TOPOLOGY OF IRRATIONALLY INDIFFERENT ATTRACTORS

DAVOUD CHERAGHI

Abstract. We study the post-critical set of a class of holomorphic systems with an irrationally indifferent fixed point. We prove a trichotomy for the topology of the post-critical set based on the arithmetic of the rotation number at the fixed point. The only options are Jordan curves, a *one-sided hairy Jordan curves*, and *Cantor bouquet*. This explains the degeneration of the closed invariant curves inside the Siegel disks, as one varies the rotation number.

1. Introduction

1.1. **Irrationally indifferent attractors.** Let f be a rational map of the Riemann sphere, or an entire function on the complex plane, with an **irrationally indifferent fixed** point at 0. That is, near 0,

$$f(z) = e^{2\pi i\alpha} z + O(z^2)$$

for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The local dynamics of f near 0 depends on the arithmetic nature of α in a delicate fashion. By classical results of Siegel [Sie42] and Brjuno [Brj71], if α satisfies an arithmetic condition, now called **Brjuno numbers**, f is conformally conjugate to the rotation by $2\pi\alpha$ on a neighbourhood of 0. The maximal domain of linearisation (conjugacy) is called the **Siegel disk** of f about 0. When this happens, the local dynamics is rather trivial, any orbit starting near 0 becomes dense in an invariant closed analytic curve. On the other hand, Yoccoz [Yoc88] showed that if α is not a Brjuno number, the quadratic polynomial

$$P_\alpha(z) = e^{2\pi i\alpha} z + z^2$$

is not linearisable at 0. However, when the map is not linearisable near 0, the local dynamics is not explained. In this paper, for the first time, we explain the delicate topological structure of the (local) attractor for some non linearisable maps.

The presence of an irrationally indifferent fixed point influences the global dynamics of the map. In a pioneering work on the iteration of holomorphic maps in 1910s, Fatou [Fat19] showed that there must be a critical point c_f of f which "interacts" with the fixed point 0. Let $\Lambda(c_f)$ denote the closure of the orbit of c_f, that is,

$$\Lambda(c_f) = \overline{\cup_{i\geq 0} f^{\circ i}(c_f)}.$$

Fatou showed that if f is linearisable at 0, the boundary of the Siegel disk at 0 is contained in $\Lambda(c_f)$, and if f is not linearisable at 0, $0 \in \Lambda(c_f)$. The set $\Lambda(c_f)$ is part of the post-critical set of f, which is defined as the closure of the orbits of all critical points of f. By a general result in holomorphic dynamics, the post-critical set of f is the measure theoretic attractor of the action of f on its Julia set, [Lyu83]. For P_α, $\Lambda(c_{P_\alpha})$ is the post-critical set, and hence it is the measure theoretic attractor of P_α on its Julia set. For arbitrary f, $\Lambda(c_f)$ has its own basin of attraction in the Julia set of f, which *a priori*, may or may not have zero area.

The structure of $\Lambda(c_f)$ for "badly approximable" rotation numbers α is well developed over the last four decades. For many classes of maps and rotation numbers, c_f lies on the boundary of the Siegel disk, and $\Lambda(c_f)$ is a Jordan curve, with some limitations on its geometry. These

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studies make use of the Siegel disk, through ingenious surgery procedures, which were introduced by Douady [Dou87] for quadratic polynomials, Zakeri [Zak99] for cubic polynomials, and Shishikura for all polynomials, in an unpublished work. Through these surgeries, the problem was linked to the dynamics of analytic circle homeomorphisms. Combining with the work of Yoccoz, Swianek and Herman [Yoc84, Swi98, Her79] on linearisation of analytic circle homeomorphisms, Douady concluded that if α is a rotation number of bounded type, $\Lambda(c_{P_\alpha})$ is a quasi-circle, equal to the boundary of the Siegel disk. The result of Douady was extended to cubic polynomials by Zakery [Zak99], to all polynomials by Shishikura, to all rational functions by Zhang [Zha11], and to a wide class of entire functions by Zakeri [Zak10]. On the other hand, in [McM98], McMullen successfully combined these ideas with renormalisation techniques, and among other results, concluded that when α is an algebraic number, $\Lambda(c_f)$ enjoys rescaling self-similarity at c_p. In a far reaching generalisation, Petersen and Zakery [Pet96, PZ04] employed trans quasi-conformal surgery to prove that if the entries a_n in the continued fraction of α satisfy $\log a_n = \mathcal{O}(\sqrt{n})$, $\Lambda(c_{P_\alpha})$ is a David circle (a Jordan curve with some control on its geometry). Moreover, they also show that under the same condition, the Julia set of P_α has 0 area. This arithmetic condition holds for almost every α. In light of these results, there is a satisfactory understanding of $\Lambda(c_f)$ when there is some control on the growth of the entries in the continued fraction of α. In contrast, at the other end of the spectrum, for rotation numbers with arbitrarily large entries, the structure of $\Lambda(c_f)$ remained less developed. That is the main focus of this paper.

It is known that large entries in the continued fraction of α result in oscillations of the invariant curves in the Siegel disk. The size of an entry, and its location in the continued fraction, influences the shape of the oscillation. A large entry at the beginning of the continued fraction results in oscillations with large amplitude but small frequency, while the same large entry appearing later in the continued fraction results in oscillations with smaller amplitude but large frequency. See Fig 1. There are infinitely many entries in the continued fraction to play with. For an irrational number with many extremely large entries, the consecutive oscillations may build up and cause the degeneration of the closed invariant curves.

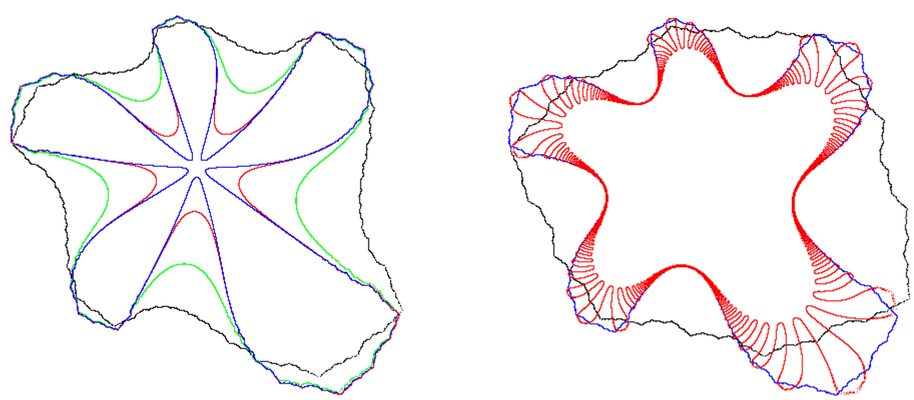

FIGURE 1. The figure on the left hand side shows the orbit of c_{P_α} for $\alpha = [2, 2, \overline{2}]$, $[2, 2, 10^2, \overline{2}]$, $[2, 2, 10^4, \overline{2}]$, and $[2, 2, 10^8, \overline{2}]$. The figure on the right hand side show the orbit of c_{P_α} for $\alpha = [2, 2, \overline{2}]$, $[2, 2, 10^2, \overline{2}]$, and $[2, 2, 10^8, \overline{2}]$.

In 1997 Perez-Marco [PM97] showed that the closed invariant sets within Siegel disks do not disappear under perturbation of the rotation number. That is, for every f with an irrationally indifferent fixed point at 0, there is a non-trivial compact, connected, invariant set containing 0, celled Siegel compacta or hedgehog. By successfully analysing these objects, he was able to build examples of non linearisable holomorphic germs with pathological behaviour, such as examples with no small cycles property [PM93], and examples with uncountably many conformal symmetries [PM95]. See [Bis08, Bis16, Che11] for more examples with unexpected features. But, these behaviours are not expected for rational functions, see for instance [ACE]. However, because those invariant sets are obtained through a limiting process, this approach provides limited control on those invariant sets for specific families of maps such as polynomials and rational functions. Also, the method does not apply near the boundary of the Siegel disks in order to study $\Lambda(c_f)$.

Inou and Shishikura in 2006 [IS06] introduced a renormalisation scheme for the study of near parabolic maps. The scheme consists of an infinite dimensional class of maps $\mathcal{F}$, and a renormalisation operator which preserves $\mathcal{F}$. All maps in $\mathcal{F}$ have a certain covering structure on a region containing 0, and in particular, have a (preferred) critical point. The set $\mathcal{F}$ contains some rational functions of arbitrarily large degrees. This scheme requires α to be of sufficiently **high type**, that is, α belongs to the set

$$\mathrm{HT}_N = \{\varepsilon_0/(a_0 + \varepsilon_1/(a_1 + \varepsilon_2/(a_2 + \dots))) \mid \forall n \geq 0, a_n \geq N, \varepsilon_n = \pm 1\},$$

for a suitable N. This arithmetic class contains a Cantor set of rotation numbers; including some rotation numbers of bounded type, as well as some rotation numbers with arbitrarily large entries. Inou and Shishikura used this renormalisation scheme to trap the orbit of the critical point in a dynamically defined region about 0. They showed that for every $f \in \mathcal{F}$, the orbit of c_f is infinite, $\Lambda(c_f)$ does not contain any periodic points, and in particular, $\Lambda(c_f)$ is not equal to the Julia set of f. This renormalisation scheme was employed to prove the upper simi-continuity of $\Lambda(c_f)$, at all bounded type rotation numbers in HT_N in [BC12] and all rotation numbers in HT_N in [Che19]. The former property was a main ingredient in the remarkable work of Buff and Cheritat [BC12] to build value of α such that the Julia set of P_α has positive area. These examples include both Brjuno and non Brjuno values of α. In light of these developments, we understand that the unbounded regime of rotation numbers exhibit non trivial dynamics.

1.2. **Statements of the results.** In this paper we improve the control on $\Lambda(c_f)$, and explain its delicate topological structure.

Theorem A (Trilogy of the post-critical set)**.** *There is $N \geq 2$ such that for every $\alpha \in \mathrm{HT}_N$ and every $f(z) = e^{2\pi i\alpha}z + O(z^2)$ in the Inou-Shishikura class $\mathcal{F}$, one of the following holds:*

(i) *α is a Herman number, and $\Lambda(c_f)$ is a Jordan curve,*
(ii) *α is a Brjuno but not a Herman number, and $\Lambda(c_f)$ is a one-sided hairy Jordan curve,*
(iii) *α is not a Brjuno number, and $\Lambda(c_f)$ is a Cantor bouquet.*

This also holds for the quadratic polynomials P_α, when $\alpha \in \mathrm{HT}_N$.

The set of **Herman numbers** was discovered by Herman and Yoccoz [Her79, Yoc02] in their landmark studies of the dynamics of analytic diffeomorphisms of the circle. Our approach in this paper does not make any connections to the dynamics of such circle maps. We find out that the Herman numbers appear in this setting due to a shared phenomenon. The set of Herman numbers is complicated to characterise in terms of the arithmetic of α; see Section 2. The set of Herman numbers is contained in the set of Brjuno numbers, and both sets have full Lebesgue measure in $\mathbb{R}$. On the other hand, the set of non-Brjuno numbers, and the set of Brjuno but not Herman numbers are uncountable and dense in $\mathbb{R}$. The density properties also hold on HT_N, that is, the set of rotation numbers corresponding to each of the cases in Thm A are dense in HT_N.

Cantor bouquets and hairy Jordan curves are universal topological objects like the Cantor sets; they are characterised by some topological axioms [AO93]. Roughly speaking, a Cantor bouquet is a collection of arcs landing at a single point, such that every arc is accumulated from both sides by arcs in the collection. A (one-sided) hairy Jordan curve is a collection of arcs landing on a dense subset of a Jordan curve such that every arc in the collection is accumulated from both sides by arcs in the collection. Both sets have empty interior, and necessarily have complicated topologies;

have uncountably many hairs and are not locally connected. See Sec. 3.3 for the precise definitions of these objects.

In case (i) of Thm A, the region inside the Jordan curve $\Lambda(c_f)$ is invariant under f and must be the Siegel disk of f. In particular c_f lies on the boundary of the Siegel disk. Case (i) applies to some rotation numbers outside the Petersen-Zakeri class. In case (ii), the region inside the unique Jordan curve in $\Lambda(c_f)$ is the Siegel disk of f. In this case, c_f cannot lie on the boundary of the Siegel disk. Indeed, we show that in cases (ii) and (iii) c_f lies at the end of one of the hairs in $\Lambda(c_f)$. The proof of the above theorem explains some geometric properties of $\Lambda(c_f)$ as well; see Sec. 8.4. For instance, in case (iii), the arcs in $\Lambda(c_f)$ land at 0 at well-defined (distinct) angles. In [Che22] we further analyse the proof of the above theorem, and show that the arcs in $\Lambda(c_f)$ are C^1 curves, except at the end points.

The above theorem explains the degeneration of the boundary of the Siegel disk when one varies the rotation number in HT_N. As the entries become large, either the boundaries of the Siegel disks make arbitrarily large oscillations reaching 0 in the limit, and collapse onto uncountably many arcs landing at 0 (cases (iii)); or the boundaries of the Siegel disks make large oscillations short of 0, and collapse onto uncountably many arcs landing on a closed invariant curve (case (ii)). The same incident occurs to the many invariant curves within the Siegel disks, degenerating to closed invariant sets within $\Lambda(c_f)$. However, after the degeneration, things become simpler. The many closed invariant sets in the Siegel disks, give rise to only a one-parameter family of closed invariant sets, all with the same topology. This is stated more precisely in the next theorem.

Theorem B (Degeneration of closed invariant curves)**.** *For every $\alpha \in \mathrm{HT}_N$ there is $r_\alpha \geq 0$ such that for every $f(z) = e^{2\pi i\alpha}z + O(z^2)$ in the class $\mathcal{F}$, there is a map*

$$\phi_f : [0, r_\alpha] \to \{X \subseteq \Lambda(c_f) \mid X \textit{ is non-empty, closed and invariant}\},$$

which is a homeomorphism with respect to the Hausdorff metric on the range. Moreover,

(i) *if α is a Herman number $r_\alpha = 0$, and otherwise $r_\alpha > 0$;*
(ii) *ϕ_f is strictly increasing on $[0, r_\alpha]$, with respect to the inclusion in the range;*
(iii) *if α is not a Brjuno number, for every $t \in (0, r_\alpha]$, $\phi_f(t)$ is a Cantor bouquet;*
(iv) *if α is a Brjuno but not a Herman number, for all $t \in (0, r_\alpha]$, $\phi_f(t)$ is a hairy Jordan curve.*

We also explain the dynamics of f on $\Lambda(c_f)$.

Theorem C (Dynamics on the attractor)**.** *For every $\alpha \in \mathrm{HT}_N$ and every $f(z) = e^{2\pi i\alpha}z + O(z^2)$ in $\mathcal{F}$, $f : \Lambda(c_f) \to \Lambda(c_f)$ is a topologically recurrent homeomorphism. Moreover, for every non-empty closed invariant set $X \subseteq \Lambda(c_f)$, there is $z \in \Lambda(c_f)$ such that X is equal to the closure of the orbit of z.*

A partial result in the direction of Thm A is obtained by Shishikura and Yang [SY18] around the same time. They prove that if α is a Brjuno number of high type, the boundary of the Siegel disk of $f \in \mathcal{F}$ is a Jordan curve, and c_f belongs to the boundary of the Siegel disk if and only if α is a Herman number. These results also follow immediately from the statement of Thm A.

Corollary D. *For any $\alpha \in \mathrm{HT}_N$ and any $f(z) = e^{2\pi i\alpha}z + O(z^2)$ in $\mathcal{F}$, the boundary of the Siegel disk of f at 0 is a Jordan curve.*

Corollary E. *For any $\alpha \in \mathrm{HT}_N$ and any $f(z) = e^{2\pi i\alpha}z + O(z^2)$ in $\mathcal{F}$, the boundary of the Siegel disk of f at 0 contains a critical point of f if and only if α is a Herman number.*

The above corollaries partially confirm conjectures of Herman and Douady on the Siegel disks of rational functions. In [Her85], Herman employs a conformal welding argument of Ghys [Ghy84] to show that if α is a Herman number, there must be a critical point on the boundary of the Siegel disk of P_α. On the other hand, Ghys and Herman [Ghy84, Her86] gave the first examples of polynomials having a Siegel disk with no critical point on the boundary. Based on these results, Herman conjectured in 1985 that Cor E holds for all rational function f of degree ≥ 2 and all irrational numbers α. Using an elegant Schwarzian derivative argument, Graczyk and Swiatek in [GS03] proved a general result, which implies in particular that if f is a rational function or an entire function with degree ≥ 2 and α is bounded type, then there must be a critical point on the boundary of the Siegel disk. It is proved in [CR16] that if f is a cubic polynomial and α is a Herman number, there must be a critical point on the boundary of the Siegel disk. On the other hand, Douady [Dou87] has conjectured that Cor D holds for all rational functions f of degree ≥ 2 and all irrational numbers α.

