# Efficient Numerical Computation of Schwarz-Christoffel Transformations for Multiply Connected Domains

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# Schwarz-Christoffel

(simply connected)

- Recall the conformal map from the unit disk in $\mathbb{C}$ to the interior of a polygon $P \subset \mathbb{C}$ is:

$$f(z) = A \int^{z} \prod_{k=1}^{n} (\zeta - z_k)^{-\beta_k} \, d\zeta + B.$$

Notation:

- polygon vertices are $w_k$,
- prevertices on the circle are $z_k$, s.t. $w_k = f(z_k)$,
- polygon tangent turning angles are $\beta_k \pi$.

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  Notation:
  - polygon vertices are $w_k$,
  - prevertices on the circle are $z_k$, s.t. $w_k = f(z_k)$,
  - polygon tangent turning angles are $\beta_k \pi$.
- The factors of the product are such that

$$\arg\left\{ \frac{\partial}{\partial\theta} f(e^{i\theta}) \right\} = \text{p.w. const.}, \quad \text{with jumps of } \beta_k\pi \text{ at } z_k.$$

# Schwarz-Christoffel

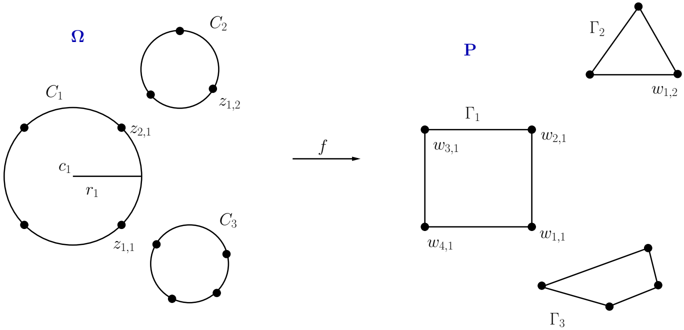
(simply connected)

$$f(z) = A \int^{z} \prod_{k=1}^{n} (\zeta - z_k)^{-\beta_k} \, d\zeta + B$$

- Given $P$, the parameter problem is to find the correct values for $z_k = e^{i\theta_k}$, $A$, and $B$ such that the side lengths, position, and orientation of the polygon is correct under this formula.
- SCPACK, Trefethen (1980); SC Toolbox, Driscoll (1996).

# Schwarz-Christoffel

## Multiply connected (unbounded)




We conformally map an unbounded domain $\Omega$ with $m$ circular holes to an unbounded domain $P$ with $m$ polygonal holes.

# Form of the map

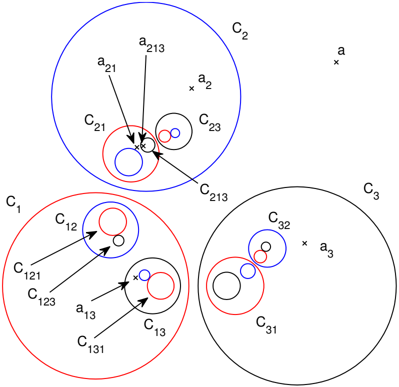
(unbounded case)

- (DeLillo, Elcrat and Pfaltzgraff, 2004) The map to an unbounded domain $P$ with $m$ polygonal holes from a conformally equivalent unbounded domain $\Omega$ with $m$ circular holes is given by

$$f(z) = A \int^{z} \prod_{j=1}^{m} \prod_{k=1}^{K_j} \left[ \prod_{\substack{n=0 \\ \nu \in \sigma_n(j)}}^{\infty} \left( \frac{\zeta - z_{k,\nu j}}{\zeta - s_{\nu j}} \right) \right]^{\beta_{k,j}} d\zeta + B.$$

  - $z_{k,\nu j}$ are reflections (through circles) of prevertices $z_{k,j}$
  - $s_{\nu j}$ are reflections (through circles) of circle centers $s_j := c_j$
  - $\nu$ is a multi-index which tracks reflections
  - $\beta_{k,j}\pi$ are the turning angles of the tangent vectors on the polygons

# Reflection example




- Example of a reflected circle domain with $m = 3$ and $N = 2$.

