Geodesic equation on the Universal Teichmller space, Teichons and Imaging

Sergey Kushnarev (Akil Narayan, Umass Dartmouth)

Singapore University of Technology and Design

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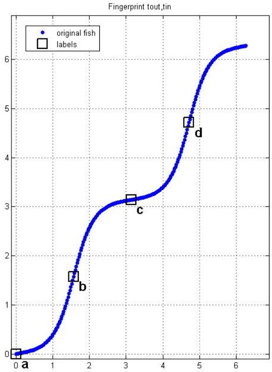
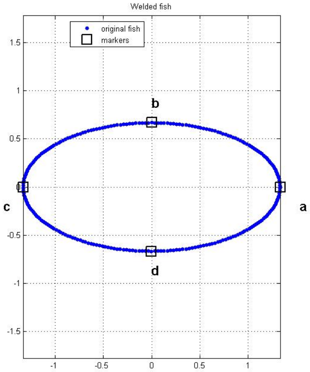
Outline

- Fingerprints – representation of the 2D shapes
- Universal Teichmüller space
- Weil-Petersson metric
- EPDiff(S^1)
- Geodesic finding
- Teichons
- Application

Shape parametrization

$$\mathrm{PSL}_2(\mathbb{R})\backslash \mathbf{Diff}(S^1) \cong \text{ set of shapes/translations, scalings}$$





Beltrami differential μ

Unit disk $\Delta \subset \mathbb{C}$.

$$L^{\infty}(\Delta)_1 = \{\mu \in L^{\infty}(\Delta) : \|\mu\|_{\infty} < 1\}.$$

K-quasiconformal map f:

$$\partial_{\bar{z}} f = \mu(z) \partial_z f,$$

where $K = \frac{1+\|\mu\|_{\infty}}{1-\|\mu\|_{\infty}}$.

Universal Teichmüller space

Model A: $\mathbf{Homeo}_{qs}(S^1)$. $\mu \in L^\infty(\Delta)_1$ is the Beltrami differential.

$$\mu\left(\frac{1}{\bar{z}}\right) = \bar{\mu}(z)\frac{z^2}{\bar{z}^2}, z \in \Delta$$

f_μ is the unique solution to the Beltrami equation:

$$\partial_{\bar{z}} f_\mu = \mu(z)\partial_z f_\mu, \quad (\mu(z) = \mu(z)\frac{\partial_z}{\partial_{\bar{z}}})$$

$\mu, \nu \in L^\infty(\Delta)_1$: $\mu \sim \nu$ iff $f_\mu \mid_{S^1} = f_\nu \mid_{S^1}$.

$$Teich(\Delta) = L^\infty(\Delta)_1 / \sim .$$

$$Teich(\Delta) = \mathbf{Homeo}_{qs}(S^1)/\mathrm{PSL}_2(\mathbb{R}).$$

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Universal Teichmüller space

Model B: quasicircles. $\mu \in L^\infty(\Delta)_1$ is the Beltrami differential. $\mu(z) = 0$ for $z \in \Delta^*$. f^μ is the solution to the Beltrami equation:

$$\partial_{\bar{z}} f^\mu = \mu(z) \partial_z f^\mu,$$

$f^\mu : \Delta^* \to D^\mu := f(\Delta^*)$ is conformal.

$$\mu \sim \nu \text{ iff } f^\mu |_{\Delta^*} = f^\nu |_{\Delta^*} .$$

$$Teich(\Delta) = L^\infty(\Delta)_1 / \sim .$$

In other words $Teich(\Delta) = \{\text{quasicircles}\}/\{\text{translations, scaling}\}$

Universal Teichmüller space

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Universal Teichmüller space

Model A $\Leftrightarrow$ **Model B**

$$f_\mu \mid_{S^1} = f_\nu \mid_{S^1} \Leftrightarrow f^\mu \mid_{\Delta^*} = f^\nu \mid_{\Delta^*} .$$

Therefore

$$\mathbf{Homeo}_{qs}(S^1)/\mathrm{PSL}_2(\mathbb{R}) \cong \{\text{quasicircles}\}/\{\text{translations, scaling}\}$$

$$\mathbf{Diff}(S^1)/\mathrm{PSL}_2(\mathbb{R}) \cong \{\text{smooth shapes}\}/\{\text{translations, scaling}\}$$

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$$\mathbf{Diff}(S^1)/\mathrm{PSL}_2(\mathbb{R}) \cong \{\text{smooth shapes}\}/\{\text{translations, scaling}\}$$

Shape $\to$ fingerprint?

Now, without the scary words:

$$\mathrm{PSL}_2(\mathbb{R})\backslash \mathbf{Diff}(S^1) \cong \text{ set of shapes/translations, scalings}$$

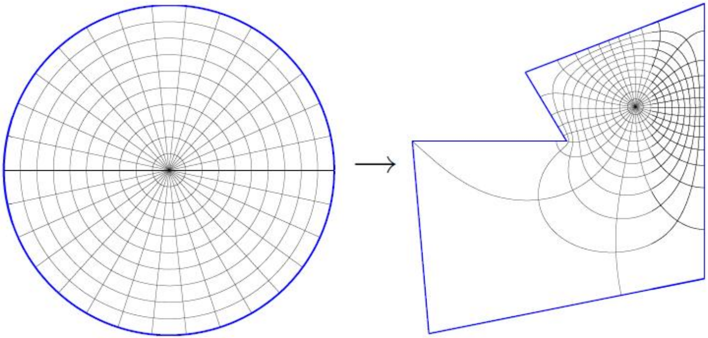
By Riemann mapping theorem

$$\phi_{\mathrm{int}} : \mathbb{D}_{\mathrm{int}} \to \Gamma_{\mathrm{int}},$$

$$\phi_{\mathrm{ext}} : \mathbb{D}_{\mathrm{ext}} \to \Gamma_{\mathrm{ext}}.$$

Shape $\rightarrow$ fingerprint?

$$\phi_{\text{int}} : \mathbb{D}_{\text{int}} \rightarrow \Gamma_{\text{int}},$$



How to get conformal maps?

- Schwarz-Christoffel toolbox for Matlab, Toby Driscoll.

  http://www.math.udel.edu/~driscoll/software/SC/

- Zipper algorithm, Don Marshall.

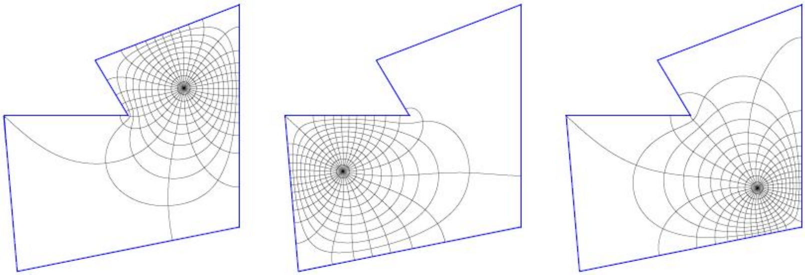
  http://www.math.washington.edu/~marshall/zipper.html

Shape $\to$ fingerprint?