Shishikura and Yang in [SY18] have a fundamentally different approach to the proofs of Corollaries D and E. They study the convergence of the closed invariant curves inside the Siegel disk towards the boundary, and are able to conclude that the limiting set is a Jordan curve. They also show that when the Herman condition is not satisfied, the closed invariant curves inside the Siegel disks stay uniformly away from the critical point. These require a detailed analysis of the long compositions of the changes of coordinates in the renormalisation tower. In particular, they make use of the geometric properties of the tower and distortion estimates on the changes of coordinates established in [Che13, Che19]. As they directly target these corollaries, the proofs in [SY18] are naturally shorter overall. But, strictly speaking, the above corollaries are slightly more general. In [SY18], the notion of high type in terms of the standard continued faction is used. The set of high type numbers in terms of the modified continued faction is strictly larger than the set of high type numbers in terms of the standard continued fraction. For any value of N, there are elements in HT_N with infinitely many $+1$ in their standard continued fraction. The modified notion of continued fraction naturally arises in the near parabolic renormalisation scheme. The reason for this difference is that the arithmetic condition of Herman obtained by Yoccoz is presented in terms of the standard continued fraction, which is readily employed in [SY18]. The equivalent form of the condition in terms of the modified continued fraction is established in [Che21]. Evidently, the above corollaries were not the main purpose of this paper, but an immediate bi-product of a work mainly aimed at explaining the global dynamics of a non linearisable P_α.

Combining with earlier results on the topic, we now understand the topological dynamics of the quadratic polynomials P_α, for $\alpha \in \mathrm{HT}_N$. The measure theoretic properties of $\Lambda(c_{P_\alpha})$ are studied in [Che13, Che19], and in particular, it is proved that $\Lambda(c_{P_\alpha})$ has zero area. Moreover, it is proved that for Lebesgue almost every z in the Julia set of P_α, the set of accumulation points of the orbit of z is equal to $\Lambda(c_{P_\alpha})$. Thus, the basins of attraction of all those closed invariant sets strictly contained in $\Lambda(c_{P_\alpha})$ have zero area. The statistical behaviour of the orbits of P_α is explained in [AC18], where it is proved that $P_\alpha : \Lambda(c_{P_\alpha}) \to \Lambda(c_{P_\alpha})$ is uniquely ergodic. In the study of the measurable structure of $\Lambda(c_f)$ in [Che13, Che19], the most difficult case to deal with was when $\Lambda(c_f)$ is a Jordan curve. That required a very fine distortion estimate on the Fatou coordinates. We do not employ that fine estimate here. We hope that the the puzzle pieces with equivariant properties in the renormalisation tower constructed in this paper pave the way towards explaining the mysterious measurable dynamics of maps with non-linearisable fixed points.

1.3. **Outline of the proofs.** The proofs of the above theorems make use of a toy model for the renormalisation of irrationally indifferent fixed points we built in [Che21]. The toy model consists of a one-parameter family of maps parametrised by the rotation number, and a renormalisation operator which preserves that family of maps. Each map in the family sends straight rays landing at 0 to straight rays landing at 0, tangentially at 0 acts as rotation by $2\pi\alpha$, and radially mimics the behaviour of a generic holomorphic map of the form $e^{2\pi i\alpha}z + \mathcal{O}(z^2)$. The toy model was used to build a topological model for $\Lambda(c_f)$ and $f : \Lambda(c_f) \to \Lambda(c_f)$, for all irrational numbers α. The dynamics of the toy model was explained as well. In Sec. 3 we briefly summarise the construction of the model in a self contained fashion, leaving the technical steps in [Che21]. In this paper we make a conjugation between the toy model for the renormalisation and the near-parabolic renormalisation scheme. This allows us to transfer many features of the toy models to the maps. We discuss the main ideas of the proofs below.

Roughly speaking, the main argument is in the spirit of rigidity results in complex dynamics. That is, one builds nests of partitions (Yoccoz puzzle pieces) shrinking to single points for a given pair of maps, and then builds partial conjugacies between them by matching the corresponding pieces up to some depth. Then, a global conjugacy is obtained by passing to the limit of those partial conjugacies. However, we need to overcome some major obstacles in order to implement this approach. One issue is having puzzle pieces with matching boundary markings (Bötther coordinates) and with equivariant properties in the renormalisation tower. Another issue is that the puzzle pieces do not necessarily shrink to points (only after the proof is completed we realise that the nests shrink to the hairs). Also, the collections of partial conjugacies do not lie in a pre-comapct class of maps, due to the degeneration of the complex structure in the toy model for the renormalisation.

There is a suitable collection of puzzle pieces in the renormalisation tower of the toy model, due to the maps and changes of coordinates preserving straight rays. We present a construction of puzzle pieces for the renormalisation tower of f similar to the construction of external rays for polynomials. In this construction, the changes of coordinates in the renormalisation of the toy model play the role of the power map (when building the Böttcher coordinates), and the changes of coordinates for the renormalisations of f play the role of the map for which an external ray is built. It turns out that the collections of the external rays built in this fashion enjoys an equivariant property with respect to the changes of coordinates in the renormalisation tower of f. There is an alternative geometric interpretation of this construction. The domain of the n-th renormalisation of f about the critical value is a topological sector landing at 0. There is a unique hyperbolic geodesic in that sector, which starts at the critical value, and approaches 0 on the boundary of that sector. As n tends to infinity, these geodesics, with suitable re-parametrisations, converge to a Jordan arc, with a unique parametrisation. The limiting arc and its parametrisation is an external ray in the renormalisation tower of f. For example, when f has a Siegel disk with the critical value on its boundary, the limiting arc is the internal ray in the Siegel disk landing at the critical value. When f is not linearisable at 0, only after the proof is completed, we realise that the limiting arc is the hair of $\Lambda(c_f)$ containing the critical value of f. This provides an alternative characterisation of $\Lambda(c_f)$ as the collections of rays in the renormalisation tower of f.

A puzzle piece at a deep level in the tower for f is brought to the shallow level by applying successive changes of coordinates. The equivariant property of the puzzle pieces tell us that we obtain a puzzle piece of shallow level for some suitable renormalisation of f. The same thing happens for the puzzle pieces in the toy model. Those puzzle pieces on the top level have similar shapes, and can be matched accordingly. The composition of the changes of coordinates bringing the deep puzzle piece to the shallow level for f is conformal, but highly distorting. The corresponding composition for the toy model is far from conformal, and indeed, degenerates the conformal structure, as one moves to the deeper levels. Because of this, the partial conjugacies obtain in this fashion leave any compact class of maps. However, the degeneration occurs in the direction transverse to the rays, and in spite of this degeneration, we show that the sequence of partial conjugacies is Cauchy with respect to suitable hyperbolic metrics. This provides us with a limiting map, which not only links the dynamics of f to the dynamics of the model, but also satisfies equivariant properties with respect to the renormalisation changes of coordinates. Then, the injectivity of the conjugacy is driven from its equivariant property, and its surjectivity is driven from the special topology of the post-critical set for the model. That is, the set of end points in any Cantor bouquet, and any hairy Cantor set, are dense.

The starting point of the above argument is showing that the corresponding changes of coordinates in the Inou-Shishikura renormalisation and the toy model are uniformly close. The rest of the argument does not make any reference to specific properties of the Inou-Shishikura renormalisation scheme, in particular, the detailed information about the locations and geometries of relevant dynamical pieces near the fixed point. More precisely, given a renormalisation scheme of similar nature, one only needs to verify Propositions 5.2 and 5.3 about the changes of coordinates in the renormalisation in order to conclude the results stated in this paper. For this reason, in [Che21] we conjecture that the trichotomy presented in Thm A holds for all irrationally indifferent fixed points of all rational functions of the Riemann sphere.

There are a number of advantages in explaining the dynamics of f through a topological model. Instead of simultaneously dealing with the arithmetic properties of α and the distorting behaviour of the large iterates of f, our approach allows us to investigate those phenomena in separate stages. The arithmetic properties are studied in the setting of the model where the nonlinear analysis is much simpler. The arithmetic conditions of Herman and Brjuno naturally emerge in that setting. The link between the model and the map made in this paper does not involve any arithmetic arguments; it provides a unified approach to all arithmetic types.

Contents

2. Arithmetic classes of Brjuno and Herman

Here we define the arithmetic classes of Brjuno and Herman. The definition requires the action of the modular group $\mathrm{PGL}(2,\mathbb{Z})$ on the real line, which leads to continued fraction expansion of irrational numbers. To study the action of this group, one may choose a fundamental interval for the action of $z \mapsto z+1$ and study the action of $z \mapsto 1/z$ on that interval. Due to the nature of the near parabolic renormalisation, it is natural to work with the fundamental interval $(-1/2, 1/2)$ for the translation. That is because the scheme works for rotation numbers close to 0; see Sec. 4. This choice of the fundamental interval leads to a modified notion of continued fraction for irrational numbers.

2.1. **Modified continued fraction.** For x in $\mathbb{R}$, define $d(x,\mathbb{Z}) = \min_{k\in\mathbb{Z}} |x-k|$. Let us fix an irrational number $\alpha \in \mathbb{R}$. Define the numbers $\alpha_n \in (0,1/2)$, for $n \geq 0$, according to

$$\alpha_0 = d(\alpha,\mathbb{Z}), \quad \alpha_{n+1} = d(1/\alpha_n,\mathbb{Z}), \tag{2.1}$$

Then, there are unique integers a_n, for $n \geq -1$, and $\varepsilon_n \in \{+1,-1\}$, for $n \geq 0$, such that

$$\alpha = a_{-1} + \varepsilon_0\alpha_0, \quad 1/\alpha_n = a_n + \varepsilon_{n+1}\alpha_{n+1}. \tag{2.2}$$

Evidently, for all $n \geq 0$,

$$1/\alpha_n \in (a_n - 1/2, a_n + 1/2), \quad a_n \geq 2, \tag{2.3}$$

and

$$\varepsilon_{n+1} = \begin{cases} +1 & \text{if } 1/\alpha_n \in (a_n, a_n+1/2), \\ -1 & \text{if } 1/\alpha_n \in (a_n - 1/2, a_n). \end{cases} \tag{2.4}$$

For convenience, we also defined $\alpha_{-1} = +1$.

The sequences a_n and ε_n provide us with the infinite continued fraction

$$\alpha = a_{-1} + \cfrac{\varepsilon_0}{a_0 + \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + \dots}}}.$$

The best rational approximants of α, or the convergents of α, are defined as

$$\frac{p_n}{q_n} = a_{-1} + \cfrac{\varepsilon_0}{a_0 + \cfrac{\varepsilon_1}{\ddots + \cfrac{\varepsilon_n}{a_n}}}, \text{ for } n \geq -1.$$

where p_n and q_n are relatively prime, and $q_n > 0$.

2.2. **Brjuno numbers.** By a careful study of the Siegel's approach in [Sie42], Brjuno in [Brj71] showed that if the series

$$\sum_{n=-1}^{+\infty} q_n^{-1} \log q_{n+1}$$

converges to a finite value for a given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then any holomorphic map of the form $f(z) = e^{2\pi i \alpha} z + O(z^2)$ near 0 is locally conformally conjugate to the rigid rotation by $2\pi\alpha$. The work of Siegel and Brjuno is based on estimating the coefficients of the formal power series which conjugates the map to the rigid rotation. It involves formidable calculations, but does not involve any notion of renormalisation.

Later in [Yoc95b] Yoccoz carried out a geometric approach to the linearisation problem based on renormalisation. In particular, he further investigated the irrational numbers satisfying the condition of Brjuno. Thanks to his work, the natural way to look into this condition is through a function which enjoys remarkable equivariant properties with respect to the action of $\mathrm{PGL}(2, \mathbb{Z})$. That is, the Brjuno function is defined as

$$\mathcal{B}(\alpha) = \sum_{n=0}^{\infty} \beta_{n-1} \log \alpha_n^{-1} \tag{2.5}$$

where

$$\beta_{-2} = \alpha, \ \beta_{-1} = +1, \ \beta_n = \beta_n(\alpha) = \prod_{i=0}^{n} \alpha_i, \text{ for } n \geq 0.$$

The function $\mathcal{B}$ is defined on the set of irrational numbers, and takes values in $(0, +\infty]$. It satisfies the remarkable relations

$$\begin{gathered} \mathcal{B}(\alpha) = \mathcal{B}(\alpha + 1) = \mathcal{B}(-\alpha), \text{ for } \alpha \in \mathbb{R}, \\ \mathcal{B}(\alpha) = \alpha \mathcal{B}(1/\alpha) + \log 1/\alpha, \text{ for } \alpha \in (0, 1/2). \end{gathered} \tag{2.6}$$

The difference $|\sum_{n=-1}^{+\infty} q_n^{-1} \log q_{n+1} - \tilde{\mathcal{B}}(\alpha)|$ is uniformly bounded from above independent of α. Thus, an irrational number α is a **Brjuno number** iff $\mathcal{B}(\alpha) < +\infty$.

Using the renormalisation approach, Yoccoz in [Yoc88, Yoc95b] proved that the Brjuno condition is optimal for the linearisation of the quadratic maps $e^{2\pi i \alpha} z + z^2$, i.e. if α is not a Brjuno umber, then $e^{2\pi i \alpha} z + z^2$ is not linearisable near 0. The optimality of this condition has been (re)confirmed for several classes of maps [PM93, Gey01, BC04, Oku04, Oku05, BC06, CC15, FMS18, Che19] but in its general form for rational and entire functions remains a significant challenge in the field of holomorphic dynamics.

The Brjuno function naturally appears in a number of settings, and has been extensively studied since their appearance in the work of Brjuno. For instance, for the higher dimensional linearisation problem one may refer to [Sto00, Gen07, Ron08, YG08, Rai10, BZ13, GLS15], for twist maps refer to [BG01, Pon10], see also [CM00, Lin04, MS11].

The function $\mathcal{B}$ is a highly irregular function; $\mathcal{B}(\alpha) = +\infty$ for generic choice of $\alpha \in \mathbb{R}$. One may refer to [MMY97, MMY01, JM18], and the extensive list of references therein, for detailed analyses of the regularity properties of this function.

2.3. **Herman numbers.** In [Her79], Herman presented a systematic study of the problem of linearisation for orientation-preserving diffeomorphisms of the circle $\mathbb{R}/\mathbb{Z}$ with irrational rotation numbers. In particular, he presented a rather technical arithmetic condition which guaranteed the analytic linearisation of such analytic diffeomorphisms. Although the linearisation problem for analytic circle diffeomorphisms close to rigid rotations was successfully studied earlier by Arnold [Arn61], no progress had been made in between. Shortly later, enhancing the work of Herman, Yoccoz in [Yoc95a, Yoc02] identified the optimal arithmetic condition for the analytic linearisation of analytic circle diffeomorphisms. The name, Herman numbers, was suggested by Yoccoz in honour of the work of Herman on this problem.

The set of Herman numbers is defined in a different fashion. To that end, we need to consider the functions $h_r : \mathbb{R} \to (0, +\infty)$, for $r \in (0, 1)$:

$$h_r(y) = \begin{cases} r^{-1}(y - \log r^{-1} + 1) & \text{if } y \geq \log r^{-1}, \\ e^y & \text{if } y \leq \log r^{-1}. \end{cases}$$

The function h_r is C^1 on $\mathbb{R}$, and satisfies

$$\begin{aligned} h_r(\log r^{-1}) = h'_r(\log r^{-1}) = r^{-1}, \\ e^y \geq h_r(y) \geq y + 1, \forall y \in \mathbb{R}, \\ h'_r(y) \geq 1, \forall y \geq 0. \end{aligned} \tag{2.7}$$

An irrational number α is a Herman number if and only if for all $n \geq 0$ there is $m \geq n$ such that

$$h_{\alpha_{m-1}} \circ \cdots \circ h_{\alpha_n}(0) \geq \mathcal{B}(\alpha_m).$$

In the above definition, the composition $h_{\alpha_{m-1}} \circ \cdots \circ h_{\alpha_n}$ is understood as the identity map when $m = n$, and as h_{α_n} when $m = n + 1$.

The arithmetic characterisation of the Herman numbers by Yoccoz in [Yoc02] uses the standard continued fraction. That is, he works with the interval $(0, 1)$ for the action of $z \mapsto z + 1$. The above equivalent form of the Herman numbers in terms of the modified continued fractions is established in [Che21].