# Convergence conditions

$$f(z) = A \int^{z} \prod_{j=1}^{m} \prod_{k=1}^{K_j} \left[ \prod_{\substack{n=0 \\ \nu \in \sigma_n(j)}}^{\infty} \left( \frac{\zeta - z_{k,\nu j}}{\zeta - s_{\nu j}} \right) \right]^{\beta_{k,j}} d\zeta + B$$

- A sufficient condition for convergence of the infinite product is $\Delta < (m-1)^{-1/4}$ where

$$\Delta := \max_{j,p;\ j \neq p} \frac{r_j + r_p}{|c_j - c_p|} < 1, \quad 1 \leq j, p \leq m.$$

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- Far from necessary in practice.
- A better indication of convergence: $\sum_{j=1}^{m} \sum_{\nu \in \sigma_N(j)} r_{\nu j}$

  (the sum of the radii of the reflected circles at the $N^{\text{th}}$ level of reflection).

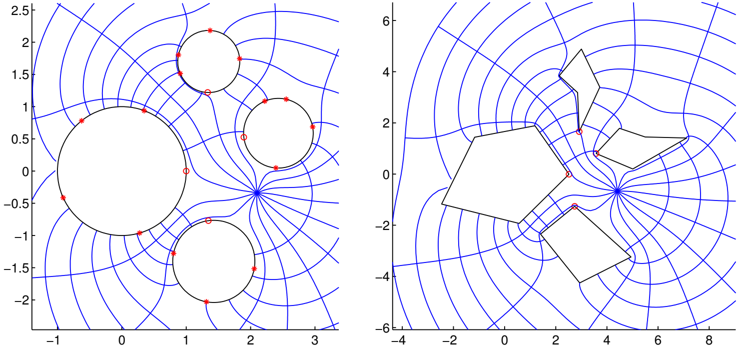
# Problem statement

$$f(z) = A \int^{z} \prod_{j=1}^{m} \prod_{k=1}^{K_j} \left[ \prod_{\substack{n=0 \\ \nu \in \sigma_n(j)}}^{\infty} \left( \frac{\zeta - z_{k,\nu j}}{\zeta - s_{\nu j}} \right) \right]^{\beta_{k,j}} d\zeta + B$$

- Given $P$, the parameter problem is to find centers $c_j$, radii $r_j$, and prevertices $z_{k,j} = c_j + r_j e^{i\theta_{k,j}}$, along with $A$ and $B$ such that the side lengths, positions, and orientations of the polygons are correct under this formula.

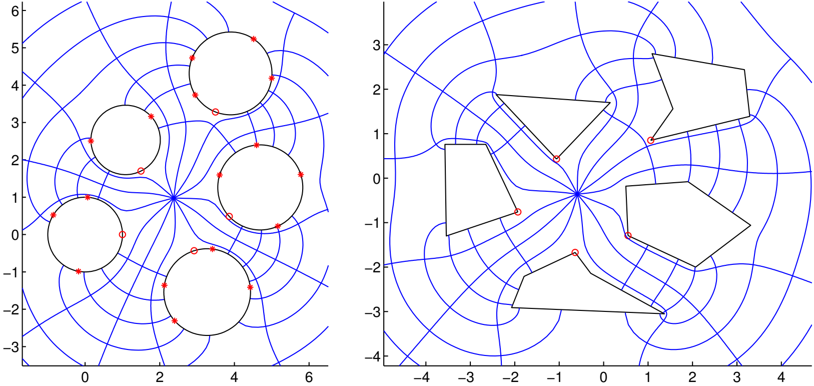
# Map preview

(unbounded)



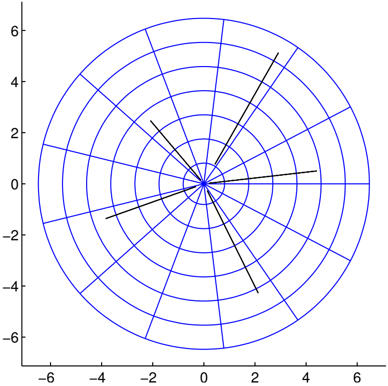

Given $P$, the numerical problem is to find centers $c_j$, radii $r_j$, and prevertices $z_{k,j} = c_j + r_j e^{i\theta_{k,j}}$, along with $A$ and $B$ such that the side lengths, positions, and orientations of the polygons are correct under this formula.

