HAIRY CANTOR SETS

DAVOUD CHERAGHI AND MOHAMMAD PEDRAMFAR

Abstract. We introduce a topological object, called *hairy Cantor set*, which in many ways enjoys the universal features of objects like Jordan curve, Cantor set, Cantor bouquet, hairy Jordan curve, etc. We give an axiomatic characterisation of hairy Cantor sets, and prove that any two such objects in the plane are ambiently homeomorphic.

Hairy Cantor sets appear in the study of the dynamics of holomorphic maps with infinitely many renormalisation structures. They are employed to link the fundamental concepts of polynomial-like renormalisation by Douady-Hubbard with the arithmetic conditions obtained by Herman-Yoccoz in the study of the dynamics of analytic circle diffeomorphisms.

1. Introduction

We introduce a topological object which enjoys a similar level of universal features as objects like Jordan curve, Cantor set, Cantor bouquet, hairy Jordan curve, Lelek fan, etc [Jor93, MK19, Lel61, Cha89, BO90, Nad92, HO16]. These are topological objects uniquely determined, up to ambient homeomorphisms, by some simple axioms. These objects frequently appear in dynamical systems, in particular, as the Julia sets or the attractors of holomorphic maps on complex spaces. The object presented here has a delicate fine-scale structure, for instance, it is not locally connected. However, due to the way these emerge in iterations of holomorphic mappings, they turn out ubiquitous.

We start by presenting simple examples of our favourite object.

Definition 1.1. A set X in the plane $\mathbb{R}^2$ is called a **straight hairy Cantor set**, if there are a Cantor set $C \subset \mathbb{R}$ and a function $l : C \to [0, +\infty)$ such that

$$X = \{(x, y) \in \mathbb{R}^2 \mid x \in C, 0 \leq y \leq l(x)\},$$

and the function l satisfies the following properties:

(i) the set of $x \in C$ with $l(x) > 0$ is dense in C;

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(ii) if $x \in C$ is an end point [1], then

$$l(x) = 0 = \limsup_{t \to x,\, t \in C} l(t);$$

(iii) if $x \in C$ is not an end point, then

$$\limsup_{t \to x^-,\, t \in C} l(t) = \limsup_{t \to x^+,\, t \in C} l(t) = l(x).$$

By property (ii) in the above definition, $l = 0$ on a dense subset of C. Also, by properties (ii) and (iii), any straight hairy Cantor set is compact. In particular, $\sup_{x \in C} l(x)$ is finite. On the other hand, as every arc in X is accumulated from both sides by arcs, any straight hairy Cantor set is not locally connected.

It is not immediately clear, but true, that for any closed interval I with $I \cap C \neq \emptyset$, the set $\{l(x) \mid x \in I \cap C\}$ is a closed interval in $\mathbb{R}$. Due to such features, straight hairy Cantor sets in the plane are topologically unique.

Theorem 1.2. *All straight hairy Cantor sets in the plane are ambiently homeomorphic.*

We are pursuing topological objects in the plane which are ambiently homeomorphic to a straight hairy Cantor set. However, for practical reasons, one requires an axiomatic description of such objects. Let $X \subset \mathbb{R}^2$, and consider the following axioms:

(A1) any connected component of X is either a single point or a Jordan arc;
(A2) the closure of the set of point components of X is a Cantor set, say B;
(A3) any arc component of X meets B at one of its end points;
(A4) whenever $x_i \to x$ within X, the unique arc in X connecting x_i to B converges to the unique arc in X connecting x to B, with the convergence in the Hausdorff topology;
(A5) for every arc component γ of X, and every x in γ minus the end points of γ, x is not accessible from $\mathbb{R}^2 \setminus X$;
(A6) $X \setminus B$ is dense in B;
(A6') the set of the end points of the arc components of X is dense in X;

All of the above axioms, except A5, are well-defined for any metric space.

Theorem 1.3. *Let X be a compact metric space. If X satisfies axioms A1 to A4, and A6', then X is homeomorphic to a straight hairy Cantor set.*

Theorem 1.4. *Let $X \subset \mathbb{R}^2$ be a compact set. If X satisfies axioms A1 to A6, then X is ambiently homeomorphic to a straight hairy Cantor set.*

[1] $x \in C$ is called an end point, if there is $\delta > 0$ such that either $(x, x+\delta) \cap C = \emptyset$ or $(x-\delta, x) \cap C = \emptyset$.

Definition 1.5. A compact set $X \subset \mathbb{R}^2$ is called a **hairy Cantor set**, if it satisfies axioms A1 to A6.

Combining Theorems 1.2 and 1.4 we conclude that all hairy Cantor sets in the plain are ambiently homeomorphic.

The proof of Theorem 1.2 is based on a careful analysis of bump-functions associated to straight hairy sets, similar to the argument in [AO93]. In contrast, the proof of Theorem 1.4 is more involved. The main technical difficulty in the proof is to build a Jordan curve which meets X only on the set B, and all components of X lie on the same side of it. The proof of this appears in Section 4, and only uses Axioms A1 to A4. Then, we build a one dimensional topological foliation of a punctured plane, equivalent to the foliation of $\mathbb{R}^2 \setminus \{(0,0)\}$ by straight rays, such that each component of X lies in a single leaf of that foliation. The (extra portions of) leaves come from certain hyperbolic geodesics. The topology of the leaves/geodesics are studied in the framework of Caratheodory's prime ends [Car13], and using classical complex analysis, in particular, Gehring-Hayman theorem [GH62]. We form a uniformisation of the hairy Cantor set to an straight hairy Cantor set using hyper-spaces and Whitney maps [Whi33].

Hairy Cantor sets have emerged in the study of the dynamics of holomorphic maps. In the counterpart paper [CP21] (see also [Ped20]) we explain the appearance of hairy Cantor sets as the attractors of a wide class of holomorphic maps with infinitely many renormalisation structures. Roughly speaking, successive perturbations of holomorphic maps with parabolic cycles leads to infinite renormalisation structures, discovered by Douady and Hubbard in early 80s [DH85]. Although such systems were the subject of profound studies in the 80s and 90s, recent significant advances on the topic fuels interest in these objects. In [CP21], hairy Cantor sets are employed to make a striking connection between the satellite renormalisation structures, and the arithmetic conditions which appeared in the study of the linearisation of analytic circle diffeomorphisms by Herman-Yoccoz [Her79, Yoc02]. Our results further explain the non-locally connected Julia sets by Sorensen [Sr00] and Levin [Lev11], as well as the infinitely renormalisable maps without *a priori* bounds in [CS15]. Also, the hairy Cantor sets introduced here shed light on the remarkable examples of positive area Julia sets by Buff-Chéritat in [BC12]. That is, one of the three types of maps identified in [BC12] has infinitely many renormalisation structures of satellite type. When the combinatorics of the successive renormalisations involve very high denominators, the attractor of the map with positive area Julia set is a hairy Cantor set. See [CP21], and the references therein, for further details on this.

Conjecturally, for a dense set of maps on the bifurcation locus of any non-trivial family of rational functions, the attractor of the map contains a hairy Cantor set.

One may compare this to the appearance of Cantor bouquets in the dynamics of exponential maps [DK84, AO93, Six18], and hairy Jordan curves in the attractors of holomorphic maps with an irrationally indifferent fixed/periodic point [Che17].

2. Straight hairy Cantor sets

At the beginning of this section we prove Theorem 1.2. In the second part of this section we look at some topological features of straight hairy Cantor sets, which will be used in the proofs of Theorems 1.3 and 1.4 in the later sections. First we present an example of a straight hairy Cantor set, which essentially represents the general form of a straight hairy Cantor set.

Lemma 2.1. *There exists a straight hairy Cantor set in the plane.*

Proof. For $n \geq 1$, consider the set τ_n of multi-indexes $(i_1, i_2, \dots, i_n)$ such that for each $1 \leq j \leq n$, $1 \leq i_j \leq 2j-1$. We aim to define a Cantor set

$$C = \cap_{n \geq 1} \cup_{t \in \tau_n} I_t,$$

where each I_t is a closed interval in $\mathbb{R}$, and $I_t \subseteq I_{t'}$, for $t \in \tau_n$ and $t' \in \tau_m$, whenever $n \geq m$ and the first m entries of t and t' are identical. For $t \in \tau_1$, we let $I_t = [0, 1]$. Assume that for some $n \geq 2$, I_t is defined for all $t \in \tau_{n-1}$. In order to define I_s, for $s \in \tau_n$, we divide each I_t, $t \in \tau_{n-1}$, into $4n-3$ intervals with equal lengths. Then, starting from the left most interval, we alternate discarding one interval, and keeping the next interval, until the last interval. In other words, with the order on the real line, we keep the odd numbered intervals. Thus, we are left with $2n-1$ closed intervals in I_t. We label these intervals as I_s, such that the first $n-1$ entries of s and t are identical, and use the last entry to label these $2n-1$ intervals in an order preserving fashion. This completes the definition of I_s, for $s \in \tau_n$. With this construction, every $x \in C$ has a unique address $\tau(x) = (t_1(x), t_2(x), t_3(x), \dots)$, where

$$\{x\} = \cap_{n \geq 1} I_{t_1(x), t_2(x), \dots, t_n(x)}.$$

By an inductive process, we define a sequence of continuous functions $l_n : \mathbb{R} \to [0, +\infty)$ as follows. For $n = 0$, let $l_0(x) = 1$ for all $x \in \mathbb{R}$. Now assume that $l_{n-1} : \mathbb{R} \to [0, +\infty)$ is defined for some $n \geq 1$. For $x \in \cup_{t \in \tau_n} I_t$, we let

$$l_n(x) = (1 - |t_n(x)/n - 1|) l_{n-1}(x).$$

For $x \in (-\infty, 0)$ we let $l_n(x) = l_n(0)$, for $x \in (1, +\infty)$ we let $l_n(x) = l_n(1)$, and on any bounded component (a, b) of $\mathbb{R} \setminus \cup_{t \in \tau_n} I_t$, we use a linear interpolation of $l_n(a)$ and $l_n(b)$. This gives a continuous function l_n on $\mathbb{R}$. See Figure 1.

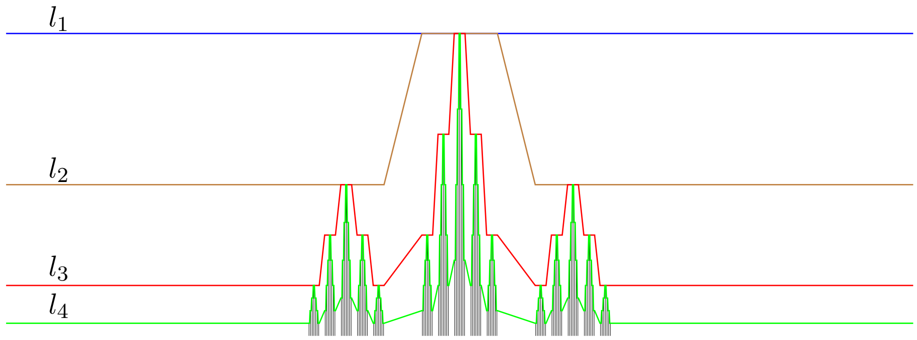

FIGURE 1. The graphs of the functions l_1, l_2, l_3, and l_4 from Lemma 2.1. The vertical lines in black form the straight hairy Cantor set determined by the functions $(l_n)_{n\geq 1}$.

Note that for all $n \geq 1$, $1 \leq t_n(x) \leq 2n-1$, which implies $|t_n(x) - n| \leq n-1$. Therefore, $l_{n-1}(x)/n \leq l_n(x) \leq l_{n-1}(x)$, for all $n \geq 1$ and $x \in \mathbb{R}$. In particular, we may define

$$l(x) = \lim_{n\to\infty} l_n(x).$$

For $x \in C$, we have

$$l(x) = \textstyle\prod_{n\geq 1}(1 - |t_n(x)/n - 1|). \tag{1}$$

Since each l_n is continuous, the set

$$X_n = \{(x,y) \in \mathbb{R}^2 \mid x \in \mathbb{R}, 0 \leq y \leq l_n(x)\}$$

is closed in $\mathbb{R}^2$. Therefore,

$$\cap_{n\geq 0} X_n = \{(x,y) \in \mathbb{R}^2 \mid x \in \mathbb{R}, 0 \leq y \leq l(x)\}$$

is also closed in $\mathbb{R}^2$. This implies that $l : \mathbb{R} \to [0, +\infty)$ is upper semi-continuous, that is, for all $x \in \mathbb{R}$,

$$\limsup_{t\to x} l(t) \leq l(x). \tag{2}$$

We claim that the set

$$X = \{(x,y) \in \mathbb{R}^2 \mid x \in C, 0 \leq y \leq l(x)\}$$

is a straight hairy Cantor set.

By Equation (1), for any $x \in C$ such that $t_n(x) = n$ for large values of n, we have $l(x) > 0$. As the set of such points is dense in C, we obtain property (i) in Definition 1.1.

If x is an end point of C, we must have either $t_n(x) = 1$ or $t_n(x) = 2n - 1$, for large values of n. Then, Equation (1) implies that $l(x) = 0$. By Equation (2), $0 \leq \limsup_{t \to x, t \in C} l(t) \leq l(x) = 0$. Thus, property (ii) in Definition 1.1 holds.

Now assume that $x \in C$ is not an end point. If $l(x) = 0$, by Equation (2), we must have $0 \leq \limsup_{t \to x^+, t \in C} l(t) \leq l(x) = 0$ and $0 \leq \limsup_{t \to x^-, t \in C} l(t) \leq l(x) = 0$. If $l(x) > 0$, by Equation (1), there must be n_0 such that for all $n \geq n_0$ we have $n/2 < t_n(x) < 3n/2$. For $n \geq \max\{n_0, 3\}$ we may consider the unique point $p_n \in C$ whose address is

$$(t_1(x), \cdots, t_{n-1}(x), t_n(x) - 1, t_{n+1}(x), t_{n+2}(x), \cdots).$$

Evidently, (p_n) is a strictly increasing sequence converging to x. Moreover,

$$\begin{aligned}
\lim_{n \to \infty} \left| \frac{l(p_n)}{l(x)} - 1 \right| &= \lim_{n \to \infty} \left| \frac{1 - |(t_n(x) - 1)/n - 1|}{1 - |t_n(x)/n - 1|} - 1 \right| \\
&= \lim_{n \to \infty} \left| \frac{|t_n(x)/n - 1| - |t_n(x)/n - 1/n - 1|}{1 - |t_n(x)/n - 1|} \right| \\
&\leq \lim_{n \to \infty} \frac{1/n}{1 - |t_n(x)/n - 1|} = 0.
\end{aligned}$$

Therefore, $l(p_n) \to l(x)$. Similarly, we may define a strictly decreasing sequence $\{q_n\}$ in C converging to x such that $l(q_n) \to l(x)$. This implies part (iii) of Definition 1.1. □

Let us introduce some basic definitions and notations which will be used in this section. Given a straight hairy Cantor set X, the corresponding function $l : C \to [0, +\infty)$ determining X in Definition 1.1 is unique. We refer to l as the **length function** of X, and often denote it by l^X. Also, we refer to the Cantor set $C \subset X$ as the **base** Cantor set of X, or equivalently, say that X is **based** on the Cantor set C. Note that C is the unique minimal Cantor set in X which contains all point components of X.

Let $C \subset \mathbb{R}$ be a Cantor set. A set $P \subset \mathbb{R}$ is called a **Cantor partition** for C, if P is the union of a finite number of closed intervals in $\mathbb{R}$ with $C \subset P$ and $\partial P \subset C$. A **nest of Cantor partitions shrinking** to C, by definition, is a nest of Cantor partitions $P_0 \supset P_1 \supset P_2 \supset \dots$ for C such that $C = \cap_{n \geq 0} P_n$. When we write a Cantor partition P for C as $P = \cup_{i=1}^k J_i$, we mean that each J_i is a non-empty closed interval in $\mathbb{R}$ and $J_i \cap J_j = \emptyset$, for $1 \leq i < j \leq k$. Also, we assume that the intervals J_i are labelled in a way that $\inf J_i < \inf J_j$ if and only if $i < j$.

Let X be a straight hairy Cantor set based on the Cantor set C. For any Cantor partition P for C, by property (ii) in Definition 1.1, $l^X(t) = 0$, for all $t \in \partial P$.

Let C be a Cantor set, and let $l : C \to [0, \infty)$ be an upper semi-continuous function. For any closed interval $[u, v] \subset \mathbb{R}$ with $[u, v] \cap C \neq \emptyset$, we use the notation

$$\operatorname{Max}(l, [u, v]) = \max\{l(x) \mid x \in [u, v] \cap C\}.$$

By the upper semi-continuity of l, there is $m_{[u,v]} \in [u, v] \cap C$ such that

$$\operatorname{Max}(l, [u, v]) = l(m_{[u,v]}).$$

Clearly, $m_{[u,v]}$ with this property might not be unique. For any such choice of $m_{[u,v]}$, we define

$$B^l_{[u,v]} : [u, v] \to [0, \operatorname{Max}(l, [u, v])]$$

as

$$B^l_{[u,v]}(x) = \begin{cases} \operatorname{Max}(l, [u, x]) & x \leq m_{[u,v]}, \\ \operatorname{Max}(l, [x, v]) & x \geq m_{[u,v]}. \end{cases}$$

We refer to $B^l_{[u,v]}$ as the **bump function** of l on the interval $[u, v]$ associated to $m_{[u,v]}$.

Assume that $P = \cup_{i=1}^k J_i$ is a Cantor partition for C. We may consider a bump function on each interval J_i, say $B^l_{J_i}$, and combine them to define a bump function for l associated to the partition P,

$$B^l_P : \mathbb{R} \to [0, +\infty),$$

according to

$$B^l_P(x) = \begin{cases} B^l_{J_i}(x) & \text{if } x \in J_i, \\ 0 & \text{if } x \notin P. \end{cases}$$

The following lemma is fairy easy, but will be used several times in the upcoming arguments.

Lemma 2.2. *Let C be a Cantor set, and $l : C \to [0, \infty)$ be an upper semi-continuous function. Also assume that $(I_n)_{n \geq 0}$ be a sequence of closed intervals in $\mathbb{R}$ shrinking to a point $z \in C$. Then*

$$l(z) = \lim_{n \to \infty} \operatorname{Max}(l, I_n).$$

Proof. For all $n \geq 0$, we have $l(z) \leq \operatorname{Max}(l, I_n)$, which gives

$$l(z) \leq \liminf_{n \to \infty} \operatorname{Max}(l, I_n).$$

On the other hand, since l is upper semi-continuous, we have

$$l(z) \geq \limsup_{n \to \infty} \operatorname{Max}(l, I_n). \qquad \square$$

The key property of bump functions is stated in the following lemma.

Lemma 2.3. *Let X be a straight hairy Cantor set, based on a Cantor set C, and with length function $l^X : C \to [0, \infty)$. Assume that $P = \cup_{i=1}^k J_i$ is a Cantor partition for C. Any bump function $B_P^{l^X} : \mathbb{R} \to [0, +\infty)$ is continuous and piecewise monotone.*

Proof. Recall that for any $u \in \partial P$, $l^X(u) = 0$. Therefore, it is enough to show that each $B_{J_i}^{l^X}$ is continuous and piecewise monotone. Fix an arbitrary i, and let $J_i = J = [u, v]$ and $B_J^X = B_J^{l^X}$. Assume that B_J^X is associated to some $m_J \in J$ satisfying $\mathrm{Max}(l^X, J) = l^X(m_J)$.

By definition, B_J^X is increasing on the interval $[u, m_J]$ and is decreasing on $[m_J, v]$.

Since P is a Cantor partition for C, $(u, v) \cap C \neq \emptyset$. Then, by property (i) in Definition 1.1, l^X is positive at some point in J, and therefore, $m_J \in (u, v)$.

By the increasing property of B_J^X on $[u, m_J]$, and properties (ii) and (iii) in Definition 1.1, we note that

$$l^X(u) = B_J^X(u) \leq \lim_{t \to u^+} B_J^X(t) \leq \limsup_{t \to u^+} l^X(t) = l^X(u).$$

This implies the continuity of B_J^X at u. Similarly, one obtains the continuity of B_J^X at v.

