LACUNARY SERIES, RESONANCES, AND AUTOMORPHISMS OF $\mathbb{C}^2$ WITH A ROUND SIEGEL DOMAIN

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ABSTRACT. We construct transcendental automorphims of $\mathbb{C}^2$ having an unbounded and regular Siegel domain.

1. INTRODUCTION

As shown by Cremer in 1927, a holomorphic germ $f(z) = e^{2\pi i\theta}z + a_2 z^2 + \cdots$, with irrational rotation number θ, is not always linearisable at the origin [Cre38]. However, Siegel proved in 1940 that linearisability occurs when θ is Diophantine [Sie42]. Thirty years later Brjuno extended Siegel's result showing that the arithmetic condition $\sum_n q_n^{-1} \log q_{n+1} < +\infty$ on the best rational approximants $(p_n/q_n)_{n\geq 0}$ of θ is a sufficient condition for linearisability. In 1995, Yoccoz proved that if the Brjuno condition does not hold for some θ, then the quadratic map $e^{2\pi i\theta}z + z^2$ is not linearisable [Yoc95].

When f is a rational function or entire, any such linearisable fixed point gives rise to a Siegel domain, i.e. a Fatou component of f on which f is holomorphically conjugate to the irrational rotation $z \mapsto e^{2\pi i\theta}z$. Boundaries of Siegel discs are usually very irregular [McM98, PZ04] but, Perez-Marco (unpublished manuscript [PM97]) and Avila-Buff-Cheritat [ABC04] proved that Siegel domains with smooth boundaries do actually exist.

Siegel domains also exist in higher dimension and the canonical class of biholomorphic models for them is that of Reinhardt domains. This is a rather large class where one does not expect a Siegel domain to be holomorphically equivalent to a familiar Reinhardt domain, such as the Euclidean ball or the poly-disc. It is very likely that, generically, two distinct Siegel domains in $\mathbb{C}^{k\geq 2}$ are not bi-holomorphic equivalent. These questions remain widely open and very few explicit examples are known [FsS92, Bed18].

Here, we exhibit examples of shear automorphisms of $\mathbb{C}^2$ for which the product of the complex line with the Euclidean unit disc $\mathbb{C} \times \Delta$ is a Siegel domain. Indeed, the Siegel domain is the unique Fatou component of the map. These automorphisms are tangent to a rotation by an irrational number $R_\theta := (e^{2\pi i\theta}w, e^{2\pi i\theta}z)$, where θ does not satisfy the Brjuno's arithmetic condition. This is in strong contrast with Yoccoz's theorem in dimension one, and is related to the resonance in the tangent map R_θ.

Assume that $\theta \in \mathbb{R} \setminus \mathbb{Q}$ with best rational approximants $(p_n/q_n)_{n \geq 0}$. Consider the formal power series

$$\varphi_{\theta,q}(z) := z e^{2\pi i\theta} \sum_{n=0}^{\infty} (1 - e^{2\pi i q_n \theta}) z^{q_n},$$

and let

$$A_{\theta,q}(w, z) := (e^{2\pi i\theta} w + \varphi_{\theta,q}(z), e^{2\pi i\theta} z).$$

Theorem 1.1. *Assume that θ is an irrational number whose best rational approximants $(p_n/q_n)_{n \geq 0}$ satisfies $\lim_{n \to \infty} q_n^{-1} \log q_{n+1} = \infty$. Then,*

  (i) *$A_{\theta,q}$ is an automorphism of $\mathbb{C}^2$,*
  (ii) *$\mathbb{C} \times \Delta$ is a Fatou component of $A_{\theta,q}$, where $A_{\theta,q}$ on $\mathbb{C} \times \Delta$ is analytically conjugate to a rotation,*
  (iii) *for every $(w, z) \in \mathbb{C}^2$ with $|z| > 1$ the orbit of (w, z) under $A_{\theta,q}$ is unbounded.*

See Theorem 2.4 for a generalisation of the above theorem. Moreover, by carefully choosing the coefficients in a power series of the above form, we obtain examples of automorphisms of $\mathbb{C}^2$ enjoying additional dynamical properties. See Theorem 2.5.