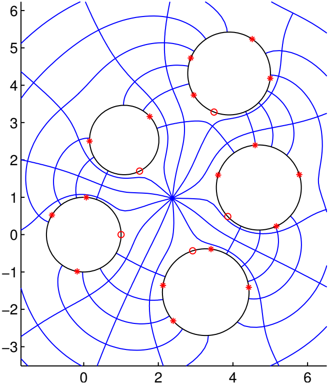
# Orthogonal grid




An orthogonal grid is plotted using the resultant map from the solution of the parameter problem.

# Polar grid




- Slit map from circle domain is constructed using Laurent series based on boundary behavior.
- Polar grid is mapped to circle domain by numerical inversion.

# Prevertices determine map

- Consider the truncated integrand

$$p(\zeta) = \prod_{j=1}^{m} \prod_{k=1}^{K_j} \left[ \prod_{\substack{n=0 \\ \nu \in \sigma_n(j)}}^{N} \left( \frac{\zeta - z_{k,\nu j}}{\zeta - s_{\nu j}} \right) \right]^{\beta_{k,j}}$$

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- Write $z_{k,j} = s_j + r_j e^{i\theta_{k,j}}$.
- The map is then determined by

$$K_1 + K_2 + \cdots + K_m + 3m$$

unknown real parameters.

# Parameter count

## further determined by normalization

- Relax normalization from

$$f(z) = z + O(1/z), \quad z \to \infty$$

(determines circle domain uniquely (Henrici, 1986))

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where $C$ and $D$ are determined implicitly by setting $c_1 = 0$, $r_1 = 1$, and $\theta_{1,1} = 0$ (one circle is fixed).

- This leaves

$$(K_1 - 1) + K_2 + \cdots + K_m + (3m - 3) = K_1 + \cdots + K_m + 3m - 4$$

real parameters to determine.

# Nonlinear conditions

- There are $(K_1 - 1) + K_2 + \cdots + K_m$ side-length conditions,

$$\left| A \int_{z_{k,j}}^{z_{k+1,j}} p(\zeta)\, d\zeta \right| = |w_{k+1,j} - w_{k,j}|, \quad (k,j) \neq (1,1)$$

where

$$A := \frac{w_{2,1} - w_{1,1}}{\int_{z_{1,1}}^{z_{2,1}} p(\zeta)\, d\zeta}$$

(this fixes one side length, and orientation of $\Gamma_1$; setting $B = w_{1,1}$ fixes its position).

Note: for convenience we will write

$$f(z_{k+1,j}) - f(z_{k,j}) = A \int_{z_{k,j}}^{z_{k+1,j}} p(\zeta)\, d\zeta.$$

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- There are 2(m-1) real equations

$$f(z_{1,j}) - f(z_{1,1}) = w_{1,j} - w_{1,1}, \quad 2 \leq j \leq m$$

to determine the positions of polygons $\Gamma_2, \ldots, \Gamma_m$ wrt $\Gamma_1$.

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- There are (m-1) real equations

$$\arg(f(z_{2,j}) - f(z_{1,j})) = \arg(w_{2,j} - w_{1,j})$$

to determine the orientation of polygons $\Gamma_2, \ldots, \Gamma_m$.

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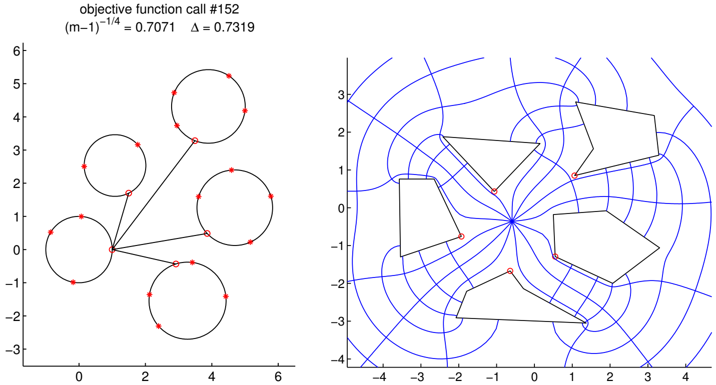
  to determine the orientation of polygons $\Gamma_2, \ldots, \Gamma_m$.
- Total of $K_1 + \cdots + K_m + 3m - 4$ equations to match the parameter count.

# Integration paths




Evaluating the side length, position, and orientation conditions involves integrating around and between the circles.

