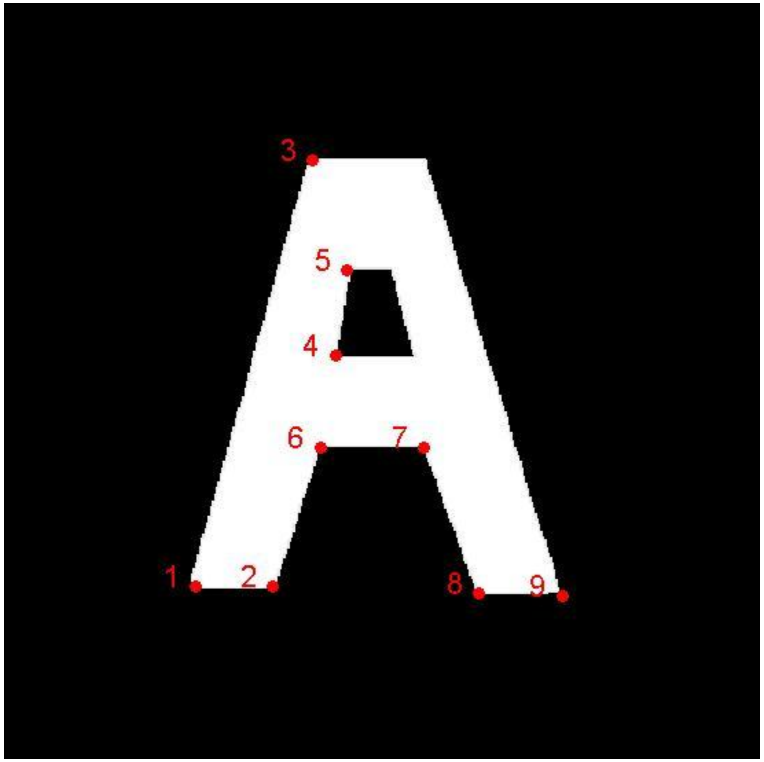
Solve by Alternating Direction Method of multipliers (ADMM)

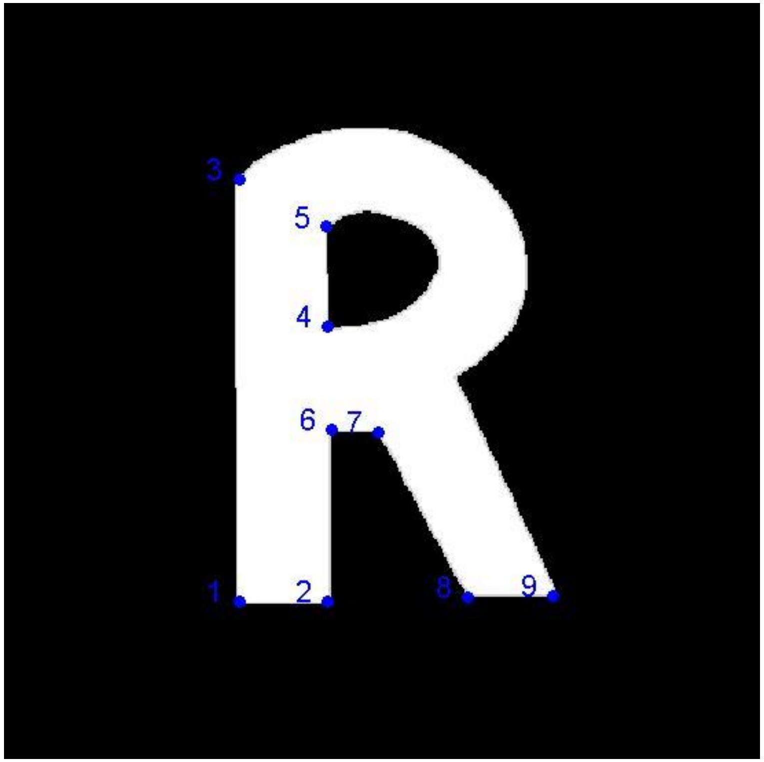
QC iterations with intensity matching

$$\mu_{n+1} := \mathcal{I}_{matching}(\nu_n);$$
$$f_{n+1} := \mathbf{LBS}_{LM}(\mu_{n+1});$$
$$\nu_{n+1} := \mu(f_{n+1}).$$

QC iterations with intensity matching

Example 1: 'A' to 'R'