It follows from the definition that every Herman number is a Brjuno number. That is because, if α is not a Brjuno number, then $\mathcal{B}(\alpha) = \mathcal{B}(\alpha_0) = +\infty$. Repeatedly using the functional equations in (2.6), one concludes that $\mathcal{B}(\alpha_m) = +\infty$ for all $m \geq 0$. In particular, the inequality in the definition of Herman numbers never holds.

In the definition of Herman numbers, one may only require that for large n there is m such that the inequality holds. That is because, if m' works for some n', then the same m' works for all $n \leq n'$. This shows that α_0 is a Herman number if and only if α_1 is a Herman number. On the other hand, since α and $\alpha + 1$ produce the same sequence of α_i, we see that the set of Herman numbers is invariant under $z \mapsto z + 1$. These show that the set of Herman numbers is invariant under the action of $\mathrm{PGL}(2, \mathbb{Z})$.

Recall that α is a Diophantine number, if there are $\tau \geq 0$ and $c > 0$ such that for all $p/q \in \mathbb{Q}$ with $q \geq 1$ we have $|\alpha - p/q| \geq c/q^{2+\tau}$. Any Diophantine number is of Herman type. Since the set of Diophantine numbers has full Lebesgue measure in $\mathbb{R}$, the set of Herman numbers and the set of Brjuno numbers have full Lebesgue measure in $\mathbb{R}$. On the other hand, there is a dense set of irrational numbers in $\mathbb{R}$ which are of Brjuno but not Herman type. See [Yoc02] for alternative characterisations of the set of Herman numbers, and see [Che21] for more details on these.

Although Herman did not have the optimal characterisation for the linearisation of analytic circle diffeomorphisms, he used the linearisation property of circle maps to show that if α satisfies that optimal condition, the critical point of $e^{2\pi i\alpha}z + z^2$ must lie on the boundary of the Siegel disk [Her85]. His argument also applies to polynomials with a single critical point of higher orders. This result has been extended to cubic polynomials in [CR16]. On the other hand, Herman built the first examples of holomorphic maps with an irrationally indifferent fixed point such that there is no critical point on the boundary of the Siegel disk containing that fixed point. Until the present work, it was not known how the arithmetic condition of Herman is related to the presence of a critical point on the boundary of the Siegel disk.

Thanks to the relations in (2.6), one may think of the Brjuno function as a $\mathrm{PGL}(2,\mathbb{Z})$-cocycle. This point of view drives the arguments in [Che21] to explain the topology and dynamics of a toy model for the dynamics near irrationally indifferent fixed points. In this paper we do not need to consider separate arithmetic classes and their properties. In this paper, our unified approach works for rotations numbers of different type at the same time.

Remark 2.1. The set of hight type numbers HT_N in the modified continued fraction may be strictly larger than the set of high type numbers in the standard continued fraction. To be precise, let HT^s_N denote the set of irrational numbers α whose entries $\tilde{a}_n$ in the standard continued fraction are at least N, for all $n \geq 0$. If $\tilde{a}_n \geq 2$ for all $n \geq 0$, then $a_n = 2$ and $\varepsilon_n = +1$, for all $n \geq 0$. This shows that for $N \geq 2$, $\mathrm{HT}^s_N \subseteq \mathrm{HT}_N$. But in general, HT_N is not contained in HT^s_{N-1}, or in HT^s_{N-2}, etc. Indeed, for any N, an element of HT_N may have infinitely many $+1$ entries in its standard continued fraction. In this sense, the theorems stated in the introduction are stronger.

3. Topological model for the post-critical set

In this section we present a topological model for the post-critical set, and a map on this model which will serve as a model for the map on the post-critical set. This is based on the construction in [Che21], and here we mainly summarise the key properties that will be used in this paper.

3.1. **Model for the changes of coordinates.** The starting point for building the model is a model for the changes of coordinates that appear in the sector renormalisation of irrationally indifferent fixed points. This is presented in this section.

Consider the set

$$\mathbb{H}' = \{w \in \mathbb{C} \mid \operatorname{Im} w > -1\}.$$

For $r \in (0, 1/2)$, we define the map $Y_r : \overline{\mathbb{H}'} \to \mathbb{C}$ as [1]

$$Y_r(w) = r \operatorname{Re} w + \frac{i}{2\pi} \log \left| \frac{e^{-3\pi r} - e^{-\pi r i} e^{-2\pi r i w}}{e^{-3\pi r} - e^{\pi r i}} \right|.$$

This map is continuous on $\mathbb{H}'$, and real analytic in the variables $\operatorname{Re} w$ and $\operatorname{Im} w$. It sends vertical lines in $\mathbb{H}'$ to vertical lines. But, it maps horizontal lines in $\mathbb{H}'$ to non-straight curves which are periodic in $\operatorname{Re} w$ of period $1/r$. In particular, Y_r is not conformal for any value of $r \in (0, 1/2]$. As we shall see in a moment, this map degenerates the conformal structure as $r \to 0$. In spite of this, it is proved fundamentally useful when compared to conformal changes of coordinates which appear in the renormalisation. Fig 2 shows the behaviour of Y_r on horizontal and vertical lines.

We have $Y_r(0) = 0$.

Lemma 3.1. *For every $r \in (0, 1/2]$, Y_r is injective on $\overline{\mathbb{H}'}$ and its image is contained in $\mathbb{H}'$.*

Proof. Using $0 \leq r \leq 1/2$, $3 \leq \pi \leq 4$, and $10 \leq e^{12/5}$, we obtain

$$\log(4 + 3\pi r) + 2\pi r \leq \log(e^{12/5}) + \pi \leq \log e^{4\pi/5} + \pi \leq 9\pi/5.$$

Thus, for all $r \in [0, 1/2]$, we have $e^{9\pi/5} \geq (4 + 3\pi r)e^{2\pi r}$, and hence

$$\pi e^{\pi r}(e^{9\pi/5} - (4 + 3\pi r)e^{2\pi r})) \geq 0.$$

[1] $\overline{X}$ denotes the topological closure of a given set X.

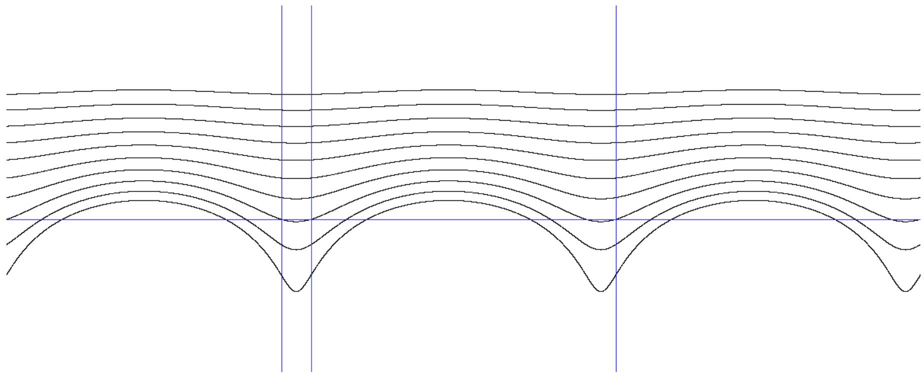

FIGURE 2. The black curves are the images of some horizontal lines by Y_r. The vertical lines in blue, from left to right, are the images of the vertical lines $\operatorname{Re} w = -1$, $\operatorname{Re} w = 0$, and $\operatorname{Re} w = 1/\alpha$, under Y_r. Here, $r = 1/(10 + 1/(1 + 1/(1 + \dots)))$.

Now, fix $r \in [0, 1/2]$. One may integrate the above inequality from 0 to r, to obtain

$$e^{9\pi/5}(e^{\pi r} - 1) - (\pi r + 1)e^{3\pi r} + 1 \geq 0.$$

This implies that for all $r \in (0, 1/2]$,

$$\frac{e^{\pi r} - 1}{(\pi r + 1)e^{3\pi r} - 1} \geq e^{-9\pi/5}. \tag{3.1}$$

On the other hand, for $w \in \overline{\mathbb{H}'}$, by the triangle inequality,

$$|e^{-3\pi r} - e^{-\pi r i} e^{-2\pi r i w}| \geq |e^{-\pi r i} e^{-2\pi r i w}| - |e^{-3\pi r}| \geq e^{-2\pi r} - e^{-3\pi r},$$

and

$$|e^{-3\pi r} - e^{\pi r i}| \leq |e^{-3\pi r} - 1| + |1 - e^{\pi r i}| \leq (1 - e^{-3\pi r}) + \pi r.$$

Combining the above three inequalities, we obtain

$$\left| \frac{e^{-3\pi r} - e^{-\pi r i} e^{-2\pi r i w}}{e^{-3\pi r} - e^{\pi r i}} \right| \geq \frac{e^{-2\pi r} - e^{-3\pi r}}{\pi r + 1 - e^{-3\pi r}} = \frac{e^{\pi r} - 1}{(\pi r + 1)e^{3\pi r} - 1} \geq e^{-9\pi/5}.$$

This implies that for all $w \in \mathbb{H}'$ we have

$$\operatorname{Im} Y_r(w) \geq \frac{1}{2\pi} \log e^{-9\pi/5} > -1.$$

In particular, Y_r is well-defined, and maps $\overline{\mathbb{H}'}$ into $\mathbb{H}'$.

Let $w_1 \neq w_2$ be arbitrary points in $\overline{\mathbb{H}'}$. If $\operatorname{Re} w_1 \neq \operatorname{Re} w_2$, then $\operatorname{Re} Y_r(w_1) \neq \operatorname{Re} Y_r(w_2)$. If $\operatorname{Re} w_1 = \operatorname{Re} w_2$ but $\operatorname{Im} w_1 \neq \operatorname{Im} w_2$, then

$$|e^{-3\pi r} - e^{-\pi r i} e^{-2\pi r i w_1}| \neq |e^{-3\pi r} - e^{-\pi r i} e^{-2\pi r i w_2}|.$$

This implies that $\operatorname{Im} Y_r(w_1) \neq \operatorname{Im} Y_r(w_2)$. These show that Y_r is injective. □

Lemma 3.2. *For every $r \in (0, 1/2)$, we have*

(i) *for every $w \in \overline{\mathbb{H}'}$,*

$$Y_r(w + 1/r) = Y_r(w) + 1$$

(ii) *for every $t \geq -1$,*

$$Y_r(it + 1/r - 1) = Y_r(it) + 1 - r.$$

Proof. Part (i) of the lemma readily follows from the formula defining Y_r.

To prove part (ii) of the lemma, first note that

$$\begin{aligned}\left|e^{-3\pi r} - e^{-\pi ri}e^{-2\pi ri(it+1/r-1)}\right| &= \left|e^{-3\pi r} - e^{-\pi ri}e^{2\pi rt}e^{2\pi ri}\right| \\ &= \left|e^{-3\pi r} - e^{\pi ri}e^{2\pi rt}\right| = \left|e^{-3\pi r} - e^{-\pi ri}e^{2\pi rt}\right|.\end{aligned}$$

Above, the first and second "=" are simple multiplications of complex numbers, while for the third "=" we have used that $|x - z| = |x - \overline{z}|$, for real x and complex z. Thus,

$$Y_r(it + 1/r - 1) = r(1/r - 1) + \frac{i}{2\pi}\log\left|\frac{e^{-3\pi r} - e^{-\pi ri}e^{2\pi rt}}{e^{-3\pi r} - e^{\pi ri}}\right| = (1 - r) + Y_r(it). \qquad \square$$

Lemma 3.3. *For every $r \in (0, 1/2)$, and every w_1, w_2 in $\overline{\mathbb{H}'}$, we have*

$$|Y_r(w_1) - Y_r(w_2)| \leq 0.9|w_1 - w_2|.$$

The precise contraction factor 0.9 in the above lemma is not crucial; any constant less than 1 suffices.

Proof. Let $g(w) = (e^{-3\pi r} - e^{-\pi ri}e^{-2\pi riw})(e^{-3\pi r} - e^{\pi ri}e^{2\pi ri\overline{w}})$. Then, $g(w)$ is of the form $\zeta\overline{\zeta}$, for some $\zeta \in \mathbb{C}$, and hence it produces positive real values for $w \in \overline{\mathbb{H}'}$. We have

$$\partial g(w)/\partial w = 2\pi rie^{-\pi ri}e^{-2\pi riw}(e^{-3\pi r} - e^{\pi ri}e^{2\pi ri\overline{w}}),$$

and

$$\partial g(w)/\partial \overline{w} = -2\pi rie^{\pi ri}e^{2\pi ri\overline{w}}(e^{-3\pi r} - e^{-\pi ri}e^{-2\pi riw}).$$

Therefore, by the complex chain rule,

$$\frac{\partial}{\partial w}\big(\log g(w)\big) = \frac{1}{g(w)}\frac{\partial g}{\partial w} = \frac{2\pi rie^{-\pi ri}e^{-2\pi riw}}{e^{-3\pi r} - e^{-\pi ri}e^{-2\pi riw}} = \frac{2\pi ri}{e^{-3\pi r}e^{\pi ri}e^{2\pi riw} - 1},$$

and

$$\frac{\partial}{\partial \overline{w}}\big(\log g(w)\big) = \frac{1}{g(w)}\frac{\partial g}{\partial \overline{w}} = \frac{-2\pi rie^{\pi ri}e^{2\pi ri\overline{w}}}{e^{-3\pi r} - e^{\pi ri}e^{2\pi ri\overline{w}}} = \frac{-2\pi ri}{e^{-3\pi r}e^{-\pi ri}e^{-2\pi ri\overline{w}} - 1}.$$

We rewrite Y_r in the following form

$$Y_r(w) = r\cdot\frac{w + \overline{w}}{2} + \frac{i}{2\pi}\cdot\frac{1}{2}\log g(w) - \frac{i}{2\pi}\log|e^{-3\pi r} - e^{\pi ri}|.$$

Then, by the above calculations,

$$\frac{\partial Y_r}{\partial w}(w) = \frac{r}{2} + \frac{i}{4\pi}\cdot\frac{2\pi ri}{e^{-3\pi r}e^{\pi ri}e^{2\pi riw} - 1} = \frac{r}{2}\Big(1 - \frac{1}{e^{-3\pi r}e^{\pi ri}e^{2\pi riw} - 1}\Big),$$

and

$$\frac{\partial Y_r}{\partial \overline{w}}(w) = \frac{r}{2} + \frac{i}{4\pi}\cdot\frac{-2\pi ri}{e^{-3\pi r}e^{-\pi ri}e^{-2\pi ri\overline{w}} - 1} = \frac{r}{2}\Big(1 + \frac{1}{e^{-3\pi r}e^{-\pi ri}e^{-2\pi ri\overline{w}} - 1}\Big).$$

Let $\xi = e^{-3\pi r}e^{\pi ri}e^{2\pi riw}$. For $w \in \overline{\mathbb{H}'}$, $|\xi| \leq e^{-\pi r}$. For the maximum size of the directional derivative of Y_r we have

$$\begin{aligned}\max_{\theta\in[0,2\pi]}\big|\operatorname{D}Y_r(w)\cdot e^{i\theta}\big| &= \Big|\frac{\partial Y_r}{\partial w}(w)\Big| + \Big|\frac{\partial Y_r}{\partial \overline{w}}(w)\Big| \\ &\leq \frac{r}{2}\cdot\Big|1 - \frac{1}{\xi - 1}\Big| + \frac{r}{2}\cdot\Big|1 + \frac{1}{\overline{\xi} - 1}\Big| \\ &\leq \frac{r}{2}\cdot\frac{2 + e^{-\pi r}}{1 - e^{-\pi r}} + \frac{r}{2}\cdot\frac{e^{-\pi r}}{1 - e^{-\pi r}} = r\cdot\frac{e^{\pi r} + 1}{e^{\pi r} - 1}.\end{aligned}$$

For $r \geq 0$, $e^{\pi r} - 1 \geq \pi r + \pi^2 r^2/2$, (the first two terms of the Taylor series with positive terms). This gives us

$$r \cdot \frac{e^{\pi r}+1}{e^{\pi r}-1} = r\left(1+\frac{2}{e^{\pi r}-1}\right) \leq r\left(1+\frac{2}{\pi r+\pi^2 r^2/2}\right) = \frac{2\pi r+\pi^2 r^2+4}{2\pi+\pi^2 r}.$$

The last function in the above equation is increasing on $(0,1/2)$, because it has a non-negative derivative $(4\pi r+\pi^2 r^2)/(2+\pi r)^2$. Then, it is bounded by its value at $1/2$, which, using $\pi \geq 3$, gives us

$$\frac{2\pi r+\pi^2 r^2+4}{2\pi+\pi^2 r} \leq \frac{\pi+\pi^2/4+4}{2\pi+\pi^2/2} = \frac{1}{2}+\frac{4}{2\pi+\pi^2/2} \leq \frac{1}{2}+\frac{4}{6+4} = \frac{9}{10}. \qquad \square$$

Recall the map h_r defined in Sec. 2 , that is, a diffeomorphism from $\mathbb{R}$ onto $(0,+\infty)$. The map Y_r closely traces the behaviour of h_r^{-1}. By some elementary calculations, one can see that for all $r \in (0,1/2]$ and all $y \geq 1$, we have

$$|2\pi \operatorname{Im} Y_r(iy/(2\pi)) - h_r^{-1}(y)| \leq \pi. \tag{3.2}$$

Also, Y_r captures the remarkable functional relation for the Brjuno function. By elementary calculations, one can see that for all $r \in (0,1/2]$, and all $y \geq 0$, we have

$$2\pi r y + \log(1/r) - 4 \leq 2\pi \operatorname{Im} Y_r(1/(2r)+iy) \leq 2\pi r y + \log(1/r) + 2. \tag{3.3}$$

Compare this with the second functional equation in (2.6). In particular, $Y_r(1/(2r))$ is uniformly close to $1/2 + i(\log 1/r)/(2\pi)$.