# Transformation

## to unconstrained coordinates

- The prevertex angles must satisfy the constraints $\theta_{1,j} < \theta_{2,j} < \cdots < \theta_{K_j,j}$, and $\sum_{k=1}^{K_j}(\theta_{k+1} - \theta_k) = 2\pi$ where $\theta_{K_j+1,j} = \theta_{1,j} + 2\pi$.

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  - set $\phi_{k,j} = \theta_{k+1,j} - \theta_{k,j}$, $k = 1, \ldots, K_j$,
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  - Given $\theta_{1,j}$, we have $\theta_{k,j} = \theta_{1,j} + 2\pi \frac{1+\sum_{\mu=1}^{k-2} e^{\psi_{\mu,j}}}{1+\sum_{\mu=1}^{K_j-1} e^{\psi_{\mu,j}}}$.

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- The unconstrained angle variables are $\psi_{1,1}, \ldots, \psi_{K_1-1,1}$ and $\theta_{1,j}, \psi_{1,j}, \ldots, \psi_{K_j-1,1}$ for $2 \le j \le m$.

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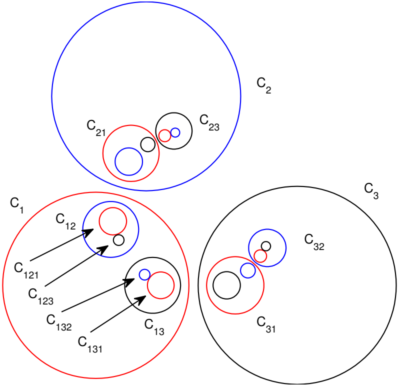
- Since also $r_j > 0$, the unconstrained radii are $\log r_j$.

# System of equations

- The above conditions, in terms of the transformed variables, are expressed as a nonlinear system of equations, $F(x) = 0$.
- We use the numerical continuation algorithm (homotopy method) CONTUP, program 3, from a book by Allgower and Georg to solve this system.
- In testing, this algorithm was shown to be more robust – less sensitive to the initial guess – than the nonlinear equation solvers in MATLAB.

# Reflection complexity




Note that for $m$ boundary circles and $N$ levels of reflection, there are a total of $\sum_{n=1}^{N} m(m-1)^n$ reflected circles!

# MCSC factors

- Write the MCSC map as a finite product of factors

$$f(z) = A\int^{z}\prod_{j=1}^{m}\prod_{k=1}^{K_j}\left(f_{k,j}(\zeta)\right)^{\beta_{k,j}}\,d\zeta + B$$

where

$$f_{k,j}(z) = \prod_{\substack{n=0\\ \nu\in\sigma_n(j)}}^{\infty}\left(\frac{z - z_{k,\nu j}}{z - s_{\nu j}}\right).$$

- We will consider evaluating $f_{k,j}$ without using reflections.

# Boundary behavior

- In general let $a_j$ be a point (prevertex) on a boundary circle $C_j$ and write

$$f_{a_j}(z) := \prod_{\substack{n=0 \\ \nu\in\sigma_n(j)}}^{\infty} \left(\frac{z - a_{\nu j}}{z - s_{\nu j}}\right)$$

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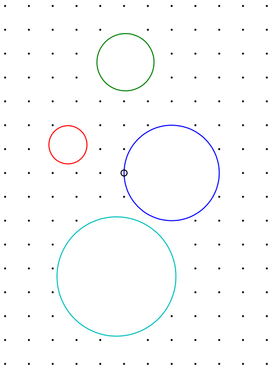
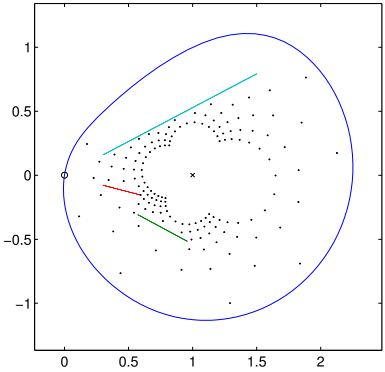
- It can be shown this function satisfies the boundary conditions
  - $\dfrac{\partial}{\partial \theta} \arg f_{a_j}(z) = -\dfrac{1}{2}$ for $z = c_j + r_j e^{i\theta} \in C_j$.
  - $\arg f_{a_j}(z) = \text{const.}$ for $z \in C_p$, $p \neq j$, and