ϕ_{int} is unique up to $\phi_{\text{int}} \circ A$
$A : \mathbb{D}_{\text{int}} \to \mathbb{D}_{\text{int}}$, subgroup of Möbius transformations
$A(z) \in \mathrm{PSL}_2(\mathbb{R})$,

$$A(z) = \frac{az + b}{\bar{b}z + \bar{a}}, |a|^2 - |b|^2 = 1$$

.

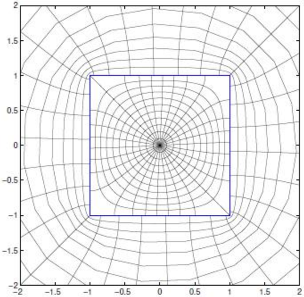


Shape $\rightarrow$ fingerprint?

$$\phi_{\text{ext}}(S^1) = \Gamma, \; \phi_{\text{int}}^{-1}(\Gamma) = S^1$$

$$\psi = \phi_{\text{int}}^{-1} \circ \phi_{\text{ext}} \in \mathrm{PSL}_2(\mathbb{R}) \backslash \mathbf{Diff}(S^1),$$

defined on the circle S^1



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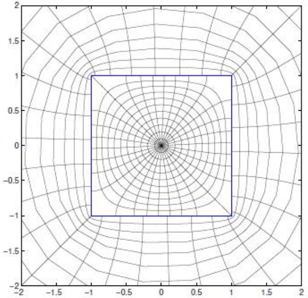
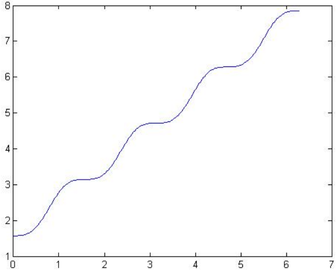
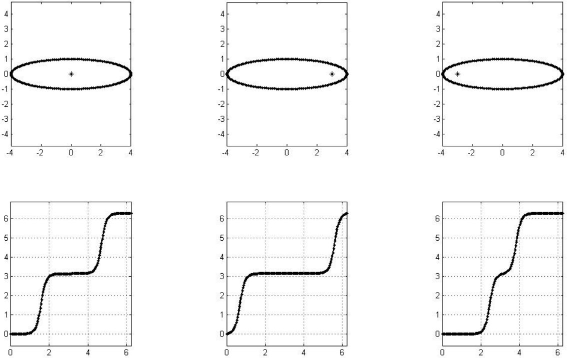




Möbius ambiguity

One shape, different fingerprints: $A_1 \circ \psi, A_2 \circ \psi, A_3 \circ \psi$. Where $A_k \in \mathrm{PSL}_2(\mathbb{R})$.



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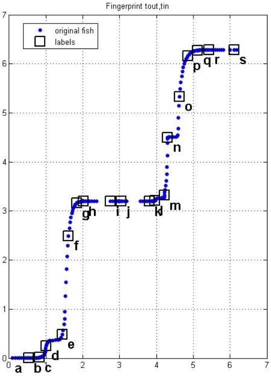

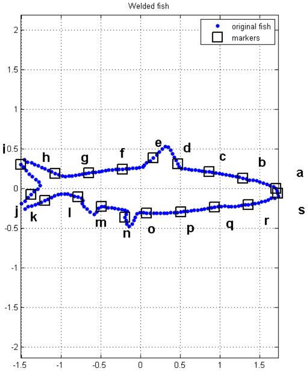

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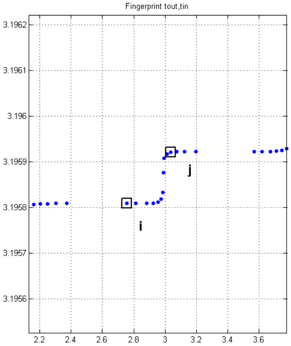

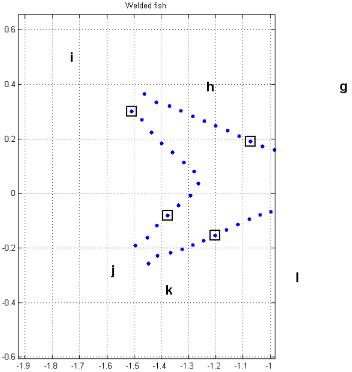

Fingerprint $\rightarrow$ shape?

Welding: $F(\theta) := f_{\text{ext}}(e^{i\theta}) = f_{\text{int}}(e^{i\theta}) \circ \Psi$.

$$K(F) + F = e^{i\theta},$$

$$K(F)(\theta_1) = \frac{i}{2} \int \left(\cot\left(\frac{\theta_1 - \theta_2}{2}\right) - \Psi'(\theta_2) \cot\left(\frac{\Psi(\theta_1) - \Psi(\theta_2)}{2}\right)\right) F(\theta_2) d\theta_2$$

$$K_{\alpha,\beta} = 2 \log \left| \frac{\sin(\theta^{\alpha} - \theta^{\beta+1/2}) \sin(\Psi(\theta^{\alpha}) - \Psi(\theta^{\beta-1/2}))}{\sin(\theta^{\alpha} - \theta^{\beta-1/2}) \sin(\Psi(\theta^{\alpha}) - \Psi(\theta^{\beta+1/2}))} \right|,$$

E. Sharon and D. Mumford, 2D-Shape Analysis using Conformal Mapping, IJCV, V. 70, 1.

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So far...

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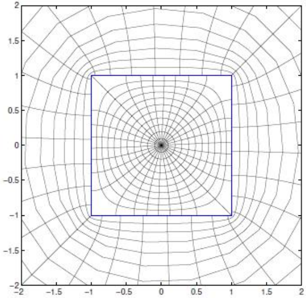

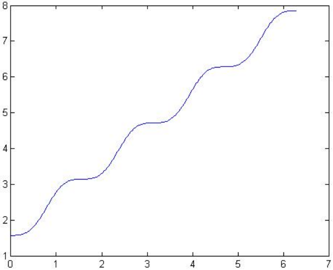

Weil-Petersson metric

Traditionally, $\mu, \nu \in T(1)$

$$\langle \mu, \nu \rangle_{WP} = \int_{\Delta} \mu \bar{\nu} d\rho$$

Nag, Verjovsky* showed, that for fingerprints, given $v(\theta) = \sum_{n=-\infty}^{\infty} v_n e^{in\theta} \in \textbf{Vec}(S^1)$:

$$\|v\|_{WP}^2 = \sum_{n \in \widehat{\mathbb{Z}}} |n^3 - n||v_n|^2$$

or

$$\|v\|_{WP}^2 = \int_{S^1} Lv.vd\theta,$$
$$= -\int_{S^1} \mathcal{H}(v_{\theta\theta\theta} + v_\theta).vd\theta.$$

* **Diff**(S^1) and the Teichmuller spaces, S Nag and A Verjovsky. Comm. Math. Phys. V 130, N1, 1990, 123-138.