3.2. **The straight topological model.** Recall the sequence $\{\alpha_n\}_{n=0}^{\infty}$ introduced in Sec. 2.1. Let $s(w) = \overline{w}$ denote the complex conjugation map. For $n \geq 0$ we define the maps $Y_n : \overline{\mathbb{H}'} \to \mathbb{H}'$ as

$$Y_n(w) = \begin{cases} Y_{\alpha_n}(w) & \text{if } \varepsilon_n = -1, \\ -s \circ Y_{\alpha_n}(w) & \text{if } \varepsilon_n = +1. \end{cases} \tag{3.4}$$

Each Y_n is either orientation preserving or reversing, depending on the sign of ε_n. For $n \geq 0$, we have[2]

$$Y_n(i[-1,+\infty)) \subset i(-1,+\infty),\ Y_n(0) = 0. \tag{3.5}$$

It follows from Lem 3.2 that for all $n \geq 0$ and all $w \in \overline{\mathbb{H}'}$,

$$Y_n(w+1/\alpha_n) = \begin{cases} Y_n(w)+1 & \text{if } \varepsilon_n = -1, \\ Y_n(w)-1 & \text{if } \varepsilon_n = +1. \end{cases} \tag{3.6}$$

Also, by the same lemma, for all $n \geq 0$, and all $t \geq -1$,

$$Y_n(it+1/\alpha_n-1) = \begin{cases} Y_n(it)+(1-\alpha_n) & \text{if } \varepsilon_n = -1, \\ Y_n(it)+(\alpha_n-1) & \text{if } \varepsilon_n = +1. \end{cases} \tag{3.7}$$

Lem 3.3 implies that for all $n \geq 0$ and all w_1, w_2 in $\overline{\mathbb{H}'}$, we have

$$|Y_n(w_1) - Y_n(w_2)| \leq 0.9|w_1 - w_2|. \tag{3.8}$$

For $n \geq 0$ let

$$\begin{aligned} M_n^0 &= \{w \in \overline{\mathbb{H}'} \mid \operatorname{Re} w \in [0,1/\alpha_n]\}, \\ J_n^0 &= \{w \in M_n^0 \mid \operatorname{Re} w \in [1/\alpha_n - 1, 1/\alpha_n]\}, \\ K_n^0 &= \{w \in M_n^0 \mid \operatorname{Re} w \in [0, 1/\alpha_n - 1]\}. \end{aligned} \tag{3.9}$$

[2]We define $iX = \{ix \mid x \in X\}$, for a given set $X \subset \mathbb{C}$.

We inductively defined the sets M_n^j, J_n^j, and K_n^j, for $j \geq 1$ and $n \geq 0$. Assume that M_n^j, J_n^j, and K_n^j are defined for some $j \geq 0$ and all $n \geq 0$. We define these sets for $j+1$ and all $n \geq 0$ as follows. Fix an arbitrary $n \geq 0$. If $\varepsilon_{n+1} = -1$, let

$$M_n^{j+1} = \bigcup_{l=0}^{a_n-2} \big(Y_{n+1}(M_{n+1}^j) + l\big) \bigcup \big(Y_{n+1}(K_{n+1}^j) + a_n - 1\big). \tag{3.10}$$

If $\varepsilon_{n+1} = +1$, let

$$M_n^{j+1} = \bigcup_{l=1}^{a_n} \big(Y_{n+1}(M_{n+1}^j) + l\big) \bigcup \big(Y_{n+1}(J_{n+1}^j) + a_n + 1\big). \tag{3.11}$$

Regardless of the sign of ε_{n+1}, define

$$J_n^{j+1} = \{w \in M_n^{j+1} \mid \operatorname{Re} w \in [1/\alpha_n - 1, 1/\alpha_n]\},$$
$$K_n^{j+1} = \{w \in M_n^{j+1} \mid \operatorname{Re} w \in [0, 1/\alpha_n - 1]\}.$$

Fig 3 presents two generations of these domains.

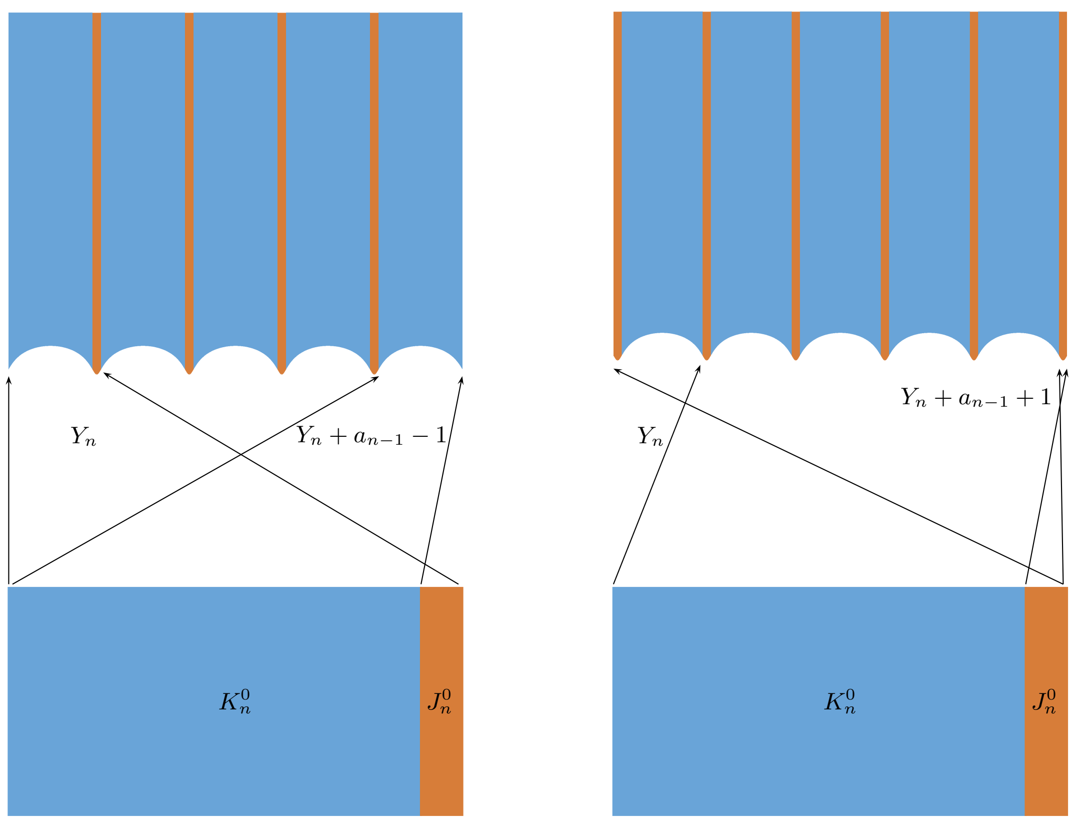

FIGURE 3. In the left hand column $\varepsilon_n = -1$ and in the right hand column $\varepsilon_n = +1$. The sets K_n^0 and J_n^0 are on the lower row, and the set M_{n-1}^1 is on the upper row.

For all $n \geq 0$ and $j \geq 0$, the sets M_n^j, J_n^j, and K_n^j are closed and connected subsets of $\mathbb{C}$, and are bounded by piece-wise analytic curves. Moreover,

$$\{\operatorname{Re} w \mid w \in M_n^j\} = [0, 1/\alpha_n].$$

The functional relations in (3.6) and (3.7) allow us to align together the pieces in the unions (3.10) and (3.11). More precisely, we have the following property of the sets M_n^j.

Corollary 3.4. *For every $n \geq 0$ and $j \geq 0$, the following hold:*

(i) *for all $w \in \mathbb{C}$ satisfying $\operatorname{Re} w \in [0, 1/\alpha_n - 1]$, $w \in M_n^j$ if and only if $w + 1 \in M_n^j$;*
(ii) *for all $t \in \mathbb{R}$, $it \in M_n^j$ if and only if $it + 1/\alpha_n \in M_n^j$.*

Recall that $\alpha_{-1} = +1$. Let $M_{-1}^0 = \{w \in \overline{\mathbb{H}'} \mid \operatorname{Re} w \in [0, 1/\alpha_{-1}]\}$, and for $j \geq 1$, consider the sets

$$M_{-1}^j = \mathrm{Y}_0(M_0^{j-1}) + (\varepsilon_0 + 1)/2.$$

By Lem 3.1, $M_n^1 \subset M_n^0$, for $n \geq -1$. By an inductive argument, this implies that for all $n \geq -1$ and all $j \geq 0$,

$$M_n^{j+1} \subset M_n^j. \tag{3.12}$$

For $n \geq -1$, we define

$$M_n = \bigcap_{j \geq 1} M_n^j.$$

Each M_n consists of closed half-infinite vertical lines tending to $+i\infty$. However, M_n may or may not be connected. We note that $\operatorname{Re} M_{-1} \subset [0, 1]$. Indeed, by Cor 3.4, for real t, $it \in M_{-1}$ if and only if $(it + 1) \in M_{-1}$. We may define the set

$$\begin{aligned} A_\alpha &= \left\{ s(e^{2\pi i w}) \mid w \in M_{-1}, \operatorname{Im} w \leq p_{-1}(\operatorname{Re} w) \right\}, \quad \text{if } \alpha \in \mathscr{B}, \\ A_\alpha &= \left\{ s(e^{2\pi i w}) \mid w \in M_{-1} \right\} \cup \{0\}, \quad \text{if } \alpha \notin \mathscr{B}. \end{aligned} \tag{3.13}$$

The set A_α is our topological model for the post-critical set. It is defined for irrational values of α, and depends only on the arithmetic of α.

3.3. **Hairy Cantor sets and Cantor bouquets.** We shall describe the topology of A_α in Section 3.4.

The simplest scenario is when A_α is a Jordan curve. However, there are other possibilities, which we present below.

A **Cantor bouquet** is any subset of the plane which is ambiently homeomorphic to a set of the form

$$\{re^{2\pi i\theta} \in \mathbb{C} \mid 0 \leq \theta \leq 1, 0 \leq r \leq R(\theta)\}$$

where $R : \mathbb{R}/\mathbb{Z} \to [0, 1]$ satisfies the following:

(a) $R = 0$ on a dense subset of $\mathbb{R}/\mathbb{Z}$, and $R > 0$ on a dense subset of $\mathbb{R}/\mathbb{Z}$,
(b) for each $\theta_0 \in \mathbb{R}/\mathbb{Z}$ we have

$$\limsup_{\theta \to \theta_0^+} R(\theta) = R(\theta_0) = \limsup_{\theta \to \theta_0^-} R(\theta).$$

A **one-sided hairy Jordan curve** is any subset of the plane which is ambiently homeomorphic to a set of the form

$$\{re^{2\pi i\theta} \in \mathbb{C} \mid 0 \leq \theta \leq 1, 1 \leq r \leq 1 + R(\theta)\}$$

where $R : \mathbb{R}/\mathbb{Z} \to [0, 1]$ satisfies properties (a) and (b) in the above definition.

Remark 3.5. The Cantor bouquet and one-sided hairy Jordan curve enjoy similar topological features as the standard Cantor set. Under an additional mild condition (topological smoothness) they are uniquely characterised by some topological axioms, see [AO93].

An alternative approach for building a topological model for $\Lambda(f)$ was studied by Buff and Chéritat in 2009 [BC09]. In their model, they use rational approximation of α, and some conformal changes of coordinates, to build topological models for some invariant sets for maps with a parabolic fixed point. Then, the model for irrational values of α is obtained from taking Hausdorff limits of those objects. Naturally, one looses control along the limit. This has been the main reason behind the construction in [Che21], which is summarised here.

3.4. **Topology of the model.** [3] Recall that the sets M_n^j and M_n consist of closed half-infinite vertical lines. Thus, each of these sets lies above the graph of a function, which may be conveniently used to explain the topological structure of the sets M_n. For $n \geq -1$, and $j \geq 0$, define the function $b_n^j : [0, 1/\alpha_n] \to [-1, +\infty)$ as

$$b_n^j(x) = \min\{y \mid x + iy \in M_n^j\}. \tag{3.14}$$

Since each Y_n preserves vertical lines, it follows that

$$M_n^j = \{w \in \mathbb{C} \mid 0 \leq \operatorname{Re} w \leq 1/\alpha_n, \operatorname{Im} w \geq b_n^j(\operatorname{Re} w)\}.$$

It follows from the definition of the sets M_n^j, and the functional equations (3.6)–(3.7), one may see that for all $n \geq -1$ and $j \geq 0$, $b_n^j : [0, 1/\alpha_n] \to [-1, +\infty)$ is continuous. Moreover, by (3.12), we must have $b_n^{j+1} \geq b_n^j$ on $[0, 1/\alpha_n]$. Thus, for $n \geq -1$, we may define $b_n : [0, 1/\alpha_n] \to [-1, +\infty]$ as

$$b_n(x) = \lim_{j \to +\infty} b_n^j(x) = \sup_{j \geq 0} b_n^j(x).$$

Note that b_n is allowed to attain $+\infty$. Evidently, the function b_n describes the set M_n as follow

$$M_n = \{w \in \mathbb{C} \mid 0 \leq \operatorname{Re} w \leq 1/\alpha_n, \operatorname{Im} w \geq b_n(\operatorname{Re} w)\}. \tag{3.15}$$

By Cor 3.4, $b_n^j(0) = b_n^j(1/\alpha_n)$, and $b_n^j(x+1) = b_n^j(x)$ for all $x \in [0, 1/\alpha_n - 1]$. Taking limits as $j \to +\infty$, we conclude that for all $n \geq -1$

$$b_n(0) = b_n(1/\alpha_n), \quad \text{and} \quad b_n(x+1) = b_n(x), \forall x \in [0, 1/\alpha_n - 1]. \tag{3.16}$$

The collection of the functions b_n^j and b_n enjoy an equivariant relation induced by the maps Y_n. That is, the graph of b_n is obtained from the graph of b_{n+1} by applying the map Y_{n+1} and its translations. Each of the maps Y_n exhibits two distinct behaviour. Above the horizontal line $\operatorname{Im} w = 1/\alpha_n$ it nearly acts as the linear map multiplication by α_n. Below that horizontal line, there is a logarithmic behaviour in imaginary direction. The functions b_n mostly capture the behaviour of the collection of maps Y_n near the lower end of their domains. There is another collection of functions which captures the behaviour of the collection of maps Y_n for the top regions. Indeed, near the bottom end of the top regions. As we shall see in a moment, these are only relevant for Brjuno values of α. We present these functions below.

Assume that α is a Brjuno number. For $n \geq -1$ and $j \geq 0$ we inductively define the functions

$$p_n^j : [0, 1/\alpha_n] \to [-1, +\infty).$$

For all $n \geq -1$, we set $p_n^0 \equiv (\mathcal{B}(\alpha_{n+1}) + 5\pi)/(2\pi)$. Assuming that p_n^j is defined for some $j \geq 0$ and all $n \geq -1$, we define p_n^{j+1} on $[0, 1/\alpha_n]$ as follows. For $x_n \in [0, 1/\alpha_n]$, we find $x_{n+1} \in [0, 1/\alpha_{n+1}]$ and $l_n \in \mathbb{Z}$ such that $-\varepsilon_{n+1}\alpha_{n+1}x_{n+1} = x_n - l_n$, and define

$$p_n^{j+1}(x_n) = \operatorname{Im} Y_{n+1}\big(x_{n+1} + i p_{n+1}^j(x_{n+1})\big).$$

In other words, the graph of p_n^{j+1} is obtained from applying Y_{n+1} to the graph of p_{n+1}^j on $[0, 1/\alpha_{n+1}]$, and translating it by some integers.