2. PROOFS

Let θ be an irrational real number, with continued fraction expansion

$$\theta = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}},$$

with integers $a_i \geq 1$, for all $i \geq 1$, and $a_0 \in \mathbb{Z}$. The *best rational approximants* of θ are defined as the sequence of rational numbers

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{\cdot^{\cdot^{\cdot}} + \cfrac{1}{a_n}}{}}}, \quad n \geq 0.$$

We shall assume that p_n and q_n are relatively prime, and $q_n \geq 0$, for all $n \geq 0$. It is known that the sequence of integers p_n and q_n, for $n \geq 0$, may be defined according to the following recursive relations

$$p_{-2} = q_{-1} = 0, \qquad p_{-1} = q_{-2} = 1,$$
$$q_{n+1} = a_{n+1} q_n + q_{n-1}, \qquad p_{n+1} = a_{n+1} p_n + p_{n-1}.$$

The sequence $(q_n)_{n \geq 0}$ has exponential growth. For instance, since $a_i \geq 1$, the sequence q_n grows at least as fast as the Fibonacci sequence. Indeed, by an inductive argument, one may see that

(1) $$q_n \geq 2^{(n-1)/2}, \quad \forall n \geq 0.$$

Moreover, one has the following estimate on the size of approximation

(2) $$\left| \theta - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n q_{n+1}}, \quad \forall n \geq 0.$$

One may consult [Khi64] or [Sch80] for basic properties of continued fractions.

Given $z_0 \in \mathbb{C}$ and $r > 0$, we define

$$\Delta(z_0, r) := \{z \in \mathbb{C} \, : \, |z - z_0| < r\}, \quad \Delta := \Delta(0, 1).$$

Let us consider the (complex) Banach space $(l^\infty, \| \, \|_\infty)$, where $\|(u_n)_{n \geq 0}\|_\infty$ is defined as $\sup_{n \geq 0} |u_n|$. We assume $u_n \in \mathbb{C}$, for $n \geq 0$.

Proposition 2.1. *Let θ be an irrational number whose best rational approximants $(p_n/q_n)_{n \geq 0}$ satisfy $\lim_{n \to \infty} q_n^{-1} \log q_{n+1} = +\infty$. For every subsequence $q' := (q'_n)_{n \geq 0}$ of $(q_n)_{n \geq 0}$, the following hold:*

(i) *the Taylor series*

$$h_{q'}(z) := z \sum_{n=0}^{\infty} z^{q'_n}$$

is holomorphic on Δ, but does not have holomorphic extension across any $z \in \partial\Delta$;

(ii) *for any $u = (u_n)_{n \geq 0} \in l^\infty$ the Taylor series*

$$h_{q',u}(z) := z \sum_{n=0}^{\infty} u_{q'_n} z^{q'_n}$$

is holomorphic on Δ;

(iii) *there exists a dense G_δ subset E_1 of l^∞ (which only depends on q') such that for every $u \in E_1$, $h_{q',u}(z)$ does not have holomorphic extension across any $z \in \partial\Delta$.*

The series in the above proposition are special cases of Lacunary series, studied first by Weierstrass; see [Kah64] or [EM54].

Proof. By the hypothesis $\lim_{n \to \infty} q_n^{-1} \log q_{n+1} = +\infty$, the series $h_{q',u}$ satisfies Hadamard's lacunarity condition [Had92], that is, $\liminf_{n \geq 0}(q'_{n+1}/q'_n) > 1$. This implies that, if the radius of convergence of the series $h_{q',u}(z)$ about 0 is equal to some $r > 0$ then $h_{q',u}(z)$ does not extend holomorphically across any z in $\partial\Delta(0, r)$.

Evidently, the radius of convergence of $h_{q'}$ is equal to $+1$. The radius of convergence of $h_{q',u}(z)$ is at least $+1$. Below we show that the radius of convergence is indeed equal to $+1$ for many series of the form $h_{q',u}(z)$.

For $k \geq 1$, consider the set

$$E^k := \left\{(u_n)_{n \geq 0} \in l^\infty \, : \, \exists N \in \mathbb{N}, \left|\sum_{n=0}^{N} u_{q'_n}\right| > k\right\}.$$

For every $k \in \mathbb{N}$, E^k is open and dense in l^∞. By Baire's theorem, $E_1 = \cap_{k \in \mathbb{N}} E^k$ is a dense subset of l^∞. For every $u \in E_1$, the series $\sum_{n \geq 0} u_{q'_n}$ is divergent. Therefore, the radius of convergence of the series $h_{q',u}$ is equal to $+1$. Thus, by Hadamard's condition mentioned above, $h_{q',u}$ has no extension across any points on the circle of convergence. $\square$

Let $h_{q',u} : \Delta \to \mathbb{C}$ be a map as in Proposition 2.1, with $u \in E_1$. We may consider the shear map

$$S_{h_{q',u}}(w, z) = (w + h_{q',u}(z), z).$$

This is an automorphism of $\mathbb{C} \times \Delta$, with inverse $S_{h_{q',u}}^{-1} = S_{-h_{q',u}}$. For $\mu \in \mathbb{R}$, let us consider the two dimensional rotation $R_\mu : \mathbb{C}^2 \to \mathbb{C}^2$,

$$R_\mu(w, z) = (e^{2\pi i\mu}w, e^{2\pi i\mu}z).$$

We define the automorphism $A_{\mu,q',u} : \mathbb{C} \times \Delta \to \mathbb{C} \times \Delta$ as