Since $l(m_J) \neq 0$, by property (ii), m_J is not an end point of C, and by property (iii), we have

$$\limsup_{t \to m_J^-,\, t \in C} l^X(t) = \limsup_{t \to m_J^+,\, t \in C} l^X(t) = l^X(m_J).$$

Hence,

$$\begin{aligned} l^X(m_J) &= \limsup_{t \to m_J^-,\, t \in C} l^X(t) \leq \lim_{t \to m_J^-} B_J^X(t) \leq B_J^X(m_J) = l^X(m_J), \\ l^X(m_J) &= B_J^X(m_J) \geq \lim_{t \to m_J^+} B_J^X(t) \geq \limsup_{t \to m_J^+,\, t \in C} l^X(t) = l^X(m_J). \end{aligned}$$

These imply the continuity of B_J^X at m_J.

For an arbitrary $x \in (u, m_J)$, if $l(x) \neq B_J^X(x)$, then B_J^X is constant near x, and in particular, it is continuous at x. If $l^X(x) = B_J^X(x)$, then $l^X(x) \neq 0$ and we have

$$\begin{aligned} l^X(x) = \limsup_{t \to x^-,\, t \in C} l^X(t) &\leq \lim_{t \to x^-} B_J^X(t) \\ &\leq B_J^X(x) \leq \lim_{t \to x^+} B_J^X(t) \leq \limsup_{t \to x^+,\, t \in C} l^X(t) = l^X(x). \end{aligned}$$

This implies the continuity of B_J^X at x. Similarly, one proves the continuity of B_J^X on (m_J, v). □

The proof of Theorem 1.2 will be based on the following two propositions.

Proposition 2.4. *Let X and Y be straight hairy Cantor sets based on the same Cantor set C and with length functions l^X and l^Y. There are nests of Cantor partitions $P_n^X = \cup_{i=1}^{k_n} J_{n,i}^X$ and $P_n^Y = \cup_{i=1}^{k_n} J_{n,i}^Y$, for $n \geq 0$, shrinking to C, such that for all $n \geq 0$ we have*

(i) *for all $1 \leq i \leq k_n$ we have $\left|J_{n,i}^X\right| < 2^{-n}|C|$ and $\left|J_{n,i}^Y\right| < 2^{-n}|C|$;* [2]

(ii) *for all $1 \leq i \leq k_{n+1}$ and $1 \leq j \leq k_n$, $J_{n+1,i}^X \subset J_{n,j}^X$ if and only if $J_{n+1,i}^Y \subset J_{n,j}^Y$;*

(iii) *whenever $J_{n+1,i}^X \subset J_{n,j}^X$ for some i and j, then*

$$1 - 2^{-(n+2)} < \frac{\operatorname{Max}(l^Y, J_{n+1,i}^Y)/\operatorname{Max}(l^Y, J_{n,j}^Y)}{\operatorname{Max}(l^X, J_{n+1,i}^X)/\operatorname{Max}(l^X, J_{n,j}^X)} < 1 + 2^{-(n+2)}.$$

Proof. We shall use the notations

$$J_{n,i}^X = [u_{n,i}^X, v_{n,i}^X], \quad J_{n,i}^Y = [u_{n,i}^Y, v_{n,i}^Y], \quad \text{for } n \geq 0 \text{ and } 1 \leq i \leq k_n,$$

with

$$u_{n,1}^X < v_{n,1}^X < u_{n,2}^X < \cdots < u_{n,k_n}^X < v_{n,k_n}^X,$$

and

$$u_{n,1}^Y < v_{n,1}^Y < u_{n,2}^Y < \cdots < u_{n,k_n}^Y < v_{n,k_n}^Y.$$

By an inductive argument on n, we shall simultaneously define both partitions P_n^X and P_n^Y. For $n = 0$, we set $k_0 = 1$ and $J_{0,1}^X = J_{0,1}^Y = [\inf C, \sup C]$. That is, each of P_0^X and P_0^Y consists of one closed interval. Property (i) in the proposition holds, and there is nothing to verify for items (ii) and (iii).

Now assume that for some $n \geq 0$, the partitions P_m^X and P_m^Y are defined for $m = 0, 1, 2, \ldots, n$ and satisfy the properties in items (i)-(iii) of the proposition. Below we defined P_{n+1}^X and P_{n+1}^Y.

Let us assume that n is even (the odd case is mentioned at the end of the proof). Let $P_{n+1}^Y = \cup_{i=1}^{k_{n+1}} J_{n+1,i}^Y$ be an arbitrary Cantor partition of C such that P_{n+1}^Y is a subset of P_n^Y and

$$\left|J_{n+1,i}^Y\right| < 2^{-(n+2)}|C|, \quad \text{for } 1 \leq i \leq k_{n+1}. \tag{3}$$

We shall use P_{n+1}^Y to construct P_{n+1}^X. To that end, it is enough to identify all the intervals $J_{n+1,i}^X$ which are contained in $J_{n,j}^X$, for each j with $1 \leq j \leq k_n$. Before identifying those intervals, we note that the number of such intervals for each j must be equal to the number of the intervals $J_{n+1,i}^Y$ which are contained in $J_{n,j}^Y$. This will clearly guarantee property (ii). To ensure property (iii) we need more detailed analysis.

[2]We use the notation $|J|$ to denote the Euclidean diameter of a given $J \subset \mathbb{R}$.

Fix an arbitrary j with $1 \leq j \leq k_n$. Recall that

$$l^X(u^X_{n,j}) = l^X(v^X_{n,j}) = l^Y(u^Y_{n,j}) = l^Y(v^Y_{n,j}) = 0.$$

There are $m^X_{n,j} \in (u^X_{n,j}, v^X_{n,j})$ and $m^Y_{n,j} \in (u^Y_{n,j}, v^Y_{n,j})$ such that

$$l^X(m^X_{n,j}) = \mathrm{Max}(l^X, J^X_{n,j}), \quad l^Y(m^Y_{n,j}) = \mathrm{Max}(l^Y, J^Y_{n,j}).$$

Since $P^Y_{n+1} \subseteq P^Y_n$ there are $0 \leq p \leq q \leq k_{n+1}$ such that

$$u^Y_{n+1,p} = u^Y_{n,j}, \quad v^Y_{n+1,q} = v^Y_{n,j}.$$

We set

$$u^X_{n+1,p} = u^X_{n,j}, \quad v^X_{n+1,q} = v^X_{n,j}.$$

If $p = q$, then we are done. Below we assume that $0 \leq p < q \leq k_{n+1}$. We need to identify $u^X_{n+1,i}$ and $v^X_{n+1,i-1}$, for $p < i \leq q$.

There is a unique integer l with $p \leq l \leq q$ such that

$$u^Y_{n+1,l} < m^Y_{n,j} < v^Y_{n+1,l}.$$

We shall determine $v^X_{n+1,i}$ and $u^X_{n+1,i+1}$ recursively by going from $i = p$ to $l-1$, if there are any such i, and independently going from $i = q-1$ to l, if there are any such i, so that

$$u^X_{n+1,p} < v^X_{n+1,p} < \cdots < u^X_{n+1,l} < m^X_{n,j} < v^X_{n+1,l} < \cdots < u^X_{n+1,q} < v^X_{n+1,q}.$$

We only explain the process for $p \leq i \leq l-1$, the other case being similar.

Let us assume that for some $p \leq i < l-1$, we have already defined

$$u^X_{n+1,p} < v^X_{n+1,p} < \cdots < u^X_{n+1,i} < m^X_{n,j}.$$

Since

$$\mathrm{Max}(l^X, [u^X_{n+1,p}, v^X_{n+1,q}]) = \mathrm{Max}(l^X, J^X_{n,j}) = l^X(m^X_{n,j}),$$

and $m^X_{n,j} > u^X_{n+1,i}$, we have

$$\mathrm{Max}\left(l^X, [u^X_{n+1,i}, v^X_{n+1,q}]\right) = l^X(m^X_{n,j}).$$

Thus, we may use $m^X_{n,j}$ to define a bump function $B^{l^X}_{[u^X_{n+1,i}, v^X_{n+1,q}]}$ on the set $[u^X_{n+1,i}, v^X_{n+1,q}]$. By Lemma 2.3, as we move v from $u^X_{n+1,i}$ to $m^X_{n,j}$, the value of $B^{l^X}_{[u^X_{n+1,i}, v^X_{n+1,q}]}(v)$ continuously increases from 0 to $\mathrm{Max}(l^X, J^X_{n,j})$. Therefore, since

$$0 < \mathrm{Max}(l^Y, J^Y_{n+1,i}) / \mathrm{Max}(l^Y, J^Y_{n,j}) \leq 1,$$

there must be $v' \in (u^X_{n+1,i}, m^X_{n,j})$ such that

$$1 - 2^{-(n+3)} < \frac{\operatorname{Max}(l^Y, J^Y_{n+1,i}) / \operatorname{Max}(l^Y, J^Y_{n,j})}{l^X(v') / \operatorname{Max}(l^X, J^X_{n,j})} < 1 + 2^{-(n+3)}.$$

Now, by slightly moving v' to the right, so that it becomes an end point of C, we may find an interval $(v^X_{n+1,i}, u^X_{n+1,i+1})$ such that

$$u^X_{n+1,i} < v' < v^X_{n+1,i} < u^X_{n+1,i+1} < m^X_{n,j},$$
$$v^X_{n+1,i}, u^X_{n+1,i+1} \in C, \quad (v^X_{n+1,i}, u^X_{n+1,i+1}) \cap C = \emptyset,$$

and

$$1 - 2^{-(n+2)} < \frac{\operatorname{Max}(l^Y, J^Y_{n+1,i}) / \operatorname{Max}(l^Y, J^Y_{n,j})}{\operatorname{Max}(l^X, [u^X_{n+1,i}, v^X_{n+1,i}]) / \operatorname{Max}(l^X, J^X_{n,j})} < 1 + 2^{-(n+2)}.$$

Here, $v^X_{n+1,i} = v'$ and $u^X_{n+1,i} = \inf\{t \in C \mid t > v'\}$. We have used that $v' \mapsto \operatorname{Max}(l^X, [u^X_{n+1,i}, v'])$ depends continuously on v'. This completes the induction step to identify $v^X_{n+1,i}$ and $u^X_{n+1,i+1}$. Note that at the end of this construction, $m^X_{n,j} \in [u^X_{n+1,l}, v^X_{n+1,l}]$, thus,

$$\frac{\operatorname{Max}(l^Y, J^Y_{n+1,l}) / \operatorname{Max}(l^Y, J^Y_{n,j})}{\operatorname{Max}(l^X, J^X_{n+1,l}) / \operatorname{Max}(l^X, J^X_{n,j})} = 1.$$

We assumed at the induction step that n is even. For odd n, we interchange the role of X and Y in the above process, that is, we start with an arbitrary partition P^X_{n+1} for C which is contained in P^X_n and satisfies Equation (3), and build P^Y_{n+1} using P^X_{n+1}, in the same fashion. Property (iii) guarantees that for all $n \geq 0$ and $1 \leq i \leq k_n$ we have $\left|J^X_{n,i}\right| \leq 2^{-n}|C|$ and $\left|J^Y_{n,i}\right| \leq 2^{-n}|C|$. $\square$

Proposition 2.5. *Let X and Y be straight hairy Cantor sets based on the same Cantor set C and let $P_n = \bigcup_{i=1}^{k_n} J_{n,i}$, for $n \geq 0$, be a nest of Cantor partitions shrinking to C. Assume that whenever $J_{n,i} \subseteq J_{n-1,j}$ for some integers n, i, and j, we have*

$$1 - 1/2^{n+1} < \frac{\operatorname{Max}(l^Y, J_{n,i}) / \operatorname{Max}(l^Y, J_{n-1,j})}{\operatorname{Max}(l^X, J_{n,i}) / \operatorname{Max}(l^X, J_{n-1,j})} < 1 + 1/2^{n+1}.$$

Then, X and Y are ambiently homeomorphic.[3]

[3] Recall that two subsets of the plane are ambiently homeomorphic if there is a homeomorphism of the plain which maps one bijectively onto the other.

Let us denote the Euclidean ball of radius r about $z \in \mathbb{R}^2$ with $\mathbb{D}(z,r)$. In the same fashion, given $K \subset \mathbb{R}^2$, let

$$\mathbb{D}(K,r) = \cup_{z \in K} \mathbb{D}(z,r).$$

Proof. Without loss of generality we may assume that $C \subset [0,1]$, with 0 and 1 in C. This may be achieved by applying a translation and then a linear rescaling. Also, by applying two rescalings in the second coordinate in $\mathbb{R}^2$, we may assume that

$$\operatorname{Max}(l^X, [0,1]) = \operatorname{Max}(l^Y, [0,1]) = 1/2.$$

These changes do not alter the ratios of the maximums in the hypotheses of the proposition.

For $n \geq 1$, we define the function $\eta_n : \mathbb{R} \to \mathbb{R}$ as follows. If $t \in P_n$, we identify the unique integers i and j with $t \in J_{n,i} \subseteq J_{n-1,j}$, and set

$$\eta_n(t) = \frac{\operatorname{Max}(l^Y, J_{n,i}) / \operatorname{Max}(l^Y, J_{n-1,j})}{\operatorname{Max}(l^X, J_{n,i}) / \operatorname{Max}(l^X, J_{n-1,j})}.$$

For $t \notin P_n$, we set $\eta_n(t) = 1$. Clearly, η_n is constant on each closed interval of the partition P_n.

By the assumption in the proposition, for all $n \geq 1$ and all $t \in \mathbb{R}$, we have

$$|\eta_n(t) - 1| < 1/2^{n+1}.$$

By an inductive argument, we aim to build homeomorphisms $\varphi_n : \mathbb{R}^2 \to \mathbb{R}^2$, for $n \geq 0$, such that each $\varphi_n(X)$ is a straight hairy Cantor set based on C and its length function $l^{\varphi_n(X)}$ satisfies

$$l^{\varphi_n(X)}(t) = \frac{\operatorname{Max}(l^Y, J_{n,i})}{\operatorname{Max}(l^X, J_{n,i})} \cdot l^X(t), \tag{4}$$

whenever $t \in J_{n,i}$ for some i.

For $n = 0$, we let φ_0 be the identity map on $\mathbb{R}^2$. Assume that for some $n \geq 1$, φ_{n-1} is defined and satisfies Equation (4). Below we define φ_n.

Let B_n be a bump function for $l^{\varphi_{n-1}(X)}$ associated to the Cantor partition P_n. Since η_n is constant on each interval of P_n, we have

$$\begin{aligned}
&\max\{\eta_n(t) B_n(t) \mid t \in [0,1]\} \\
&\qquad = \max\{\eta_n(t) l^{\varphi_{n-1}(X)}(t) \mid t \in C\} \\
&\qquad = \max\left\{ \frac{\operatorname{Max}(l^Y, J_{n,i})}{\operatorname{Max}(l^X, J_{n,i})} \cdot l^X(t) \;\middle|\; \begin{matrix} 1 \leq i \leq k_n \\ 1 \leq j \leq k_{n-1} \end{matrix}, t \in J_{n,i} \cap C \subset J_{n-1,j} \right\} \\
&\qquad = \max\{\operatorname{Max}(l^Y, J_{n,i}) \mid 1 \leq i \leq k_n\} = 1/2.
\end{aligned}$$

In particular, for all $t \in [0,1]$,

$$B_n(t) \leq 1/(2\eta_n(t)) \leq (1/2)(4/3) = 2/3.$$

Define $H_n : \mathbb{R}^2 \to \mathbb{R}^2$ as the identity map outside of $[0,1]^2$, and for $(x,y) \in [0,1]^2$, let

$$H_n(x,y) = \begin{cases} (x, \eta_n(x)y) & \text{if } y \leq B_n(x), \\ \left(x, 1 - \frac{1-\eta_n(x)B_n(x)}{1-B_n(x)}(1-y)\right) & \text{if } y > B_n(x). \end{cases}$$

Evidently, H_n is identity outside $P_n \times [0,1]$, and also on the real slice $(\mathbb{R}, 0)$. For $x_0 \in P_n$, H_n maps the line segment $\{x_0\} \times [0,1]$ onto itself in a piecewise linear fashion, with

$$H_n\left(\{x_0\} \times [0, B_n(x_0)]\right) = \{x_0\} \times [0, \eta_n(x_0)B_n(x_0)],$$
$$H_n\left(\{x_0\} \times [B_n(x_0), 1]\right) = \{x_0\} \times [\eta_n(x_0)B_n(x_0), 1].$$

In particular,

$$H_n\left(\{(x,y) \mid x \in [0,1], y = B_n(x)\}\right) = \{(x,y) \mid x \in [0,1], y = \eta_n(x)B_n(x)\}.$$

It follows from the continuity of B_n in Lemma 2.3 that H_n is continuous, and hence, a homeomorphism of the plane. We define $\varphi_n = H_n \circ \varphi_{n-1}$.

The set $\varphi_n(X)$ is a straight hairy Cantor set. That is because, the homeomorphism H_n sends each vertical line $x = x_0$ into itself, and is the identity on the horizontal line $(\mathbb{R}, 0)$.

To prove that φ_n satisfies Equation (4), choose $t \in J_{n,i} \subseteq J_{n-1,j}$, and note that $\varphi_n = H_n \circ \varphi_{n-1}$ implies

$$\begin{aligned} l^{\varphi_n(X)}(t) &= \eta_n(t) l^{\varphi_{n-1}(X)}(t) \\ &= \frac{\operatorname{Max}(l^Y, J_{n,i}) / \operatorname{Max}(l^Y, J_{n-1,j})}{\operatorname{Max}(l^X, J_{n,i}) / \operatorname{Max}(l^X, J_{n-1,j})} \cdot \frac{\operatorname{Max}(l^Y, J_{n-1,j})}{\operatorname{Max}(l^X, J_{n-1,j})} \cdot l^X(t) \\ &= \frac{\operatorname{Max}(l^Y, J_{n,i})}{\operatorname{Max}(l^X, J_{n,i})} \cdot l^X(t). \end{aligned}$$

This completes the process of defining the maps φ_n, for $n \geq 0$.

Next we show that the maps φ_n converge uniformly to a homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$. To prove this, first note that when $0 \leq y \leq B_n(x)$,

$$|H_n(x,y) - (x,y)| = |(\eta_n(x) - 1)y| \leq 1/2^{n+1},$$

and when $B_n(x) \leq y \leq 1$,

$$|H_n(x,y) - (x,y)| = \left|(1-y)(\eta_n(x)-1)\frac{B_n(x)}{1-B_n(x)}\right| \leq \frac{1}{2^{n+1}} \frac{2/3}{1/3} = 1/2^n.$$

These imply that

$$\begin{aligned}&\max\left\{|\varphi_n(x,y)-\varphi_{n-1}(x,y)| \mid (x,y)\in\mathbb{R}^2\right\}\\&\quad=\max\left\{|H_n(x,y)-(x,y)| \mid (x,y)\in\mathbb{R}^2\right\}\leq 1/2^n.\end{aligned}$$

Hence, the sequence $(\varphi_n)_{n=0}^\infty$ is uniformly Cauchy. In particular, the limiting map φ exists and is continuous.

We claim that ϕ is injective. To see this, fix $x\in[0,1]$. For any $n\geq 1$, and all y and y' in $[0,B_n(x)]$, we have

$$|H_n(x,y)-H_n(x,y')|=\eta_n(x)|y-y'|\geq(1-2^{-(n+1)})|(x,y)-(x,y')|.$$

For y and y' in $[B_n(x),1]$, we have

$$\begin{aligned}|H_n(x,y)-H_n(x,y')|&=\frac{1-\eta_n(x)B_n(x)}{1-B_n(x)}|y-y'|\\&=\left(1-\frac{B_n(x)}{1-B_n(x)}(1-\eta_n(x))\right)|y-y'|\\&\geq\left(1-\frac{B_n(x)}{1-B_n(x)}2^{-(n+1)}\right)|y-y'|\\&\geq(1-2^{-n})|(x,y)-(x,y')|.\end{aligned}$$

Finally, since H_n is a homeomorphism of $\{x\}\times[0,1]$, for $0\leq y'<B_n(x)<y\leq 1$, we have

$$\begin{aligned}|H_n(x,y)-H_n(x,y')|&=|H_n(x,y)-H_n(x,B_n(x))|+|H_n(x,B_n(x))-H_n(x,y')|\\&\geq(1-2^{-n})|(x,y)-(x,y')|.\end{aligned}$$

Combining the above inequalities, we conclude that for all x, y, and y' in $[0,1]$, as well as all $n\geq 0$, we have

$$\begin{aligned}|\phi_n(x,y)-\phi_n(x,y')|&=|H_n\circ H_{n-1}\circ\cdots\circ H_1(x,y)-H_n\circ H_{n-1}\circ\cdots\circ H_1(x,y')|\\&\geq(1-2^{-n})|H_{n-1}\circ\cdots\circ H_1(x,y)-H_{n-1}\circ\cdots\circ H_1(x,y')|\\&\;\;\vdots\\&\geq\left(\textstyle\prod_{k=1}^n(1-2^{-k})\right)|(x,y)-(x,y')|\\&\geq\left(\textstyle\prod_{k=1}^\infty(1-2^{-k})\right)|y-y'|.\end{aligned}$$

We note that $\prod_{k=1}^\infty(1-2^{-k})>0$. By virtue of the above inequality, ϕ is injective.