Evidently, for all $n \geq -1$ and all $j \geq 0$, we have $p_n^{j+1}(x+1) = p_n^{j+1}(x)$, for $x \in [0, 1/\alpha_n - 1]$. Moreover, it follows from (3.6) and (3.7) that each $p_n^j : [0, 1/\alpha_n] \to \mathbb{R}$ is continuous, and $p_n^j(0) = p_n^j(1/\alpha_n)$.

[3] The letter b stands for "base" and "p" for "pinnacle"; the reason for these will become clear in a moment.

Because of the choice of the constant $(5\pi)/(2\pi)$ in the definition of p_n^0, some calculations of the formula for Y_n may be used to show that $p_n^1 \leq p_n^0$, for all $n \geq -1$. Then, using an induction argument, one may show that for all $n \geq -1$ and $j \geq 0$ we have

$$p_n^{j+1}(x) \leq p_n^j(x), \quad \forall x \in [0, 1/\alpha_n].$$

Therefore, we may define the functions

$$p_n(x) = \lim_{j \to +\infty} p_n^j(x), \quad \forall x \in [0, 1/\alpha_n].$$

A main difference with the functions b_n^j is that the convergence in the above equation is uniform on all of $[0, 1/\alpha_n]$. This is mainly because the maps Y_n behave better near the top end of M_n^0; they are close to multiplication by α_n. In particular, we conclude that for $n \geq -1$, $p_n : [0, 1/\alpha_n] \to [1, +\infty)$ is continuous. Moreover, the functional relations for p_n^j imply that

$$p_n(0) = p_n(1/\alpha_n), \quad p_n(x) = p_n(x+1), \forall x \in [0, 1/\alpha_n - 1]. \tag{3.17}$$

By definition, $p_n^0 \geq b_n^0$, for all $n \geq -1$. Since the graphs of b_{n+1}^0 and p_{n+1}^0 are mapped to the graphs of b_n^1 and p_n^1, respectively, under Y_{n+1} and its translations, we must have $p_n^1 \geq b_n^1$, for all $n \geq -1$. By induction, this implies that for all $n \geq -1$ and all $j \geq 0$, $p_n^j(x) \geq b_n^j(x)$, for all $x \in [0, 1/\alpha_n]$. In particular,

$$p_n(x) \geq b_n(x), \quad \forall x \in [0, 1/\alpha_n]. \tag{3.18}$$

Because for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $Y_n(0) = 0$ for all $n \geq 0$, and all of Y_n are uniformly contracting, one may see that for all $n \geq -1$,

$$b_n(0) = 0. \tag{3.19}$$

Employing the periodicity of the functions b_n, and their equivariant relations induced from the maps Y_n, one concludes that or every $n \geq -1$, $b_n < +\infty$ holds on a dense subset of $[0, 1/\alpha_n]$. In the same fashion, if some b_m attains $+\infty$ at a single point, then every b_n attains $+\infty$ on a dense subset of $[0, 1/\alpha_n]$. The question of whether this happens or not depends on the arithmetic of α. Indeed, because of the functional relations in (2.6) and (3.3), an explicit calculation show that for all $n \geq -1$ we have

$$\Big| 2\pi \sup_{x \in [0, 1/\alpha_n]} b_n(x) - \mathcal{B}(\alpha_{n+1}) \Big| \leq 5\pi. \tag{3.20}$$

In the above relation, $\infty - \infty$ is assumed to be 0. In particular, if α is a Brjuno number, every b_n is bounded, and if α is not a Brjuno number, every b_n attains $+\infty$ on a dense set of points.

The estimate in (3.20) also implies that when α is a Brjuno number, b_n is uniformly close to p_n at some points. By the uniform contraction of the maps Y_n, and the equivariant property of the collections of functions b_n and p_n, we must have $p_n(x) = b_n(x)$ on a dense set of points in $[0, 1/\alpha_n]$. Alternatively, one may see that these equalities occur due to the particular contracting factor of each map Y_n near the vertical line $\operatorname{Re} w = 1/(2\alpha_n)$. Among the vertical lines in the domain of Y_n, the least amount of contraction occurs near the vertical line $\operatorname{Re} w = 0$. So, 0 is the least likely place where b_n and p_n become equal. Indeed, the answer to this question depends on the arithmetic of α as we explain below.

Because of the uniform contraction of the maps h_{α_n}, the criterion for the Herman numbers is stable under uniform changes to the maps h_{α_n}. More precisely, if one replaces h_{α_n} by uniformly nearby maps, say Y_n^{-1}, the corresponding set of rotation numbers stays the same. Indeed, one may employ the estimate in (3.2) to show that for integers $m > n \geq 0$ and $y \in (1, +\infty)$,

$$\big| 2\pi \operatorname{Im} Y_n \circ \cdots \circ Y_m(iy/(2\pi)) - h_{\alpha_n}^{-1} \circ \cdots \circ h_{\alpha_m}^{-1}(y) \big| \leq 10\pi,$$

provided $h_{\alpha_n}^{-1} \circ \cdots \circ h_{\alpha_m}^{-1}(y)$ is defined. The uniform estimate above, and the uniform contraction of the maps Y_n may be used to show that an irrational number α belongs to $\mathscr{H}$, if and only if, for all $x > 0$ there is $m \geq 1$ such that

$$\operatorname{Im} Y_0 \circ \cdots \circ Y_{m-1}(i\mathcal{B}(\alpha_m)/(2\pi)) \leq x.$$

In particular, this implies that for Brjuno values of α, α is a Herman number if and only if $p_n(0) = 0$ for all $n \geq -1$. Combining with earlier arguments, we conclude that when α is a Herman number, we have $b_n \equiv p_n$, and when α is a Brjuno but not a Herman number, $b_n < p_n$ holds on a dense subset of $[0, 1/\alpha_n]$.

Because each M_n is a closed set, for every $x \in [0, 1/\alpha_n)$, $\liminf_{s \to x^+} b_n(s) \geq b_n(x)$, and for all $x \in (0, 1/\alpha_n]$, $\liminf_{s \to x^-} b_n(s) \geq b_n(x)$. In fact, both of "$\geq$" are "$=$". That is because, for large values of m, b_{n+m} is periodic of period $+1$. Employing the equivariant property of the maps b_j, and the uniform contraction of the maps Y_j, one may obtain a sequence of points on the graph of b_n which converges to $(x, b_n(x))$. With a detailed analysis of the trajectories of points in the tower of maps Y_j, one can see that this can be done from both sides. These relations imply the property (b) in the definition of the hairy Jordan curve and the Cantor bouquet.

Combining the properties we summarised in this section, we obtain the following classification.

Theorem 3.6 ([Che21]). *Let α be an irrational number. We have,*

(i) *if α is a Herman number, then A_α is a closed Jordan curve;*
(ii) *if α is a Brjuno but not a Herman number, then A_α is a one-sided hairy Jordan curve;*
(iii) *if α is not a Brjuno number, then A_α is a Cantor Bouquet.*

3.5. **Dynamics on the topological model.** In this section we introduce a map

$$T_\alpha : A_\alpha \to A_\alpha, \tag{3.21}$$

which serves as the model for the map f on $\Lambda(f)$. We summarise the dynamical properties of this map which were obtained in [Che21].

Let us fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and let M_{-1} be the corresponding topological model defined in Sec. 3.2. Given $w_{-1} \in M_{-1}$, we inductively identify the integers l_i, and then the points $w_{i+1} \in M_{i+1}$ such that

$$\begin{gathered} 0 \leq \operatorname{Re}(w_i - l_i) < 1, \quad \text{if } \varepsilon_{i+1} = -1, \\ -1 < \operatorname{Re}(w_i - l_i) \leq 0, \quad \text{if } \varepsilon_{i+1} = +1, \\ Y_{i+1}(w_{i+1}) + l_i = w_i. \end{gathered} \tag{3.22}$$

It follows that for all $n \geq 0$, we have

$$w_{-1} = (Y_0 + l_{-1}) \circ (Y_1 + l_0) \circ \cdots \circ (Y_n + 1_{n-1})(w_n). \tag{3.23}$$

Also, by the definition of M_i in (3.10) and (3.11), for all $i \geq 0$,

$$0 \leq l_i \leq a_i + \varepsilon_{i+1}, \quad \text{and} \quad 0 \leq \operatorname{Re} w_i < 1/\alpha_i. \tag{3.24}$$

We refer to the sequence $(w_i; l_i)_{i \geq -1}$ as the **trajectory** of w_{-1}, with respect to α, or simply, as the trajectory of w_{-1}, when it is clear from the context what irrational number is used.

Consider the map

$$\tilde{T}_\alpha : M_{-1} \to M_{-1}, \tag{3.25}$$

defined as follows. For an arbitrary point w_{-1} in M_{-1} with trajectory $(w_i; l_i)_{i \geq -1}$,

(i) if there is $n \geq 0$ such that $w_n \in K_n$, and for all $0 \leq i \leq n-1$, $w_i \in M_i \setminus K_i$, then

$$\tilde{T}_\alpha(w_{-1}) = \left(Y_0 + \frac{\varepsilon_0 + 1}{2}\right) \circ \left(Y_1 + \frac{\varepsilon_1 + 1}{2}\right) \circ \cdots \circ \left(Y_n + \frac{\varepsilon_n + 1}{2}\right)(w_n + 1);$$

(ii) if for all $n \geq 0$, $w_n \in M_n \setminus K_n$, then

$$\tilde{T}_\alpha(w_{-1}) = \lim_{n\to+\infty} \left(Y_0 + \frac{\varepsilon_0+1}{2}\right) \circ \left(Y_1 + \frac{\varepsilon_1+1}{2}\right) \circ \cdots \circ \left(Y_n + \frac{\varepsilon_n+1}{2}\right) (w_n + 1 - 1/\alpha_n).$$

Evidently, item (i) leads to continuous maps on pieces of M_{-1}. There might be a vertical half-infinite line where item (ii) above applies. On that set, the uniform contraction of the maps Y_j implies that the maps converge to a well-defined map. It follows from the functional equations (3.6) and (3.7) that these pieces match together and produce a well-define homeomorphism

$$\tilde{T}_\alpha : M_{-1}/\mathbb{Z} \to M_{-1}/\mathbb{Z}.$$[4]

One may compare the above definition of the map $\tilde{T}_\alpha$ to the action of the map f on its renormalisation tower in Sec. 7.4. By the definition of A_α in (3.13), $\tilde{T}_\alpha$ induces a homeomorphism

$$T_\alpha : A_\alpha \to A_\alpha.$$

Recall that a map $f : X \to X$, of a topological space X, is called **topologically recurrent**, if for every $x \in X$ there is a strictly increasing sequence of positive integers $(m_i)_{i\geq 0}$ such that $f^{\circ m_i}(x) \to x$ as $i \to +\infty$. Also, recall that a set $K \subset A_\alpha$ is called **invariant** under T_α, if $T_\alpha(K) = K = T_\alpha^{-1}(K)$.

Define $r_\alpha \in [0,1]$ according to

$$[r_\alpha, 1] = \{z \in A_\alpha \mid \operatorname{Im} z = 0, \operatorname{Re} z \geq 0\}.$$

Below we summarise the dynamical behaviour of T_α on A_α which are obtained in [Che21].

In the next theorem, $\omega(z)$ denotes the set of accumulation points of the orbit of the point z.

Theorem 3.7. *For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ the map $T_\alpha : A_\alpha \to A_\alpha$ satisfies the following properties.*

(i) *$T_\alpha : A_\alpha \to A_\alpha$ is topologically recurrent.*

(ii) *The map*

$$\omega : [r_\alpha, 1] \to \{X \subseteq A_\alpha \mid X \text{ is non-empty, closed and invariant}\}$$

is a homeomorphism with respect to the Hausdorff metric on the range. In particular, every non-empty closed invariant subset of A_α is equal $\omega(z)$, for some $z \in A_\alpha$.

(iii) *The map ω on $[r_\alpha, 1]$ is strictly increasing with respect to the linear order on $[r_\alpha, 1]$ and the inclusion on the range.*

(iv) *If α is not a Brjuno number, for every $t \in (r_\alpha, 1]$, $\omega(t)$ is a Cantor bouquet.*

(v) *If α is a Brjuno number, for every $t \in (r_\alpha, 1]$, $\omega(t)$ is a hairy Jordan curve.*

4. Near-parabolic renormalisation scheme

In this section we present the rear-parabolic renormalisation scheme introduced by Inou and Shishikura [IS06]. This consists of a class of maps discussed in Section 4.1, and a renormalisation operator acting on that class discussed in Section 4.2. Our presentation of the renormalisation process is slightly different from the one by Inou and Shishikura, but produces the same map.

4.1. **Inou-Shishikura class of maps.** Let $\hat{\mathbb{C}}$ denote the Riemann sphere. Consider the filled-in ellipse

$$E = \left\{x + iy \in \mathbb{C} \;\Big|\; \left(\frac{x+0.18}{1.24}\right)^2 + \left(\frac{y}{1.04}\right)^2 \leq 1\right\},$$

and the domain

$$U = g(\hat{\mathbb{C}} \setminus E), \text{ where } g(z) = \frac{-4z}{(1+z)^2}. \tag{4.1}$$

The domain U is simply connected and contains 0.

[4]Here, $\mathbb{Z}$ acts on M_{-1} through horizontal translations by integers.

The restriction of the polynomial $P(z) = z(1+z)^2$ on U has a specific covering structure which plays a central role in this renormalisation scheme. The polynomial P has a parabolic fixed point at 0 with multiplier $P'(0) = 1$. It has a simple critical point at $\mathrm{cp}_P = -1/3 \in U$ and a critical point of order two at $-1 \in \mathbb{C} \setminus \overline{U}$. The critical point $-1/3$ is mapped to $\mathrm{cv}_P = -4/27 \in U$, and -1 is mapped to 0.

Let $\mathcal{IS}$ denote the class of all maps of the form

$$h = P \circ \varphi^{-1} \colon U_h \to \mathbb{C}$$

where

(i) $\varphi \colon U \to U_h$ is holomorphic, one-to-one, onto; and
(ii) $\varphi(0) = 0$ and $\varphi'(0) = 1$.

By (i), every map in $\mathcal{IS}$ has the same covering structure on its domain as the one of P on U. By (ii), every map in $\mathcal{IS}$ has a fixed point at 0 with multiplier $+1$, and a unique critical point at $\mathrm{cp}_h = \varphi(-1/3) \in U_h$ which is mapped to $\mathrm{cv}_h = -4/27$.

For $\alpha \in \mathbb{R}$, let $R_\alpha(z) = e^{2\pi\alpha i} z$. Define

$$\mathcal{IS}_\alpha = \{h \circ R_\alpha \mid h \in \mathcal{IS}\}.$$

We continue to use the notation U_h to denote the domain of definition of $h \in \mathcal{IS}_\alpha$. That is, if $h = f \circ R_\alpha$ with $f \in \mathcal{IS}$, then $U_h = R_\alpha^{-1}(U_f)$.

We equip $\cup_{\alpha \in \mathbb{R}} \mathcal{IS}_\alpha$ with the topology of uniform convergence on compact sets. That is, given a map $h : U_h \to \mathbb{C}$, a compact set $K \subset U_h$, and an $\varepsilon > 0$, a neighbourhood of h (in the compact-open topology) is defined as the set of maps $g \in \cup_{\alpha \in \mathbb{R}} \mathcal{IS}_\alpha$ such that $K \subset U_g$ and for all $z \in K$ we have $|g(z) - h(z)| < \varepsilon$. There is a one-to-one correspondence between $\mathcal{IS}$ and the space of normalised univalent maps on the unit disk. By the Koebe distortion theorem [Dur83, Thm 2.5], for any compact set $A \subset \mathbb{R}$, the set $\cup_{\alpha \in A} \mathcal{IS}_\alpha$ is pre-compact in this topology.

We normalise the family of quadratic polynomials by placing a fixed point at 0 and the finite critical value at $-4/27$;

$$Q_\alpha(z) = e^{2\pi\alpha i} z + \frac{27}{16} e^{4\pi\alpha i} z^2.$$

Then, Q_α has a fixed point at 0 with multiplier $e^{2\pi\alpha i}$, and a critical point at $-8e^{-2\pi\alpha i}/27$ which is mapped to $-4/27$. We shall use the notation

$$\mathcal{QIS}_\alpha = \mathcal{IS}_\alpha \cup \{Q_\alpha\}.$$

When $h = Q_\alpha$, we set $U_h = \mathbb{C}$.