(3) $$A_{\mu,q',u} = S_{h_{q',u}}^{-1} \circ R_\mu \circ S_{h_{q',u}} = S_{-h_{q',u}} \circ R_\mu \circ S_{h_{q',u}}.$$

We note that

$$
\begin{aligned}
A_{\mu,q',u}(w, z) &= S_{-h_{q',u}} \circ R_\mu \circ S_{h_{q',u}}(w, z) \\
&= S_{-h_{q',u}} \circ R_\mu(w + h_{q',u}(z), z) \\
&= S_{-h_{q',u}}(e^{2\pi i\mu}w + e^{2\pi i\mu}h_{q',u}(z), e^{2\pi i\mu}z) \\
&= \left(e^{2\pi i\mu}w + e^{2\pi i\mu}h_{q',u}(z) - h_{q',u}(e^{2\pi i\mu}z), e^{2\pi i\mu}z\right).
\end{aligned}
$$

Let us define the map $\varphi_{\mu,q',u} : \Delta \to \mathbb{C}$ as

$$\varphi_{\mu,q',u}(z) = e^{2\pi i\mu}h_{q',u}(z) - h_{q',u}(e^{2\pi i\mu}z),$$

so that

$$A_{\mu,q',u}(w, z) = (e^{2\pi i\mu}w + \varphi_{\mu,q',u}(z), e^{2\pi i\mu}z).$$

By Equation (3), for all $N \in \mathbb{N}$,

(4) $$A_{\mu,q',u}^{\circ N}(w, z) = A_{N\mu,q',u}(w, z) = (e^{2\pi i N\mu}w + \varphi_{N\mu,q',u}(z), e^{2\pi i N\mu}z).$$

When the sequence $u \in l^\infty$ is the constant sequence $u_n = 1$ we shall simply denote the above maps by $A_{\mu,q'}$ and $\varphi_{\mu,q'}$, that is,

$$\varphi_{\mu,q'}(z) = e^{2\pi i\mu}h_{q'}(z) - h_{q'}(e^{2\pi i\mu}z),$$

and

$$A_{\mu,q'}(w, z) = (e^{2\pi i\mu}w + \varphi_{\mu,q'}(z), e^{2\pi i\mu}z).$$

Evidently,

$$\varphi_{\mu,q',u}(z) = ze^{2\pi i\mu} \sum_{n=0}^\infty u_{q'_n}(1 - e^{2\pi i q'_n \mu})z^{q'_n}.$$

A priori, the above series is only defined and holomorphic on the unit disk Δ. However, when $\mu = \theta$ and for special choices of θ, the function $\varphi_{\theta,q',u}$ becomes entire. This phenomenon is crucial in our study.

Proposition 2.2. *Assume that the best rational approximants $(p_n/q_n)_{n \geq 0}$ of $\theta \in \mathbb{R} \setminus \mathbb{Q}$ satisfy $q_n^{-1} \log q_{n+1} \to \infty$, as $n \to \infty$. Then, for any subsequence q' of $(q_n)_{n \geq 0}$ and any $u \in l^\infty$, the Taylor series of $\varphi_{\theta,q',u}$ defines an entire holomorphic map on $\mathbb{C}$.*

In particular, $A_{\theta,q',u}$ is an automorphism of $\mathbb{C}^2$.

By choosing a_n large enough, one may identify values of θ such that q_n grows arbitrarily fast. In particular, there are irrational numbers θ for which $q_n^{-1} \log q_{n+1} \to +\infty$ as $n \to +\infty$.

Proof. By the estimate in Equation (2), we have $|q_n\theta - p_n| \leq 1/q_{n+1}$. On the other hand, for any $\beta \in [-1/2, 1/2]$, $|1 - e^{2\pi i\beta}| \leq 2\pi|\beta|$. In particular, $|1 - e^{2\pi i q_n \theta}| \leq 2\pi/q_{n+1}$. By the hypothesis of the proposition on the growth of q_n, we conclude that the radius of convergence of the Taylor series for $\varphi_{\theta,q,u}$ is ∞. The Taylor

series of $\varphi_{\theta,q',u}$ is a subseries of the Taylor series of $\varphi_{\theta,q,u}$, and hence, its radius of convergence is also ∞. $\qquad\square$

We shall also need to identify some $\mu \in \mathbb{R}$ for which the convergence radius of the Taylor series of $\varphi_{\mu,q',u}$ is precisely equal to 1.