Every H_n (and therefore every ϕ_n) sends vertical lines to vertical lines in a monotone fashion, and is identity on the boundary of $[0,1]^2$. Therefore ϕ also has this property. This implies that ϕ is surjective. Moreover, the convergence of φ_n to φ implies that for all $t \in C$, we have

$$l^{\varphi(X)}(t) = \lim_{n\to\infty} l^{\varphi_n(X)}(t).$$

We claim that $\varphi(X) = Y$. To prove this, it is enough to show that $l^{\varphi(X)}(t) = l^Y(t)$, for all $t \in C$. Fix $t \in C$. For each $n \geq 0$, choose i_n with $t \in J_{n,i_n}$. By Lemma 2.2,

$$l^X(t) = \lim_{n\to\infty} \mathrm{Max}(l^X, J_{n,i_n}), \quad l^Y(t) = \lim_{n\to\infty} \mathrm{Max}(l^Y, J_{n,i_n}).$$

Therefore, if $l^X(t) \neq 0$, using Equation (4), we obtain

$$l^{\varphi(X)}(t) = \lim_{n\to\infty} l^{\varphi_n(X)}(t) = \lim_{n\to\infty} \left(\frac{\mathrm{Max}(l^Y, J_{n,i_n})}{\mathrm{Max}(l^X, J_{n,i_n})} \cdot l^X(t) \right) = \frac{l^Y(t)}{l^X(t)} \cdot l^X(t) = l^Y(t).$$

On the other hand, from the hypothesis, and using $\log(1+x) \leq x$ for $x \geq 0$, we note that for all $m \geq 1$, we have

$$\begin{aligned}\frac{\mathrm{Max}(l^Y, J_{m,i_m})}{\mathrm{Max}(l^X, J_{m,i_m})} &= \prod_{n=1}^{m} \frac{\mathrm{Max}(l^Y, J_{n,i_n})\,\mathrm{Max}(l^X, J_{n-1,i_{n-1}})}{\mathrm{Max}(l^Y, J_{n-1,i_{n-1}})\,\mathrm{Max}(l^X, J_{n,i_n})} \\ &\leq \textstyle\prod_{n=1}^{m}(1+1/2^{n+1}) \leq e^{1/2}.\end{aligned}$$

Similarly, using $\log(1-x) \geq -2x$ for $x \in (0, 1/4)$, we obtain

$$\frac{\mathrm{Max}(l^Y, J_{m,i_m})}{\mathrm{Max}(l^X, J_{m,i_m})} \geq \prod_{n=1}^{m}(1-1/2^{n+1}) \geq e^{-1}.$$

Thus, $l^X(t) = 0$ if and only if $l^Y(t) = 0$. However, since φ is a homeomorphism, $l^X(t) = 0$ if and only if $l^{\varphi(X)}(t) = 0$. $\square$

Proof of Theorem 1.2. First we note that it is enough to show that any two straight hairy Cantor sets based on the same Cantor set $C \subset \mathbb{R}$ are ambiently homeomorphic. That is because, for any two Cantor sets C and C' in $\mathbb{R}$, there is a homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(C) = C'$. The map φ may be extended to a homeomorphism of $\mathbb{R}^2$ through $\varphi(x, y) = (\varphi(x), y)$. Given a straight hairy Cantor set X based on C', $\varphi^{-1}(X)$ is a straight hairy Cantor set based on C.

Now consider two straight hairy Cantor sets X and Y based on the same Cantor set $C \subset \mathbb{R}$ and with length functions l^X and l^Y, respectively. By Proposition 2.4 there are nests of Cantor partitions $P_n^X = \cup_{i=1}^{k_n}[u_{n,i}^X, v_{n,i}^X]$ and $P_n^Y = \cup_{i=1}^{k_n}[u_{n,i}^Y, v_{n,i}^Y]$ shrinking to C, which enjoy the three properties in that proposition. In particular, by properties (i) and (ii) in that proposition, there is a homeomorphism $\psi : \mathbb{R} \to \mathbb{R}$ such that for all $n \geq 0$ and all $1 \leq i \leq k_n$,

$$\psi(u_{n,i}^X) = u_{n,i}^Y, \quad \psi(v_{n,i}^X) = v_{n,i}^Y.$$

Evidently, ψ maps C onto C. We extend ψ to a homeomorphism of $\mathbb{R}^2$ through

$$\psi(x, y) = (\psi(x), y).$$

It follows that $\psi^{-1}(Y)$ is a straight hairy Cantor set based on C. Moreover, by part (iii) of Proposition 2.4, the straight hairy Cantor sets X and $\psi^{-1}(Y)$, as well as the nest of Cantor partitions $(P_n^X)_{n\geq 0}$ satisfy the hypothesis of Proposition 2.5. Therefore, we obtain a homeomorphism of $\mathbb{R}^2$ which maps X to $\psi^{-1}(Y)$. This completes the proof. $\square$

Remark 1. It is possible to give an alternative proof of Theorem 1.2 using a result in [AO93]. One may collapse the bounded intervals (and the vertical lines above them) in the complement of the base Cantor set, so that the straight hairy Cantor set becomes a "hairy arc", as defined in that paper. It is proved in [AO93] that any two straight hairy arcs are ambiently homeomorphic (with a homeomorphism which maps vertical lines to vertical lines, and preserves the real line containing the base Cantor set). One can show that the resulting homeomorphism can be lifted to an ambient homeomorphism of the straight hairy Cantor sets. However, that approach for hairy arcs cannot be used for arbitrary hairy Cantor sets in the plane, due to the technical issue we discuss in Section 4.

In the following two propositions we look at some topological properties of straight hairy Cantor sets, which will be used in the proofs of Theorems 1.3 and 1.4.

Proposition 2.6. *Let $C \subseteq \mathbb{R}$ be a Cantor set, and let $l : C \to [0, \infty)$ be a function such that the set*

$$Z = \{(x, y) \in \mathbb{R}^2 \mid x \in C, 0 \leq y \leq l(x)\}$$

satisfies the following properties:

(i) *Z is compact;*
(ii) *$\{(x, l(x)) \mid x \in C, l(x) \neq 0\}$ is dense in Z.*

Then, Z is homeomorphic to a straight hairy Cantor set.

A set Z which satisfies the conditions in the above proposition may not be ambiently homeomorphic to a straight hairy Cantor set. For instance, if X is a straight hairy Cantor set based on C, and $c \in C$ is a point where $l(c) \neq 0$, then the set of components of X which meet $[\inf C, c]$ satisfies the properties in the above proposition, but that set is not (ambiently homeomorphic to) a straight hairy Cantor set.

Proof. We aim to modify Z, by successively applying homeomorphisms $H_n : C\times\mathbb{R} \to C \times \mathbb{R}$ close to the identity, so that in the limit we obtain a straight hairy Cantor set. That is, we build a chain of homeomorphisms as in

$$Z \xrightarrow{H_0} Z_0 \xrightarrow{H_1} Z_1 \xrightarrow{H_2} Z_2 \xrightarrow{H_3} \dots \xrightarrow{H_{n-1}} Z_{n-1} \xrightarrow{H_n} Z_n \xrightarrow{H_{n+1}} Z_{n+1} \xrightarrow{H_{n+2}} \dots .$$

Each H_n is a piecewise translation, shuffling the arcs in Z_{n-1} so that any bump function of Z_n has jump discontinuities of sizes at most $1/n$. We present the details below.

Since all Cantor sets in the real line are homeomorphic, without loss of generality, we may assume that C is the middle-third Cantor set. Let $P_0 = [0,1]$, and for $n \geq 1$, recursively define[4]

$$P_n = P_{n-1}/3 \cup (2/3 + P_{n-1}/3).$$

Then, $(P_n)_{n=0}^{\infty}$ is a shrinking sequence of Cantor partitions for C. For $n \geq 0$, let $P_n = \bigcup_{i=1}^{2^n} I_{n,i}$, where each $I_{n,i}$ is a connected component of P_n. We may label the subscripts so that $\sup I_{n,i} < \inf I_{n,j}$, whenever $1 \leq i < j \leq 2^n$. For $p > q \geq 0$, and $1 \leq j \leq 2^q$, consider the set

$$K(p;q,j) = \{1 \leq i \leq 2^p \mid I_{p,i} \subseteq I_{q,j}\} = \{i \in \mathbb{N} \mid 2^{p-q}(j-1)+1 \leq i \leq 2^{p-q}j\}.$$

We divide the proof into three steps.

Step 1. There are a sequence of integers $m_0 = 0 < m_1 < m_2 < m_3 < \dots$, homeomorphisms $H_n : C \times \mathbb{R} \to C \times \mathbb{R}$, and functions $l_n : C \to [0,\infty)$, for $n \geq 0$, such that for all $n \geq 1$, and all $1 \leq j \leq 2^{m_{n-1}}$ we have

(i)
$$H_n(I_{m_{n-1},j} \cap C) = I_{m_{n-1},j} \cap C;$$

(ii) for all $x \in C$, $l_0(x) = l(x)$, and $l_n(x) = l_{n-1} \circ H_n^{-1}(x)$;

(iii) for every $k \in \{\min K(m_n; m_{n-1}, j), \max K(m_n; m_{n-1}, j)\}$,
$$\mathrm{Max}(l_n, I_{m_n,k}) < 1/n;$$

(iv) for all i satisfying $\min K(m_n; m_{n-1}, j) \leq i < \max K(m_n; m_{n-1}, j)$, we have
$$|\,\mathrm{Max}(l_n, I_{m_n,i}) - \mathrm{Max}(l_n, I_{m_n,i+1})| < 1/n.$$

Let $m_0 = 0$, H_0 be the identity map, and $l_0 \equiv l$. Fix an arbitrary $n \geq 1$. Assume that m_i, H_i, and l_i are define for all $i \leq n-1$, and we aim to define m_n, H_n, and l_n.

Consider the set

$$Z_{n-1} = \{(x,y) \in \mathbb{R}^2 \mid x \in C, 0 \leq y \leq l_{n-1}(x)\}.$$

It follows that $H_{n-1} \circ H_{n-2} \circ \cdots \circ H_0$ is a homeomorphism from Z onto Z_{n-1}. Since Z is compact, Z_{n-1} must be compact. This implies that l_{n-1} is upper semi-continuous.

[4]Given $A \subseteq \mathbb{R}$ and $k \in \mathbb{R}$, we define $A + k = \{a + k \mid a \in A\}$ and $kA = \{ka \mid a \in A\}$.

By virtue of the homeomorphism $H_{n-1} \circ \cdots \circ H_0$ from Z onto Z_{n-1}, and the hypothesis of the proposition, the set $\{(x, l_{n-1}(x)) \mid x \in C, l_{n-1}(x) \neq 0\}$ is dense in Z_{n-1}. Therefore, for every j with $1 \leq j \leq 2^{m_{n-1}}$, there is a finite set $A_j \subseteq I_{m_{n-1},j} \cap C$, such that

$$\mathbb{D}(l_{n-1}(A_j), 1/(5n)) \supseteq [0, \mathrm{Max}(l_{n-1}, I_{m_{n-1},j})]. \tag{5}$$

By Lemma 2.2, for every $x \in A_j$ we have

$$l_{n-1}(x) = \lim_{m \to \infty, x \in I_{m,i}} \mathrm{Max}(l_{n-1}, I_{m,i}).$$

Therefore, we may find an integer $m_n > m_{n-1}$ such that for all j with $1 \leq j \leq 2^{m_{n-1}}$, we have

$$\bigcup_{i \in K(m_n; m_{n-1}, j)} \mathbb{D}\big(\mathrm{Max}(l_{n-1}, I_{m_n,i}), 1/(4n)\big) \supseteq \mathbb{D}(l_{n-1}(A_j), 1/(5n)). \tag{6}$$

We define H_n on each $I_{m_{n-1},j} \cap C$, for $1 \leq j \leq 2^{m_{n-1}}$, so that it induces a homeomorphism of $I_{m_{n-1},j} \cap C$. Fix such an integer j. By virtue of Equations (5) and (6), there is a permutation

$$\sigma_j : K(m_n; m_{n-1}, j) \to K(m_n; m_{n-1}, j)$$

such that

- for $k \in \{\min K(m_n; m_{n-1}, j), \max K(m_n; m_{n-1}, j)\}$, we have
$$\mathrm{Max}(l_{n-1}, I_{m_n,\sigma_j(k)}) < 1/n,$$
- for all integers i satisfying $\min K(m_n; m_{n-1}, j) \leq i < \max K(m_n; m_{n-1}, j)$, we have
$$|\,\mathrm{Max}(l_{n-1}, I_{m_n,\sigma_j(i)}) - \mathrm{Max}(l_{n-1}, I_{m_n,\sigma_j(i+1)})| < 1/n.$$

To identify such σ_j, one may first list the numbers $\mathrm{Max}(l_{n-1}, I_{m_n,i})$, for $i \in K(m_n; m_{n-1}, j)$, in an increasing fashion, then discard those numbers in the even places from the list, and finally re-list the discarded numbers in a decreasing fashion at the end.

Once we have σ_j, there is a unique homeomorphism H_n such that for all i in $K(m_n; m_{n-1}, j)$

$$H_n(I_{m_n,i} \cap C) = I_{m_n,\sigma_j^{-1}(i)} \cap C,$$

and each H_n is a translation by a constant on each of $I_{m_n,i} \cap C$.

Carrying out the above process for all $1 \leq j \leq 2^{m_{n-1}}$, we obtain a homeomorphism H_n. Then, we define

$$l_n(x) = l_{n-1}(H_n^{-1}(x)), \quad \forall x \in C.$$

This completes the proof of Step 1.

Consider the homeomorphisms $\phi_n : C \times \mathbb{R} \to C \times \mathbb{R}$, defined according to

$$\phi_0 = H_0, \quad \phi_1 = H_1 \circ H_0, \quad \phi_n = H_n \circ H_{n-1} \circ \cdots \circ H_0, \ \forall n \geq 2.$$

Step 2. The sequence ϕ_n, for $n \geq 0$, uniformly converges to a homeomorphism

$$\phi : C \times \mathbb{R} \to C \times \mathbb{R}.$$

By property (i) in Step 1, for all $n \geq 1$ and all $x \in C$, we have

$$|H_n(x) - x| \leq \max\{\operatorname{diam}(I_{m_{n-1},j}) \mid 1 \leq j \leq 2^{m_{n-1}}\} = 3^{-m_{n-1}}.$$

Hence,

$$\|\phi_n - \phi_{n-1}\|_\infty = \|H_n - \operatorname{Id}\|_\infty \leq 3^{-m_{n-1}}.$$

Therefore, ϕ_n forms a Cauchy sequence, and converges to a continuous map $\phi : C \times \mathbb{R} \to C \times \mathbb{R}$.

To prove that $\phi : C \times \mathbb{R} \to C \times \mathbb{R}$ is surjective, note that $\phi(C) \subseteq C$ is compact. Therefore, if there is z in $C \setminus \phi(C)$, then there is $\epsilon > 0$ such that

$$\mathbb{D}(z, \epsilon) \cap C \subseteq C \setminus \phi(C).$$

Choose $n \geq 0$ so that $\|\phi_n - \phi\|_\infty < \epsilon$. Then, we have $\phi(\phi_n^{-1}(z)) \in D_\epsilon(z)$, which is a contradiction.

To prove that ϕ is injective, fix arbitrary points $x \neq y$ in C. There are $n > 0$ and $1 \leq i < i' \leq 2^{m_n}$ such that $x \in I_{m_n,i}$ and $y \in I_{m_n,i'}$. By the above construction, there are distinct integers j and j' in $[1, 2^{m_n}]$ such that $\phi_n(x) \in I_{m_n,j}$ and $\phi_n(y) \in I_{m_n,j'}$. On the other hand, for all $n' > n$, we have

$$H_{n'}(I_{m_n,j} \cap C) = I_{m_n,j} \cap C, \quad H_{n'}(I_{m_n,j'} \cap C) = I_{m_n,j'} \cap C.$$

Therefore, $\phi(x) = \cdots \circ H_{n+2} \circ H_{n+1} \circ \phi_n(x) \in I_{m_n,j}$ and similarly $\phi(y) \in I_{m_n,j'}$. In particular, $\phi(x) \neq \phi(y)$.

Step 3. The set $\phi(Z)$ is a straight hairy Cantor set.

Consider the function $l_* = l \circ \phi^{-1} : C \to [0, \infty)$, and the set

$$Z_* = \{(x, y) \in \mathbb{R}^2 \mid x \in C, 0 \leq y \leq l_*(x)\}.$$

By Step 2, $\phi : Z \to Z_*$ is a homeomorphism.

We claim that for all $n \geq 0$ and $1 \leq j \leq 2^{m_n}$, we have

$$\operatorname{Max}(l_*, I_{m_n,j}) = \operatorname{Max}(l_n, I_{m_n,j}).$$

Recall that $l_n = l \circ \phi_n^{-1}$. To prove the above property, it is sufficient to show that $\phi^{-1}(I_{m_n,j} \cap C) = \phi_n^{-1}(I_{m_n,j} \cap C)$, or equivalently, $I_{m_n,j} \cap C = \phi \circ \phi_n^{-1}(I_{m_n,j} \cap C)$. However, by property (i) in Step 1, for all $n' > n$, $H_{n'}(I_{m_n,j} \cap C) = I_{m_n,j} \cap C$. Using $\phi \circ \phi_n^{-1} = \cdots \circ H_{n+2} \circ H_{n+1}$, we conclude that $\phi \circ \phi_n^{-1}$ preserves $I_{m_n,j} \cap C$.

Since $\phi : Z \to Z_*$ is a homeomorphism, the set $\{x \in C \mid l_*(x) \neq 0\}$ is dense in C. This implies property (i) in Definition 1.1.

Let z be an end point of C. There are $p \geq 0$ and $1 \leq j_p \leq 2^{m_p}$ such that $z \in \partial I_{m_p,j_p}$. Then, for all $q > p$, $z \in I_{m_q,j_q}$ with $j_q \in \{\min K(m_q; m_{q-1}, j_{q-1}), \max K(m_q; m_{q-1}, j_{q-1})\}$. By property (iii) in Step 1, we have

$$\mathrm{Max}(l_*, I_{m_q,j_q}) < 1/q.$$

Then, by Lemma 2.2 we have $l_*(z) = \lim_{q\to\infty} \mathrm{Max}(l_*, I_{m_q,j_q}) = 0$. This completes the proof of property (ii) in Definition 1.1.