Let $h = h_0 \circ R_\alpha \in \mathcal{IS}_\alpha$ with $h_0 \in \mathcal{IS}$ and $\alpha \in \mathbb{R}$. The map h_0 has a double fixed point at 0. For α small enough and non-zero, h is a small perturbation of h_0, and hence, it has a non-zero fixed point near 0 which has split from 0 at $\alpha = 0$. We denote this fixed point by σ_h. It follows that σ_h depends continuously on h_0 and α, with asymptotic expansion $\sigma_h = -4\pi\alpha i / h_0''(0) + o(\alpha)$, as α tends to 0. Evidently, $\sigma_h \to 0$ as $\alpha \to 0$.

Proposition 4.1 ([IS06]). *There is $r_1 > 0$ such that for every $h \colon U_h \to \mathbb{C}$ in $\mathcal{QIS}_\alpha$ with $\alpha \in (0, r_1]$, there exist a simply connected domain $\mathcal{P}_h \subset U_h$ and a univalent map $\Phi_h \colon \mathcal{P}_h \to \mathbb{C}$ satisfying the following properties:*

(a) *$\mathcal{P}_h$ is bounded by piecewise smooth curves and is compactly contained in U_h;*
(b) *cp_h, 0, and σ_h belong to the boundary of $\mathcal{P}_h$, while cv_h belongs to the interior of $\mathcal{P}_h$;*
(c) *$\Phi_h(\mathcal{P}_h)$ contains the set $\{w \in \mathbb{C} \mid \operatorname{Re} w \in (0, 1]\}$;*
(d) *$\Phi_h(\mathrm{cv}_h) = 1$, $\operatorname{Im} \Phi_h(z) \to +\infty$ as $z \to 0$ in $\mathcal{P}_h$, and $\operatorname{Im} \Phi_h(z) \to -\infty$ as $z \to \sigma_h$ in $\mathcal{P}_h$;*
(e) *If z and $h(z)$ belong to $\mathcal{P}_h$, then*

$$\Phi_h(h(z)) = \Phi_h(z) + 1;$$

(f) *the induced map $\Phi_h : \mathcal{P}_h/\sim \to \mathbb{C}/\mathbb{Z}$, where $z \sim h(z)$, is a biholomorphism;*

(g) *Φ_h is unique, and depends continuously on h.*

The class $\mathcal{IS}$ is denoted by $\mathcal{F}_1$ in [IS06]. One may refer to Theorem 2.1 as well as Main Theorems 1 and 3 in [IS06], for further details on the above proposition. A precise value for r_1 is not known to date.

There are fundamental geometric properties of $\mathcal{P}_h$ and Φ_h that are crucial for applications of near-parabolic renormalisation scheme. We state these in the next proposition.

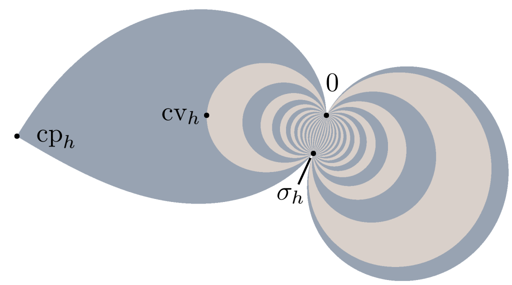

FIGURE 4. The domain $\mathcal{P}_h$ and the special points associated to some $h \in \mathcal{IS}_\alpha$. The alternating coloured croissants are the pre-images of vertical strips of width one by Φ_h.

Proposition 4.2. *There exist $r_2 \in (0, r_1]$, as well as integers $c_1 \leq 1/r_2 - 3/2$ and c_2 such that for every map $h\colon U_h \to \mathbb{C}$ in $\mathcal{QIS}_\alpha$ with $\alpha \in (0, r_2]$, the domain $\mathcal{P}_h \subset U_h$ in Prop 4.1 may be chosen to satisfy the additional properties:*

(a) *there exists a continuous branch of argument defined on $\mathcal{P}_h$ such that*
$$\max_{w,w' \in \mathcal{P}_h} |\arg(w) - \arg(w')| \leq 2\pi c_2,$$

(b) $\Phi_h(\mathcal{P}_h) = \{w \in \mathbb{C} \mid 0 < \mathrm{Re}(w) < \alpha^{-1} - c_1\}$.

See [Che13, Prop. 2.4] or [BC12, Prop. 12] for proofs. The map $\Phi_h : \mathcal{P}_h \to \mathbb{C}$ is called the perturbed Fatou coordinate or simply the **Fatou coordinate** of h. See Figure 4.

4.2. **Near-parabolic renormalisation operator.** Let $h\colon U_h \to \mathbb{C}$ be a map in $\mathcal{QIS}_\alpha$ with $\alpha \in (0, r_2]$, and let $\Phi_h \colon \mathcal{P}_h \to \mathbb{C}$ denote the Fatou coordinate of h introduced in the previous section. Let

$$\begin{aligned} A_h &= \{z \in \mathcal{P}_h : 1/2 \leq \mathrm{Re}(\Phi_h(z)) \leq 3/2\,,\ -2 \leq \mathrm{Im}\,\Phi_h(z) \leq 2\}, \\ B_h &= \{z \in \mathcal{P}_h : 1/2 \leq \mathrm{Re}(\Phi_h(z)) \leq 3/2\,,\ 2 \leq \mathrm{Im}\,\Phi_h(z)\}. \end{aligned} \tag{4.2}$$

By Prop 4.1, cv$_h$ belongs to the interior of A_h and 0 belongs to the boundary of B_h. See Figure 5.

It follows from [IS06] (see Rem 4.4 below) that there is a positive integer k_h, depending on h, such that the following four properties hold:

(i) For every integer k, with $1 \leq k \leq k_h$, there exists a unique connected component of $h^{-k}(B_h)$ which is compactly contained in U_h and contains 0 on its boundary. We denote this component by B_h^{-k}.

(ii) For every integer k, with $1 \leq k \leq k_h$, there exists a unique connected component of $h^{-k}(A_h)$ which has non-empty intersection with B_h^{-k}, and is compactly contained in U_h. This component is denoted by A_h^{-k}.

(iii) The sets $A_h^{-k_h}$ and $B_h^{-k_h}$ are contained in
$$\{z \in \mathcal{P}_h \mid 1/2 < \operatorname{Re}\Phi_h(z) < 1/\alpha - c_1\}.$$

(iv) The maps $h : A_h^{-k} \to A_h^{-k+1}$, for $2 \leq k \leq k_h$, and $h : B_h^{-k} \to B_h^{-k+1}$, for $1 \leq k \leq k_h$, are one-to-one. The map $h : A_h^{-1} \to A_h$ is a degree two proper branched covering.

Assume that k_h is the smallest positive integer for which the above properties hold. Define
$$S_h = A_h^{-k_h} \cup B_h^{-k_h}.$$

Proposition 4.3. *The is a constant $k \in \mathbb{Z}$ such that for all $h \in \mathcal{QIS}$, $k_h \leq k$.*

See [Che13] or [Che19] for the proof of the above proposition.

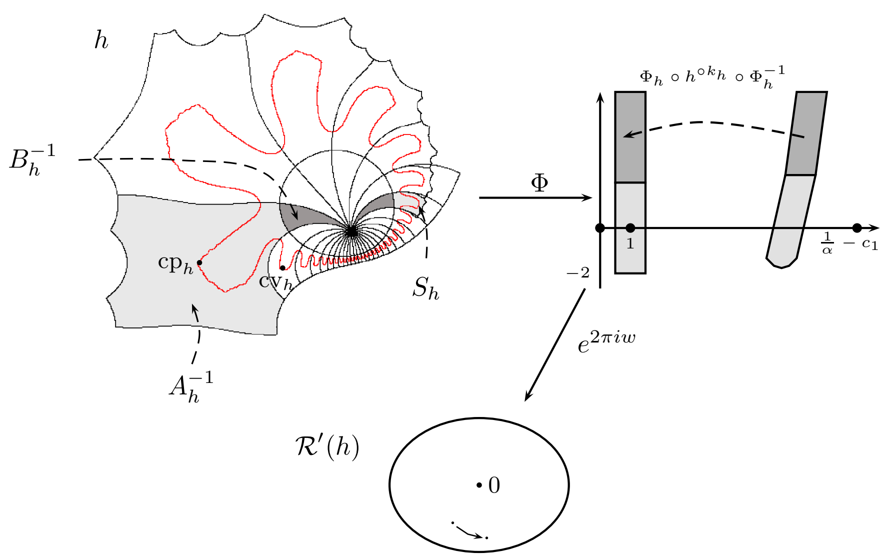

FIGURE 5. Illustration of the sets A_h, B_h,..., $A_h^{-k_h}$, $B_h^{-k_h}$, and the sector S_h. The amoeba shaped red curve denotes a large number of iterates of cp_h under h.

Since $h^{\circ k_h} : S_h \to A_h \cup B_h$, the composition

$$E_h = \Phi_h \circ h^{\circ k_h} \circ \Phi_h^{-1} : \Phi_h(S_h) \to \mathbb{C} \tag{4.3}$$

is a well-defined map. Moreover, by Prop 4.1-(e), $E_h(w+1) = E_h(w) + 1$, when both w and $w+1$ belong to the closure of $\Phi_h(S_h)$.

Consider the covering map

$$\mathbb{E}\mathrm{xp}(w) = (-4/27)e^{2\pi i w}. \tag{4.4}$$

Since E_h commutes with the translation by $+1$, it induces via $\mathbb{E}\mathrm{xp}$ a unique map $\mathcal{R}(h)$ defined on a set containing a punctured neighbourhood of 0. Since $\mathcal{R}(h)(z) \to 0$ as $z \to 0$, 0 is a removable singularity of $\mathcal{R}(h)$. Basic calculation shows that near 0,
$$\mathcal{R}(h)(z) = e^{2\pi \frac{-1}{\alpha} i} z + O(z^2).$$

The map $\mathcal{R}(h)$, restricted to the interior of $\frac{-4}{27} e^{2\pi i(\Phi_h(S_h))}$, is called the **near-parabolic renormalisation** of h. We may simply refer to this operator as **renormalisation**.

Note that $\Phi_h(\mathrm{cv}_h) = +1$, and the projection $w \mapsto \frac{-4}{27} e^{2\pi i w}$ maps integers to $-4/27$. Hence, the critical value of $\mathcal{R}(h)$ is placed at $-4/27$. See Figure 5. It is also worth noting that the action of the renormalisation on the asymptotic rotation number at 0 is

$$\alpha \mapsto -1/\alpha \mod \mathbb{Z}.$$

Remark 4.4. Inou and Shishikura give a somewhat different definition of this renormalisation operator using slightly different regions A_h and B_h compared to the ones here. However, the two processes produce the same map $\mathcal{R}(h)$ modulo their domains of definition. More precisely, there is a natural extension of Φ_h onto the sets $A_h^{-k} \cup B_h^{-k}$, for $0 \leq k \leq k_h$, such that each set $\Phi_h(A_h^{-k} \cup B_h^{-k})$ is contained in the union

$$D^\sharp_{-k} \cup D_{-k} \cup D''_{-k} \cup D'_{-k+1} \cup D_{-k+1} \cup D^\sharp_{-k+1}$$

in the notations used in [IS06, Section 5.A].

Consider the domain

$$V = P^{-1}\big(B(0, 4e^{4\pi}/27)\big) \setminus \big((-\infty, -1] \cup B\big) \tag{4.5}$$

where B is the component of $P^{-1}(B(0, 4e^{-4\pi}/27))$ containing -1. By an explicit calculation (see [IS06, Prop. 5.2]) one can see that the closure of U is contained in the interior of V. See Figure 6.

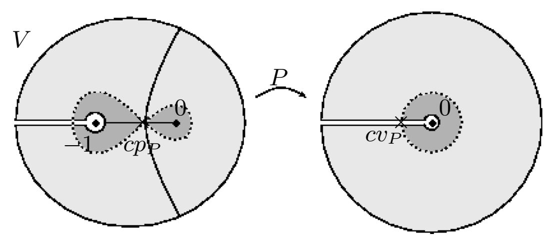

FIGURE 6. Covering structure of the polynomial P; similar colors and line styles are mapped on one another.

A remarkable result by Inou and Shishikura [IS06, Main thm 3] guarantees that the renormalisation $\mathcal{R}$ is defined at all maps in $\mathcal{QIS}_\alpha$, provided α is small enough. Moreover, it produces a map with the same covering structure as the one of P on U.

Theorem 4.5 (Inou-Shishikura)**.** *There exist $r_3 \in (0, r_2]$ such that if $h \in \mathcal{QIS}_\alpha$ with $\alpha \in (0, r_3]$, then $\mathcal{R}(h)$ is well-defined and belongs to the class $\mathcal{IS}_{-1/\alpha}$. That is, there exists a one-to-one holomorphic map $\psi : U \to \mathbb{C}$ with $\psi(0) = 0$ and $\psi'(0) = 1$ so that*

$$\mathcal{R}(h)(z) = P \circ \psi^{-1}(e^{-2\pi i/\alpha} z),\ \forall z \in \psi(U) \cdot e^{2\pi i/\alpha}.$$

Furthermore, $\psi : U \to \mathbb{C}$ extends to a univalent map on V.

Thm 4.5 is a refinement of earlier constructions by Shishikura [Shi98] and Lavaurs [Lav89]. The theorem provides a powerful tool to study the dynamics of the underlying maps. It has been used recently to obtain a number of significant results, notably, [BC12, Che19, Che13, AC18, CC15, CS15, SY18]. All the main results in the introduction, along many other technical statements proved along the way apply to all the maps in $\mathcal{IS}_\alpha$, provided α is of high type.

Slightly modified renormalisation schemes have been constructed for uni-singular maps in [Che14], and for cubic maps in [Yan15]. It is likely that the same line of argument presented here can be applied in those cases, in order to prove similar results for those classes of maps.

5. Comparing the changes of coordinates

Given $\alpha \in (0, r_3]$ and $h \in \mathcal{QIS}_\alpha$, the change of coordinate $\mathbb{E}\mathrm{xp} \circ \Phi_h$ relates the iterates of h to the iterates of $\mathcal{R}(h)$. When studying repeated renormalisations, one needs to study long compositions of such changes of coordinates. It is convenient to consider the inverse maps, that is, maps of the form $\mathbb{E}\mathrm{xp}^{-1} \circ \Phi_h^{-1}$, where $\mathbb{E}\mathrm{xp}^{-1}$ is a suitable inverse branch of $\mathbb{E}\mathrm{xp}$ on $\mathcal{P}_h$. In this section we aim to show that $\mathbb{E}\mathrm{xp}^{-1} \circ \Phi_h^{-1}$ behaves like Y_α.

5.1. **Change of coordinates.** Let $\alpha \in (0, r_3]$ and $h \in \mathcal{QIS}_\alpha$. Recall that by Propositions 4.1 and 4.2, there is a simply connected Jordan domain $\mathcal{P}_h \subset \mathrm{Dom}\, h$ and a univalent map

$$\Phi_h : \mathcal{P}_h \to \{w \in \mathbb{C} \mid 0 \leq \mathrm{Re}\, w \leq 1/\alpha - c_1\}.$$

The fundamental functional equation for Φ_h in Prop 4.1-(e) allows one to extend Φ_h^{-1} onto larger domains, using the iterates of h. We discuss this below.

Recall that by the definition of renormalisation in Section 4.2, there is a domain S_h and an integer k_h such that $S_h \subset \mathcal{P}_h$ and $h^{\circ k_h}(S_h) \subset \mathcal{P}_h$. Consider the set

$$\Pi_h = \{w \in \mathbb{C} \mid 0 < \mathrm{Re}\, w \leq 1/\alpha - c_1, \mathrm{Im}\, w \geq -2\} \cup \{\Phi_h(S_h) + l \mid l \in \mathbb{Z}, 0 \leq l \leq k_h\}.$$

There is a holomorphic map $\Phi_h^{-1} : \Pi_h \to \mathbb{C} \setminus \{0\}$, which matches Φ_h^{-1} on $\Phi_h(\mathcal{P}_h)$. For $w \in \Phi_h(S_h) + l$ one defines $\Phi_h^{-1}(w) = h^{\circ l} \circ \Phi_h^{-1}(w - l)$. Since, $\Phi_h^{-1}(w - l) \in S_h$ and $0 \leq l \leq k_h$, $h^{\circ l} \circ \Phi_h^{-1}(w - l)$ is defined. It follows from the functional equation in Prop 4.1-(e) that this is a well-defined holomorphic map, which satisfies $\Phi_h^{-1}(w + 1) = h \circ \Phi_h^{-1}(w)$ whenever both sides are defined. However, $\Phi_h^{-1} : \Pi_h \to \mathbb{C} \setminus \{0\}$ is not univalent any more. It has a critical point which is mapped to $-4/27$.