Lemma 2.3. *Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ with best rational approximants $(p_n/q_n)_{n\geq 0}$ and let $q' := (q'_n)_{n\geq 0}$ be a subsequence of $(q_n)_{n\geq 0}$. There exist $\mu \in \mathbb{R}$ and a dense G_δ-subset E_1 of l^∞ such that for any $u \in E_1$, the convergence radii of the Taylor series of $\varphi_{\mu,q',u}$ and $\varphi_{\mu,q'}$ are equal to 1.*

Proof. Let $(q''_n)_{n\geq 0}$ be a subsequence of q' such that for all $n \geq 0$ we have $q''_{n+1} \geq 4q''_n$. We may inductively choose a nest of closed connected intervals I_n on $[0,1]$, for $n \geq 0$, such that each interval I_n has length $1/q''_n$ and each set $q''_n I_{n+1}$ is contained in $[1/4, 3/4] + \mathbb{Z}$. We define μ as the unique element in the nest I_n. It follows that for all $n \geq 0$, $|1 - e^{2\pi i q''_n \mu}| \geq \sqrt{2}$.

According to Proposition 2.1, there exists a dense G_δ-subset E_1 of l^∞ such that for any $u \in E_1$ the radius of convergence of $h_{q'',u}(z) = z \sum_{n=0}^\infty u_{q''_n} z^{q''_n}$ is equal to 1. By Cauchy-Hadamard's formula, we must have $\limsup_n |u_{q''_n}|^{1/q''_n} = 1$. Since $2 \geq |1 - e^{2\pi i q''_n \mu}| \geq \sqrt{2}$, it follows that for $u \in E_1$,

$$\limsup_n |u_{q''_n}(1 - e^{2\pi i q''_n \mu})|^{1/q''_n} = 1.$$

As q'' is a subsequence of q', and $\limsup_n |u_{q'_n}(1 - e^{2\pi i q'_n \mu})|^{1/q'_n} \leq 1$, we conclude that

$$\limsup_n |u_{q'_n}(1 - e^{2\pi i q'_n \mu})|^{1/q'_n} = 1.$$

Therefore, for $u \in E_1$, the radius of convergence of $\varphi_{\mu,q',u}(z)$ is equal to 1.

The above arguments also work when the sequence u is constant equal to 1 and thus the convergence radius of $\varphi_{\mu,q'}(z)$ equals 1 as well. $\qquad\square$

Assume that θ is an irrational number which satisfies the hypothesis of Proposition 2.2. For any subsequence q' of q and any $u \in l^\infty$,

$$A_{\theta,q',u}(w,z) = (e^{2\pi i\theta} w + \varphi_{\theta,q',u}(z), e^{2\pi i\theta} z)$$

is an automorphism of $\mathbb{C}^2$. The map $A_{\theta,q',u}$ has a constant Jacobian of size 1. So, it is area preserving. Moreover, $A_{\theta,q',u}$ preserves the origin, with derivative

$$\mathrm{D}A_{\theta,q',u}(0,0) = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{2\pi i\theta} \end{pmatrix}.$$

Any irrational number θ satisfying $\lim_{n\to\infty} q_n^{-1} \log q_{n+1} = \infty$ does not satisfy the Brjuno's arithmetic condition for the linearisation [Sie42, Brj71, Her87, Yoc95]. However, here $DA_{\theta,q',u}(0,0)$ enjoys a resonance condition. By virtue of the commutative relation in Equation (3), $A_{\theta,q',u}$ is conjugate to the rotation R_θ on the round domain $\mathbb{C} \times \Delta$, via the map $S_{h_{q',u}}$. We would like to understand the global dynamics of $A_{\theta,q',u}$. In particular, if the Siegel disk of $A_{\theta,q',u}$ centred at $(0,0)$ extends beyond $\mathbb{C} \times \Delta$.

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic map. We say that Ω is a Fatou component for the iterates of F, if the family $\{F^{\circ n}\}_{n\geq 0}$ forms a locally equi-continuous family of maps near any point in Ω. One may consult [Mil06, CG93, MNTU00] for classic books in complex dynamics in dimension one, and refer to [CGSY03, Sib99] for general theory of complex dynamics in higher dimensions.

Theorem 2.4. *Assume that θ is an irrational number whose best rational approximants $(p_n/q_n)_{n\geq 0}$ satisfies $\lim_{n\to\infty} q_n^{-1} \log q_{n+1} = \infty$. Let q' be a subsequence of $(q_n)_{n\geq 0}$. There exists a dense G_δ-subset E_1 of l^∞ such that for every $u \in E_1$, $\mathbb{C} \times \Delta$ is a Fatou component of $A_{\theta,q',u}$, and for every $(w,z) \in \mathbb{C}^2$ with $|z| > 1$ the orbit of (w,z) under $A_{\theta,q',u}$ is unbounded.*

Moreover, the same conclusions hold for $A_{\theta,q'}$, that is, for the constant sequence $u_n \equiv 1$.