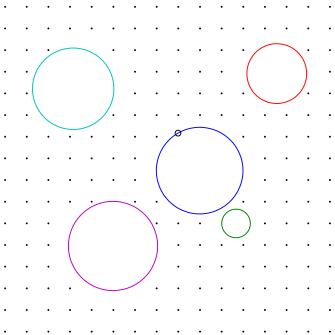
# Example MCSC factor

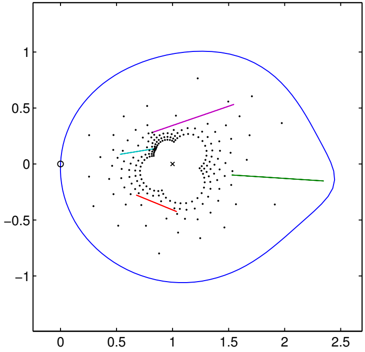





- An example of $f_{a_j}$ computed with $N = 5$ levels of reflection.
- Maps the prevertex to the origin and the point at infinity to 1.

# Example MCSC factor






- Note these factors are *not* maps to canonical slit domains.

# Series representation

- Given

$$f_{a_j}(z) := \prod_{\substack{n=0 \\ \nu \in \sigma_n(j)}}^{\infty} \left( \frac{z - a_{\nu j}}{z - s_{\nu j}} \right)$$

write

$$\log f_{a_j}(z) = \log(z - a_j) - \log(z - s_j) + g(z)$$

where

$$g(z) = \sum_{j=1}^{m} \sum_{n=1}^{\infty} \frac{d_{n,j}}{(z - s_j)^n}$$

(sum of Laurent expansions around each circle).

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(sum of Laurent expansions around each circle).

- Then

$$f_{a_j}(z) = \frac{z-a_j}{z-s_j} e^{g(z)}$$

gives $f_{a_j}(a_j) = 0$ and $f_{a_j}(\infty) = 1$ as required.

# Boundary condition

## Image of $C_j$ (outer boundary)

- Consider $z = s_j + r_j e^{i\theta}$, then

$$\frac{\partial}{\partial\theta} \arg f_{a_j}(z) = \frac{\partial}{\partial\theta} \operatorname{Im}\left\{\log f_{a_j}(z)\right\}$$
$$= \operatorname{Re}\left\{\frac{z - s_j}{z - a_j} - 1 + (z - s_j) g'(z)\right\} = -\frac{1}{2},$$

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$$= \operatorname{Re}\left\{\frac{z - s_j}{z - a_j} - 1 + (z - s_j)g'(z)\right\} = -\frac{1}{2},$$

- and for $a_j = s_j + r_j e^{i\theta_j}$,

$$\operatorname{Re}\left\{\frac{z - s_j}{z - a_j}\right\} = \operatorname{Re}\left\{\frac{e^{i\theta}}{e^{i\theta} - e^{i\theta_j}}\right\} = \operatorname{Re}\left\{\frac{1}{2} - i\cot\frac{\theta - \theta_j}{2}\right\} = \frac{1}{2}.$$

# Boundary condition

## Image of $C_j$ (outer boundary)

- Then, for $z \in C_j$,

$$\frac{\partial}{\partial \theta} \arg f_{a_j}(z) = -\frac{1}{2}$$

  becomes

$$\text{Re}\left\{(z - s_j)g'(z)\right\} = 0.$$

# Boundary condition

## Image of $C_p$, $p \neq j$ (radial slit)

- For $z \in C_p$, $p \neq j$,

$$\arg f_{a_j}(z) = \arg \frac{z - a_j}{z - s_j} + \operatorname{Im}\{g(z)\} = \text{const.}$$

- This gives

$$\operatorname{Im}\{g(z)\} = \text{const.} - \arg \frac{z - a_j}{z - s_j}.$$

# Boundary conditions

- Thus in terms of $g$ we have, for

$$f_{a_j}(z) = \frac{z - a_j}{z - s_j} e^{g(z)},$$

the boundary conditions

  - $\operatorname{Re}\left\{(z - s_j) g'(z)\right\} = 0 \qquad$ for $z \in C_j$, and
  - $\operatorname{Im}\left\{g(z)\right\} = \text{const.} - \arg \dfrac{z - a_j}{z - s_j} \qquad$ for $z \in C_p$, $p \neq j$.