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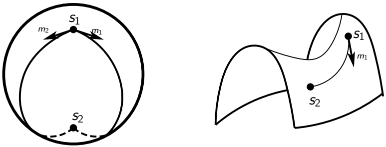
Why WP metric?

Unique geodesics



Recap

- Shape S (modulo translations and scaling) represented by fingerprint $\phi \in \mathrm{PSL}_2(\mathbb{R})\backslash \mathbf{Diff}(S^1)$.
- Metric on $\mathrm{PSL}_2(\mathbb{R})\backslash \mathbf{Diff}(S^1)$: Weil-Petersson.

$$\|v\|_{WP}^2 = \sum_{n \in \widehat{\mathbb{Z}}} |n^3 - n||v_n|^2.$$

- Goal: find geodesics between shapes, i.e. geodesics in space $\mathrm{PSL}_2(\mathbb{R})\backslash \mathbf{Diff}(S^1)$.

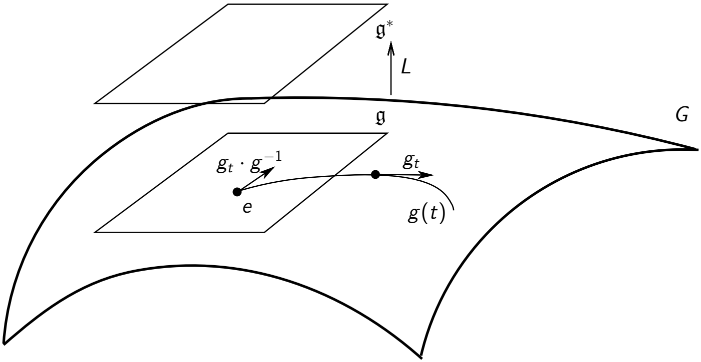
Theorem (Arnold, 1966*)

Let G be any Lie group. Let $\langle v_1, v_2 \rangle_L = (L(v_1), v_2)$ be a positive-definite symmetric inner product on the Lie algebra $\mathfrak{g}$, defined by a symmetric linear map $L : \mathfrak{g} \to \mathfrak{g}^*$. For $v_1, v_2 \in T_g G$: $\langle v_1, v_2 \rangle_g = \langle D_g R_{g^{-1}} v_1, D_g R_{g^{-1}} v_2 \rangle_e = \langle v_1 \cdot g^{-1}, v_2 \cdot g^{-1} \rangle$. Let $g(t)$ be any path in G and define $u(t) = g_t \cdot g^{-1}$ to be its tangent path in $\mathfrak{g}$. Then:

$$g(t) \subset G \text{ is a geodesic } \iff Lu_t = -\mathrm{ad}^*_u(Lu) \text{ in } \mathfrak{g}^*$$

where $\mathrm{ad}^*_u : \mathfrak{g}^* \to \mathfrak{g}^*$ is the adjoint of $\mathrm{ad}_u, u \in \mathfrak{g}$.

*Vladimir Arnold. *Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluides parfaits.* Annales de l'institut Fourier, 16 no. 1 (1966), p. 319-361



$v_1, v_2 \in T_g G$:

$\langle v_1, v_2 \rangle_g = \langle v_1 \cdot g^{-1}, v_2 \cdot g^{-1} \rangle = \Big(L(v_1 \cdot g^{-1}), v_2 \cdot g^{-1} \Big)$.

$g(t) \subset G$ is a geodesic $\iff Lu_t = -\mathrm{ad}_u^*(Lu)$,

where $u(t) = g_t \cdot g^{-1}$

Rotating body

Example

Group $G = SO(3)$, a group of rotations of a 3D rigid body with a fixed point,
The path $t \mapsto g(t)$ is the motion of the body.
Metric $E = \frac{1}{2}(\mathbb{I}\omega, \omega)$, $\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3)$.

$$Lu_t = -\mathrm{ad}_u^*(Lu) \Longleftrightarrow \begin{cases} \dot{m}_1 & = \left(\frac{1}{I_3} - \frac{1}{I_2}\right) m_2 m_3, \\ \dot{m}_2 & = \left(\frac{1}{I_1} - \frac{1}{I_3}\right) m_3 m_1, \\ \dot{m}_3 & = \left(\frac{1}{I_2} - \frac{1}{I_1}\right) m_1 m_2. \end{cases}$$

Ideal fluid

Example

Take $G = SDiff(M)$, a group of volume-preserving diffeomorphisms of $M \subset \mathbb{R}^3$.
The path $t \mapsto g(t)$ is the motion of the incompressible fluid in M.
Metric $E = \frac{1}{2} \int_M v^2 d^3x$.

$$Lu_t = -\mathrm{ad}_u^*(Lu) \Longleftrightarrow \partial v_t + v \cdot \nabla v = -\nabla p.$$

$$Lu_t = -\mathrm{ad}^*_u(Lu).$$

Group	*Metric*	*Equation*
$SO(3)$	$<\omega, I\omega>$	Euler top
$SO(3) \ltimes \mathbf{R}^3$	quadratic forms	Kirchhoff equations for a body in a fluid
$SO(n)$	Manakov's metrics	$n-$dimensional top
$\mathrm{Diff}(S^1)$	L^2	Hopf (or, inviscid Burgers) equation
Virasoro	L^2	KdV equation
Virasoro	H^1	Camassa – Holm equation
Virasoro	$\dot{H}^1$	Hunter – Saxton (or Dym) equation
$\mathrm{SDiff}(M)$	L^2	Euler ideal fluid
$\mathrm{SDiff}(M)$	H^1	Averaged Euler flow
$\mathrm{SDiff}(M) \ltimes \mathrm{SVect}(M)$	$L^2 + L^2$	Magnetohydrodynamics
$\mathrm{Maps}(S^1, SO(3))$	H^{-1}	Heisenberg magnetic chain

B.Khesin, Topological Fluid Dynamics, Notices of the AMS, January 2005.