Fix an arbitrary $z \in C$ with $l_*(z) > 0$, and let $\epsilon > 0$ be arbitrary. Assume that $(j_n)_{n\geq 0}$ are the sequence of integers with $z \in I_{m_n,j_n}$, for all $n \geq 0$. Using Lemma 2.2, there exists $n > \max(2/\epsilon, 1/l_*(z))$ such that $3^{-m_{n-1}} < \epsilon$, and

$$|\,\mathrm{Max}(l_*, I_{m_n,j_n}) - l_*(z)| < \epsilon/2.$$

On the other hand, since $\mathrm{Max}(l_*, I_{m_n,j_n}) \geq l_*(z) > 1/n$,

$$j_n \notin \{\min K(m_n; m_{n-1}, j_{n-1}), \max K(m_n; m_{n-1}, j_{n-1})\}.$$

Therefore,

$$I_{m_n,j_n-1} \cup I_{m_n,j_n+1} \subseteq I_{m_{n-1},j_{n-1}},$$

and by property (iv) in Step 1,

$$|\,\mathrm{Max}(l_*, I_{m_n,j_n-1}) - \mathrm{Max}(l_*, I_{m_n,j_n})| < 1/n,$$

$$|\,\mathrm{Max}(l_*, I_{m_n,j_n}) - \mathrm{Max}(l_*, I_{m_n,j_n+1})| < 1/n.$$

Let α be a point in $I_{m_n,j_n-1} \cap C$ with $l_*(\alpha) = \mathrm{Max}(l_*, I_{m_n,j_n-1})$. Then, $\alpha < z$,

$$|\alpha - z| < \mathrm{diam}\,(I_{m_{n-1},j_{n-1}}) = 3^{-m_{n-1}} < \epsilon,$$

and

$$|l_*(z) - l_*(\alpha)| = |l_*(z) - \mathrm{Max}(l_*, I_{m_n,j_n}) + \mathrm{Max}(l_*, I_{m_n,j_n}) - \mathrm{Max}(l_*, I_{m_n,j_n-1})| < \epsilon/2 + 1/n < \epsilon.$$

Similarly, we may find $\beta \in C$ such that

$$\beta > z, \quad |\beta - z| < \epsilon, \quad |l_*(\beta) - l_*(z)| < \epsilon.$$

This completes the proof of property (iii) in Definition 1.1. □

Proposition 2.7. *Let $C \subset \mathbb{R}$ be a Cantor set, $l : C \to [0, +\infty)$, and*

$$X = \{(x, y) \in \mathbb{R}^2 \mid x \in C, 0 \leq y \leq l(x)\}.$$

Then, X is a straight hairy Cantor set if and only if the following properties hold:

(i) *X is compact;*
(ii) *the set $\{x \in C \mid l(x) \neq 0\}$ is dense in C;*
(iii) *any point in $\{(x, y) \in X \mid x \in C, 0 < y < l(x)\}$ is not accessible from $\mathbb{R}^2 \setminus X$.*

Proof. Assume that X is a straight hairy Cantor set. By the uniqueness of the length function of X, l^X, we must have $l^X = l$. Thus, properties (i), (ii), and (iii) in the proposition hold.

Assume that X satisfies the three properties in the proposition. We get property (i) in Definition 1.1 for free. To see property (ii) in Definition 1.1, let $x \in C$ be an end point. If $l(x) \neq 0$, any $(x, y) \in X$ with $0 < y < l(x)$ will be accessible from $\mathbb{R}^2 \setminus X$. This contradiction with property (iii) in the proposition shows that we must have $l(x) = 0$.

Now assume that $x \in C$ is not an end point. Define

$$l^+(x) = \limsup_{t \to x^+, t \in C} l(t), \quad l^-(x) = \limsup_{t \to x^-, t \in C} l(t).$$

By the compactness of X, $l^+(x)$ and $l^-(x)$ are finite values. Moreover, we must have $l(x) \geq l^+(x)$ and $l(x) \geq l^-(x)$. If $l(x) > l^+(x)$, then any $(x, y) \in X$ with $l(x) > y > l^+(x)$ will be accessible from the right hand side. Similarly, if $l(x) > l^-(x)$, then any $(x, y) \in X$ with $l(x) > y > l^-(x)$ will be accessible from the left hand side. These contradict property (iii) in the proposition, so we must have $l(x) = l^+(x) = l^-(x)$. Thus, we also have property (iii) in Definition 1.1. $\square$

We present the following corollary for the future reference purposes.

Corollary 2.8. *Assume that*

$$X = \{(x, y) \in \mathbb{R}^2 \mid x \in C, 0 \leq y \leq l(x)\}$$

is a straight hairy Cantor set, where $C \subset \mathbb{R}$ is a Cantor set, and $l : C \to [0, +\infty)$ is the length function of X. Then, we have

(i) $\{l(x) \mid x \in C\} = [0, \sup_{x \in C} l(x)]$;
(ii) $\{(x, l(x)) \mid x \in C\}$ *is dense in* X.

Proof. Item (i) immediately follows from the continuity of the bump functions.

To prove item (ii), let (x, y) be an arbitrary point in X, and fix an arbitrary $\epsilon > 0$. There are δ_1 and δ_2 in $(0, \epsilon)$ such that $(x - \delta_1, x + \delta_2) \cap C$ is a Cantor set in $\mathbb{R}$. Let $D = (x - \delta_1, x + \delta_2) \cap C$. From Definition 1.1, one may see that the set

$$\{(x, y) \in \mathbb{R}^2 \mid x \in D, 0 \leq y \leq l(x)\},$$

is a straight hairy Cantor set. As (x, y) belongs to the above set, $y \leq \sup_{t \in D} l(t)$. Applying item (i) to the above straight hairy Cantor set, we conclude that there is $x' \in D$ with $|l(x') - y| < \epsilon$. Therefore,

$$|(x, y) - (x', l(x'))| \leq |x - x'| + |y - l(x')| \leq 2\epsilon. \qquad \square$$

3. Height functions

In this section we use a general theory of Whitney maps to build a notion of length for the components of X. To that end, we only use axioms A1 to A4. Within Section 3, unless otherwise stated, X is a compact metric space which satisfies axioms A1 to A4. The unique minimal Cantor set $B \subset X$ which contains all point components of X is called the **base** of X. We shall often say that X is **based** on the Cantor set B.

Any arc [5] component γ of X has two distinct end points, one of which belongs to B. For any such component γ, the end point which belongs to B is called the **base** of γ. The other end point of γ, which does not belong to B, is called the **peak** of γ. Abusing the notation, if c is a point component of X, c is both called the base of c and the peak of c. For $x \in X$, the **base** of x is defined as the base of the component of X containing x, and similarly, the **peak** of x is defined as the peak of that component. We may define the **base map** and the **peak map**

$$b : X \to B, \quad p : X \to X,$$

where $b(x)$ is the base of x and $p(x)$ is the peak of x.

Let us say that $x \in X$ is a **base point**, if $b(x) = x$, and say that $x \in X$ is a **peak point** if $p(x) = x$. Since b is the identity map on B, any element of B is a base point. On the other hand, if x is a point component of X, then x is a peak point of X. Axioms A3 and A4 imply that b is continuous on X. However, p is far from continuous.

The base points of X and the peak points of X are defined solely using the topology of X. It follows that any homeomorphism of a hairy Cantor set must send base points to base points, and peak points to peak points.

A function $h : X \to [0, +\infty)$ is called a **height function** on X, if the following properties hold:

(H1) h is continuous on X,
(H2) $B = h^{-1}(0)$
(H3) h is injective on any connected component of X.

For instance, if X is a straight hairy Cantor set, $h(x, y) = y$ is a height function on X. To prove the existence of a height function on X in general, we employ a theory of Whitney maps, which we briefly explain below.

Assume that K is a compact metric space, and let $\mathrm{CL}(K)$ denote the set of all non-empty closed subsets of K. The space $\mathrm{CL}(K)$ may be equipped with the **Hausdorff topology** induced from the Hausdorff metric. That is, given non-empty

[5] A Jordan arc, or simply an arc, in $\mathbb{R}^2$ is the image of a continuous and injective map $\gamma : [0, 1] \to \mathbb{R}^2$. The *end points* of this Jordan arc are $\gamma(0)$ and $\gamma(1)$.

closed sets K_1 and K_2 in K, the Hausdorff distance between K_1 and K_2 is defined as the infimum of all $\epsilon > 0$ such that ϵ neighbourhood of K_1 contains K_2 and ϵ neighbourhood of K_2 contains K_1.

A function $\mu : \mathrm{CL}(K) \to [0, +\infty)$ is called a **Whitney map** for K, if

(W1) μ is continuous on $\mathrm{CL}(K)$;
(W2) $\mu(\{x\}) = 0$ for every $x \in K$;
(W3) for every K_1 and K_2 in $\mathrm{CL}(K)$ with $K_1 \subsetneq K_2$, we have $\mu(K_1) < \mu(K_2)$.

In [Whi33], Whitney proves the existence of maps μ satisfying the above properties. The existence of a Whitney map for a compact subset of an Euclidean space is a classical result in topology. One may refer to the general reference [Nad92, Exercise 4.33], or the extensive monograph on the hyperspaces [IN99].

Given x and y in the same arc component γ of X, we use the notation

$$[x, y]$$

to denote the unique arc in γ connecting x to y. In particular, $\{x, y\} \subseteq [x, y]$.

Proposition 3.1. *If X is a compact metric space which satisfies axioms A1 to A4, there is a height function on X.*

Proof. Let μ be a Whitney map for X, that is, $\mu : \mathrm{CL}(X) \to [0, +\infty)$ enjoys W1-W3. Consider the function $h : X \to [0, +\infty)$, defined as

$$h(x) = \mu([b(x), x]).$$

To see H1, let $x_i \to x$ within X. By axiom A4, $[b(x_i), x_i] \to [b(x), x]$, in the Hausdorff topology. Thus, by the continuity of μ on $\mathrm{CL}(X)$, we conclude that $h(x_i) \to h(x)$.

Property W3 implies $H3$. Properties W2 and W3 together imply H2, that is,

$$h^{-1}(0) = \{x \in X \mid b(x) = x\} = B. \qquad \square$$

Proof of Theorem 1.3. Fix a compact metric space X which satisfies axioms A1 to A4, and A6'. Let B denote the base Cantor set of X, and $b : X \to B$ its base map. By Proposition 3.1, there is a height function $h : X \to [0, \infty)$.

Let $C \subset \mathbb{R}$ be a Cantor set, and let $g : B \to C$ be an arbitrary homeomorphism. Define the map $\Phi : X \to C \times [0, \infty)$ as

$$\Phi(x) = (g(b(x)), h(x)).$$

The map Φ is injective. That is because, if $g(b(x_1)) = g(b(x_2))$ and $h(x_1) = h(x_2)$, for some x_1 and x_2 in X, then x_1 and x_2 belong to the same connected component of X, and we have $[x_1, b(x_1)] = [x_2, b(x_2)]$. This implies that $x_1 = x_2$.

The map Φ is continuous and injective on a compact set. Therefore, it is a homeomorphism onto its image. We claim that $\Phi(X)$ is homeomorphic to a straight hairy Cantor set. To show that, we apply Proposition 2.6 to the set $Z = \Phi(X)$.

Consider the function $l : C \to [0, \infty)$ defined according to

$$\Phi(X) = \{(x, y) \mid x \in C, 0 \le y \le l(x)\}.$$

As Φ is a homeomorphism, and X is compact, $\Phi(X)$ must be compact.

Define the sets

$$E_b = \{(x, 0) \mid x \in C, l(x) \neq 0\}, \quad E_p = \{(x, l(x)) \mid x \in C, l(x) \neq 0\}.$$

By axiom A6', $E_b \cup E_p$ is dense in $\Phi(X)$. For any $\epsilon > 0$,

$$\overline{E_b} \cap \{(x, y) \mid x \in C, \epsilon \le y \le l(x)\} = \emptyset,$$

which implies that

$$\overline{E_p} \supseteq \{(x, y) \mid x \in C, \epsilon \le y \le l(x)\}.$$

Therefore,

$$\overline{E_p} \supseteq \bigcup_{\epsilon > 0} \{(x, y) \mid x \in C, \epsilon \le y \le l(x)\} = \{(x, y) \mid x \in C, 0 < y \le l(x)\},$$

which implies that

$$\overline{E_p} \supseteq \{(x, y) \mid x \in C, 0 \le y \le l(x), l(x) \neq 0\} \supseteq E_b.$$

Therefore, E_p is dense in X. □

4. Base curves

Unless otherwise stated, in Section 4 we assume that X is a compact subset of $\mathbb{R}^2$ which satisfies axioms A1 to A4.

We say that a Jordan curve $\omega \subset \mathbb{R}^2$ is a **base curve** for X, if $X \cap \omega = B$, and the bounded connected component of $\mathbb{R}^2 \setminus \omega$ does not intersect X. In particular, all arcs of X lie on the same side of ω. A main ingredient of the proof of Theorem 1.4 is the existence of a base curve for X. To identify such a curve, the main technical issue is to find a Jordan curve ω which satisfies $X \cap \omega = B$. Having all arcs on the same side can be achieved by carefully modifying such ω. We present an approach which simultaneously takes care of both properties. This section is devoted to the proof of the following statement.

Proposition 4.1. *Any compact set in $\mathbb{R}^2$, which satisfies axioms A1 to A4, has a base curve.*

To build a base curve for X, we first build an infinite tree in $\mathbb{R}^2 \setminus X$ such that the vertices of the tree accumulate on B. Then we turn such an infinite tree into a base curve for X by slightly thickening the edges of the tree. See Figure 2. Let us formally present such trees.

We say that T is an **ideal tree** for X, if

$$T = \cup_{n\geq 0} T_n$$

where for every $n \geq 0$, $T_n \subset \mathbb{R}^2 \setminus X$ is a (finite) tree, such that the following properties hold:

(T1) There are pairwise disjoint finite sets V_n, for $n \geq 0$, such that for every $m \geq 0$, the set of vertices of T_m is equal to $\cup_{n=0}^m V_n$. Moreover, $T_0 = V_0$ consists of a single point.

(T2) for every $m \geq 1$, T_{m+1} is obtained from T_m, by adding a collection of edges joining elements of V_m to elements of V_{m+1}.

(T3) for every $m \geq 0$ and every $z \in V_m$, there is an edge of T_{m+1} joining z to a point in V_{m+1}.

(T4) the maximum of the diameters of the edges in $T_{m+1} \setminus T_m$ tends to 0, as $m \to \infty$.

(T5) the set of accumulation points of $\cup_{n\geq 0} V_n$ is equal to B.

It follows that every ideal tree is a connected subset of the plane, and for every $x \in B$ there is an infinite branch of the tree which grows towards x.

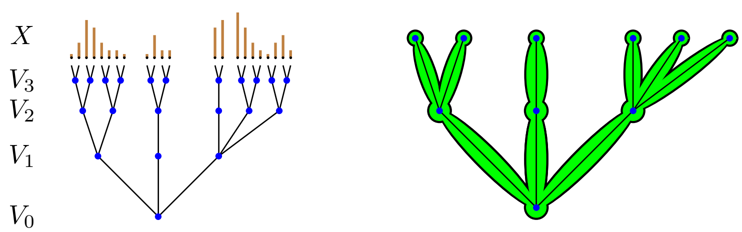

FIGURE 2. The figure on the left Illustrates an ideal tree for X, and the figure on the right illustrates the thickening of a tree into a Jordan curve.

Proposition 4.2. *Every compact set in $\mathbb{R}^2$, which satisfies axioms A1 to A4, has an ideal tree.*

Roughly speaking, our strategy for building an ideal tree for X is to form a suitable nest U_n of open sets shrinking to B, where the vertices V_n of the ideal tree lie on ∂U_n and the edges of $T_{n+1} \setminus T_n$ lie in $U_n \setminus U_{n+1}$. See Figure 3. Below we introduce the basic objects for this approach.

Recall that a curve $\gamma : [0, 1) \to \mathbb{R}^2$ **lands** at $z \in \mathbb{R}^2$, if $\lim_{t\to 1} \gamma(t)$ exists and is equal to z. A point $z \in \mathbb{R}^2$ is called **accessible** from $\Omega \subset \mathbb{R}^2$, if there is a curve

$\gamma : [0,1) \to \Omega$ which lands at z. By a **topological disk** in $\mathbb{R}^2$ we mean a connected and simply connected open subset of $\mathbb{R}^2$.

An open set $U \subseteq \mathbb{R}^2$ is called a **nice partition** for B, if the following properties hold:

(P1) U consists of a finite number of pairwise disjoint topological disks, and $B \subseteq U$;
(P2) for all $x \in X \cap U$, $[b(x), x] \subseteq U$;
(P3) for all $x \in \partial U \setminus X$, x is accessible from both of $U \setminus X$ and $\mathbb{R}^2 \setminus \overline{U}$;
(P4) if U_1 and U_2 are distinct connected components of U, then $\overline{U_1} \cap \overline{U_2} = \emptyset$.

If X is a straight hairy Cantor set, one may consider a finite union of squares with horizontal and vertical sides to form a nice partition for B. Ideally, one may wish to consider nice partitions consisting of Jordan domains. However, due to technical reasons which will be clear in a moment, it is convenient to work with nice partitions consisting of topological disks. Because of this, we have imposed P3, which guarantees that each point on the boundary of $U \setminus X$ is accessible from both sides (a property that is always true for Jordan domains). Indeed, the only remarkable property in the above list is P2, which requires that arcs of X may not leave and return back to U.

Let U be a nice partition for B. By a **marking** for U, we mean a set

$$\mathfrak{M}_U \subset \partial U \setminus X$$

such that for every connected component U' of U, $\mathfrak{M}_U \cap \partial U'$ is a single point. Let

$$\mathfrak{M}_{U'} = \mathfrak{M}_U \cap \partial U'.$$

Note that by property P3 of nice partitions, the point $\mathfrak{M}_{U'}$ is accessible from both sets $U' \setminus X$ and $\mathbb{R}^2 \setminus \overline{U'}$. We refer to the pair $(U, \mathfrak{M}_U)$ as a **marked nice partition** for B.

Let $(U, \mathfrak{M}_U)$ and $(V, \mathfrak{M}_V)$ be marked nice partitions for B with $\overline{V} \subset U$. Let V' be a component of V, which is contained in a component U' of U. We say that V' is **reachable** in $(U, \mathfrak{M}_U)$, if there is a curve

$$\gamma : (0,1) \to U' \setminus (\overline{V} \cup X)$$

such that

$$\lim_{t \to 0} \gamma(t) \in \mathfrak{M}_{U'}, \qquad \text{and} \qquad \lim_{t \to 1} \gamma(t) \in \mathfrak{M}_{V'}.$$

Given marked nice partitions $(V, \mathfrak{M}_V)$ and $(U, \mathfrak{M}_U)$ with $\overline{V} \subset U$, we say that $(V, \mathfrak{M}_V)$ is **reachable** in $(U, \mathfrak{M}_U)$, if every component of V is reachable in $(U, \mathfrak{M}_U)$.

To construct an ideal tree, we shall first build a nest of marked nice partitions shrinking to B, so that each marked nice partition is reachable in the previous marked nice partition. The mark points will become the vertices of the ideal tree.

The curves in the definition of reachable will become the edges of the ideal tree, after a minor modification to make them pair-wise disjoint. See Figure 3.

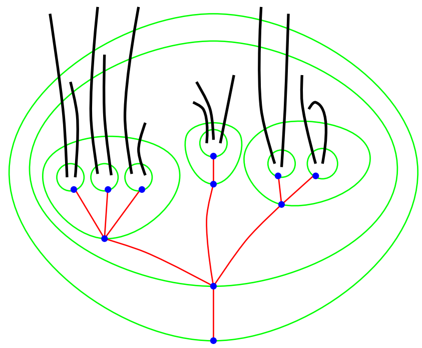

FIGURE 3. A nest of nice partitions, the markings, and the three T_2.

We present the process of building a nest of marked nice partitions in a number of technical lemmas. It is not even immediately obvious that a non-trivial nice partition for B exists. Let us first show that once we have properties P1 to P3 of nice partitions, property P4 can be achieved by slightly shrinking the components.

Lemma 4.3. *Assume that U is an open set which satisfies properties P1 to P3 in the definition of nice partitions for B, and let U_i, for $1 \le i \le n$, be the connected components of U. There are topological disks $V_i \subseteq U_i$, for $1 \le i \le n$, such that $\cup_{1\le i\le n} V_i$ is a nice partition for B.*

Proof. If $n = 1$, we let $V_1 = U_1$, and there is nothing to prove. Below we assume that $n \ge 2$. Let $J = \{1, 2, \dots, n\}$. We claim that for distinct i and j in J,

$$(\overline{X \cap U_i}) \cap (\overline{X \cap U_j}) = \emptyset.$$

To prove this, it is enough to show that for every $i \in J$, if $x \in \overline{X \cap U_i}$ then $b(x) \in U_i$. Fix an arbitrary $x \in \overline{X \cap U_i}$. There is a sequence $(x_l)_{l\ge1}$ in $X \cap U_i$ which converges to x. By P2, $b(x_l) \in U_i$, and hence $b(x) = \lim_{l\to\infty} b(x_l) \in \overline{U_i}$. Since $B \cap \partial U_i = \emptyset$, we must have $b(x) \in U_i$.

The above property allows us to slightly move ∂U_i inside U_i, in places away from X, so that property P4 holds as well. We shall do this is two steps, first shrink each component U_i to some W_i, and then shrink each W_i to some V_i, so that the conclusion in the lemma holds.