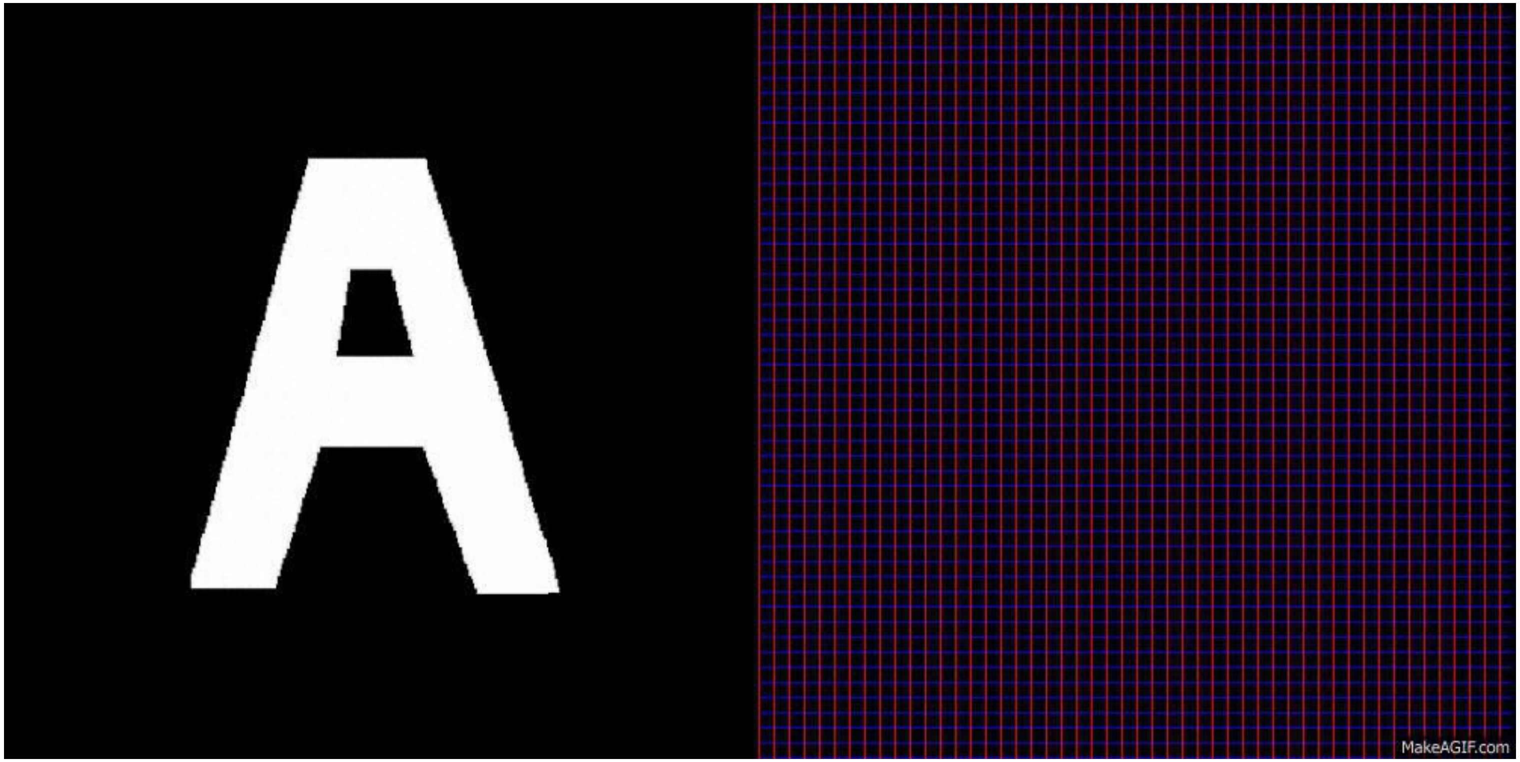




Intensity and landmark matching registration