Theorem (DM, SK 2010*)

Let G be any Lie group and H a subgroup. Let $\langle v_1, v_2\rangle_L = (L(v_1), v_2)$ be a non-negative symmetric inner product on $\mathfrak{g}$ with null space $\mathfrak{h}$, defined by a non-negative symmetric linear map $L : \mathfrak{g} \to \mathfrak{g}^*$. Assume this inner product is invariant under $Ad_h,\ h \in H$. Let $g(t)$ be any path in G and define $u(t) = g_t \cdot g^{-1}$ to be its tangent path in $\mathfrak{g}$. Then:

$$\{H \cdot g(t)\} \subset H\backslash G \textit{ is a geodesic } \iff Lu_t = -\mathrm{ad}_u^*(Lu) \textit{ in } \mathfrak{g}^*.$$

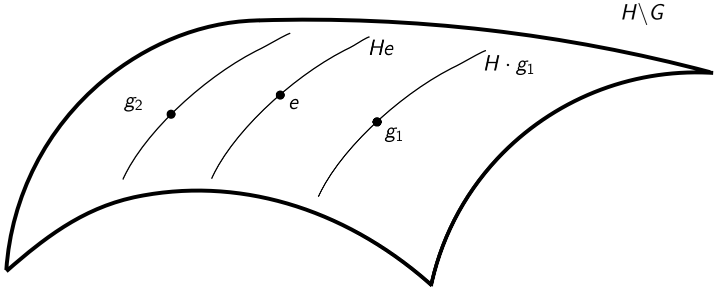
where $\mathrm{ad}_u^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is the adjoint of $\mathrm{ad}_u, u \in \mathfrak{g}$.

* Euler equations on homogeneous spaces and Virasoro orbits. B Khesin, G Misiolek, Advances in Math. 176 (2003), 116-144;

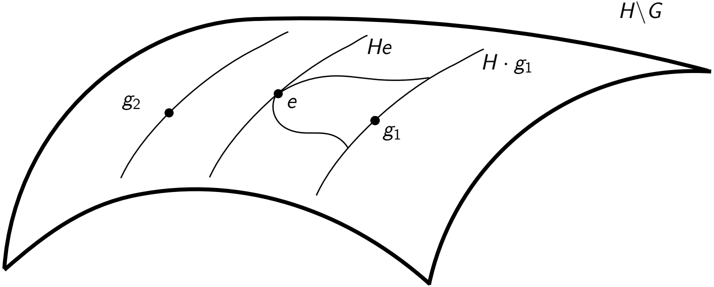
2D shapes

Take $H\backslash G = \mathrm{PSL}_2(\mathbb{R})\backslash\textbf{Diff}(S^1)$, the space of fingerprints (or 2D shapes).
The path $t \mapsto g(t)$ is the path in the space of 2D shapes.
Metric $E = \int_{S^1} Lv(\theta).v(\theta)d\theta$, where $L = -\mathcal{H}(\partial_\theta^3 + \partial_\theta)$:

$$Lu_t = -\mathrm{ad}_u^*(Lu) \Longleftrightarrow \partial(Lv)_t + 2Lv.v_\theta + Lv_\theta.v = 0.$$



$H\backslash G$
He
$H \cdot g_1$
g_2
e
g_1



$$\langle v_1, v_2 \rangle_L = (L(v_1), v_2)$$

EPDiff

Euler-Poincaré equation for the group of diffeomorphisms

$$Lv_t = -\mathrm{ad}^*_v(Lv).$$

$$\Updownarrow$$

$$(Lv)_t + v.(Lv)_\theta + 2v_\theta.Lv = 0,\ \ v \in T_{Id}\mathbf{Diff}(S^1).$$

$$m_t + v.m_\theta + 2v_\theta.m = 0,\ \ m \in T_{Id}\mathbf{Diff}(S^1)^*.$$

$$\Big(m = Lv, v = G * m, L = -\mathcal{H}(\partial_\theta^3 + \partial_\theta)\Big).$$

Minimization

Equivalent to minimizing the energy:

$$E(\psi) = \int_0^1 \left\| \frac{\partial \psi}{\partial t}(\psi^{-1}(x,t),t) \right\|_L^2 dt,$$

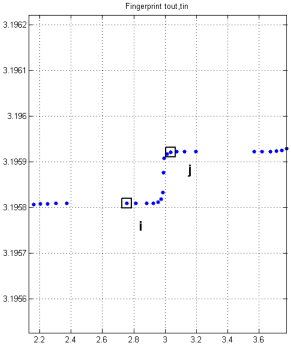
$$\begin{aligned} \text{Shape } S_1: &\quad \psi(\theta, t=0) = \psi_0(\theta), \\ \text{Shape } S_2: &\quad \psi(\theta, t=1) = \psi_1(\theta). \end{aligned}$$

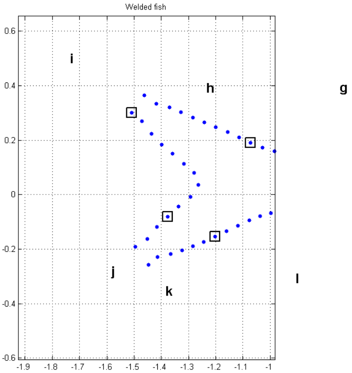
$$\begin{aligned} \psi_t \circ \psi^{-1} &= v, \\ (\psi^{-1})_t &= -v.(\psi^{-1})_\theta. \end{aligned}$$

Minimizing numerically

$$\int_0^1 \|\psi_t \circ \psi^{-1}\|_{WP}^2 dt = \int_0^1 \|(\psi^{-1})_t/(\psi^{-1})_\theta\|_{WP}^2 dt$$

0002*b_marked.fig* zoom





What shall we do?

Solution:

Teichons* = solitons + Teichmüller

*Name credit D. Holm.

Teichons (= solitons + Teichmüller)

- $$m_t + v.m_\theta + 2v_\theta.m = 0$$

- $$m(\theta, t) = \sum_{j=1}^{N} a_j \delta(\theta - b_j),$$
  $$v(\theta, t) = \sum_{j=1}^{N} a_j G(\theta - b_j).$$

- $$\begin{cases} \dot{a}_k = -a_k \sum_{j=1}^{N} a_j G'(b_k - b_j), \\ \dot{b}_k = \sum_{j=1}^{N} a_j G(b_k - b_j). \end{cases}$$