Step 1. For $i \in J$, if for all $j \ne i$ we have $\partial U_i \cap (\overline{X \cap U_j}) = \emptyset$, then we let $W_i = U_i$. If for some $i \in J$ there is $j \in J$ with $\partial U_i \cap (\overline{X \cap U_j}) \ne \emptyset$, we shrink U_i to W_i so that for all $j \neq i$ in J we have

$$\partial W_i \cap (\overline{X \cap U_j}) = \emptyset.$$

To do that, first note that $\overline{X \cap U_i}$ and $\cup_{j\in J\setminus\{i\}}(\overline{X \cap U_j}) \cap \partial U_i$ are compact, and by the above paragraph, are disjoint. Let P_i be a finite union of Jordan domains such that P_i contains $\cup_{j\in J\setminus\{i\}}(\overline{X \cap U_j}) \cap \partial U_i$, but $\overline{P_i}$ does not meet $\overline{X \cap U_i}$. We may choose P_i (for example, a finite union of cross-cut domains) such that the set

$$W_i = U_i \setminus \overline{P_i}$$

is a topological disk. Using $\partial W_i \subseteq \overline{U_i}$ and $W_i \subseteq U_i$, we have

$$\partial W_i \cap (\overline{X \cap U_j}) \subseteq \overline{U_i} \cap (\overline{X \cap U_j}) = (\partial U_i \cap (\overline{X \cap U_j})) \cup (U_i \cap (\overline{X \cap U_j})) \subseteq P_i \cup \emptyset = P_i,$$

and since P_i is an open set, we have

$$\partial W_i \cap (\overline{X \cap U_j}) \subseteq \partial W_i \cap P_i = \emptyset.$$

Moreover, we note that since for all $i \in J$, $\overline{P_i} \cap (\overline{X \cap U_i}) = \emptyset$, we have $W_i \cap X = U_i \cap X$.

We repeat the above process for all $i \in J$ and obtain the topological disks $W_i \subset U_i$. With these modifications, for distinct i and j in J, we have

$$\partial W_i \cap (\overline{X \cap W_j}) \subseteq \partial W_i \cap (\overline{X \cap U_j}) = \emptyset.$$

Step 2. Fix an arbitrary $i \in J$. If ∂W_i does not meet ∂W_j for all $j \neq i$, we let $V_i = W_i$. If there is $j \neq i$ such that $\partial W_i \cap \partial W_j \neq \emptyset$, we further modify W_i as follows. By the above paragraph, the sets $\partial W_i \cap (\cup_{j\in J\setminus\{i\}} \partial W_j)$ and $\overline{X \cap W_i}$ are disjoint. Then, there is a finite union Q_i of Jordan domains such that Q_i contains $\partial W_i \cap (\cup_{j\in J\setminus\{i\}} \partial W_j)$ and $\overline{Q_i}$ does not meet $\overline{X \cap W_i}$. We may choose the set Q_i (for example, a finite union of cross-cut domains in W_i) such that in addition,

$$V_i = W_i \setminus Q_i$$

is connected and simply connected. Repeating the above process for all $i \in J$ we obtain the desired topological disks V_i.

By construction, $\overline{V_i} \cap \overline{V_j} = \emptyset$, for distinct i and j. Moreover, since $\overline{Q_i} \cap (\overline{X \cap W_i}) = \emptyset$, we must have $V_i \cap X = W_i \cap X = U_i \cap X$. This implies that $B \subset \cup_{i\in J} V_i$ and P2 holds. As $\cup_{i\in J} U_i$ satisfies P3, and each of P_i and Q_i is a finite union of Jordan domains, $\cup_{i\in J} V_i$ also satisfies property P3. $\square$

Lemma 4.4. *For any $\epsilon > 0$, there is a nice partition for B which is contained in the ϵ neighbourhood of B.*

Proof. Fix an arbitrary $\epsilon > 0$. Let U be a finite union of Jordan domains such that $B \subset U$, U is contained in ϵ neighbourhood of B, and the closures of the connected components of U are pairwise disjoint. Below, we modify U so that it becomes a nice partition for B.

By Proposition 3.1 there is a height function $h : X \to [0, +\infty)$. Since U is open, $B \subset U$, and $h^{-1}(0) = B$, there is $t > 0$ such that $h^{-1}([0, t)) \subset U$. Let us define

$$X' = \{x \in X \mid h(x) \geq t\}.$$

This is a closed set in $\mathbb{R}^2$ with $X' \cap B = \emptyset$. Define

$$V = U \setminus X'.$$

There are several possibilities for the components of X'. Each component of X' may be either completely contained in U, completely contained outside U, leave U and never return back to U, or leave U and rerun back to U. The set V may not be simply connected, for instance, due to removing the components of X' which are contained in U. We can add those holes back into V. That is, let Q denote the union of V and the bounded components of $\mathbb{R}^2 \setminus V$. Then, every component of Q is full, and hence is simply connected. The set Q is the smallest open set which contains V and its components are simply connected. Since $V \subset U$ and every component of U is simply connected, Q is contained in U. As we only need the nice partition to cover B, let W denote the components of Q which meet B. Because B is compact, W has a finite number of connected components. These show that W satisfies property P1.

Fix an arbitrary $x \in X \cap W$. Then, either $x \in X \cap V$ or $x \in X \cap (W \setminus V)$. First assume that $x \in X \cap V$. Then $h(x) < t$, and hence $[b(x), x] \subset h^{-1}([0, t)) \subset U \setminus X' = V$. In particular, since $[x, b(x)]$ is connected and meets B, $[x, b(x)]$ is contained in W. Now assume that $x \in X \cap (W \setminus V)$. We have $x \notin V$, $b(x) \in V$, and for every $y \in [x, b(x)] \cap V$, $[y, b(x)] \subset V$. This implies that there is $z \in [x, b(x)]$ such that $[x, b(x)] \cap V = [z, b(x)] \setminus \{z\}$. Now, $[x, z] \cap V = \emptyset$, and since $x \in W$, the connected set $[x, z]$ must be contained in a bounded component of $\mathbb{R}^2 \setminus V$. Thus,

$$[x, b(x)] = [x, z] \cup ([z, b(x)] \setminus \{z\}) \subset W.$$

This shows that W satisfies property P2.

Because X has no interior points, we have $\overline{U \setminus X} = \overline{U}$. Then, using $U \setminus X \subseteq V \subseteq Q \subseteq U$, we conclude that $\overline{U \setminus X} = \overline{Q} = \overline{U}$. Also, since $U \setminus Q \subseteq X$, we have $(\mathbb{R}^2 \setminus Q) \setminus X = (\mathbb{R}^2 \setminus U) \setminus X$. Combining these we obtain

$$\partial Q \setminus X = (\overline{Q} \cap (\overline{\mathbb{R}^2 \setminus Q})) \setminus X = \overline{Q} \cap ((\mathbb{R}^2 \setminus Q) \setminus X) = \overline{U} \cap ((\mathbb{R}^2 \setminus U) \setminus X) = \partial U \setminus X.$$

Because, ∂U is a finite union of Jordan domains, every point in $\partial U \setminus X$ is accessible from both $U \setminus X$ and $\mathbb{R}^2 \setminus U$. By the above equation, $\partial Q \setminus X = \partial U \setminus X$, we conclude that every point in $\partial Q \setminus X$ is accessible from both $Q \setminus X$ and $\mathbb{R}^2 \setminus Q$. In particular, every point on $\partial W \setminus X$ is accessible from both $W \setminus X$ and $\mathbb{R}^2 \setminus W$.

Above, we have shown that W satisfies properties P1 to P3 for the nice partitions. If necessary, we may employ Lemma 4.3 to obtain a nice partition for B which is contained in W. $\square$

Lemma 4.5. *If U is a nice partition for B, and U' is a connected component of U, then $U' \setminus X$ is path-connected.*

Proof. By the properties of the nice partitions, $B' = B \cap U'$ is a Cantor set. Let $X' = b^{-1}(B')$, that is, the set of all components of X which meet U'. We have $U' \setminus X = U' \setminus X'$ and, since b is continuous, X' is compact.

Because $U' \setminus X'$ is open, in order to show that $U' \setminus X'$ is path connected, it is enough to show that $U' \setminus X'$ is connected. To prove that $U' \setminus X'$ is connected, we use a criterion of Borsuk [ES52, Chapter XI, Theorem 3.7]. That is, for a closed set Y in the Riemann sphere $\hat{\mathbb{C}}$, the set $\hat{\mathbb{C}} \setminus Y$ is connected if and only if any continuous map from Y to the circle

$$\mathbb{S}^1 = \{z \in \mathbb{C}; |z| = 1\}$$

is null-homotopic. Here, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, so $\hat{\mathbb{C}}$ is the one point compactification of $\mathbb{R}^2$. Note that

$$\hat{\mathbb{C}} \setminus (U' \setminus X') = (\hat{\mathbb{C}} \setminus U') \cup X'.$$

Thus, it is enough to show that any continuous map

$$g : (\hat{\mathbb{C}} \setminus U') \cup X' \to \mathbb{S}^1$$

is null homotopic. Equivalently, we need to show that any such map g can be lifted to a continuous map $\hat{g} : (\hat{\mathbb{C}} \setminus U') \cup X' \to \mathbb{R}$, that is, $e^{i\hat{g}(a)} = g(a)$ for all a in $(\hat{\mathbb{C}} \setminus U') \cup X'$. Because U' is connected, by Borsuk's criterion, $g : \hat{\mathbb{C}} \setminus U' \to \mathbb{S}^1$ is null homotopic. This implies that g can be lifted to a continuous map $\hat{g} : (\hat{\mathbb{C}} \setminus U') \to \mathbb{R}$. We need to extend $\hat{g}$ onto $X' \cap U'$. On the components of X' which exit U', there is a unique choice for extending $\hat{g}$. That is because by property P2 of nice partitions, any such component, once it exits U', it stays outside U' where $\hat{g}$ is defined. There may be many other components of X' which are in U' and accumulate on the components which exit U'. We need to extend $\hat{g}$ on X' in a consistent fashion. The main idea is to decompose X' into a finite number of disjoint "thin clusters of hairs" so that we can extend $\hat{g}$ on each such cluster by treating it as a single arc.

Let $\mathrm{d}_{\hat{\mathbb{C}}}$ denote the spherical distance on $\hat{\mathbb{C}}$. Because g is continuous on the compact set X', there is $\delta_1 > 0$ such that for every a and b in X' with $\mathrm{d}_{\hat{\mathbb{C}}}(a, b) \leq \delta_1$, the arc length between $g(a)$ and $g(b)$ is at most $\pi/2$.

Let $B'' \subseteq B'$ be the set of points $z \in B'$ such that $b^{-1}(z) \nsubseteq U'$. The set B'' might be empty, in which case some of the arguments below can be skipped. We note that since X' and $\hat{\mathbb{C}} \setminus U'$ are compact, and $b : X' \to B'$ is continuous, B'' is (closed and hence) compact.

There is a unique extension of the map $\hat{g} : \hat{\mathbb{C}} \setminus U' \to \mathbb{R}$ onto B''. That is because, for every $z \in B''$, $b^{-1}(z)$ is an arc and $\hat{g}$ is already defined at one end of $b^{-1}(z)$.

Thus, we may extend $\hat{g}$ on all of $b^{-1}(z)$ in a continuous fashion. In particular, this defines $\hat{g}$ on z. In the next few paragraphs we show that $\hat{g} : B'' \to \mathbb{R}$ is continuous.

By Proposition 3.1, there is a heigh function $h : X \to [0, 1]$. Consider a closed set $E \subset B'$. There is a component γ_E of $b^{-1}(E)$ such that the maximum of h on $b^{-1}(E)$ is realised at the peak point of γ_E. There is a projection map $\pi_E : b^{-1}(E) \to \gamma_E$ defined according to $h(x) = h(\pi_E(x))$ for all $x \in b^{-1}(E)$. By the properties of the height functions, π_E is a continuous map, which is identity on the component γ_E.

There are a finite number of disjoint closed sets B''_i, for $1 \leq i \leq n$, such that $B'' = \cup_{i=1}^n B''_i$ and for every $1 \leq i \leq n$ the projection map $\pi_{B''_i} : b^{-1}(B''_i) \to \gamma_{B''_i}$ is $\delta_1/2$-close to the identity map on $b^{-1}(B''_i)$. That is because otherwise, one can identify a nest of closed sets $E_i \subset B''$ shrinking to a single point e, and a sequence of points $x_i \in b^{-1}(E_i)$ so that $\mathrm{d}_{\hat{\mathbb{C}}}(\pi_{E_i}(x_i), x_i) > \delta_1/2$. Without loss of generality, we may assume that x_i converges to some $x \in X'$, and $\pi_{E_i}(x_i)$ converges to some $y \in X'$. We must have $\mathrm{d}_{\hat{\mathbb{C}}}(x, y) \geq \delta_1/2$ and $h(x) = h(y)$. As x_i and $\pi_{E_i}(x_i)$ belong to $b^{-1}(E_i)$ and E_i shrink to e, by Axiom A4, we must have $b(y) = b(x) = e$. However, by the injectivity of h on each component of X, we cannot have $h(y) = h(x)$ and $x \neq y$.

To show that $\hat{g} : B'' \to \mathbb{R}$ is continuous, let $z_i \to z$ in B''. Without loss of generality, we may assume that all of z_i and z belong to B''_1. For each i, there is $y_i \in b^{-1}(z_i) \setminus U'$. It is enough to show that for every subsequence of $(z_i)_{i\geq 1}$, say $(z_{i_k})_{k\geq 1}$, there is a subsequence of $(z_{i_k})_{k\geq 1}$ on which $\hat{g}$ converges to $\hat{g}(z)$. Fix an arbitrary subsequence of z_i, say z_i for simplicity. Since $\hat{\mathbb{C}} \setminus U'$ is compact, there is a subsequence of y_i, say $(y_{i_k})_{k\geq 1}$, which converges to some $y \in \hat{\mathbb{C}} \setminus U'$. By the continuity of the base map b, $b(y) = z$. By the continuity of $\hat{g}$ on $\hat{\mathbb{C}} \setminus U'$, there is N_1 such that for all $k \geq N_1$, $|\hat{g}(y_{i_k}) - \hat{g}(y)| \leq \pi/2$. By the choice of B''_i, there is a map of the form $\pi_{B''_1}^{-1} \circ \pi_{B''_1}$ from $[y_{i_k}, b(y_{i_k})]$ to $[y, b(y)]$ if $h(y) \geq h(y_{i_k})$, or from $[y, b(y)]$ to $[y_{i_k}, b(y_{i_k})]$ if $h(y_{i_k}) \geq h(y)$, which is δ_1-close to the identity map. By the choice of δ_1, this implies that one can lift g along $[y_{i_k}, b(y_{i_k})]$ and $[y, b(y)]$ using the same choices of the inverse branches of the projection. In particular, $|\hat{g}(z_{i_k}) - \hat{g}(z)|$ is equal to the arc length from $g(z_{i_k})$ to $g(z)$, which tends to 0 as $k \to \infty$.

Since $B'' \cup (\hat{\mathbb{C}} \setminus U')$ is compact, there is $\delta_2 > 0$ such that for every a and b in $B'' \cup (\hat{\mathbb{C}} \setminus U')$ with $\mathrm{d}_{\hat{\mathbb{C}}}(a, b) \leq \delta_2$, $|\hat{g}(a) - \hat{g}(b)| \leq \pi/2$. Let $\delta = \min\{\delta_1, \delta_2\}$.

There are a finite number of disjoint closed sets B'_i, for $1 \leq i \leq m$, such that $B' = \cup_{i=1}^m B'_i$ and for every $1 \leq i \leq m$ the projection map $\pi_{B'_i} : b^{-1}(B'_i) \to \gamma_{B'_i}$ is $\delta/2$-close to the identity map on $b^{-1}(B'_i)$.

It follows that for each $i = 1, 2, \ldots, m$, the diameter of B'_i is less than δ. This implies that for each i, $g(B'_i)$ is contained in a single arc of length at most $\pi/2$ on $\mathbb{S}^1$. Also, if $B'_i \cap B'' \neq \emptyset$, $\hat{g}(B'_i)$ is contained in an interval of length at most $\pi/2$.

These show that there is a continuous lift of g on each B_i' which agrees with $\hat{g}$. That is, one can use a single continuous inverse branch of the projection to define $\hat{g}$ on each B_i'.

Now that $\hat{g}$ is defined on B', and we know that every connected component of $b^{-1}(B') = X'$ is an arc, we may extend $\hat{g}$ on each component of X'. If a connected component $b^{-1}(z)$ of X' exits U', by the above construction, the map $\hat{g}$ on $b^{-1}(z)$ agrees with $\hat{g}$ on $\hat{\mathbb{C}} \setminus U'$, where they meet. And if a component does not exit U', there is no obstruction to the choice of the lift.

We need to show that $\hat{g} : (\hat{\mathbb{C}} \setminus U') \cup X' \to \mathbb{R}$ is continuous. We already know that $\hat{g}$ is continuous on $\hat{\mathbb{C}} \setminus U'$. So, let $y_j \to y$ in X'. Without loss of generality, assume that all $b(y_j)$ and $b(y)$ are contained in B_1'. By the above paragraph, there is a map of the form $\pi_{B_1'}^{-1} \circ \pi_{B_1'}$ from $[y_j, b(y_j)]$ to $[y, b(y)]$ if $h(y) \geq h(y_j)$, or from $[y, b(y)]$ to $[y_j, b(y_j)]$ if $h(y_j) \geq h(y)$, which is δ-close to the identity map. This implies that one can lift g along $[y_j, b(y_j)]$ and $[y, b(y)]$ using the same choices of the inverse branch of the projection. Since $\hat{g}(b(y_j)) \to \hat{g}(b(y))$, we must have $\hat{g}(y_j) \to \hat{g}(y)$. $\square$

Lemma 4.6. *Let $(U, \mathfrak{M}_U)$ be a marked nice partition for B, and V be a nice partition for B satisfying $\overline{V} \subset U$. There exists a marked nice partition $(W, \mathfrak{M}_W)$ for B such that $W \subseteq V$ and $(W, \mathfrak{M}_W)$ is reachable in $(U, \mathfrak{M}_U)$.*

Proof. Let V' be a component of V, which is contained in a component U' of U. Let us say that V' is an accessible component of V in $(U, \mathfrak{M}_U)$, if there is a curve

$$\lambda_{V'} : (0, 1) \to U' \setminus (\overline{V} \cup X)$$

such that

$$\lim_{t \to 0} \lambda_{V'}(t) \in \mathfrak{M}_{U'}, \quad \text{and} \quad \lim_{t \to 1} \lambda_{V'}(t) \in \partial V' \setminus X.$$

If every component V' of V is an accessible component of V in $(U, \mathfrak{M}_U)$, we define $W = V$, and choose $\mathfrak{M}_W$ as a collection of $\lim_{t \to 1} \lambda_{V'}(t)$ over all components V' of V. If V has non-accessible components within $(U, \mathfrak{M}_U)$, we modify the components of V so that all components of V become accessible. We explain this process below.

Let V' be a non-accessible component of V. Let U' be the component of U which contains V'. Let Q_i, for $1 \leq i \leq m$, denote the components of V which lie in U'. Rearranging the indexes if necessary, we may assume that $Q_1 = V'$. Recall from Lemma 4.5 that $U' \setminus X$ is path-connected. Also, recall that $\mathfrak{M}_{U'}$ is accessible from $U' \setminus X$. These imply that there is a curve

$$\lambda : (0, 1) \to U' \setminus (\overline{V'} \cup X)$$

such that $\lim_{t \to 0} \lambda(t) = \mathfrak{M}_{U'}$ and $\lim_{t \to 1} \lambda(t) \in \partial V' \setminus X$.

For $2 \leq i \leq m$, let Q_i' denote the union of the connected components of $Q_i \setminus \lambda((0,1))$ which intersect X. We define

$$P = (V \setminus \cup_{2\leq i\leq m} Q_i) \bigcup (\cup_{2\leq i\leq m} Q_i') \subseteq V \setminus \lambda((0,1)).$$

Since each Q_i is a topological disk, the connected components of Q_i' are topological disks. As $X \cap \lambda([0,1]) = \emptyset$, for each i, $Q_i \cap X = Q_i' \cap X$, and $V \cap X = P \cap X$. It follows that P satisfies properties P1 to P3 of nice partitions.

At this stage, the set P might not satisfy property P4 of nice partitions. However, we may employ Lemma 4.3 to shrink the components of P so that P4 holds as well. Let L denote this modified nice partition obtain from employing that lemma. Note that the number of components of L is the same as the number of components of P.