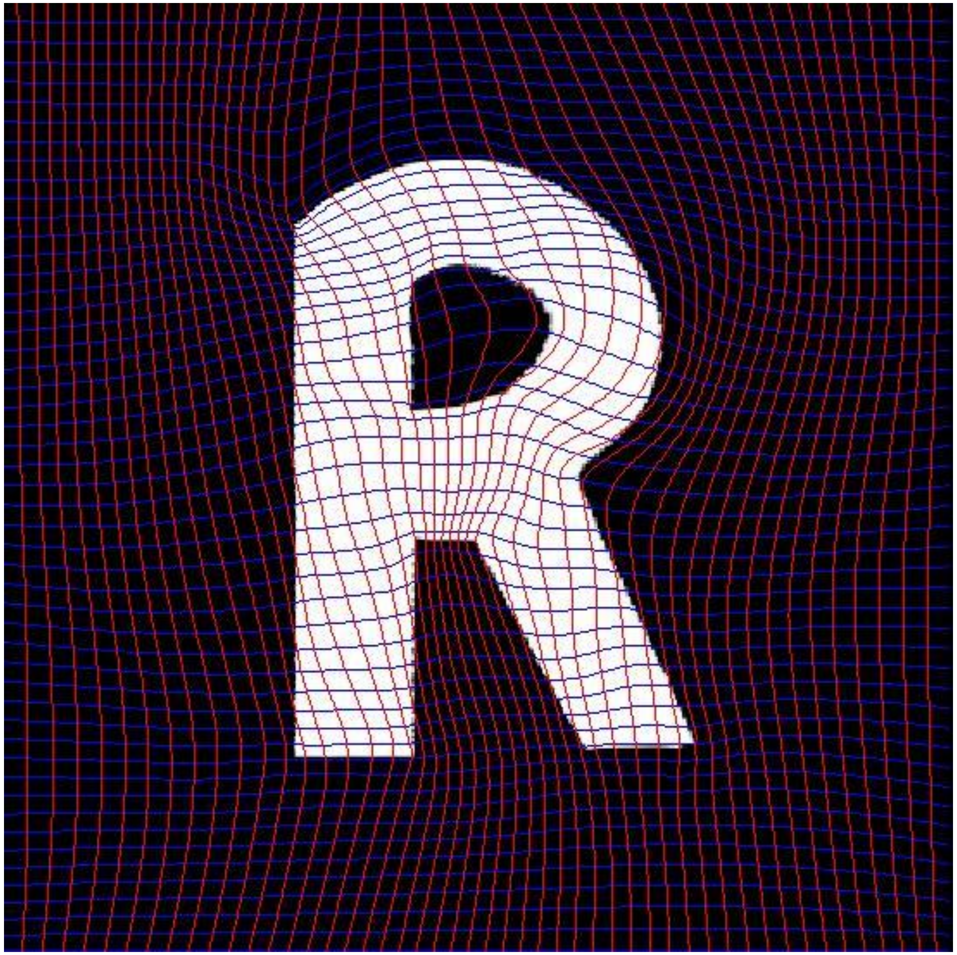
QC iterations with intensity matching

Example 1: 'A' to 'R'