We may lift the map $\Phi_h^{-1} : \Pi_h \to \mathbb{C} \setminus \{0\}$ via the covering map $\mathbb{E}\mathrm{xp} : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ to define the holomorphic map

$$\Upsilon_h = \mathbb{E}\mathrm{xp}^{-1} \circ \Phi_h^{-1} : \Pi_h \to \mathbb{C}, \quad \Upsilon_h(+1) = +1. \tag{5.1}$$

5.2. **Estimates on Fatou coordinate.** To understand the behaviour of Υ_h, we need some estimates on Φ_h. However, estimates on Φ_h are non-trivial to obtain, and often have significant consequences, see for instance [Shi98]. A general idea is to compare Φ_h to an explicit formula, as we explain below.

Recall that h has two fixed points on $\partial \mathcal{P}_h$; 0 and σ_h. Consider the covering map

$$\tau_h(\zeta) = \frac{\sigma_h}{1 - e^{-2\pi\alpha i \zeta}} : \mathbb{C} \to \hat{\mathbb{C}} \setminus \{0, \sigma_h\}.$$

Note that $\tau_h(\zeta + 1/\alpha) = \tau_h(\zeta)$. Moreover, as $\mathrm{Im}\, \zeta \to +\infty$, $\tau_h(\zeta) \to 0$, and as $\mathrm{Im}\, \zeta \to -\infty$, $\tau_h(\zeta) \to \sigma_h$.

We may lift $\Phi_h^{-1} : \Pi_h \to \mathbb{C} \setminus \{0, \sigma_h\}$ via the covering map $\tau_h : \mathbb{C} \to \hat{\mathbb{C}} \setminus \{0, \sigma_h\}$, to define a holomorphic map

$$L_h : \Pi_h \to \mathbb{C}.$$

That is, $\tau_h \circ L_h = \Phi_h^{-1}$ on Π_h. However, this map is only determined up to translation by elements of $\mathbb{Z}/\alpha$. Below we make a unique choice for this lift.

Note that $\Phi_h^{-1}(\Pi_h) \subset \mathbb{C}$, and $\tau_h(\mathbb{Z}/\alpha)$ is the point at infinity of $\hat{\mathbb{C}}$. Hence, $L_h(\Pi_h) \cap (\mathbb{Z}/\alpha) = \emptyset$. On the other hand, the simply connected region $\mathcal{P}_h$ lifts under τ_h to a periodic set of period $\mathbb{Z}/\alpha$. Each connected component of $\tau_h^{-1}(\mathcal{P}_h)$ is a simply connected region in $\mathbb{C} \setminus (\mathbb{Z}/\alpha)$, which spreads from $+i\infty$ to $-i\infty$. We choose the lift L_h so that $L_h(\Pi_h)$ separates 0 from $1/\alpha$.

Estimates on L_h lead to estimates on Φ_h through the explicit formula τ_h. In the following proposition, we collect some estimates on L_h. One may refer to [Che13, Section 5] and [Che19, Section 5] for a detailed study of estimates on L_h.

Proposition 5.1. *There is a constant C_1 such that for all $\alpha \in (0, r_3]$ and all $h \in \mathcal{QIS}_\alpha$, we have*

(i) *for all* $w \in \Pi_h$, $|L_h(w) - w| \leq C_1 \log(1 + \alpha^{-1})$,
(ii) *for all* $w \in \Pi_h$, $|L_h(w) - w| \leq C_1 \log(1 + d(w, \mathbb{Z}/\alpha))$,
(iii) *for all* $w \in \Pi_h$ *with* $\operatorname{Im} w \geq 1$, $|L'_h(w) - 1| \leq C_1/\operatorname{Im} w$, [5]
(iv) *as* $\operatorname{Im} w \to +\infty$ *in* Π_h, $L_h(w) - w$ *tends to a constant,*
(v) *for all* $w \in \Pi_h$ *with* $\operatorname{Re} w \geq 1/2$, $|L'_h(w)| \leq C_1$.

By part (iii) of the above Proposition, and differentiation of the explicit formula τ_h, we get

$$\lim_{\operatorname{Im} w \to +\infty, w \in \Pi_h} \Upsilon'_h(w) = \alpha. \tag{5.2}$$

5.3. **Dropping the non-linearity.** We aim to compare Υ_h to Y_α, but a priori these maps have different domains of definition. Below, we state a general form of such estimates, and later apply it to more specific domains. Recall that Y_α is defined on $\mathbb{H}' = \{w \in \mathbb{C} \mid \operatorname{Im} w > -1\}$.

Proposition 5.2. *There is a constant* C_3 *such that for all* $\alpha \in (0, r_3]$, *all* $h \in \mathcal{QIS}_\alpha$, *all* $w_1 \in \mathbb{H}'$, *and all* $w_2 \in \Pi_h$, *we have*

$$|\Upsilon_h(w_2) - Y_\alpha(w_1)| \leq C_3 \max\{1, |w_1 - w_2|\}.$$

Proof. We shall use the decomposition of Υ_h as $\mathbb{E}\mathrm{xp}^{-1} \circ \tau_h \circ L_h$ on Π_h. Let $g_h = \mathbb{E}\mathrm{xp}^{-1} \circ \tau_h$; the explicit part of Υ_h. First we compare g_h to Y_α. That is, there is a constant D_1, independent of α and h, such that for every $w_1 \in \mathbb{H}'$, we have

$$|g_h(w_1 + 1/2 + 3i/2) - Y_\alpha(w_1)| \leq D_1. \tag{5.3}$$

We estimate the imaginary and real parts separately. For the imaginary part we have

$$|\operatorname{Im} g_h(w_1 + 1/2 + 3i/2) - \operatorname{Im} Y_\alpha(w_1)| = \frac{1}{2\pi} \log \Big| \frac{27 e^{-3\pi\alpha} \sigma_h}{4(e^{-3\pi\alpha} - e^{\pi\alpha i})} \Big|$$

Using the Koebe distortion theorem, one may see that $\{h''(0) \mid h \in \mathcal{IS}\}$ is relatively compact in $\mathbb{C} \setminus \{0\}$, see [IS06] for more details. This implies that there is a constant D, independent of α and h, such that $\alpha/D \leq |\sigma_h| \leq D\alpha$. Note that for all $\alpha \in (0, 1/2]$, we have

$$|e^{-3\pi\alpha} - e^{\pi\alpha i}| \leq |e^{-3\pi\alpha} - 1| + |1 - e^{\pi\alpha i}| \leq 3\pi\alpha + \pi\alpha = 4\pi\alpha, \tag{5.4}$$

and

$$|e^{-3\pi\alpha} - e^{\pi\alpha i}| \geq |\operatorname{Im}(e^{-3\pi\alpha} - e^{\pi\alpha i})| = \sin(\pi\alpha) \geq \pi\alpha/2. \tag{5.5}$$

These imply that $1/(4\pi D) \leq |\sigma_h/(e^{-3\pi\alpha} - e^{\pi\alpha i})| \leq 2D/\pi$. Note that $e^{-3\pi/2} \leq e^{-3\pi\alpha} \leq 1$. Combining these estimates, we note that $|\operatorname{Im} g_h(w_1 + 1/2 + 3i/2) - \operatorname{Im} Y_\alpha(w_1)|$ is uniformly bounded from above, independent of α, h, and w_1.

On the other hand, g_h maps the set $\{w \in \mathbb{C} \mid 0 \leq \operatorname{Re} w \leq 1/\alpha, \operatorname{Im} w \geq 1/2\}$ into a vertical strip of width $+1$ whose projection onto the real axis contains $+1$. Similarly, Y_α maps $\{w \in \mathbb{H}' \mid 0 \leq \operatorname{Re} w \leq 1/\alpha\}$ into the vertical strip $\{w \in \mathbb{C} \mid 0 \leq \operatorname{Re} w \leq 1\}$. Using $Y_\alpha(w + 1/\alpha) = Y_\alpha(w) + 1$ and $g_h(w + 1/\alpha) = g_h(w) + 1$, one concludes that

$$|\operatorname{Re} g_h(w_2) - \operatorname{Re} Y_\alpha(w_1)| = |\operatorname{Re} g_h(w_2) - \operatorname{Re} Y_\alpha(w_2)| + |\operatorname{Re} Y_\alpha(w_2) - \operatorname{Re} Y_\alpha(w_1)| \leq 2 + \alpha/2 \leq 9/4.$$

This completes the proof of the first step; the existence of D_1.

The next step is to show that there is a constant D_2, independent of α and h, such that for all $w_3 \in \Pi_h$ and all $w_4 \in \mathbb{C}$ with $\operatorname{Im} w_4 \geq 1/2$ we have

$$\big| g_h \circ L_h(w_3) - g_h(w_4) \big| \leq D_2 \max\{1, |w_3 - w_4|\}. \tag{5.6}$$

[5] Indeed, an exponentially decaying estimate on $L'_h(w) - 1$ is established in [Che13]. The estimate $O(1/\operatorname{Im} w)$ may be obtained using the classical Koebe distortion theorem.

The above inequality follows from the bounds on $L_h(w) - w$ in Prop 5.1 and elementary estimates on τ_h. We break the details into three cases. Recall the constant C_1 from Prop 5.1, and choose a constant D such that $D/\alpha - C_1 \log(1 + 1/\alpha) \geq 1/\alpha$.

Assume that $\operatorname{Im} w_3 \geq D/\alpha$. By Prop 5.1-(i), $|L_h(w_3) - w_3| \leq C_1 \log(1 + 1/\alpha)$, and hence $\operatorname{Im} L_h(w_3) \geq 1/\alpha$. By elementary calculations one may see that for $\operatorname{Im} w \geq 1/\alpha$, $|\tau'(w)| = O(\alpha)$, and for $\operatorname{Im} w \geq 1/2$ we have $|\tau_h'(w)| = O(1)$. Then, using $\alpha \leq 1/2$, we obtain

$$\begin{aligned}|g_h \circ L_h(w_3) - g_h(w_4)| &\leq |g_h \circ L_h(w_3) - g_h(w_3)| + |g_h(w_3) - g_h(w_4)| \\ &\leq O(\alpha) \cdot |L_h(w_3) - w_3| + O(1) \cdot |w_3 - w_4| \\ &\leq O(\alpha) \cdot O(\log(1/\alpha)) + O(|w_3 - w_4|) \leq O(1) + O(|w_3 - w_4|).\end{aligned}$$

Assume that $1/2 \leq \operatorname{Im} w_3 \leq D/\alpha$. By Prop 5.1, $|L_h(w_3) - w_3| \leq C_1 \log(1 + d(w_3, \mathbb{Z}/\alpha))$. On the other hand, by elementary calculations, for $1/2 \leq \operatorname{Im} w \leq D/\alpha$ we have $|\tau'(w)| = O(1/d(w, \mathbb{Z}/\alpha))$. Since, $\log(1 + d(w, \mathbb{Z}/\alpha)) \cdot 1/d(w, \mathbb{Z}/\alpha)$ is uniformly bounded from above, we conclude that $|g_h \circ L_h(w_3) - g_h(w_3)|$ is uniformly bounded from above. As in the above equation, one obtains the desired inequality in this case as well.

Assume that $\operatorname{Im} w_3 \leq 1/2$. Let $w_3' = \operatorname{Re} L_h(w_3) + i/2$. Then, $|w_3 - w_3'|$ is uniformly bounded, and hence

$$|g_h \circ L_h(w_3)| - g_h(w_4)| \leq |g_h \circ L_h(w_3)| - g_h(w_3')| + |g_h(w_3') - g_h(w_4)| = O(1) + O(|w_3 - w_4|).$$

We combine (5.3) and (5.6) with the triangle inequality

$$\left|\Upsilon_h(w_2) - Y_\alpha(w_1)\right| \leq \left|g_h \circ L_h(w_2) - g_h(w_1 + 1/2 + 3i/2)\right| + \left|g_h(w_1 + 1/2 + 3i/2) - Y_\alpha(w_1)\right|$$

to deduce the inequality in the proposition. □

Proposition 5.3. *For all* $\alpha \in (0, r_3]$ *and all* $h \in \mathcal{QIS}_\alpha$,

$$\lim_{\operatorname{Im} w \to +\infty; w \in \Pi_h} (\Upsilon_h(w) - Y_\alpha(w)),$$

exists and is finite.

Proof. As in the proof of the previous proposition, we use the decomposition of Υ_h as $\mathbb{E}\mathrm{xp}^{-1} \circ \tau_h \circ L_h$ on Π_h. Let $g_h = \mathbb{E}\mathrm{xp}^{-1} \circ \tau_h$. By elementary calculations one may see that $g_h(w) - Y_\alpha(w)$ tends to a finite constant as $\operatorname{Im} w \to +\infty$. Also, for all w_1 and $w_2 \in \mathbb{C}$, $g_h(w_1 + w_2) - g_h(w_1)$ tends to a finite constant, as $\operatorname{Im} w_1 \to +\infty$. On the other hand, by Prop 5.1-(iv), $L_h(w) - w$ tends to a constant, as $\operatorname{Im} w \to +\infty$ within Π_h. □

6. Marked critical curve

Later in Sec. 7 we will build a nest of domains shrinking to the post-critical set in the same fashion that we built the sets M_n^j shirking to the topological model in Section 3. The invariance of the vertical line $i[0, +\infty)$ under the model maps Y_n plays a crucial role in building the sets M_n^j. In this section we identify a simple curve, with a special parametrisation (marking), which satisfies an equivariant property under the renormalisation change of coordinates Υ_h. This curve connects $+1$ to $+i\infty$ and plays the role of the vertical line $i[0, +\infty)$ for the maps Y_n. As we shall see, this curve is mapped by Φ_h^{-1} to a Jordan curve connecting the critical value of h at $-4/27$ to the fixed point at 0, and projects under $\mathbb{E}\mathrm{xp}$ to a Jordan curve connecting the critical value of $\mathcal{R}(h)$ at $-4/27$ to 0. All iterates of these curves under h (and $\mathcal{R}(h)$, respectively) are pairwise disjoint.

6.1. **Repeated renormalisations.** Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, and α_i, for $i \geq 0$, denote the sequence generated by the modified continued fraction algorithm in Sec. 2.1. Recall the complex conjugation map $s(z) = \overline{z}$.

Proposition 6.1. *There exists a positive integer N such that for all $\alpha \in \mathrm{HT}_N$ and all $f \in \mathcal{QIS}_\alpha$, there is a sequence of maps f_n, for $n \geq 0$, satisfying*

$$f_0 = \begin{cases} f & \text{if } \varepsilon_0 = +1, \\ s \circ f \circ s & \text{if } \varepsilon_0 = -1, \end{cases} \qquad f_{n+1} = \begin{cases} \mathcal{R}(f_n) & \text{if } \varepsilon_{n+1} = -1, \\ s \circ \mathcal{R}(f_n) \circ s & \text{if } \varepsilon_{n+1} = +1. \end{cases}$$

Moreover, $f_0 \in \mathcal{QIS}_{\alpha_0}$ and $f_n \in \mathcal{IS}_{\alpha_n}$, for $n \geq 1$;

$$f_n : \mathrm{Dom}\, f_n \to \mathbb{C}, \quad f_n(0) = 0, \quad f_n'(0) = e^{2\pi i \alpha_n}.$$

Proof. Let $N \geq 1/r_3 + 1/2$, where r_3 is the constant in Thm 4.5. Fix an arbitrary $\alpha = a_{-1} + \varepsilon_0/(a_0 + \varepsilon_1/(a_1 + \dots)) \in \mathrm{HT}_N$, that is, $a_i \geq N$, for all $i \geq 0$. Then, for all $i \geq 0$,

$$1/\alpha_i = a_i + \varepsilon_{i+1}\alpha_{i+1} \geq N - 1/2 \geq 1/r_3.$$

Thus, $\alpha \in \mathrm{HT}_N$ implies that for all $i \geq 0$, $\alpha_i \in (0, r_3]$.

We note that if $h \in \mathcal{QIS}_\alpha$, $s \circ h \circ s \in \mathcal{QIS}_{-\alpha}$. To see this, first assume that $h \in \mathcal{IS}_\alpha$, with $h(z) = P \circ \psi^{-1}(e^{2\pi i \alpha} z)$. Since, $s \circ P = P \circ s$,

$$s \circ h \circ s(z) = s \circ P \circ \psi^{-1}(e^{2\pi i\alpha}\overline{z}) = P \circ s \circ \psi^{-1} \circ s(e^{-2\pi i \alpha} z).$$

Recall that the domain U in the definition of $\mathcal{IS}$ is symmetric with respect to the real line, that is, $s(U) = U$. Then, $s \circ \psi \circ s : U \to \mathbb{C}$ is holomorphic, maps 0 to 0, and has derivative +1 at 0. This shows that $s \circ h \circ s$ belongs to $\mathcal{IS}_{-\alpha}$. Evidently, $s \circ Q_\alpha \circ s = Q_{-\alpha}$.