Proof. Let us fix q', and let $\mu \in \mathbb{R}$ and $E_1 \subset l^\infty$ be the G_δ dense set obtained in Lemma 2.3. By Proposition 2.2, for every $u \in l^\infty$, $A_{\theta,q',u}$ is a well-defined holomorphic map on $\mathbb{C}^2$. From now on we assume that $u \in E_1$, or u is the constant sequence $u_n \equiv 1$.

Note that $A_{\theta,q',u}$ preserves $\mathbb{C} \times \Delta$. Indeed, by Equation 3 and Proposition 2.1, $A_{\theta,q',u}$ is conjugate to an irrational rotation on $\mathbb{C} \times \Delta$. Thus, $\mathbb{C} \times \Delta$ is contained in the Fatou set of $A_{\theta,q',u}$. Let $\Omega \subset \mathbb{C}^2$ be a Fatou component of $A_{\theta,q',u}$ which contains $\mathbb{C} \times \Delta$. Assume in the contrary that $\Omega \neq \mathbb{C} \times \Delta$.

We may choose $w_0 \in \mathbb{C}$, $z_0 \in \partial\Delta$, and $\epsilon > 0$ such that the bidisk $\Delta(w_0, \epsilon) \times \Delta(z_0, \epsilon)$ is contained in Ω. It follows from the formula

$$A_{\theta,q',u}^{\circ N}(w,z) = (e^{2\pi i N\theta}w + \varphi_{N\theta,q',u}(z), e^{2\pi i N\theta}z), \quad N \geq 0,$$

that the iterates of $A_{\theta,q',u}$ are locally equi-continuous on the tube $\mathbb{C} \times \Delta(z_0, \epsilon)$. On the other hand, since Ω is invariant under $A_{\theta,q',u}$, and $A_{\theta,q',u}$ acts as an irrational rotation in the z-coordinate, Ω contains $\mathbb{C} \times \{z \in \mathbb{C} : ||z| - 1| < \epsilon\}$. Thus, the iterates of $A_{\theta,q',u}$ are equi-continuous on the tube $T = \mathbb{C} \times \Delta(0, 1+\epsilon)$.

Since $A_{\theta,q',u}$ is conjugate to the irrational rotation R_θ on $\mathbb{C} \times \Delta$, there is an increasing sequence of positive integers m_i such that

$$A_{\theta,q',u}^{\circ m_i} \to S_{-h_{q',u}} \circ R_\mu \circ S_{h_{q',u}},$$

with the convergence uniform on compact subsets of $\mathbb{C} \times \Delta$. Since Ω is a Fatou set for $A_{\theta,q',u}$, we may extract an increasing subsequence k_i of m_i such that the iterates $A_{\theta,q',u}^{\circ k_i}$ converges uniformly on compact subsets of T to a holomorphic map $\psi : T \to \mathbb{C}^2$. Thus, we must have

$$\psi = S_{-h_{q',u}} \circ R_\mu \circ S_{h_{q',u}} \text{ on } \mathbb{C} \times \Delta.$$

However, the map $S_{-h_{q',u}} \circ R_\mu \circ S_{h_{q',u}}$ is of the form

$$(w,z) \mapsto (e^{2\pi i\mu}w + \varphi_{\mu,q',u}(z), e^{2\pi i\mu}z),$$

which, by our choice of μ and u and according to Lemma 2.3, diverges at some point on $\mathbb{C} \times \partial\Delta$. This contradicts ψ being defined on T.

We have just proved that $\mathbb{C} \times \Delta$ is a Fatou component of $A_{\theta,q',u}$. As we shall show below, this implies that for any $(w,z) \in \mathbb{C}^2$ with $|z| > 1$, the orbit of (w,z) by $A_{\theta,q',u}$ is unbounded.

Let us fix $(w_0, z_0) \in \mathbb{C}^2$ with $|z_0| > 1$. Assume in the contrary that there is $M > 0$ such that $|A_{\theta,q',u}^{\circ N}(w_0, z_0)| \leq M$, for all $N \geq 0$. Then $|A_{\theta,q',u}^{\circ(N+n)}(w_0, z_0)| \leq M$, for all $N \geq 0$ and all $n \geq 0$. Writing $A_{\theta,q',u}^{\circ n}(w_0, z_0) =: (w_n, e^{2\pi i n\theta}z_0)$, we thus have simultaneously $|w_n| \leq M$ and $|A_{\theta,q',u}^{\circ N}(w_n, e^{2\pi i n\theta}z_0)| \leq M$, for all $n \geq 0$ and all

$N \geq 0$. Then, since

$$A_{\theta,q',u}^{\circ(N+n)}(w_0, z_0) = A_{\theta,q',u}^{\circ N}(w_n, e^{2\pi in\theta} z_0)$$
$$= (e^{2\pi iN\theta} w_n + \varphi_{N\theta,q',u}(e^{2\pi in\theta} z_0), e^{2\pi i(N+n)\theta} z_0),$$

we get that $|\varphi_{N\theta,q',u}| \leq 2M$ on $\{e^{2\pi in\theta} z_0 : n \geq 0\}$, for any $N \geq 0$. Thus, the family of maps $\varphi_{N\theta,q',u}$, for $N \geq 0$, is uniformly bounded on the circle $|z| = |z_0|$ and, by the maximum principle, on the disk $\Delta(0, |z_0|)$.