If $\lambda((0,1))$ meets some Q_i, for $2 \leq i \leq m$, then every component of Q_i' is an accessible component of P in $(U, \mathfrak{M}_U)$ (via a portion of the curve $\lambda((0,1))$). Also, if some Q_i, for $2 \leq i \leq m$, is an accessible component of V within $(U, \mathfrak{M}_U)$, and $\lambda((0,1)) \cap Q_i = \emptyset$, then Q_i is an accessible component of P within $(U, \mathfrak{M}_U)$. Moreover, as $V' \cap \lambda((0,1)) = \emptyset$, V' is an accessible component of P in $(U, \mathfrak{M}_U)$, via the curve λ.

By the above paragraph, the number of non-accessible components of P within $(U, \mathfrak{M}_U)$ is strictly less than the number of non-accessible components of V within $(U, \mathfrak{M}_U)$. On the other hand, if L' is a component of L which is contained in a component P' of P, and P' is an accessible component of P within $(U, \mathfrak{M}_U)$, then L' is an accessible component of L within $(U, \mathfrak{M}_U)$. Therefore, the number of non-accessible components of L within $(U, \mathfrak{M}_U)$ is strictly less than the number of non-accessible components of V within $(U, \mathfrak{M}_U)$. Recall that V has a finite number of connected components. By repeating the above process, we may reduce the number of non-accessible components of V until all its components become accessible. □

Corollary 4.7. *Let X be a compact subset of $\mathbb{R}^2$ which satisfies axioms A1 to A4. There are marked nice partitions $(U_n, \mathfrak{M}_{U_n})$, for $n \geq 1$, such that the following hold:*

(i) $\cap_{n\geq 1} U_n = B$;
(ii) *for $n \geq 1$, U_{n+1} is compactly contained in U_n;*
(iii) *for $n \geq 1$, $(U_{n+1}, \mathfrak{M}_{U_{n+1}})$ is reachable in $(U_n, \mathfrak{M}_{U_n})$.*

Proof. For $n = 1$, let U_1 be a Jordan domain which contains X. Let $\mathfrak{M}_{U_1}$ consist of an arbitrary point on ∂U_1. Evidently, $(U_1, \mathfrak{M}_{U_1})$ is a marked nice partition for B.

Assume that $(U_m, \mathfrak{M}_{U_m})$ is defined for all $1 \leq m < n$, for some $n \geq 2$, and satisfy properties (ii) and (iii) in the corollary for values of $m \leq n-1$. Below, we build $(U_n, \mathfrak{M}_{U_n})$.

Let

$$\epsilon_n = 2^{-1} \inf\{|z - w| \mid z \in \partial U_{n-1}, w \in B\}.$$

By Lemma 4.4, there is a nice partition U'_n for B which is contained in ϵ_n neighbourhood of B. In particular, U'_n is compactly contained in U_{n-1}. By the induction hypothesis, $(U_{n-1}, \mathfrak{M}_{U_{n-1}})$ is a marked nice partition for B. Therefore, by Lemma 4.6, there is a marked nice partition $(U_n, \mathfrak{M}_{U_n})$ for B such that $U_n \subset U'_n$ and $(U_n, \mathfrak{M}_{U_n})$ is reachable within $(U_{n-1}, \mathfrak{M}_{U_{n-1}})$.

For $n \geq 2$, $\epsilon_n \leq \operatorname{diam}(U_1)/2^{n-1}$. As U_n is contained in ϵ_n neighbourhood of B, (i) holds. □

Proof of Proposition 4.2. By Corollary 4.7, we have a nest of marked nice partitions $(U_n, \mathfrak{M}_{U_n})$, for $n \geq 1$, such that U_n shrinks to B. Recall that U_1 is a Jordan domain containing X. To simplify the forthcoming presentation, let us choose a Jordan domain U_0 which contains $\overline{U_1}$, and let $\mathfrak{M}_{U_0} \subset \partial U_0$ contain a single point. For each $n \geq 0$, let $U_{n,i}$, for $1 \leq i \leq k_n$, denote the connected components of U_n. We have $k_0 = k_1 = 1$. Also, let $u_{n,i}$ denote the unique point in $\mathfrak{M}_{U_n} \cap \partial U_{n,i}$. For $n \geq 0$ and $1 \leq i \leq k_n$, define $J_{n,i}$ as the set of integers j with $1 \leq j \leq k_{n+1}$ and $U_{n+1,j} \subset U_{n,i}$.

For each $n \geq 0$, $(U_{n+1}, \mathfrak{M}_{U_{n+1}})$ is reachable in $(U_n, \mathfrak{M}_{U_n})$. Therefore, there are

$$\eta_{n,i,j} : (0,1) \to U_{n,i} \setminus (\cup_{j \in J_{n,i}} \overline{U_{n+1,j}} \cup X),$$

such that

$$\lim_{t \to 0} \eta_{n,i,j}(t) = u_{n,i}, \qquad \lim_{t \to 1} \eta_{n,i,j}(t) = u_{n+1,j},$$

for all $1 \leq i \leq k_n$ and $j \in J_{n,i}$. We may extend $\eta_{n,i,j}$ onto $[0,1]$ by setting $\eta_{n,i,j}(0) = u_{n,i}$ and $\eta_{n,i,j}(1) = u_{n+1,j}$. If necessary, we may modify these curves such that $\eta_{n,i,j}((0,1)) \cap \eta_{n,i,j'}((0,1)) = \emptyset$, for distinct values of j and j'.

For $m \geq 0$, let T_m denote the union of the curves $\eta_{n,i,j}([0,1])$, for all $0 \leq n \leq m$, $1 \leq i \leq k_n$, and $j \in J_{n,i}$. Each T_m forms a finite tree embedded in the plane. It follows that

$$T = \cup_{m \geq 0} T_m$$

is an idea tree for X. □

Proof of Proposition 4.1. By Proposition 4.2, there is an ideal tree T for X. We aim to turn T into a base curve γ for X. The curve γ will be the limit of a sequence of curves γ_n, where γ_n is obtained from slightly "thickening" the tree T_n. We present the details of this construction below. We will continue to use the notations in the proof of Proposition 4.2.

Note that the curves $\eta_{n,i,j}((0,1))$ are pairwise disjoint, for distinct values of the triple (n,i,j). Hence, there are tubular neighbourhoods of these curves, which are pairwise disjoint, and do not meet X. In other words, there are orientation preserving, continuous and injective maps

$$\tilde{\eta}_{n,i,j} : (0,1) \times [-0.1, 0.1] \to U_{n,i} \setminus (\cup_{j \in J_{n,i}} \overline{U_{n+1,j}} \cup X),$$

such that for all $(t, y) \in (0,1) \times [-0.1, 0.1]$, we have

$$\tilde{\eta}_{n,i,j}(t, 0) = \eta_{n,i,j}(t),$$
$$\lim_{t\to 0} \tilde{\eta}_{n,i,j}(t, y) = \eta_{n,i,j}(0) = u_{n,i},$$
$$\lim_{t\to 1} \tilde{\eta}_{n,i,j}(t, y) = \eta_{n,i,j}(1) = u_{n+1,j},$$

and for distinct triples (n, i, j) and (n', i', j') we have

$$\tilde{\eta}_{n,i,j}((0,1) \times [-0.1, 0.1]) \bigcap \tilde{\eta}_{n',i',j'}((0,1) \times [-0.1, 0.1]) = \emptyset.$$

We may extend $\tilde{\eta}_{n,i,j}$ onto $[0,1] \times [-0.1, 0.1]$ by setting $\tilde{\eta}_{n,i,j}(0, y) = u_{n,i}$ and $\tilde{\eta}_{n,i,j}(1, y) = u_{n+1,j}$, for all $y \in [-0.1, 0.1]$. Consider

$$M = \bigcup_{n\geq 0} \bigcup_{1\leq i\leq k_n} \bigcup_{j\in J_{n,i}} \tilde{\eta}_{n,i,j}([0,1] \times [-0.1, 0.1]) \bigcup B.$$

This is a compact set in $\mathbb{R}^2$, with $M \cap X = B$.

Fix $n \geq 1$ and $1 \leq i \leq k_n$. There are $2(|J_{n,i}| + 1)$ curves on ∂M which land at $u_{n,i}$. These are the curves $\tilde{\eta}_{n,i,j}([0,1] \times \{-0.1\})$ and $\tilde{\eta}_{n,i,j}([0,1] \times \{+0.1\})$, for $j \in J_{n,i}$, as well as $\tilde{\eta}_{n-1,i',i}([0,1] \times \{-0.1\})$ and $\tilde{\eta}_{n-1,i',i}([0,1] \times \{+0.1\})$, for some $1 \leq i' \leq k_{n-1}$ with $i \in J_{n-1,i'}$. Let $j_1, j_2, \ldots, j_m$ denote the elements of $J_{n,i}$, labelled in such a way that the curves $\tilde{\eta}_{n,i,j_l}([0,1]\times\{-0.1\})$, for $1 \leq l \leq m$, followed by $\tilde{\eta}_{n-1,i',i}([0,1]\times\{-0.1\})$ and then $\tilde{\eta}_{n-1,i',i}([0,1]\times\{+0.1\})$ land at $u_{n,i}$ in a clockwise fashion.

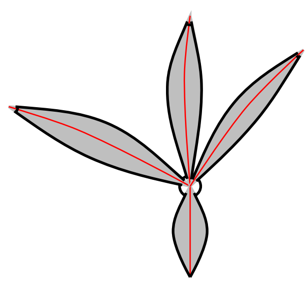

FIGURE 4. Turning the tree T_1 (red curves) to the Jordan curve γ_2 (black loop) through a thickening process.

Let $\epsilon_n \in (0, 1/4)$ be small enough, and consider pairwise disjoint curves

$$\tau_{n,i,j_l} : (0,1) \to \mathbb{R}^2 \setminus M,$$

for $1 \leq l \leq m-1$, such that

$$\lim_{t\to 0} \tau_{n,i,j_l}(t) = \tilde{\eta}_{n,i,j_l}(\epsilon_n, -0.1), \quad \lim_{t\to 1} \tau_{n,i,j_l}(t) = \tilde{\eta}_{n,i,j_l+1}(\epsilon_n, 0.1).$$

Similarly, let $\rho_{n,i,1} : (0,1) \to \mathbb{R}^2 \setminus M$ and $\rho_{n,i,2} : (0,1) \to \mathbb{R}^2 \setminus M$ be such that

$$\lim_{t\to 0} \rho_{n,i,1}(t) = \tilde{\eta}_{n-1,i',i}(1-\epsilon_n, 0.1), \quad \lim_{t\to 1} \rho_{n,i,l}(t) = \tilde{\eta}_{n,i,j_1}(\epsilon_n, 0.1),$$
$$\lim_{t\to 0} \rho_{n,i,2}(t) = \tilde{\eta}_{n,i,j_m}(\epsilon_n, -0.1), \quad \lim_{t\to 1} \rho_{n,i,2}(t) = \tilde{\eta}_{n-1,i',i}(1-\epsilon_n, -0.1).$$

Let ϵ'_n be a decreasing sequence of positive numbers tending to 0 such that the closures of the disks $\mathbb{D}(u_{n,i}, \epsilon'_n)$ are pairwise disjoint, and are disjoint from X, for all n and i. We may choose the above maps so that the union of the curves $\tilde{\eta}_{n,i,j}(\{\epsilon_n\} \times [-0.1, +0.1])$, for $j \in J_{n,i}$, $\tilde{\eta}_{n-1,i',i}(\{1-\epsilon_{n-1}\} \times [-0.1, +0.1])$, $\tau_{n,i,j_l}((0,1))$, for $1 \leq l \leq m-1$, $\rho_{n,i,1}(0,1)$, and $\rho_{n,i,2}(0,1)$ forms a closed loop which encloses the mark point $u_{n,i}$ and is contained in $\mathbb{D}(u_{n,i}, \epsilon'_n)$. See Figure 4.

Recall that $\mathbb{S}^1 \subset \mathbb{R}^2$ denotes the circle of radius 1 about $(0,0)$. We may choose closed and connected sets $C_{n,i} \subsetneq \mathbb{S}^1$, for $n \geq 1$ and $1 \leq i \leq k_n$, satisfying the following properties:

- $C_{n,i}$ is contained in the interior of $C_{m,j}$ if and only if $U_{n,i} \subsetneq U_{m,j}$;
- if $i \neq j$, then $C_{n,i} \cap C_{n,j} = \emptyset$;
- the Euclidean length of $C_{n,i}$ is bounded by 2^{-n}.

For $n \geq 1$, let $C_n = \cup_{i=1}^{k_n} C_{n,i}$ and define

$$C = \cap_{n\geq 1} C_n.$$

It follows that C is a Cantor set on $\mathbb{S}^1$. We aim to build a Jordan curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ such that $\gamma(\mathbb{S}^1) \cap X = B$, and $\gamma(t) \in B$ if and only if $t \in C$. The map γ will be defined as the limit of a sequence of maps $\gamma_n : \mathbb{S}^1 \to \mathbb{R}^2$, for $n \geq 1$. We define γ_n inductively as follows.

For $n = 1$, we define the injective map $\gamma_1 : \mathbb{S}^1 \to \mathbb{R}^2$ such that

$$\gamma_1(C_{1,1}) = \tilde{\eta}_{0,1,1}(\{1-\epsilon_1\} \times [-0.1, +0.1]),$$
$$\gamma_1(\mathbb{S}^1 \setminus C_{1,1}) = \tilde{\eta}_{0,1,1}([0, 1-\epsilon_1) \times \{-0.1\}) \cup \tilde{\eta}_{0,1,1}([0, 1-\epsilon_1) \times \{+0.1\}).$$

The choice of the parametrisation of this curve is not important; any choice will work.

Now assume that the curve γ_n is defined and injective for some $n \geq 1$. We define γ_{n+1} as follows. On $\mathbb{S}^1 \setminus C_n$ we let $\gamma_{n+1} \equiv \gamma_n$. On each $C_{n,i}$, we define γ_{n+1} in such a way that for all $j \in J_{n,i}$

$$\gamma_{n+1}(C_{n+1,j}) = \tilde{\eta}_{n,i,j}(\{1-\epsilon_n\} \times [-0.1, +0.1]),$$

and $\gamma_{n+1}(C_{n,i} \setminus \cup_{j\in J_{n,i}} C_{n+1,j})$ becomes

$$\bigcup_{j\in J_{n,i}} \tilde{\eta}_{n,i,j}((\epsilon_n, 1-\epsilon_n)\times\{-0.1\}) \bigcup_{j\in J_{n,i}} \tilde{\eta}_{n,i,j}((\epsilon_n, 1-\epsilon_n)\times\{+0.1\}])$$
$$\bigcup_{1\le l\le m-1} \tau_{n,i,j_l}([0,1]) \bigcup \rho_{n,i,1}([0,1]) \bigcup \rho_{n,i,2}([0,1]).$$

Due to the choices of these curves, γ_{n+1} can be chosen to be injective. Again, the specific parametrisation of the curve γ_{n+1} is not important. This completes the definition of γ_n for all $n \ge 1$. Note that for all $n \ge 1$ and $1 \le i \le k_n$ we have

$$\gamma_n(C_{n,i}) \subset \mathbb{D}(U_{n,i}, \epsilon'_n). \tag{7}$$

Now we show that $\gamma_n : \mathbb{S}^1 \to \mathbb{R}^2$, for $n \ge 1$, forms a Cauchy sequence. Fix $l > 0$, and assume that m and n are bigger than l. If $x \notin C_l$, we have $\gamma_n(x) = \gamma_m(x)$. If $x \in C_{l,i}$, for some $1 \le i \le k_l$, by Equation (7), $\gamma_l(x) \in \mathbb{D}(U_{l,i}, \epsilon'_l)$. It follows from the construction that $\gamma_n(x)$ and $\gamma_m(x)$ also belong to $\mathbb{D}(U_{l,i}, \epsilon'_l)$. Therefore, for all $x \in \mathbb{S}^1$, we have

$$|\gamma_n(x) - \gamma_m(x)| \le \operatorname{diam}(U_{l,i}) + 2\epsilon'_l.$$

However, as $\cap_{n\ge1} \cup_{i=1}^{k_n} U_{n,i} = B$, $\operatorname{diam}(U_{l,i})$ tends to zero as l tends to infinity. Hence, the sequence of maps γ_n converges uniformly on $\mathbb{S}^1$ to a map $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$. In particular, γ is continuous.

We claim that $\gamma(\mathbb{S}^1) \cap X = B$. If $x \notin C$, there are n and i such that $x \notin C_{n,i}$. This implies that for all $m \ge n$, $\gamma_m(x) = \gamma_n(x) \notin B$. Therefore, $\gamma(x) \notin X$. If $x \in C$, there are arbitrarily large n and $1 \le i \le k_n$ such that $x \in C_{n,i}$. By Equation (7), $\gamma_n(x) \in \mathbb{D}(U_{n,i}, \epsilon'_n)$. Since $\cap_{n\ge1} \cup_{1\le i\le k_n} \mathbb{D}(U_{n,i}, \epsilon'_n) = B$, we conclude that $\gamma(x) = \lim_{n\to\infty} \gamma_n(x) \in B$. On the other hand, for every $z \in B$ there is $x \in C$ with $\gamma(x) = z$. To see this, choose $1 \le i_n \le k_n$ such that $\cap_{n\ge1} U_{n,i_n} = z$. There is $x_n \in C_{n,i}$ with $\gamma_n(x_n) \in \mathbb{D}(U_{n,i_n}, \epsilon'_n)$. If x_{n_j} is a convergent subsequence of x_n, converging to some $x \in \mathbb{S}^1$, we obtain $\gamma(x) = \lim_{j\to\infty} \gamma_{n_j}(x_{n_j}) = z$. This completes the proof of the claim.

To see that γ is injective, let x and y be distinct elements of $\mathbb{S}^1$. If x and y belong to C, then there must be $n \ge 1$ and $i \ne j$ such that $x \in C_{n,i}$ and $y \in C_{n,j}$. By Equation (7) and the decreasing property of ϵ'_n, for all $m \ge n$, $\gamma_m(x) \in \mathbb{D}(U_{n,i}, \epsilon'_m)$ and $\gamma_m(y) \in \mathbb{D}(U_{n,j}, \epsilon'_m)$. Since $\overline{U_{n,i}} \cap \overline{U_{n,j}} = \emptyset$, we must have $\gamma(x) \ne \gamma(y)$. If x and y do not belong to C, then there is $n \ge 1$ such that $x \notin C_n$ and $y \notin C_n$. Then, $\gamma(x) = \gamma_n(x) \ne \gamma_n(y) = \gamma(y)$, by the definition of γ, and the injectivity of γ_n. If exactly one of x and y belongs to C, say $x \in C$ and $y \notin C$, by the above paragraph, $\gamma(x) \in B$ and $\gamma(y) \notin B$. Thus, $\gamma(x) \ne \gamma(y)$.

There remains to show that X does not meet the bounded connected component of $\mathbb{R}^2 \setminus \gamma(\mathbb{S}^1)$. To see this, note that $\gamma_n(\mathbb{S}^1)$ may be continuously deformed into the tree T_{n-1} in the complement of X. This implies that X does not meet the bounded component of $\mathbb{R}^2 \setminus \gamma_n(\mathbb{S}^1)$. Since γ_n converges to γ uniformly on $\mathbb{S}^1$, one infers that X does not meet the bounded component of $\mathbb{R}^2 \setminus \gamma(\mathbb{S}^1)$. □

5. Uniformisation of hairy Cantor sets

In this section we prove Theorem 1.4. Here it is convenient to work on the complex plane $\mathbb{C}$ and the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We use the notations $\mathrm{d}_{\hat{\mathbb{C}}}$ and $\mathrm{diam}_{\hat{\mathbb{C}}}$ for the distance and the diameter, respectively, with respect to the spherical metric on $\hat{\mathbb{C}}$.

Throughout this section X is a compact subsets of $\mathbb{C}$ which satisfies axioms A1 to A6.

By Proposition 4.1, there is a base curve ω for X. Recall that the bounded connected component of $\mathbb{C} \setminus \omega$ does not meet X. Let U' denote the bounded component of $\mathbb{C} \setminus \omega$.