Intensity and landmark matching registration

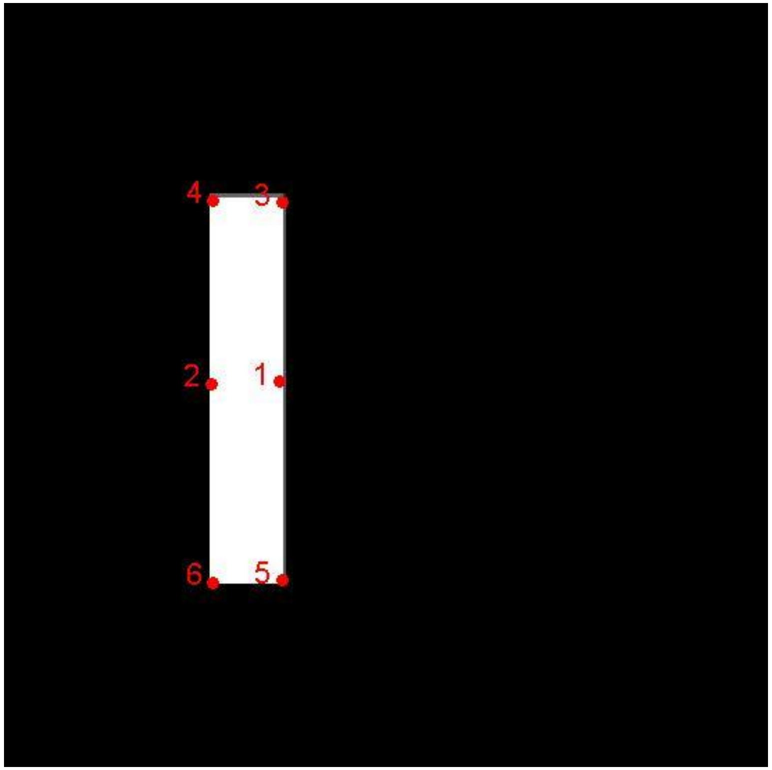
QC iterations with intensity matching

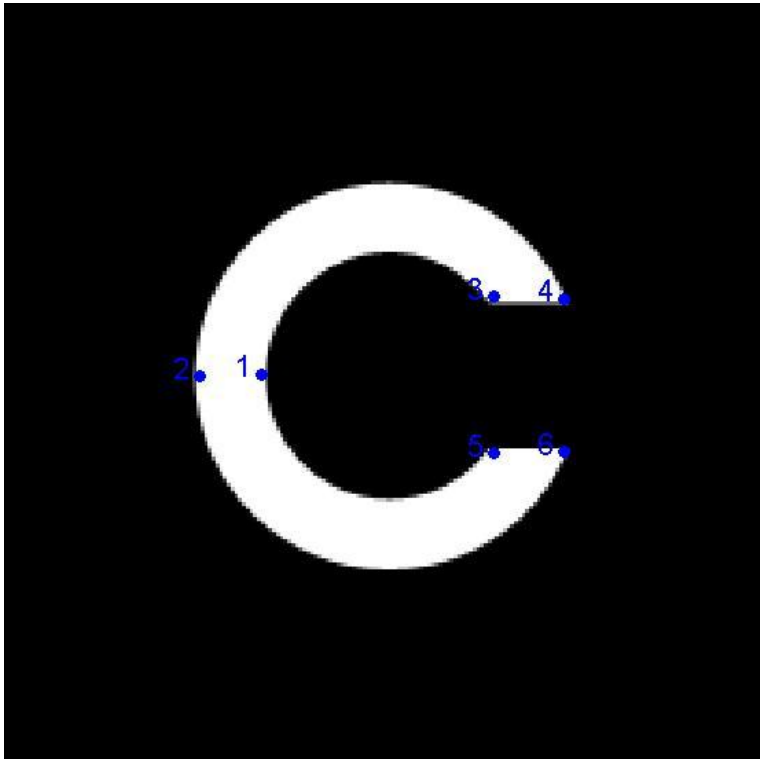


Intensity and landmark matching registration

QC iterations with intensity matching

Example 2: ‘I’ to ‘C’