As $A_{\theta,q',u}^{\circ N}(w, z) = (e^{2\pi iN\theta} w + \varphi_{N\theta,q',u}(z), e^{2\pi iN\theta} z)$, by the above paragraph, the iterates of $A_{\theta,q',u}$ are locally uniformly bounded on $\mathbb{C} \times \Delta(0, |z_0|)$. Therefore, $\mathbb{C} \times \Delta(0, |z_0|)$ must be a Fatou component of $A_{\theta,q',u}$, which strictly contains the Siegel tube $\mathbb{C} \times \Delta$. This contradicts the earlier statement that $\mathbb{C} \times \Delta$ itself is a Fatou component of $A_{\theta,q',u}$. $\qquad\square$

We now aim to show that for a certain choice of the subsequence q' and for a generic $u \in l^\infty$ the Julia set of $A_{\theta,q',u}$ coincides with the complement of the Siegel tube $\mathbb{C} \times \Delta$. More precisely, we prove the following theorem.

Theorem 2.5. *Assume that θ is an irrational number with best rational approximants $(p_n/q_n)_{n\geq 0}$ satisfying $\lim_{n\to\infty} q_n^{-1} \log q_{n+1} = \infty$. Let S be a countable dense subset of $\mathbb{C} \times \{|z| > 1\}$. Then, there exist a subsequence q' of $(q_n)_{n\geq 0}$ and a dense G_δ-subset $E_2 \subset l^\infty$ such that for any $u \in E_2$, $A_{\mu,q',u}(w, z)$ is an automorphism of $\mathbb{C}^2$ satisfying the following properties:*

i) *$A_{\theta,q',u}$ is analytically conjugate to the rotation R_θ on $\mathbb{C} \times \Delta$;*
ii) *the orbit of any $(w, z) \in \mathbb{C} \times \{|z| > 1\}$ by $A_{\theta,q',u}$ is unbounded and recurrent;*
iii) *the sequence of derivatives $(\|DA_{\theta,q',u}^{\circ N}(w, z)\|)_{N\geq 0}$ is unbounded, for any $(w, z) \in S$.*

The proof is based on the following two lemmas.

Lemma 2.6. *Assume that θ is an irrational number whose best rational approximants $(p_n/q_n)_{n\geq 0}$ satisfies $q_n^{-1} \log q_{n+1} \to \infty$, as $n \to \infty$. For any sequence of positive numbers $(\varepsilon_p)_{p\geq 1}$, there exist a subsequence $(q_n')_{n\geq 0}$ of $(q_n)_{n\geq 0}$ and an increasing sequence of integers $(N_p)_{p\geq 1}$ such that for every $p \geq 1$, we have*

$$|1 - e^{2\pi iN_p\theta}| + \sum_{n\geq 0} |1 - e^{2\pi iq_n'N_p\theta}| p^{q_n'} \leq \varepsilon_p.$$

Proof. Let us set $N_0 = 1$ and $q_0' = q_0$. We inductively identify the pair of integers $q_p' = q_{\varphi(p)}$ and N_p so that the following properties are satisfied for all $p \geq 1$:

(i) $N_p > N_{p-1}$,
(ii) $|1 - e^{2i\pi N_p\theta}| + \sum_{n=0}^{p-1} |1 - e^{2i\pi q_n'N_p\theta}| p^{q_n'} \leq \varepsilon_p/2$,
(iii) $q_p' \in \{q_n : n \geq 0, q_n > q_{p-1}', q_n \geq 2N_p\}$,
(iv) $\sum_{n; q_n \geq q_p'} \frac{p^{q_n}}{q_{n+1}} \leq \varepsilon_p/(4\pi N_p)$.

Assume that $q_0', \cdots, q_{p-1}'$ and $N_0, \cdots, N_{p-1}$ are already constructed. First we choose an integer N_p so that (i) holds, and $e^{2\pi iN_p\theta}$ is close enough to 1 in order to satisfy property (ii). Then we choose q_p' according to property (iii), while large enough to guarantee property (iv). The latter choice is possible due to our assumption on the growth of $(q_n)_{n\geq 0}$, that is, the radius of convergence of the series $\sum_{n\geq 0} z^{q_n}/q_{n+1}$ is infinity.