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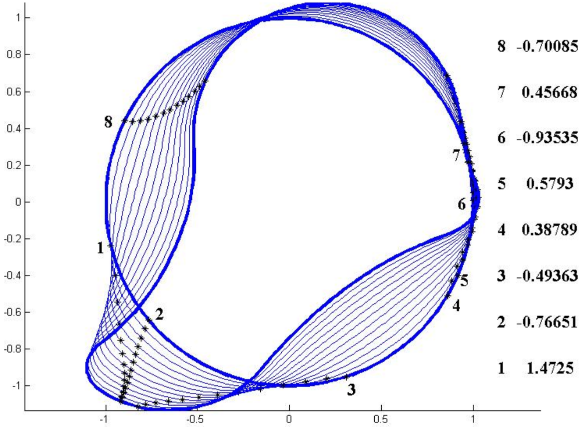
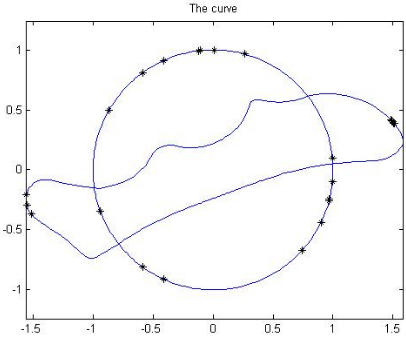
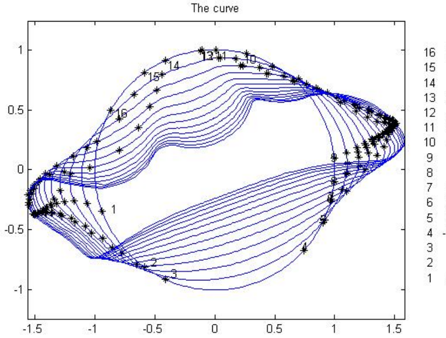
...

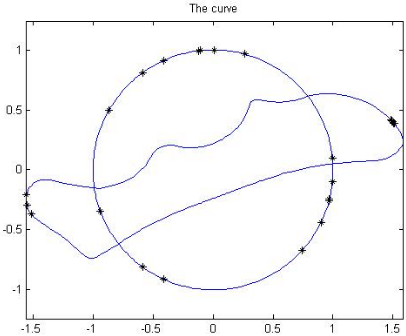


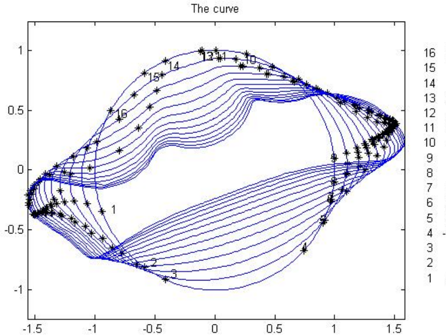
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1
8
1
2
3
4
5
6
7
8 -0.70085
7 0.45668
6 -0.93535
5 0.5793
4 0.38789
3 -0.49363
2 -0.76651
1 1.4725

....

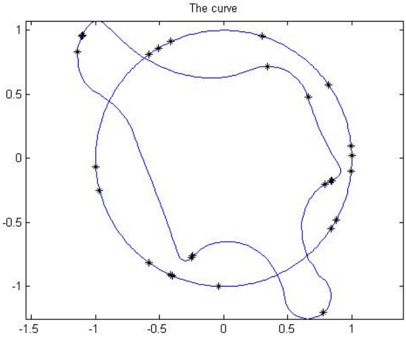

The curve

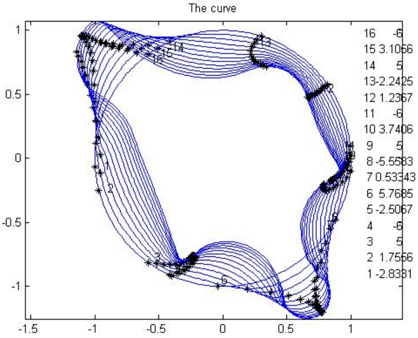

The curve

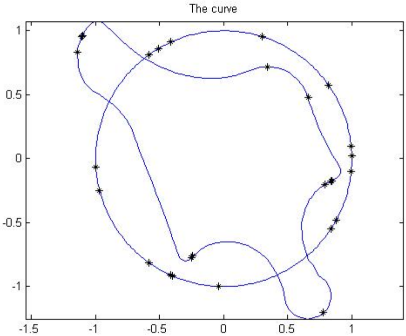

The curve
1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1
1.5

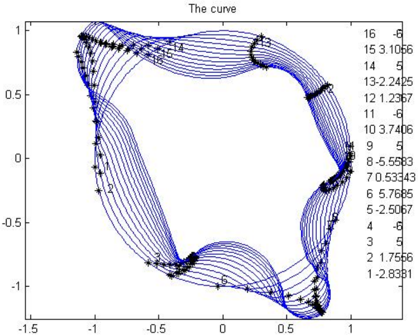

The curve
1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1
1.5
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1

Canoe


The curve
1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1

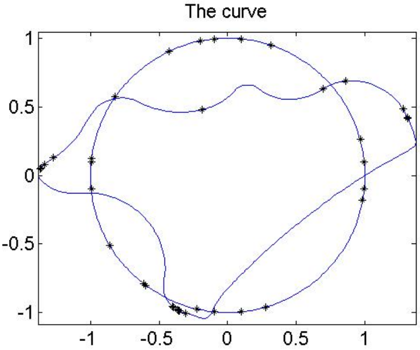

The curve
16 -6
15 3.1056
14 5
13 -2.2425
12 1.2367
11 -6
10 3.7406
9 5
8 -5.5583
7 0.53343
6 5.7685
5 -2.5067
4 -6
3 5
2 1.7566
1 -2.8331



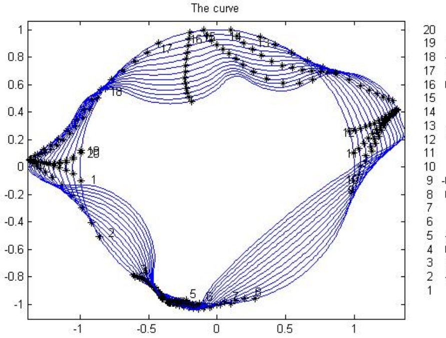


Torch with the flame.

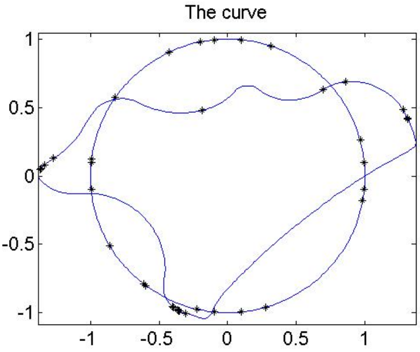
....