Briefly speaking, we plan to extend the base map $b : X \to B$ to a continuous map $b : \mathbb{C} \setminus U' \to \omega$, so that the pre-image of any point in ω is a curve (homeomorphic to $[1, +\infty)$) whose one end lands at a point in ω at the other end converges to infinity. Moreover, the collection $b^{-1}(x)$, for $x \in \omega$, forms a foliation of $\mathbb{C} \setminus U'$ homeomorphic to the trivial foliation of $\mathbb{C} \setminus \mathbb{D}(0, 1)$ by straight rays. This allows us to brush the hairs of X so that it becomes a hairy Cantor set with straight hairs. The analysis required to extend b onto $\mathbb{C} \setminus U'$ will be carried out in the framework of Carathéodory's theory of prime ends, which relates the geometry of a plane domain to the topology of its boundary. For the general theory of prime ends one may see the classic references [Car13] and [Koe15], or the modern treatment [Pom75]. However, we shall briefly recap the basic notions and statements of the theory which are employed here.

Within this section we will use the notations

$$X_\omega = X \cup \omega, \quad X^*_\omega = X_\omega \cup U', \quad U = \mathbb{C} \setminus X^*_\omega, \quad U^* = U \cup \{\infty\} \subset \hat{\mathbb{C}}. \tag{8}$$

Proposition 5.1. *The set U^* is connected and simply connected.*

Proof. We use Borsuk's criterion, as in the proof of Lemma 4.5. In order to prove that U^* is connected, it is sufficient to show that any continuous map $g : X^*_\omega \to \mathbb{S}^1$ is null-homotopic. Below we construct a homotopy between a given such map g and a constant map.

By Proposition 3.1, there is a height function h on X, which after rescaling we may assume $h : X \to [0, 1]$. Let $\psi : B \to \psi(B)$ be a homeomorphism from B onto a Cantor set in the line $\{(x, y) \in \mathbb{R}^2 \mid y = 0\}$. We extend the map ψ to

$\psi : X \to \psi(B) \times [0,1] \subset \mathbb{R}^2$, according to

$$\psi(z) = (\psi(b(z)), h(z)).$$

As ψ is continuous and injective on a compact set, it must be a homeomorphism onto its image.

Consider the family of maps $H_1 : (\psi(B) \times [0,1]) \times [0,1] \to \psi(B) \times [0,1]$, defined as

$$H_1((x,y),t) = (x, \min(y, 1-t)).$$

Define the family of maps $G_1 : X_\omega^* \times [0,1] \to \mathbb{S}^1$, as

$$G_1(z,t) = \begin{cases} g(\psi^{-1}(H_1(\psi(z),t))) & \text{if } z \in X, \\ g(z) & \text{if } z \notin X. \end{cases}$$

One may see that G_1 is continuous. Moreover, $G_1(z,0) = g(z)$, for all $z \in X_\omega^*$. Hence, G_1 is a homotopy between g and the map

$$g' : (X_\omega \cup U') \to \mathbb{S}^1, \quad g'(z) = G_1(z,1).$$

Note that $g'(z) = g(z)$, for all $z \in \overline{U'}$, and $g'(z) = g(b(z))$, for all $z \in X$.

Now we construct a homotopy between g' and a constant map. The set $\overline{U'} = U' \cup \omega$ is homeomorphic to the closed unit disk $\overline{\mathbb{D}(0,1)}$. Hence, the map $g'|_{\overline{U'}} = g|_{\overline{U'}}$ is null-homotopic. Therefore, there is a continuous map $H_2 : \overline{U'} \times [0,1] \to \mathbb{S}^1$ and a constant $\theta \in \mathbb{S}^1$ such that $H_2(z,0) \equiv g(z)$ and $H_2(z,1) \equiv \theta$. Now define $G_2 : X_\omega^* \times [0,1] \to \mathbb{S}^1$ as $G_2(z,t) = H_2(b(z)),t)$. Evidently, G_2 is continuous, and for all $z \in X_\omega^*$ we have $G_2(z,0) = g(b(z)) = g'(z)$ and $G_2(z,1) = \theta$. Hence, g is null-homotopic.

Because the complement of U^* in $\hat{\mathbb{C}}$, X_ω^*, is connected, U^* must be simply connected. □

Lemma 5.2. *We have $X_\omega = \partial U^*$.*

Proof. Let $z \in \partial U^*$. Since U^* and U' are open in $\hat{\mathbb{C}}$, and $U^* \cap U' = \emptyset$, z may not be in $U^* \cup U'$. As $\hat{\mathbb{C}} = U^* \cup U' \cup X_\omega$, $z \in X_\omega$. Therefore, $\partial U^* \subseteq X_\omega$. For the opposite inclusion, let $z \in X_\omega$ and $r > 0$. Since U' is a Jordan domain, $\mathbb{D}(z,r) \setminus \overline{U'}$ is a non-empty open set. As X_ω has empty interior, $\mathbb{D}(z,r) \setminus \overline{U'}$ may not be contained in X_ω, and hence, must meet U^*. This implies that $z \in \partial U^*$. Thus, $X_\omega \subseteq \partial U^*$. □

Let $V \subset \hat{\mathbb{C}}$ be an open, connected, and simply connected set whose boundary contains at least two points. A **fundamental chain** in V is a nest of open sets $(\Omega_i)_{i \geq 1}$ in V satisfying the following properties:

- for $i \geq 1$, $\partial\Omega_i \cap V$ is a simple curve in V, which is homeomorphic to $(0,1)$, and its closure contains two distinct points in ∂V;
- $\overline{(\partial\Omega_i \cap V)} \cap \overline{(\partial\Omega_j \cap V)} = \emptyset$, for $1 \leq i < j$;

- for $i \geq 1$, $\Omega_{i+1} \subset \Omega_i$;
- $\mathrm{diam}_{\hat{\mathbb{C}}}(\partial\Omega_i \cap V) \to 0$, as $i \to +\infty$.

Note that by the Jordan curve theorem, $\partial\Omega_i \cap V$ divides V into two connected components. Equivalently, some authors define the notion of fundamental chains using sequences of Jordan arcs in V with end points in ∂V.

Two fundamental chains $(\Omega_i)_{i\geq 1}$ and $(\Omega'_i)_{i\geq 1}$ in V are called **equivalent** if every Ω_i contains some Ω'_j and every Ω'_j contains some Ω_i. Any two fundamental chains in V are either equivalent or eventually disjoint, that is, $\Omega_i \cap \Omega'_j = \emptyset$ for large enough i and j. An equivalence class of fundamental chains in V is called a **prime end** of V.

The **impression** of a prime end of V, represented by a fundamental chain $(\Omega_i)_{i\geq 1}$, is defined as $\cap_{i\geq 1}\overline{\Omega_i}$. This is a non-empty compact and connected subset of ∂V. Evidently, the impression of a prime end is independent of the choice of the fundamental chain $(\Omega_i)_{i\geq 1}$. Any point on ∂V is contained in the impression of a prime end of V.

We say that $x \in \partial V$ is the **principal point** of a fundamental chain $(\Omega_i)_{i\geq 1}$ in V, if $\mathrm{diam}_{\hat{\mathbb{C}}}(\{x\} \cup (\partial\Omega_i \cap V)) \to 0$. In general, equivalent fundamental chains might have different principal points. The set of **principal points of a prime end** is defined as the set consisting of all principal points of the fundamental chains in the class of that prime end. Every prime end of V has at least one principal point. By the theory of prime ends, $p \in \partial V$ is accessible from V, if and only if, there is a prime end of V whose impression contains p, and p is the only point in the set of principal points of that prime end. Abusing the terminology, we say that a prime end is **accessible** if it has a unique principal point.

Recall the base map $b : X \to B$ and the peak map $p : X \to X$ defined in Section 3. We may extend these maps to

$$b : X_\omega \to \omega, \quad p : X_\omega \to X_\omega,$$

by setting $b(x) = p(x) = x$, for all $x \in X_\omega \setminus X$. Also, for $x \in X_\omega \setminus X$, we set $[b(x), p(x)] = \{x\}$. As b is continuous on both X and ω, and X and ω are closed sets, b is continuous on X_ω.

We present the properties of the prime ends of U^* in the following lemma.

Lemma 5.3. *The following properties hold:*

(i) *For any prime end P of U^* there is $z \in X_\omega$ such that the impression of P is equal to $[b(z), p(z)]$. On the other hand, for any $z \in X_\omega$, there is a prime end of U^* whose impression is equal to $[b(z), p(z)]$.*

(ii) *Every prime end of U^* is accessible. Moreover, $w \in X_\omega$ is accessible from U^* if and only if $w = p(z)$ for some $z \in X_\omega$.*

(iii) *Distinct prime ends of U^* have disjoint impressions.*

Proof. (i): By Lemma 5.2, $\partial U^* = X_\omega$. Therefore, any $z \in X_\omega$ belongs to the impression of a prime end of U^*. That is, there is a fundamental chain $(\Omega_i)_{i\geq 1}$ in U^* whose impression contains z. We aim to show that $\cap_{i\geq 1}\overline{\Omega_i} = [b(z), p(z)]$.

For $i \geq 1$, $\overline{\partial\Omega_i \cap U^*}$ is a Jordan arc connecting two points on X_ω. Let α_i and β_i denote those points. Since $\overline{\Omega_i} \cap X_\omega$ is connected, for all $i \geq 1$, $b(\overline{\Omega_i} \cap X_\omega)$ is a closed arc on ω, which is bounded by $b(\alpha_i)$ and $b(\beta_i)$. As z belongs to $\partial\Omega_i \cap X_\omega$, we must have $b(\alpha_i) \leq b(z) \leq b(\beta_i)$, with respect to a fixed cyclic order on ω. On the other hand, as $i \to \infty$, $\operatorname{diam}_{\hat{\mathbb{C}}}(\partial\Omega_i \cap U^*) \to 0$, which implies that $\mathrm{d}_{\hat{\mathbb{C}}}(\alpha_i, \beta_i) \to 0$. By the uniform continuity of $b : X_\omega \to \omega$, we conclude that

$$\lim_{i\to\infty} b(\alpha_i) = b(z) = \lim_{i\to\infty} b(\beta_i).$$

By axiom A5, the only accessible points of X from $\mathbb{C} \setminus X$ belong to $b(X) \cup p(X)$. It follows that α_i and β_i belong to $b(X_\omega) \cup p(X_\omega)$. Using $\overline{(\partial\Omega_i \cap U^*)} \cap \overline{(\partial\Omega_j \cap U^*)} = \emptyset$, for $j > i \geq 1$, one concludes that for all $i \geq 1$,

$$b(\alpha_i) < b(z) < b(\beta_i). \tag{9}$$

In particular, $[b(z), p(z)] \subset \overline{\Omega_i}$, for all $i \geq 1$, and hence, $[b(z), p(z)] \subseteq \cap_{i\geq 1}\overline{\Omega_i}$. On the other hand, for all $w \in \cap_{i\geq 1}\overline{\Omega_i}$, $b(\alpha_i) \leq b(w) \leq b(\beta_i)$, for all $i \geq 1$. By the above paragraph, $b(\alpha_i) \to b(z)$ and $b(\beta_i) \to b(z)$, which leads to $b(w) = b(z)$. Therefore, $\cap_{i\geq 1}\overline{\Omega_i} \subseteq [b(z), p(z)]$. Combining the two relations, we obtain $\cap_{i\geq 1}\overline{\Omega_i} = [b(z), p(z)]$.

(ii): Let P be a prime end in U^*, and let $(\Omega_i)_{i\geq 1}$ be a fundamental chain in U^* in the class of P which has a principal point, say $y \in X_\omega$. We aim to show that $p(y) = y$.

Assume in the contrary that $p(y) \neq y$. In particular, $b(y) \neq p(y)$, and $[b(y), p(y)]$ is a Jordan arc. We may extend $[b(y), p(y)]$ into U' so that we obtain a Jordan arc γ in $[b(y), p(y)] \cup U'$ with one of the end points of γ in U'. There is a Jordan domain D such that $[b(y), y] \subset D$, $p(y) \notin \overline{D}$, $\partial D \cap \omega$ consists of two points, and $D \setminus \gamma$ has two connected components.

Let α_i and β_i denote the landing points of $\partial\Omega_i \cap U^*$ on X_ω. Since y is the principal point of $(\Omega_i)_{i\geq 1}$, we must have $\alpha_i \to y$ and $\beta_i \to y$, as $i \to \infty$. Moreover, since y belongs to the impression of $(\Omega_i)_{i\geq 1}$, as in Equation (9), we have $b(\alpha_i) < b(y) < b(\beta_i)$, for all $i \geq 1$. By the choice of D, this implies that for large enough i, α_i and β_i belong to distinct components of $D \setminus \gamma$. On the other hand, α_i and β_i belong to $\overline{\partial\Omega_i \cap U^*}$, with $\overline{\partial\Omega_i \cap U^*} \cap \gamma = \emptyset$ and $\operatorname{diam}_{\hat{\mathbb{C}}}(\overline{\partial\Omega_i \cap U^*}) \to 0$. This implies that for large enough i, both α_i and β_i belong to the same component of $D \setminus \gamma$. This contradiction shows that we must have $p(y) = y$.

Let P be an arbitrary prime end of U^*. By part (i), the impression of P is of the form $[b(z), p(z)]$, for some $z \in X_\omega$. By the above argument, if y is a principal point of P, we must have $y = p(y)$. Since $y \in [b(z), p(z)]$, and there is a single point y in

$[b(z), p(z)]$ with $p(y) = y$, we conclude that P has a unique principle point. Thus, P is accessible. On the other hand, by Part (i), for any $z \in X_\omega$, there is a prime end of U^* whose impression is equal to $[b(z), p(z)]$. By the above argument, $p(z)$ is the unique principal point of that prime end. Therefore, the set of principal points of the prime ends of U^* is equal to $\{p(z) \mid z \in X_\omega\}$.

(iii): Assume in the contrary that there are distinct prime ends P^1 and P^2 in U whose impressions intersect. By part (i), the impressions of P_1 and P_2 must be equal to $[b(y), p(y)]$, for some $y \in X_\omega$. Let $(\Omega_i^1)_{i\geq 1}$ and $(\Omega_i^2)_{i\geq 1}$ be fundamental chains in the classes of P^1 and P^2, respectively. Since P_1 and P_2 are not equivalent, there is $j \geq 1$ such that $\Omega_j^1 \cap \Omega_j^2 = \emptyset$. For $i \geq 1$, choose $y_i \in \Omega_i^1$. We have $d(y_i, [b(y), p(y)]) \to 0$.

Let α_i and β_i denote the landing points of $\partial\Omega_i^2 \cap U^*$ on X_ω. We may relabel these points so that $b(\alpha_i) < b(y) < b(\beta_i)$, with respect to a fixed cyclic order on ω. The open set

$$W_j = \operatorname{int}(\overline{\Omega_j^2} \cup \overline{U'})$$

is a Jordan domain, which is bounded by

$$\overline{\partial\Omega_j^2 \cap U} \cup [\alpha_j, b(\alpha_j)] \cup [\beta_j, b(\beta_j)] \cup \{w \in \omega \mid b(\beta_j) \leq b(w) \leq b(\alpha_j)\}).$$

In particular, $[b(y), p(y)] \subseteq W_j$. Since $d(y_i, [b(y), p(y)]) \to 0$, as $i \to \infty$, there must be $i > j$ such that $y_i \in W_j$, and hence $y_i \in W_j \cap U = \Omega_j^2$. Therefore,

$$y_i \in \Omega_i^1 \cap \Omega_j^2 \subseteq \Omega_j^1 \cap \Omega_j^2 = \emptyset,$$

which is a contradiction. □

Let V be a connected and simply connected domain in $\hat{\mathbb{C}}$ whose boundary contains at least two points. Let $\psi : \mathbb{D}(0,1) \to V$ be a Riemann map, that is, a one-to-one and onto holomorphic map. By the general theory of prime ends, for any fundamental chain $(\Omega_i)_{i\geq 1}$ in V, $(\psi^{-1}(\Omega_i))_{i\geq 1}$ is a fundamental chain in $\mathbb{D}(0,1)$. This correspondence induces a bijection between the set of prime ends of V and the set of prime ends of $\mathbb{D}(0,1)$.

The prime ends of $\mathbb{D}(0,1)$ are easy to understand. Any prime end of $\mathbb{D}(0,1)$ is accessible, and distinct prime ends of $\mathbb{D}(0,1)$ have distinct principal points. The map which sends a prime end of $\mathbb{D}(0,1)$ to its unique principal point induces a bijection between the set of prime ends of $\mathbb{D}(0,1)$ and $\partial\mathbb{D}(0,1)$. Combining with the above paragraph, ψ induces a one-to-one correspondence between $\partial\mathbb{D}(0,1)$ and the set of the prime ends of V. Let P be a prime end of V which is the image of $e^{i\theta} \in \partial\mathbb{D}(0,1)$ under this bijection and P has a unique principal point in ∂V. Then, the radial limit $\lim_{r\to 1^-} \psi(re^{i\theta})$ exists and is equal to the principal point of P. See Corollary 2.17 in [Pom92].

By Proposition 5.1, the set U^* is connected, and simply connected. We may consider a Riemann map

$$\Phi : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}(0,1)} \to U^*,$$

with $\Phi(\infty) = \infty$. By the above paragraphs, Φ induces a bijection between the set $\partial\mathbb{D}(0,1) = \partial(\hat{\mathbb{C}} \setminus \overline{\mathbb{D}(0,1)})$ and the set of the prime ends of U^*. By Lemma 5.3, every prime end in U^* is accessible, and has a unique principal point which belongs to $\{p(z) \mid z \in X_\omega\}$. Thus, there is a bijection between $\partial\mathbb{D}(0,1)$ and $p(X_\omega)$. For every $\theta \in \mathbb{R}$, the radial limit $\lim_{r\to 1^+} \Phi(re^{i\theta})$ exists and belongs to $p(X_\omega)$. Moreover, for $\theta \neq \theta'$ in $\mathbb{R}/(2\pi\mathbb{Z})$, $\lim_{r\to 1^+} \Phi(re^{i\theta}) \neq \lim_{r\to 1^+} \Phi(re^{i\theta'})$.

We extend the base map $b : X_\omega \to \omega$ to

$$b : U \cup X_\omega \to \omega \tag{10}$$

as follows. For an arbitrary $x \in U$ we let $\theta(x) = \arg(\Phi^{-1}(x)) \in \mathbb{R}/(2\pi\mathbb{Z})$, and define

$$b(x) = b\Big(\lim_{r\to 1^+} \Phi\big(re^{i\theta(x)}\big)\Big).$$

For each $w \in \omega$, there is a unique $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ such that $\lim_{r\to 1^+} \Phi(re^{i\theta}) = p(w)$. Then, for all w' on the curve $\Phi(\{re^{i\theta} \mid r > 1\})$, $b(w') = b(p(w)) = w$. This implies that for each $w \in \omega$, the set $b^{-1}(w)$ in $U \cup X_\omega$ is a Jordan arc consisting of $[w, p(w)]$ and $\Phi(\{re^{i\theta} \mid r > 1\})$. This curve meets ω only at w.

Proposition 5.4. *The map $b : U \cup X_\omega \to \omega$ is continuous.*

Proof. We aim to prove that $b^{-1}(I)$ is closed in $U \cup X_\omega$, for every closed set $I \subseteq \omega$. Since ω is a Jordan curve, it is sufficient to prove that $b^{-1}(I)$ is closed whenever $I \subseteq \omega$ is a Jordan arc. If $w \in \omega$, $b^{-1}(w)$ is a Jordan arc tending to infinity, and therefore is closed. Let $I \subseteq \omega$ be a Jordan arc and let α and β be its end-points. We know that $b^{-1}(\alpha)$ and $b^{-1}(\beta)$ are disjoint Jordan arcs which land at α and β on ω. Therefore, $b^{-1}(\alpha) \cup b^{-1}(\beta))$ divides $U \cup X_\omega$ into two disjoint Jordan domains, denoted by V_1 and V_2. The sets $\overline{V_1} \cap \omega$ and $\overline{V_2} \cap \omega$ are Jordan arcs in ω with end points at α and β. Hence, one of these sets, say $\overline{V_1} \cap \omega$, is equal to I. On the other hand, $U \cup X_\omega$ is a disjoint union of $b^{-1}(w)$, for $w \in \omega$. Therefore, $\overline{V_1}$ is a disjoint union of such arcs. This implies that

$$\overline{V_1} = b^{-1}(b(\overline{V_1})) = b^{-1}(\overline{V_1} \cap \omega) = b^{-1}(I). \qquad \square$$

For $w \in \omega$, $b^{-1}(w)$ is a Jordan arc in $U \cup X_\omega$ which lands at a single point on ω. For x and y in $b^{-1}(w)$, we define $[x, y]$ to be the unique Jordan arc in $b^{-1}(w)$ which ends at x and y. When $x = y$, we have $[x, y] = \{x\}$. Note that if x and y belong to $X_\omega \cap b^{-1}(w)$, this notation is consistent with our earlier notation $[x, y]$.