By Property (iii), $(q_p') = (q_{\varphi(p)})_p$ is a strictly increasing subsequence of $(q_p)_p$.

Recall that $q'_n = q_{\varphi(n)}$, and let $p'_n = p_{\varphi(n)}$, for $n \geq 0$. By Equation (2), we have

$$|2\pi q'_n N_p \theta - 2\pi N_p p'_n| \leq 2\pi N_p / q_{\varphi(n)+1}.$$

On the other hand, by (the last condition in) property (iii) in the above list, for all $n \geq p$, we have

$$N_p / q_{\varphi(n)+1} \leq N_p / q_{\varphi(n)} \leq N_p / q_{\varphi(p)} \leq 1/2.$$

We now employ the inequality $|1 - e^{i\beta}| \leq |\beta|$ for $\beta \in (-\pi, \pi)$, and conclude that for all $n \geq p$,

(5) $$|1 - e^{2i\pi q'_n N_p \theta}| \leq 2\pi N_p / q_{\varphi(n)+1}.$$

Also note that since $(q'_n)_{n \geq 0}$ is a subsequence of $(q_n)_{n \geq 0}$, by property (iv) in the above list we have

(6) $$\sum_{n \geq p} \frac{p^{q'_n}}{q_{\varphi(n)+1}} = \sum_{n \geq p} \frac{p^{q_{\varphi(n)}}}{q_{\varphi(n)+1}} \leq \sum_{n; q_n \geq q'_p} \frac{p^{q_n}}{q_{n+1}} \leq \frac{\varepsilon_p}{4\pi N_p}.$$

We may now check that the conclusion of the lemma is fulfilled. Let $p \geq 1$. Using the estimate in property (ii), and the above estimates (5) and (6), we get

$$\left|1 - e^{2\pi i N_p \theta}\right| + \sum_{n=0}^{p-1} \left|1 - e^{2\pi i q'_n N_p \theta}\right| p^{q'_n} + \sum_{n \geq p} \left|1 - e^{2\pi i q'_n N_p \theta}\right| p^{q'_n}$$

$$\leq \varepsilon_p/2 + 2\pi N_p \sum_{n \geq p} \frac{p^{q'_n}}{q_{\varphi(n)+1}} \leq \varepsilon_p/2 + \varepsilon_p/2. \quad \square$$

Assume that θ is an irrational number which satisfies the hypothesis of Proposition 2.2, so that for any subsequence q' of q and any $u \in l^\infty$, $\varphi_{\theta,q',u}$ is an entire function. It follows from the formula in Equation (4) that for any integer $N \in \mathbb{N}$, $\varphi_{N\theta,q',u}$ is an entire function.

Lemma 2.7. *Assume that θ is an irrational number whose best rational approximants $(p_n/q_n)_{n \geq 0}$ satisfy $q_n^{-1} \log q_{n+1} \to \infty$, as $n \to \infty$. For any subsequence q' of $(q_n)_{n \geq 0}$ and any $z \in \mathbb{C}$ with $|z| > 1$, there exists a dense G_δ-subset $E(z)$ of l^∞ such that for every $u \in E(z)$,*

$$\sup_N |\varphi'_{N\theta,q',u}(z)| = +\infty.$$

Proof. Let us fix an arbitrary subsequence q' and a point $z \in \mathbb{C}$ with $|z| > 1$. Recall that

$$\varphi_{N\theta,q',u}(z) = z e^{2\pi i N\theta} \sum_{n=0}^{\infty} u_{q'_n}(1 - e^{2\pi i q'_n N\theta}) z^{q'_n}.$$

For each $N \in \mathbb{N}$, we consider the bounded linear operator $\Phi_{N,z} : l^\infty \to \mathbb{C}$, defined as

$$\Phi_{N,z}(u) := \varphi'_{N\theta,q',u}(z) = e^{2\pi i N\theta} \sum_{n=0}^{\infty} u_{q'_n}(1 + q'_n)(1 - e^{2\pi i q'_n N\theta}) z^{q'_n}.$$

The operator norm of $\Phi_{N,z}$ satisfies

$$\|\Phi_{N,z}\| = \sum_{n=0}^{\infty} (1 + q'_n)|1 - e^{2\pi i q'_n N\theta}||z|^{q'_n}.$$

Recall that $\varphi_{N\theta,q'}$ is the entire function corresponding to the constant sequence $u_n = 1$, and observe that