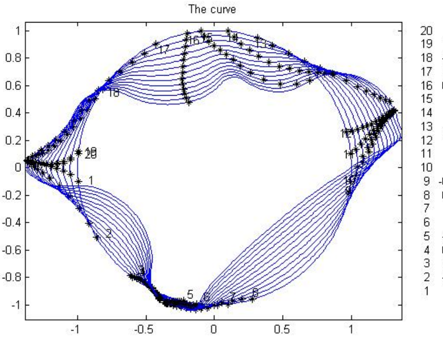


The curve
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1



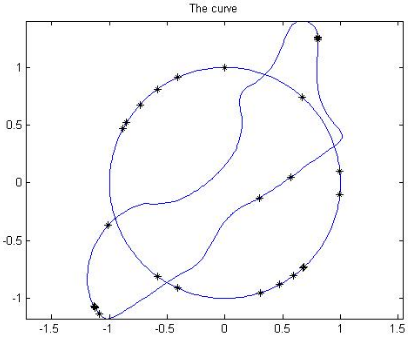
The curve
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1
20
19
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1

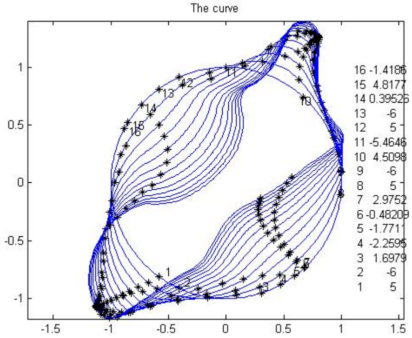




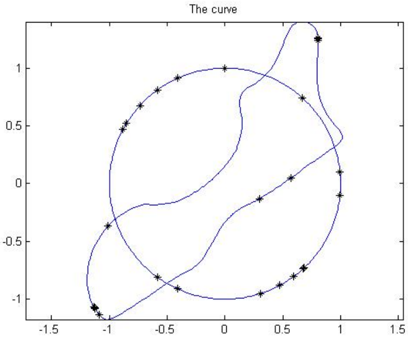
Seagull.

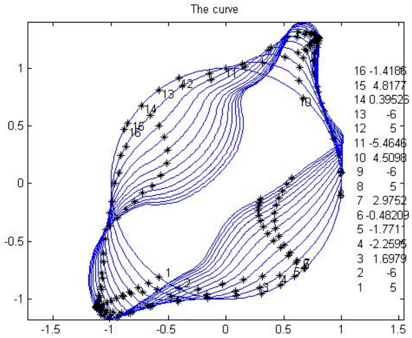
....


The curve
1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1
1.5


The curve
16 -1.4186
15 4.8177
14 0.39526
13 -6
12 5
11 -5.4646
10 4.5098
9 -6
8 5
7 2.9752
6 -0.48209
5 -1.7711
4 -2.2595
3 1.6979
2 -6
1 5

....





Some random shape.

Shooting method

Given ψ_0 and ψ_1 find v_0:

$$Lv_t + 2Lv.v_\theta + Lv_\theta.v = 0, \quad v(t=0) = v_0.$$

Reconstruction equation:

$$\psi_t = v \circ \psi,$$
$$\psi(t=0) = \psi_0,$$

such that $\psi(t=1) = \psi_1$.

Shooting method

Given ψ_0 and ψ_1 find $p_k(0), q_k(0)$:

$$\begin{cases} \dot{p}_k = -p_k \sum_{j=1}^N p_j G'(q_k - q_j), \\ \dot{q}_k = \sum_{j=1}^N p_j G(q_k - q_j). \end{cases}$$

Reconstruction equation:

$$v(\theta, t) = \sum_{k=1}^{N} p_k(t) G(\theta - q_k(t))$$

$$\partial_t \psi = v \circ \psi,$$

$$\psi(t = 0) = \psi_0,$$

such that $\psi(t = 1) = \psi_1$.

Shooting with Teichons. Outline.

Given ψ_0 and ψ_1 find $\mathbf{q}_0, \mathbf{p}_0$:

- integrate to get $\mathbf{q}(t), \mathbf{p}(t)$: $\dot{\mathbf{q}} = G(\mathbf{q})\mathbf{p}, \dot{p} = -\mathbf{p} \bullet G'(\mathbf{q})\mathbf{p}$;
- get $v(t) = \sum p_k(t) G(x - q_k(t))$;
- integrate to get $\psi(t)$: $\psi_t = v \circ \psi$;
- weld: $\psi(t) \longrightarrow$ shapes(t);
- update $\mathbf{p}_0$ via the $\nabla_{\mathbf{p}_0} E$.

Shooting with Teichons

What does it mean, $\psi(t = 1) = \psi_1$?

$$F = \int_0^{2\pi} (\psi_1(x) - \psi(x, t = 1))^2 dx.$$

Would it work for fingerprints?

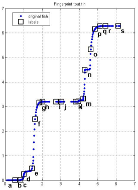
Shooting with Teichons

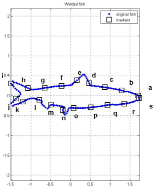
What does it mean, $\psi(t=1) = \psi_1$?

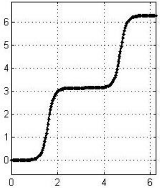
$$F = \int_0^{2\pi} (\psi_1(x) - \psi(x, t=1))^2 dx.$$

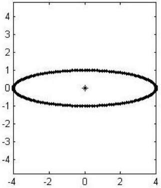
Would it work for fingerprints?

No!


Fingerprint tout,tin
original fish
labels


Welded fish
original fish
markers





New matching functional

Cross-ratios

$$C(\mathbf{z}) = \frac{z_4 - z_1}{z_1 - z_2}\frac{z_2 - z_3}{z_3 - z_4},$$

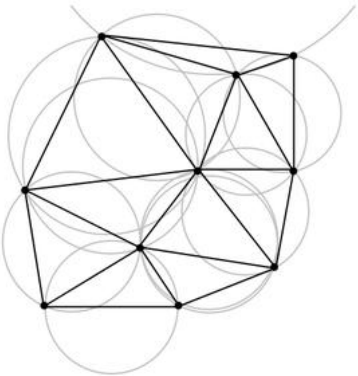
where $\mathbf{z} = (z_1, z_2, z_3, z_4)$. Main property:

$$C(A(\mathbf{z})) = C(\mathbf{z}),$$

where $A(\mathbf{z}) = (A(z_1), A(z_2), A(z_3), A(z_4))$, and A is a Möbius transformation.

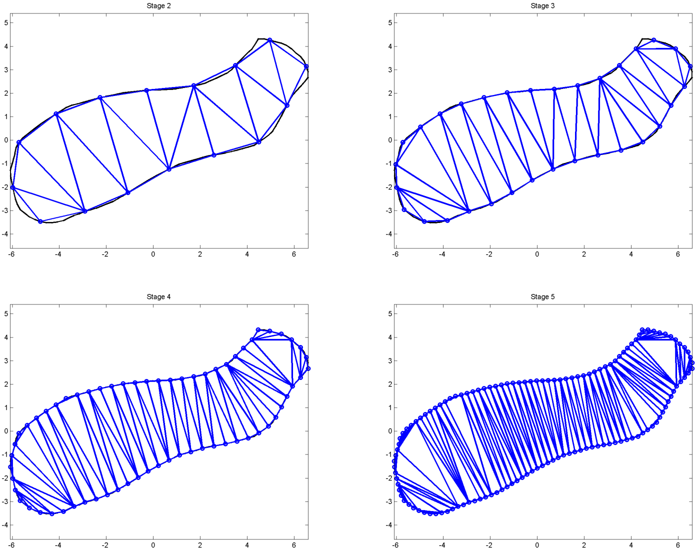
Delaunay Triangulation

Delaunay triangulation for a set P of points in a plane is a triangulation $\mathrm{DT}(P)$ such that no point in P is inside the circumcircle of any triangle in DT(P)



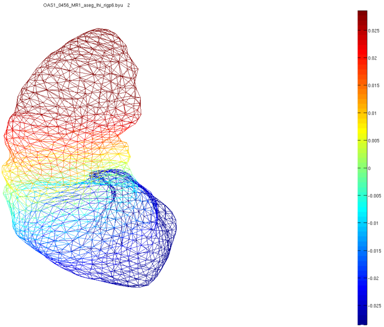
Delaunay Triangulation

Multiscale: DT on 8, 16, 32, 64, 128 points.