Proposition 5.5. *If $x_i \to x$ in $U \cup X_\omega$, we have*

$$\lim_{i\to\infty} [x_i, b(x_i)] = [x, b(x)],$$

in the Hausdorff topology.

When $x_i \in X$, for all $i \geq 1$, we already have the convergence of $[x_i, b(x_i)]$ to $[x, b(x)]$ by axiom A4. When $x_i \notin X$, $[x_i, b(x_i)]$ is the union of $[b(x_i), p(x_i)]$ and $[p(x_i), x_i] \setminus \{p(x_i)\}$, where the latter set is a hyperbolic geodesic in U^*. For the above proposition, we need to show that such a sequence of hyperbolic geodesics may not have pathological behaviour. We control the location of those hyperbolic geodesic by employing a theorem of Gehring and Hayman, see [GH62] or [Pom92, Page 88]. That is, let $V \subseteq \hat{\mathbb{C}}$ be a simply connected domain whose boundary contains at least two points, and let x and y be distinct points in V. Assume that λ is an arbitrary curve in V connecting x to y, and γ is a geodesic in V, with respect to the hyperbolic metric on V, connecting x to y. By Gehring-Hayman theorem,

$$\operatorname{diam}_{\hat{\mathbb{C}}}(\gamma) \leq C_{GH} \operatorname{diam}_{\hat{\mathbb{C}}}(\lambda),$$

where C_{GH} is a universal constant independent of V, x, y, and λ.

For $r \geq 1$, let

$$A_r = \Phi\big(\{w \in \mathbb{C} \mid 1 < |w| \leq r\}\big) \cup X_\omega. \tag{11}$$

This is a compact annulus in $\mathbb{C}$.

Proof of Proposition 5.5. For any $w \in \omega$, $b^{-1}(w) \subset U \cup X_\omega$ is a Jordan arc which is homeomorphic to $[0, 1)$. So there is a linear order on each $b^{-1}(w)$, where $w = b(w)$ is the smallest point.

Let us fix $R \geq 1$ such that for all $i \geq 1$, $x_i \in A_R$, where A_R is defined by Equation (11). Then, for all $i \geq 1$, $[x_i, b(x_i)] \subset A_R$. The set of non-empty compact subsets of A_R with respect to the Hausdorff topology is compact. Therefore, to prove the proposition, it is sufficient to prove that if $[x_i, b(x_i)])_{i\geq1}$ is a convergent sequence in the Hausdorff topology, then $[x_i, b(x_i)] \to [x, b(x)]$.

Assume that $([x_i, b(x_i)])_{i\geq1}$ converges in the Hausdorff topology, and let K denote the limit of this sequence. Then,

$$K = \{y \in A_R \mid y = \lim_{i\to\infty} y_i, y_i \in [x_i, b(x_i)], \forall i \geq 1\}.$$

By Proposition 5.4, $b : U \cup X_\omega \to \omega$ is continuous. This implies that K is contained in $b^{-1}(b(x))$. On the other hand, since each $[x_i, b(x_i)]$ is connected, K must be connected. Therefore, $K = [y, b(x)]$, for some $y \in b^{-1}(b(x))$. Moreover, since $x_i \to x$, we have $x \in K$, and hence $x \leq y$, with respect to the linear order on $b^{-1}(b(x))$. We need to show that $x = y$. Assume in the contrary that $y > x$. We shall derive a contradiction by considering two cases.

Case (i): Assume that $y > x$ and $y \in U$. Define

$$r = \begin{cases} (|\Phi^{-1}(y)| + |\Phi^{-1}(x)|)/2 & \text{if } x \in U, \\ (|\Phi^{-1}(y)| + 1)/2 & \text{if } x \in X_\omega. \end{cases}$$

Consider the compact annulus A_r defined by Equation (11). Since $x \in \mathrm{int}(A_r) \cup \omega$, and $x_i \to x$, there is an integer i_0 such that for all $i \geq i_0$, $x_i \in A_r$. In particular, for $i \geq i_0$, $[x_i, b(x_i)]$ is contained in A_r. As A_r is compact, the Hausdorff limit of $[x_i, b(x_i)]$, that is K, must be contained in A_r. But, $y \in K$ and $y \notin A_r$, since $|\Phi^{-1}(y)| > r$. This is a contradiction.

Case (ii): Assume that $y > x$ and $y \in X_\omega$. By passing to a subsequence, we may assume that $(b(x_i))_{i \geq 1}$ is monotone on ω, that is, for all $i \geq 1$, $b(x_i) < b(x_{i+1}) < b(x)$, with respect to a fixed cyclic order on ω. Define

$$\gamma = b^{-1}(b(x)) \cap A_R, \quad \gamma_i = b^{-1}(b(x_i)) \cap A_R, \ \forall i \geq 1.$$

By virtue of the continuity of $b : A_R \to \omega$ in Proposition 5.4, $\gamma_i \to \gamma$, as $i \to \infty$, with respect to the Hausdorff topology.

The Jordan arcs γ and γ_1 divide A_R into two Jordan domains. Let

$$E = \{w \in \mathrm{int}(A_R) \mid b(x_1) < b(w) < b(x)\}.^6$$

Since $x < y$, we may choose two points z and w in γ such that $x < z < w < y$. There are Jordan arcs $\lambda^z : [0, 1] \to \overline{E}$ and $\lambda^w : [0, 1] \to \overline{E}$ such that

$$\lambda^z(0) \in \gamma_1, \quad \lambda^z(1) = z \in \gamma, \quad \lambda^z((0, 1)) \subseteq E,$$
$$\lambda^w(0) \in \gamma_1, \quad \lambda^w(1) = w \in \gamma, \quad \lambda^w((0, 1)) \subseteq E,$$

and

$$\lambda^z([0, 1]) \cap \lambda^w([0, 1]) = \emptyset.$$

Let us fix $y_i \in [x_i, b(x_i)]$, for $i \geq 1$, such that $y_i \to y$ as $i \to \infty$. Let t_i denote the unique point in $\gamma_i \cap \Phi(\partial\mathbb{D}(0, R))$. Then, $\gamma_i = [t_i, b(t_i)] = [t_i, b(x_i)]$, for all $i \geq 1$.

The curve $\lambda^w((0, 1))$ divides E into two connected components. As $y_i \to y$, $x_i \to x$, and $x < w < y$, there must be $i_1 \geq 1$ such that for all $i \geq i_1$, x_i and y_i lie in distinct components of $E \setminus \lambda^w((0, 1))$. In particular, for every $i \geq i_1$, both of the curves $[b(x_i), x_i]$ and $[x_i, t_i]$ meet the curve $\lambda^w((0, 1))$. It follows that for $i \geq i_1$, the sets $C_{i,1} = [b(x_i), x_i] \cap \lambda^w((0, 1))$ and $C_{i,2} = [x_i, t_i] \cap \lambda^w((0, 1))$ are non-empty compact sets in $\lambda^w((0, 1))$. Since $C_{i,1} \cap C_{i,2} = \emptyset$, there must be a Jordan arc

$$\lambda_i^w : [0, 1] \to \lambda^w((0, 1))$$

such that

$$\lambda_i^w(0) \in C_{i,1}, \quad \lambda_i^w(1) \in C_{i,2}, \quad \lambda_i^w((0, 1)) \cap (C_{i,1} \cup C_{i,2}) = \emptyset.$$

[6] $\mathrm{int}(A)$ denotes the topological interior of a given set $A \subseteq \mathbb{C}$.

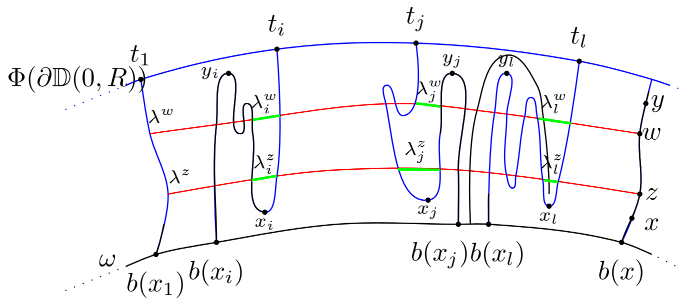

FIGURE 5. Illustration of the curves and points in the proof of Proposition 5.5. The black curves lie in X_ω while the blue curves lie in U. The green Jordan arcs denote the curve λ_i^z, λ_i^w, ...

Note that the last property in the above equation implies that $\lambda_i^w((0,1)) \cap \gamma_i = \emptyset$. Then, the union of the Jordan arcs $\lambda_i^w([0,1])$ and $[\lambda_i^w(0), \lambda_i^w(1)]$ forms a Jordan curve in the plane. Let W_i denote the bounded component of the complement of that Jordan curve. Thus, we have

$$\partial W_i = \lambda_i^w((0,1)) \cup [\lambda_i^w(0), \lambda_i^w(1)]. \tag{12}$$

and

$$W_i \cap \gamma_i = \emptyset. \tag{13}$$

The closure of the Jordan domain W_i is homeomorphic to $\overline{\mathbb{D}(0,1)}$. Moreover, as x_i belongs to $[\lambda_i^w(0), \lambda_i^w(1)]$, $\overline{W_i}$ contains x_i and $\lambda_i^w([0,1])$. Since x_i and $\lambda_i^w([0,1])$ lie on distinct components of $E \setminus \lambda^z((0,1))$, the curve $\lambda^z((0,1))$ must divide $\overline{W_i}$ into at least two connected components. It follows that there is a Jordan arc

$$\lambda_i^z : [0,1] \to \overline{W_i} \cap \lambda^z((0,1))$$

such that

$$\lambda_i^z(0) \in [\lambda_i^w(0), x_i], \quad \lambda_i^z(1) \in [x_i, \lambda_i^w(1)], \quad \lambda_i^z((0,1)) \subset W_i, \tag{14}$$

and $\lambda_i^z([0,1])$ divides $\overline{W_i}$ into two components, one of which contains x_i and the other one contains $\lambda_i^w([0,1])$.

We claim that for every $i \geq i_1$, the Jordan domain W_i enjoys the following property:

$$\text{if } \zeta \in \lambda_i^z((0,1)) \cap X, \quad \text{then} \quad [b(\zeta), \zeta] \cap \lambda_i^w((0,1)) \neq \emptyset. \tag{15}$$

To see this, fix an arbitrary $\zeta \in \lambda_i^z((0,1)) \cap X$. By Equation (14), $\zeta \in W_i$, but $b(\zeta) \notin \overline{W_i}$. Therefore, the Jordan arc $[\zeta, b(\zeta)]$ must cross $\partial W_i = \lambda_i^w((0,1)) \cup [\lambda_i^w(0), \lambda_i^w(1)]$. However, $[\zeta, b(\zeta)] \cap [\lambda_i^w(0), \lambda_i^w(1)] = \emptyset$. That is because, $[b(\zeta), \zeta] \subset b^{-1}(b(\zeta))$ and $[\lambda_i^w(0), \lambda_i^w(1)] \subset \gamma_i$, while $b^{-1}(b(\zeta))$ and $\gamma_i = b^{-1}(b(x_i)$ are either disjoint or identical. But, ζ belongs to $b^{-1}(b(\zeta))$, and by Equation (13)), $\zeta \notin \gamma_i$ (since $\zeta \in W_i$). This completes the proof of the claim.

Since all the points $\lambda_i^w(0)$, $\lambda_i^w(1)$, $\lambda_i^z(0)$, and $\lambda_i^z(1)$ lie on γ_i, we may compare them using the linear order on γ_i. We have

$$b(x_i) < \lambda_i^w(0) < \lambda_i^z(0) < x_i < \lambda_i^z(1) < \lambda_i^w(1) < t_i. \tag{16}$$

Since $\lambda_i^w(0)$ and $\lambda_i^w(1)$ belong to $\lambda^w((0,1)) \cap \gamma_i$, we must have

$$\lim_{i\to\infty} \lambda_i^w(0) = \lim_{i\to\infty} \lambda_i^w(1) = w. \tag{17}$$

Similarly, as $\lambda_i^z(0)$ and $\lambda_i^z(1)$ belong to $\lambda^z((0,1)) \cap \gamma_i$, we must have

$$\lim_{i\to\infty} \lambda_i^z(0) = \lim_{i\to\infty} \lambda_i^z(1) = z \tag{18}$$

In particular, we have

$$\lim_{i\to\infty} \operatorname{diam}_{\hat{\mathbb{C}}}(\lambda_i^z([0,1])) = 0. \tag{19}$$

Moreover, since $x_i \in [\lambda_i^z(0), \lambda_i^z(1)]$,

$$\liminf_{i\to\infty} \operatorname{diam}_{\hat{\mathbb{C}}}([\lambda_i^z(0), \lambda_i^z(1)]) \geq \mathrm{d}_{\hat{\mathbb{C}}}(z, x) > 0. \tag{20}$$

Now note that at least one of the following three possibilities occurs:

(a) there are infinitely many i with $\lambda_i^z([0,1]) \cap X = \emptyset$,
(b) there are infinitely many i with $\{\lambda_i^z(0), \lambda_i^z(1)\} \cap X \neq \emptyset$,
(c) there are infinitely many i with $\lambda_i^z((0,1)) \cap X \neq \emptyset$.

Below we show that each of the above scenarios leads to a contradiction.

If (a) occurs, let $(i_k)_{k\geq 1}$ be an increasing sequence such that $\lambda_{i_k}^z([0,1]) \cap X = \emptyset$. In particular, the points $\lambda_{i_k}^z(0)$ and $\lambda_{i_k}^z(1)$ do not belong to X. Then, the Jordan arc $[\lambda_{i_k}^z(0), \lambda_{i_k}^z(1)]$ does not meet X, and hence, it is a hyperbolic geodesic in U. Therefore, according to the Gehring-Hayman theorem discussed before the proof, for $k \geq 1$, we must have

$$\operatorname{diam}_{\hat{\mathbb{C}}}([\lambda_{i_k}^z(0), \lambda_{i_k}^z(1)]) \leq C_{GH} \operatorname{diam}_{\hat{\mathbb{C}}}(\lambda_{i_k}^z([0,1])).$$

This contradicts the limiting behaviours in Equations (19) and (20).

If (b) occurs, let $(i_k)_{k\geq 1}$ be an increasing sequence with $\{\lambda_{i_k}^z(0), \lambda_{i_k}^z(1)\} \cap X \neq \emptyset$. By Equation (16), we must have $\lambda_{i_k}^z(0) \in X$. Therefore, by axiom A4, and Equation (18), $[\lambda_{i_k}^z(0), b(x_{i_k})]$ converges to $[z, b(x)]$. On the other hand, by Equation (16), we have $\lambda^w_{i_k}(0) \in [\lambda^z_{i_k}(0), b(x_i)]$, and by Equation (17), $\lambda^w_{i_k}(0) \to w$, But $w \notin [z, b(x)]$, which is a contradiction.

If (c) occurs, let $(i_k)_{k\geq 1}$ be an increasing sequence with $\lambda^z_{i_k}((0,1)) \cap X \neq \emptyset$. Let $z_{i_k} \in \lambda^z_{i_k}((0,1)) \cap X$. By Equation (15), $[z_{i_k}, b(z_{i_k})] \cap \lambda^w_{i_k}((0,1)) \neq \emptyset$. Let $w_{i_k} \in [z_{i_k}, b(z_{i_k})] \cap \lambda^w_{i_k}((0,1))$. As $w_{i_k} \in \lambda^w_{i_k}((0,1))$, from Equation (17), we conclude that $w_{i_k} \to w$ as $k \to \infty$. On the other hand, since $z_{i_k} \in X$, by axiom A4, we have $[z_{i_k}, b(z_{i_k})] \to [z, b(z)]$. But, $w_{i_k} \in [z_{i_k}, b(z_{i_k})]$ and $w_{i_k} \to w \notin [z, b(z)]$. This is a contradiction. □

Proof of Theorem 1.4. By virtue of Theorem 1.2, it is sufficient to prove that every hairy Cantor set in the plane is ambiently homeomorphic to a straight hairy Cantor set in the plane. Let X be an arbitrary hairy Cantor set, with base Cantor set B, that is, B is the closure of the set of point components of X. By Proposition 4.1, X admits a base curve, that is, a Jordan curve ω such that $X \cap \omega = B$ and the bounded component of $\mathbb{R}^2 \setminus \omega$ does not meet X. Then we define the sets in Equation (8).

Let A_2 denote the closed annulus defined by Equation (11). The base map $b : X \to B$ extends to the map $b : A_2 \to \omega$; see Equation (10). By Proposition 5.4, $b : A_2 \to \omega$ is continuous. On the other hand, following the discussions in Section 3, we may consider a Whitney map μ for the set A_2. Then, we define

$$h : A_2 \to [0, \infty), \quad h(z) = \mu([z, b(z)]).$$

By virtue of Proposition 5.5 and properties of Whitney maps, $h : A_2 \to [0, \infty)$ is continuous. Moreover, if $b(z) = b(w)$ and $h(z) = h(w)$ for some z and w in A_2, by the properties of height functions, we have $[z, b(z)] = [w, b(z)]$, which implies $z = w$.

Let $I \subseteq \omega$ be a Jordan arc such that $b(X) = B \subset I$. Consider a homeomorphism

$$u : I \to \big\{(x, y) \in \mathbb{R}^2 \mid y = 0, x \in [0, 1]\big\}.$$

Then, we may define the map

$$\Psi : \{z \in A_2 \mid b(z) \in I\} \to \{(x, y) \in \mathbb{R}^2 \mid y \geq 0, x \in [0, 1]\}$$

according to

$$\Psi(z) = (u(b(z)), h(z)).$$

By the above paragraph, Ψ is continuous and injective. As $\{z \in A_2 \mid b(z) \in I\}$ is a compact set, Ψ must be a homeomorphism from its domain onto its image. Since the domain of Ψ is a Jordan curve, its image must be a Jordan curve. Therefore, one may extend Ψ to a homeomorphism of the plane.

There remains to show that $\Psi(X)$ is a straight hairy Cantor set. To prove that, we employ Proposition 2.7. Let $C = u(B)$. Since u is a homeomorphism, and B is a Cantor set, C must be a Cantor set. Define $l : C \to [0, \infty)$ according to

$$l(x) = \max\{y \in \mathbb{R} \mid (x, y) \in \Psi(X)\}.$$

As $\Psi(X)$ is compact, $l(x)$ is well-defined and non-negative for all $x \in C$. We have

$$\Psi(X) = \{(x, y) \in \mathbb{R}^2 \mid x \in C, 0 \le y \le l(x)\}.$$

We need to verify the three conditions in Proposition 2.7. Evidently, $\Psi(X)$ is compact, so we have condition (i) in Proposition 2.7. By axiom A6, $X \setminus B$ is dense in B. This implies that the closure of $\Psi(X \setminus B)$ contains $\Psi(B) = C$. In particular, we have condition (ii) in Proposition 2.7. Axiom A5 implies condition (iii) in Proposition 2.7. Therefore, by Proposition 2.7, Y is a straight hairy Cantor set. □

Axiom A6' is stronger than axiom A6. But it turns out that any compact set $X \subset \mathbb{R}^2$ which satisfies axioms A1 to A6, also satisfies axiom A6'. We state the following corollary for future applications.

Corollary 5.6. *For any hairy Cantor set $X \subset \mathbb{R}^2$ the set of peak points of X is dense in X.*

Proof. By Theorem 1.4, X is ambiently homeomorphic to a straight hairy Cantor set. By Corollary 2.8, for every straight hairy Cantor set, the set of peak points is dense in that straight hairy Cantor set. Therefore, the set of peak points of X must be dense in X. □

In [DR14], the authors propose a topological model for the attractor (post-critical set) of a especial class of infinitely satellite renormalisable maps. The main object in that paper is built as the intersection of a nest of plain domains. However, they do not study the topological features of the model presented in the paper. It is likely that the topological model presented in that paper is a hairy Cantor set.

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Department of Mathematics, Imperial College London, London, SW7 2AZ, UK
Email address: `d.cheraghi@imperial.ac.uk`

Department of Mathematics, Imperial College London, London, SW7 2AZ, UK
Email address: `m.pedramfar15@imperial.ac.uk`