(7) $$|\varphi'_{N\theta,q'}(z)| \leq \|\Phi_{N,z}\| = \|\Phi_{N,|z|}\|.$$

Let us consider the quantity $K_z := \sup_N \|\Phi_{N,z}\|$. We claim that for every $z \in \mathbb{C}$ with $|z| > 1$, $K_z = +\infty$. Otherwise, by Equation (7) and the maximum modulus principle, we must have $|\varphi'_{N\theta,q'}(z')| \leq K_{|z|}$, for every integer N and every $z' \in \mathbb{C}$ with $|z'| \leq |z|$. This implies that the family of maps $\{\varphi_{N\theta,q'}; N \geq 0\}$ is uniformly bounded on the disc $\Delta(0, (1 + |z|)/2)$. Then, since $A^{\circ N}_{\theta,q'}(w,z) = (e^{2\pi i N\theta}w + \varphi_{N\theta,q'}(z), e^{2\pi i N\theta}z)$, the iterates $(A^{\circ N}_{\theta,q'})_{N\geq 0}$ must be uniformly bounded on the tube $\mathbb{C} \times \Delta(0, (1 + |z|)/2)$. This contradicts the latter part of Theorem 2.4.

By the above paragraph, for $z \in \mathbb{C}$ with $|z| > 1$ we have $\sup_N \|\Phi_{N,z}\| = +\infty$. We may employ the Banach-Steinhaus theorem [Rud87, page 98] to the family of linear operators $\{\Phi_{N,z}\}_{N\geq 1}$. That gives us a a dense G_δ-subset $E(z)$ of l^∞, such that for every $u \in E(z)$, $\sup_N |\varphi'_{N\theta,q',u}(z)| = \sup_N |\Phi_{N,z}(u)| = +\infty$. □

Proof of Theorem 2.5. Let $(\varepsilon_p)_{p\geq 1}$ be a sequence of positive numbers converging to zero. With Lemma 2.6, we generate a subsequence q' of $(q_n)_{n\geq 0}$ and an increasing sequence of integers $(N_p)_{p\geq 1}$. We also employ Theorem 2.4 with q' to obtain the dense G_δ subset of l^∞ denoted by E_1. Similarly, for q' and each $z \in S$, we use Lemma 2.7 to obtain the dense G_δ set $E(z)$ in l^∞. By Baire's theorem, $E_2 := \cap_{z\in S}E(z) \cap E_1$ is a dense G_δ-subset of l^∞.

Fix an arbitrary $u \in E_2$. According to Theorem 2.4, the map $A_{\theta,q',u}$ is analytically conjugate to the rotation R_θ on $\mathbb{C} \times \Delta$. Moreover, the orbit of any $(w,z) \in \mathbb{C}^2$ with $|z| > 1$ by the map $A_{\theta,q',u}$ is unbounded.

Let $(N_p)_{p\geq 1}$ be the sequence of integers given by Lemma 2.6. Recall that

$$A^{\circ N_p}_{\theta,q',u}(w,z) = (e^{2\pi i N_p\theta}w + \varphi_{N_p\theta,q',u}(z), e^{2\pi i N_p\theta}z),$$

where

$$\varphi_{N_p\theta,q',u}(z) = ze^{2\pi i N_p\theta} \sum_{n=0}^{\infty} u_{q'_n}(1 - e^{2\pi i q'_n N_p\theta})z^{q'_n}.$$

Fix an arbitrary $(w,z) \in \mathbb{C}^2$. For integers $p \geq |z|$, we have

$$\left\| A^{\circ N_p}_{\theta,q',u}(w,z) - (w,z) \right\|$$

$$\leq \|(w,z)\| \cdot \left| 1 - e^{2\pi i N_p\theta} \right| + |z| \, \|u\|_\infty \sum_{n\geq 0} |1 - e^{2\pi i q'_n N_p\theta}| \, |z|^{q'_n}$$

$$\leq \|(w,z)\| \cdot \left| 1 - e^{2\pi i N_p\theta} \right| + |z| \|u\|_\infty \sum_{n\geq 0} |1 - e^{2\pi i q'_n N_p\theta}| \, p^{q'_n}$$

$$\leq \left(\|(w,z)\| + |z|\|u\|_\infty \right) \left(|1 - e^{2\pi i N_p\theta}| + \sum_{n\geq 0} |1 - e^{2\pi i q'_n N_p\theta}| p^{q'_n} \right)$$

$$\leq \left(\|(w,z)\| + |z|\|u\|_\infty \right) \varepsilon_p.$$

In the last line of the above equation we have used the estimate in Lemma 2.6. The above bound shows that the orbit of (w,z) by $A_{\theta,q',u}$ is recurrent.

Evidently, $\|DA^N_{\theta,q',u}(w,z)\| \geq |\varphi'_{N\theta,q',u}(z)|$. Therefore, by Lemma 2.7, for every $(w,z) \in S$, we have $\sup_N \|DA^N_{\theta,q',u}(w,z)\| = +\infty$. □

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