3D surfaces

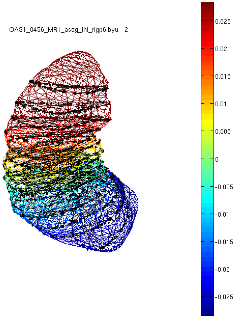
How can we compare 3D surfaces?

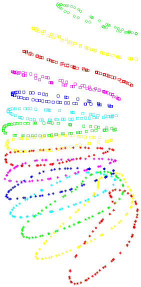


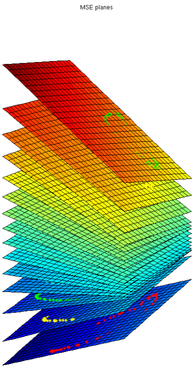
Laplace-Beltrami operator

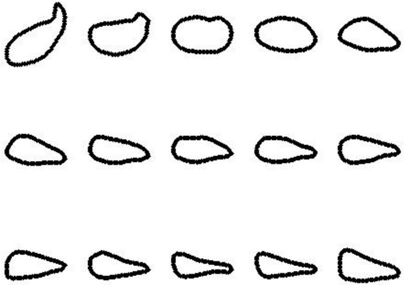
Level sets of eigenfunction f:

$$\Delta f = \lambda f$$

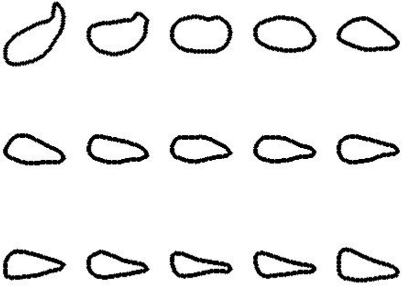




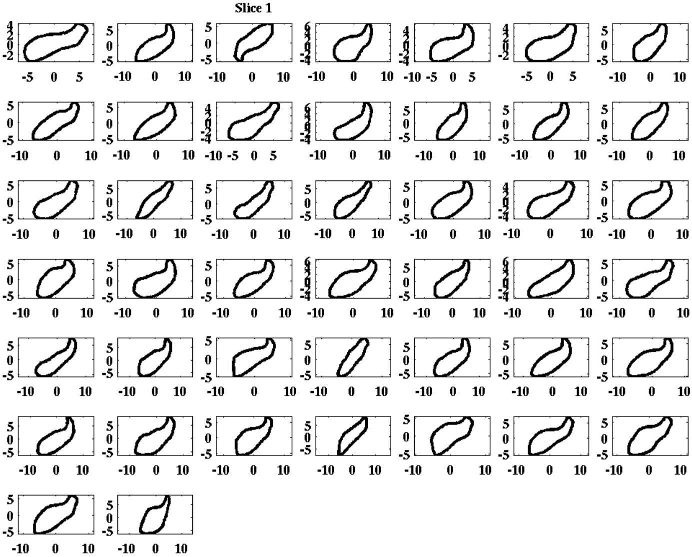

MSE planes

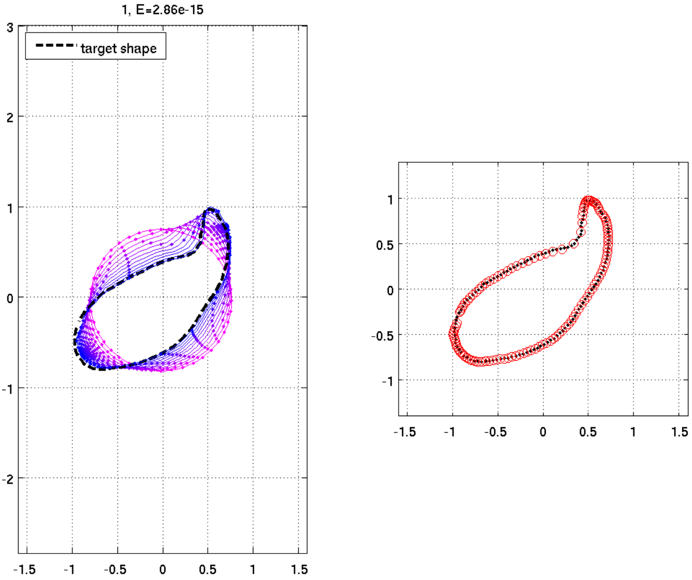


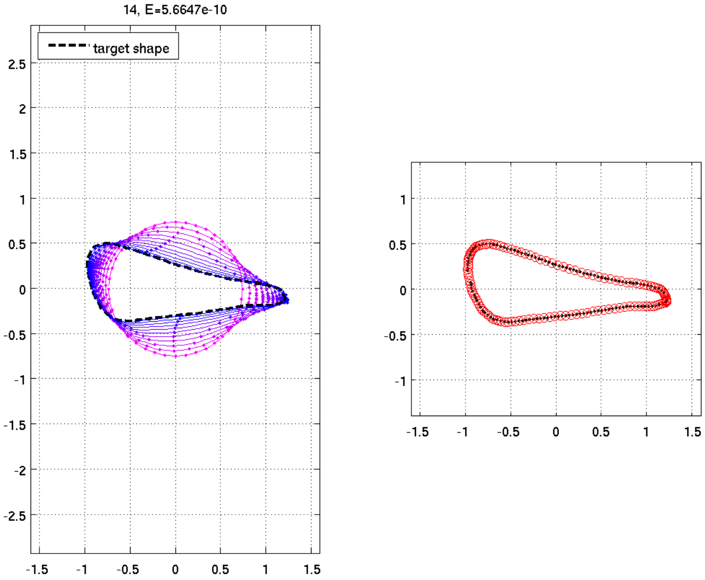
We will compare to a unit template: circle



44 subjects: CDR 0 (22), CDR 1 (22). We have $\{\mathbf{p}_k\}_{k=1}^{44}$ for each slice.