# Discretization

- Truncate: $g(z) \approx \sum_{k=1}^{m} \sum_{n=1}^{N} \frac{d_{n,k}}{(z - s_k)^n}$

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$$x = [\, d_{n,k} \,]_{mN \times 1}$$

- Using $z$ to determine rows and the double sum in $g$ to determine columns, based on $g$ define

$$F_p = [\,(z - s_k)^{-n}\,]_{M \times mN} \quad \text{for } k = 1, \ldots, m; \; z \in C_p, \; p \neq j$$

and based on $g'$ define

$$G = [\,-n(z - s_j)(z - s_k)^{-n-1}\,]_{M \times mN} \quad \text{for } z \in C_j.$$

# Approximate

## boundary conditions

- Consider
  - $F_p = F_{R_p} + iF_{I_p}$,
  - $G = G_R + iG_I$, and
  - $x = x_R + ix_I$.

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  - $F_p = F_{R_p} + iF_{I_p}$,
  - $G = G_R + iG_I$, and
  - $x = x_R + ix_I$.
- This gives
  - $\text{Im}\{g(z)\} \approx F_{I_p}x_R + F_{R_p}x_I$
  - $\text{Re}\left\{(z - s_j)g'(z)\right\} \approx G_R x_R - G_I x_I$

# Linear system

## basis matrices

- In light of $F_{I_p}x_R + F_{R_p}x_I$ and $G_R x_R - G_I x_I$, define

$$A = \begin{bmatrix} F_{I_1} & F_{R_1} \\ \vdots & \vdots \\ F_{I_{j-1}} & F_{R_{j-1}} \\ G_R & -G_I \\ F_{I_{j+1}} & F_{R_{j+1}} \\ \vdots & \vdots \\ F_{I_m} & F_{R_m} \end{bmatrix}_{mM \times 2mN}$$

# Constant argument

## for radial slits

$$\operatorname{Im}\{g(z)\} = \text{const.} - \arg \frac{z - a_j}{z - s_j}$$

- For the images of radial slits, that is for any two $z_1, z_2 \in C_p$, $p \neq j$, we can say $\operatorname{Im}\left\{f_{a_j}(z_2) - f_{a_j}(z_1)\right\} = 0$.

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- So define

$$P = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}_{(M-1)\times M}$$

# Linear system

## multiplier

- Define also

$$E = \begin{bmatrix} P & & & & & & \\ & \ddots & & & & & \\ & & P & & & & \\ & & & I & & & \\ & & & & P & & \\ & & & & & \ddots & \\ & & & & & & P \end{bmatrix}_{(M(m-1)+1)\times mM}$$

where $I$ is in the $j$th block-row.

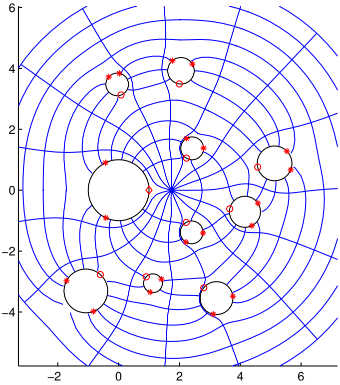
# Linear system

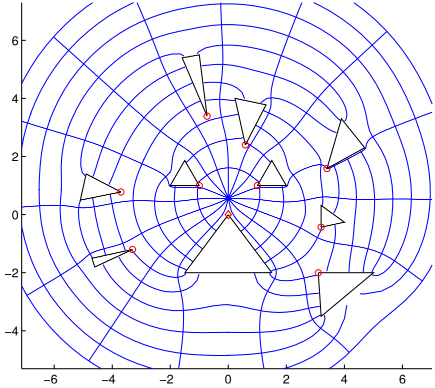
- Then

$$EA \begin{bmatrix} x_R \\ x_I \end{bmatrix} = -E \begin{bmatrix} \arg \frac{z-a_j}{z-s_j} \\ \vdots \\ 0 \\ \vdots \\ \arg \frac{z-a_j}{z-s_j} \end{bmatrix}_{mM\times 1}$$

  gives the coefficients of $g$.
- This linear system must be solved for each prevertex, $f_{k,j}(z) = \frac{z-z_{k,j}}{z-s_j} e^{g(z)}$, which seems like a lot of work, but…

# 10-connected example




- Takes over 5 hours using the reflection method, $N = 4$.
- Done in around 5 minutes using the series method.

Questions?