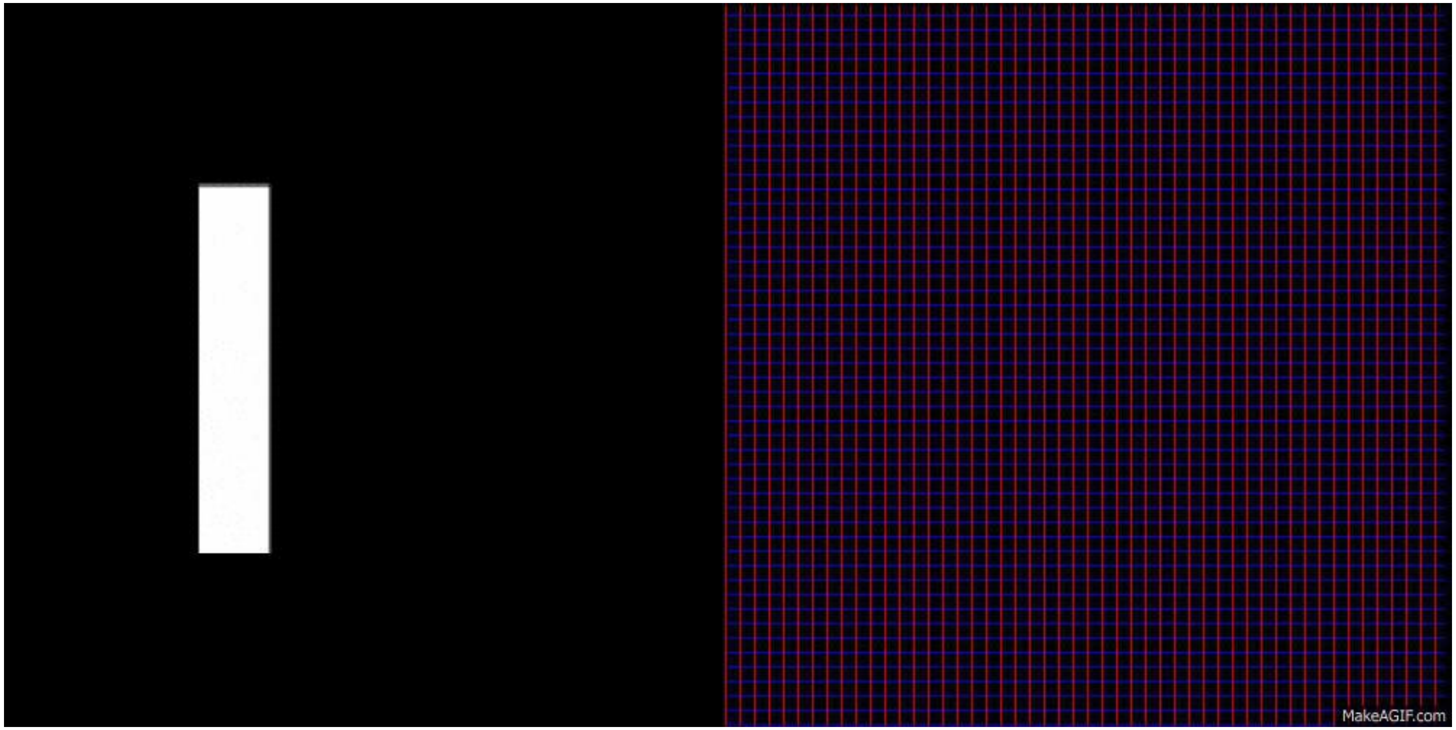




Intensity and landmark matching registration

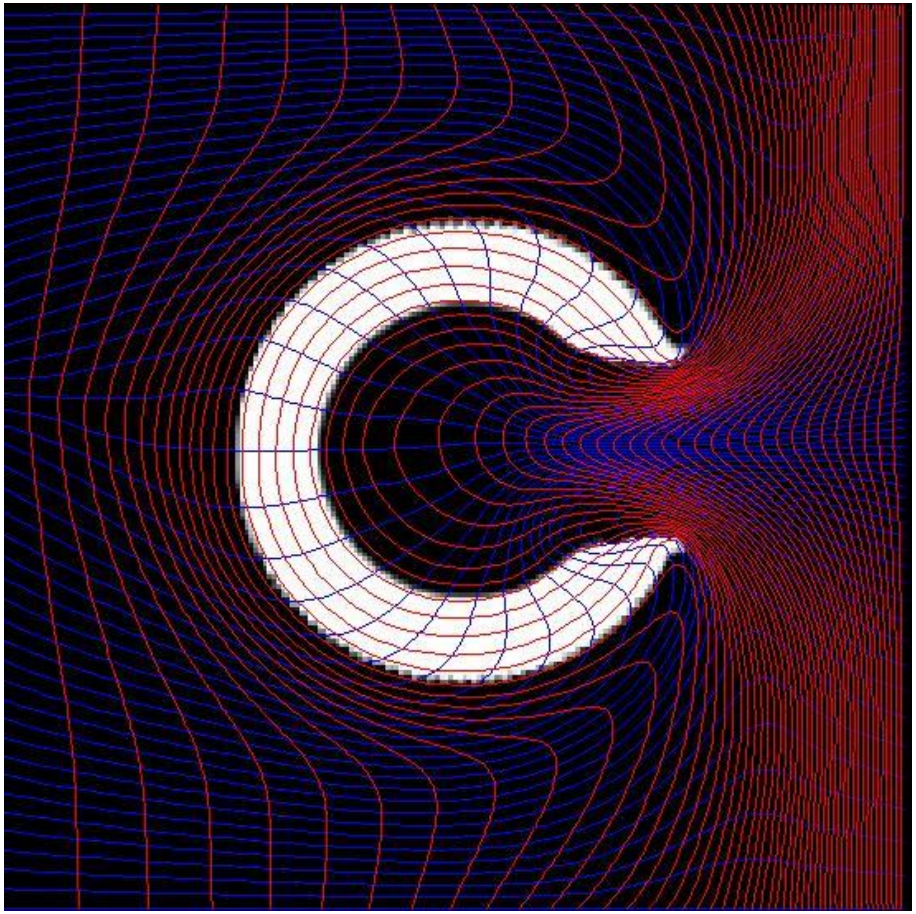
QC iterations with intensity matching

Example 2: 'I' to 'C'



Intensity and landmark matching registration

QC iterations with intensity matching



Intensity and landmark matching registration

Conclusion and Future works

- **Conclusion: T-Map**
  - Introduce T-Map: minimum and uniform local geometric distortion;
  - **QC iterations:** fast algorithm to compute T-Map
  - T-Map is suitable for landmark-matching registration: Every prescribed set of landmark constraints is associated to a UNIQUE T-Map
  - Applications: medical imaging, computer graphics and computer visions.
- **Future work:**
  - Study the convergence rate of QC iterations;
  - GPU implementation of QC iterations;
  - Extend the algorithms to point clouds;
  - Apply T-Map to medical morphometry…