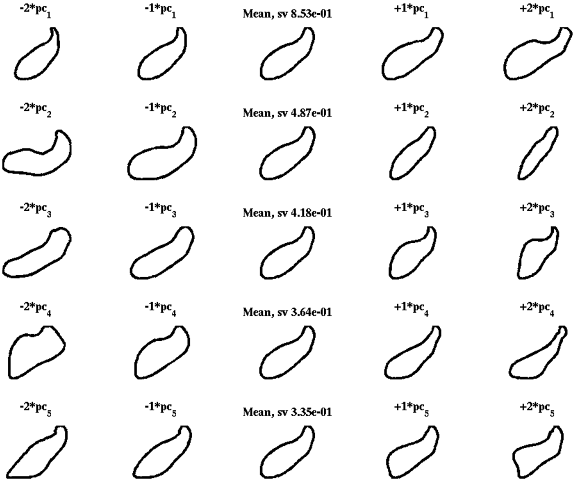



1, E=2.86e-15
target shape
3
2
1
0
-1
-2
-1.5
-1
-0.5
0
0.5
1
1.5
0.5
-0.5

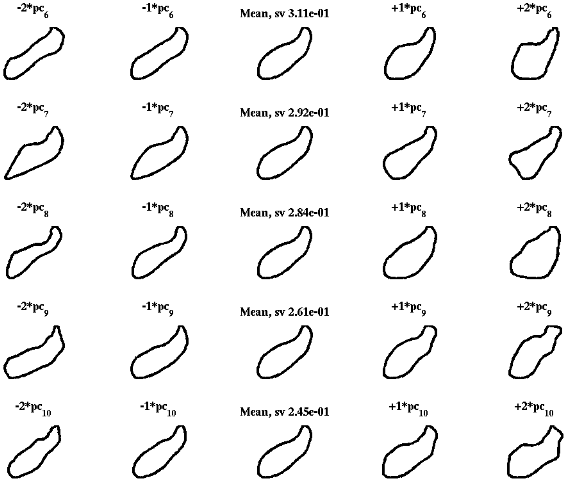


14, E=5.6647e-10
target shape
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-1.5
-1
-0.5
0
0.5
1
1.5

Slice 1: $PC_1, \ldots, PC_5$

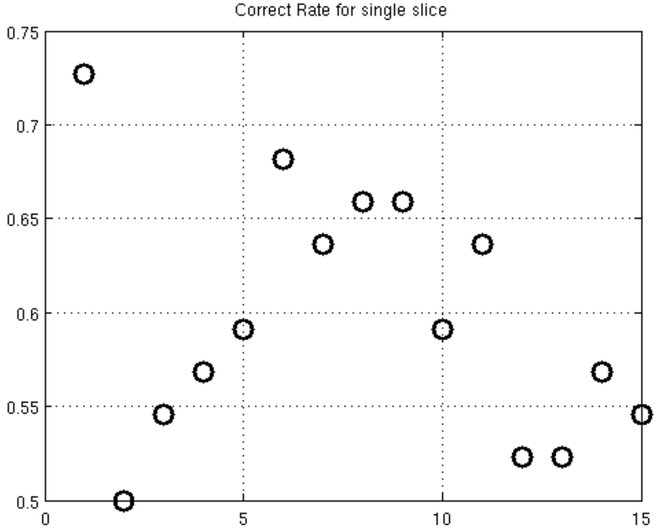


Slice 1: $PC_6, \ldots, PC_{10}$



SVM, Correct Rate of Classification

Leave-one-out cross validation on momenta projected to PC_k



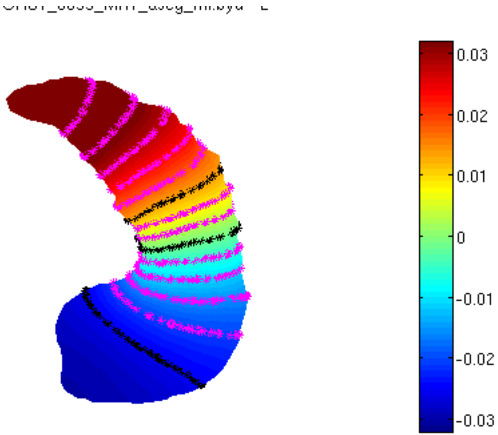
SVM, Correct Rate of Classification

Leave-one-out cross validation on momenta: All slices:

$$CorrectRate = 0.59091.$$

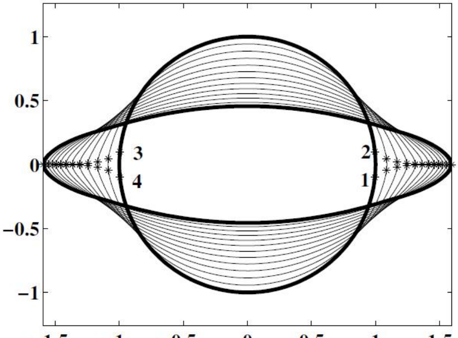
The best score: **combine slices 1, 6, 9**

$$CorrectRate = 0.81818$$



Why can't we shoot from slice to slice?

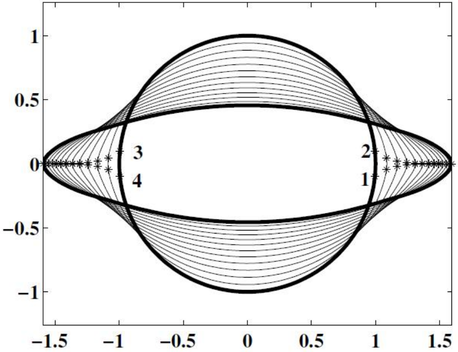
$$m(\theta, t=0) = \sum_{k=1}^{4} p_k \delta(\theta - q_k),$$
$$(p_1, p_2, p_3, p_4) = (p, -p, p, -p)$$
$$(q_1, q_2, q_3, q_4) = (2\pi - q, q, \pi - q, \pi + q).$$
$$p(t) \sim e^{t^2}/t, \quad q(t) \sim e^{-t^2}.$$



Solution (A.Narayan)

Change of variables:

$$p_1(t), p_2(t) \to \frac{p_1(t)+p_2(t)}{2}, \frac{p_1(t)-p_2(t)}{2},$$
$$q_1(t), q_2(t) \to \frac{q_1(t)+q_2(t)}{2}, \frac{q_1(t)-q_2(t)}{2}.$$



Future Directions

- extend WP to multiply connected domains
- What is the WP distance between two closed simply connected surfaces?

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- What is the WP distance between two closed simply connected surfaces?

Conclusions

T(1) as 2D shapes
Weil-Petersson metric
Teichons

Conclusions

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Weil-Petersson metric
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T(1) as 2D shapes
Weil-Petersson metric
Teichons

Thank you.

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