We define the maps f_n by induction on n. If $\varepsilon_0 = +1$, then $\alpha = a_{-1} + \alpha_0$ with $\alpha_0 \in (0, r_3]$. Hence, $f_0 = f \in \mathcal{QIS}_\alpha = \mathcal{QIS}_{\alpha_0}$. If $\varepsilon_0 = -1$, then $\alpha = a_{-1} - \alpha_0$ with $\alpha_0 \in (0, r_3]$. Then $f \in \mathcal{QIS}_\alpha = \mathcal{QIS}_{-\alpha_0}$, and by the above paragraph, $f_0 = s \circ f \circ s \in \mathcal{QIS}_{\alpha_0}$.

Now assume that f_n is defined and belongs to $\mathcal{IS}_{\alpha_n}$. Since $\alpha_n \in (0, r_3]$, by Thm 4.5, $\mathcal{R}(f_n)$ is defined and belongs to $\mathcal{IS}_{-1/\alpha_n}$. Recall that $-1/\alpha_n = -a_n - \varepsilon_{n+1}\alpha_{n+1}$, which gives $\mathcal{R}(f_n) \in \mathcal{IS}_{-\varepsilon_{n+1}\alpha_{n+1}}$. If $\varepsilon_{n+1} = +1$, by the above paragraph, $f_{n+1} = s \circ \mathcal{R}(f_n) \circ s \in \mathcal{IS}_{\alpha_{n+1}}$. If $\varepsilon_{n+1} = -1$, then $f_{n+1} = \mathcal{R}(f_n) \in \mathcal{IS}_{\alpha_{n+1}}$. □

6.2. **Successive changes of coordinates.** Recall the set Π_h, $\Phi_h^{-1} : \Pi_h \to \mathbb{C} \setminus \{0\}$, and $\Upsilon_h : \Pi_h \to \mathbb{C}$ defined in Section 5.1. For $n \geq 0$, we use the notations

$$\Pi_n = \Pi_{f_n}, \quad \Phi_n^{-1} = \Phi_{f_n}^{-1} : \Pi_n \to \mathbb{C} \setminus \{0\},$$

$$\Upsilon_n = \begin{cases} \mathbb{E}\mathrm{xp}^{-1} \circ \Phi_n^{-1} & \text{if } \varepsilon_n = -1, \\ \mathbb{E}\mathrm{xp}^{-1} \circ s \circ \Phi_n^{-1} & \text{if } \varepsilon_n = +1, \end{cases} \tag{6.1}$$

with the normalisations

$$\Upsilon_n(+1) = +1. \tag{6.2}$$

Note that each Υ_n is either holomorphic or anti-holomorphic.

Consider the set

$$\Pi = \{w \in \mathbb{C} \mid 1/2 \leq \mathrm{Re}\, w \leq 3/2, \mathrm{Im}\, w \geq -2\}.$$

Lemma 6.2. *There is a constant $\delta_1 > 0$ such that for all $\alpha \in (0, r_3]$ and all $h \in \mathcal{QIS}_\alpha$, we have*

$$B_{\delta_1}(\Upsilon_h(\Pi)) \subseteq \Pi.$$

Proof. By [IS06], for every $h \in \mathcal{QIS}_0$, the sets A_h^{-k} and B_h^{-k} are defined for all $k \geq 0$. Indeed, for large enough k, $A_h \cup B_h$ is contained in the repelling Fatou coordinate of h. Comparing to their notations, $A_h \cup B_h$ is contained in the union

$$\psi_0(D_0) \cup \psi_0(D_1) \cup \psi_0(D_0^\sharp) \cup \psi_0(D_1^\sharp),$$

where $\psi_0(z) = -4/z$. See Section 5.A–Outline of the proof in [IS06]. They prove in Propositions 5.6 and 5.7 that the closure of the set $D_0 \cup D_1 \cup D_0^\sharp \cup D_1^\sharp$ does not intersect the negative real axis. In particular, it follows that for all $z \in A_h \cup B_h$, $d(\operatorname{Re} \mathbb{E}\mathrm{xp}^{-1}(z), \mathbb{Z}) < 1/2$. By the pre-compactness of the class of maps $\mathcal{QIS}_0$, there is a constant $C < 1/2$ such that for all $h \in \mathcal{QIS}_0$ and all $z \in A_h \cup B_h$, $d(\operatorname{Re} \mathbb{E}\mathrm{xp}^{-1}(z), \mathbb{Z}) < C$. Then, by the continuous dependence of the Fatou coordinate on the map, one may guarantee that for small enough α, all $h \in \mathcal{QIS}_\alpha$, and all $z \in A_h \cup B_h$, $d(\operatorname{Re} \mathbb{E}\mathrm{xp}^{-1}(z), \mathbb{Z}) < C' < 1/2$. Since, $\Phi_h^{-1}(\Pi) = A_h \cup B_h$, we conclude that there is $\delta_1 > 0$ such that

$$B_{\delta_1}(\Upsilon_h(\Pi)) \subset \{w \in \mathbb{C} \mid 1/2 \leq \operatorname{Re} w \leq 3/2\}.$$

On the other hand, we note that $A_h \cup B_h$ is contained in the range of h, with the latter set contained well-inside the disk of radius $4e^{4\pi}/27$ centred at 0. Therefore, by making δ_1 small enough, we have the inclusion in the lemma for small enough α.

In [IS06], the constant r_3 in Thm 4.5 is obtained from a continuity property of the locations of the domains $D_0 \cup D_1 \cup D_0^\sharp \cup D_1^\sharp$ with respect to h. As such, it is implicitly assumed that the inclusion in the lemma also holds for small perturbations of the maps $h \in \mathcal{QIS}_0$. Because of this we do not introduce a new constant for small enough α, and assume that the same constant r_3 works here as well. □

Let $\rho(z)|dz|$ denote the hyperbolic metric of constant curvature -1 on Π. Let $\Upsilon_n^*\rho$ denote the pull back of ρ by Υ_n, that is, $(\Upsilon_n^*\rho)(w) = \rho(\Upsilon_n(w)) \cdot |\Upsilon_n'(w)|$.

Proposition 6.3. *There exists a constant $\delta_2 < 1$ such that for every $n \geq 0$ and every $w \in \Pi$ we have*

$$(\Upsilon_n^*\rho)(w) \leq \delta_2 \rho(w).$$

In other words, the maps $\Upsilon_n : \Pi \to \Pi$ are uniformly contracting with respect to the hyperbolic metric ρ, with a uniform contraction factor independent of n.

It is classic in complex analysis that when $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic map with $f(\mathbb{D})$ compactly contained in $\mathbb{D}$, then f is uniformly contracting with respect to the hyperbolic metric on $\mathbb{D}$. However, here $\Upsilon_n(\Pi)$ is not compactly contained in Π. But the uniform space provided in Lem 6.2 is enough to secure uniform contraction, independent of n. We present the details below.

Proof. Let $\rho_n(z)|dz|$ denote the Poincaré metric on the domain $\Upsilon_n(\Pi)$. By the Schwartz-Pick Lemma, the map $\Upsilon_n : (\Pi, \rho) \to (\Upsilon_n(\Pi), \rho_n)$ is non-expanding. It is enough to show that the inclusion map from $(\Upsilon_n(\Pi), \rho_n)$ to (Π, ρ) is uniformly contracting.

Fix an arbitrary point ξ_0 in $\Upsilon_n(\Pi)$. Recall the constant δ_1 from Lem 6.2, and consider the map $H : \Upsilon_n(\Pi) \to \mathbb{C}$ defined as

$$H(\xi) = \xi + \frac{\delta_1(\xi - \xi_0)}{\xi - \xi_0 + 2}.$$

For $\xi \in \Upsilon_n(\Pi)$ we have $|\operatorname{Re}(\xi - \xi_0)| < 1$. This implies that $|\xi - \xi_0| < |\xi - \xi_0 + 2|$, and hence $|H(\xi) - \xi| < \delta_1$. It follows from Lem 6.2 that H maps $\Upsilon_n(\Pi)$ into Π. By Schwartz-Pick Lemma, H is non-expanding, with respect to the corresponding hyperbolic metrics. In particular, at $H(\xi_0) = \xi_0$ we obtain

$$\rho(\xi_0)|H'(\xi_0)| = \rho(\xi_0)(1 + \delta_1/2) \leq \rho_n(\xi_0).$$

Hence,

$$\rho(\xi_0) \leq \big(\frac{2}{2 + \delta_1}\big)\rho_n(\xi_0).$$

The contraction factor is $\delta_2 = 2/(2+\delta_1)$. □

6.3. **Critical curve.** Inductively, we define the curves

$$u_n^j : i[0,+\infty) \to \Pi,$$

for $j \geq 0$ and $n \geq -1$. For $j=0$ and all $n \geq -1$, let

$$u_n^0(it) = 1 + it.$$

Assume that for some $j \geq 0$, and all $n \geq -1$, u_n^j is defined. We define u_n^{j+1}, for $n \geq -1$, as

$$u_n^{j+1}(it) = \Upsilon_{n+1} \circ u_{n+1}^j \circ Y_{n+1}^{-1}(it), \text{ for } t \geq 0. \tag{6.3}$$

Lemma 6.4. *For all $n \geq -1$ and $j \geq 0$, $u_n^j : i[0,+\infty) \to \Pi$ is a well-defined analytic map satisfying $u_n^j(0) = +1$.*

Proof. For $j=0$ and all $n \geq -1$, u_n^0 maps $i[0,+\infty)$ into Π, $u_n^0(0) = +1$, and u_n^0 is analytic. Assume that for some $j \geq 0$, and all $n \geq -1$, u_n^j is defined, maps $i[0,+\infty)$ into Π, $u_n^j(0) = +1$, and u_n^j is analytic.

By (3.5), for every $t \geq 0$, $\operatorname{Im} Y_{n+1}^{-1}(t) \in [0,+\infty)$. Hence, $u_{n+1}^j(Y_{n+1}^{-1}(it))$ is defined, and belongs to Π. This implies that $\Upsilon_{n+1} \circ u_{n+1}^j \circ Y_{n+1}^{-1}(it)$ is defined. By Lem 6.2, the image of u_n^{j+1} is contained in Π. Each Υ_n is either holomorphic or anti-holomorphic, and each Y_n is real analytic on $i(-1,+\infty)$. Therefore, each u_n^{j+1} is analytic. Also, since $Y_{n+1}(0) = 0$ and $\Upsilon_{n+1}(+1) = +1$, for all $n \geq -1$, the induction hypothesis $u_{n+1}^j(0) = 1$ implies that $u_n^{j+1}(0) = 1$. □

Recall the constant δ_2 introduced in Prop 6.3.

Proposition 6.5. *There is a constant C_2 such that for every $n \geq -1$, every $j \geq 0$, and every $t \geq 0$,*

$$|u_n^{j+1}(it) - u_n^j(it)| \leq C_2(\delta_2)^j.$$

In particular, for every $n \geq -1$, as $j \to +\infty$, u_n^j converges to a continuous map $u_n : i[0,+\infty) \to \Pi$.

Proof. We use Prop 5.2 with $\alpha = \alpha_{n+1}$, $h = f_{n+1}$, $w_1 = Y_{n+1}^{-1}(it)$ and $w_2 = Y_{n+1}^{-1}(it) + 1$, to conclude that for all $n \geq -1$ and all $t \geq 0$,

$$\begin{aligned} |u_n^1(it) - u_n^0(it)| &= |\Upsilon_{n+1}(Y_{n+1}^{-1}(it)+1) - (it+1)| \\ &\leq |\Upsilon_{n+1}(Y_{n+1}^{-1}(it)+1) - Y_{n+1}(Y_{n+1}^{-1}(it))| + 1 \leq C_3 + 1. \end{aligned}$$

Recall the hyperbolic metric $\rho(z)|dz|$ on Π. One has the classic inequalities $1/(2d(z,\partial\Pi)) \leq \rho(z) \leq 2/d(z,\partial\Pi)$. In particular, $\rho \geq 1$ on Π. On the other hand, by Lem 6.2, $d(u_n^1(it), \partial\Pi) \geq \delta_1$ which implies that $d_\rho(u_n^1(it), u_n^0(it)) \leq 2(C_3+1)/\delta_1$, for all $n \geq -1$ and all $t \geq 0$.

Now we apply the uniform contraction of the maps Υ_l with respect to the hyperbolic metric, Prop 6.3, to conclude that for $j \geq 1$,

$$\begin{aligned} d_\rho(u_n^{j+1}(it), u_n^j(it)) &= d_\rho\left(\Upsilon_{n+1} \circ u_{n+1}^j \circ Y_{n+1}^{-1}(it), \Upsilon_{n+1} \circ u_{n+1}^{j-1} \circ Y_{n+1}^{-1}(it)\right) \\ &\leq \delta_2 d_\rho\left(u_{n+1}^j(Y_{n+1}^{-1}(it), u_{n+1}^{j-1}(Y_{n+1}^{-1}(it)\right) \\ &\leq (\delta_2)^j d_\rho\left(u_{n+j}^1(it'), u_{n+j}^0(it')\right), \end{aligned}$$

where $it' = Y_{n+j}^{-1} \circ \cdots \circ Y_{n+1}^{-1}(it)$. Therefore, as $\rho \geq 1$ on Π,

$$|u_n^{j+1}(it) - u_n^j(it)| \leq d_\rho(u_n^{j+1}(it), u_n^j(it)) \leq (\delta_2)^j d_\rho(u_{n+j}^1(it'), u_{n+j}^0(it')) \leq (\delta_2)^j 2(C_3+1)/\delta_1. \tag{6.4}$$

By the above inequality, for each $n \geq -1$, u_n^j forms a Cauchy sequence on $i[0,+\infty)$. This implies that u_n^j converges to a continuous map u_n, as $j \to +\infty$. □

Proposition 6.6. *For every $n \geq 0$ and every $t \geq 0$ we have*

$$\Upsilon_n \circ u_n(it) = u_{n-1} \circ Y_n(it), \quad \textit{and} \;\; u_n(0) = 1.$$

Proof. These relations are obtained from taking limits as $j \to +\infty$ in (6.3) and the latter part of Lem 6.4. □

Proposition 6.7. *For every $n \geq -1$, and every $t \geq 0$, we have*

$$|u_n(it) - (1+it)| \leq C_2/(1-\delta_2).$$

Proof. Using Prop 6.5, for every $j \geq 1$ and $t \geq 0$, we have

$$\begin{aligned} |u_n^j(it) - (1+it)| &= |u_n^j(it) - u_n^0(it)| \\ &= \big| \sum_{l=1}^{j} (u_n^l(it) - u_n^{l-1}(it)) \big| \leq \sum_{l=1}^{j} C_2(\delta_2)^{l-1} \leq C_2/(1-\delta_2). \end{aligned} \tag{6.5}$$

Taking limit as $j \to +\infty$, we conclude the inequality in the proposition. □

Proposition 6.8. *For every $n \geq -1$, $u_n : i[0, +\infty) \to \Pi$ is injective.*

Proof. Fix an arbitrary $n \geq -1$. Let $0 \leq t_n < s_n$ be arbitrary real values. Define the sequence of numbers $t_{l+1} = \operatorname{Im} Y_{l+1}^{-1}(it_l)$ and $s_{l+1} = \operatorname{Im} Y_{l+1}^{-1}(is_l)$, for $l \geq n$. By (3.8), for $l \geq n$, $|t_l - s_l| \geq (10/9)^{l-n}|t_n - s_n|$. In particular, for large enough l, $|t_l - s_l| \geq 3C_2/(1-\delta_2)$. By virtue of Prop 6.7, this implies that $u_l(it_l) \neq u_l(is_l)$. Now, inductively using the commutative relation in Prop 6.6, and the injectivity of Y_k and Υ_k for all k, we conclude that $u_n(it_n) \neq u_n(is_n)$. □

Proposition 6.9. *For every $n \geq -1$, $\lim_{t \to +\infty}(u_n(it) - (1+it))$ exists and is finite.*

Proof. By Prop 5.3 and the explicit formula for Y_{n+1}, the following limit exists and is finite

$$\begin{aligned} \lim_{t \to +\infty} (u_n^1(it) - u_n^0(it)) &= \lim_{t \to +\infty} \big(\Upsilon_{n+1}(Y_{n+1}^{-1}(it) + 1) - (1+it) \big) \\ &= \lim_{t \to +\infty} \big(\Upsilon_{n+1}(Y_{n+1}^{-1}(it) + 1) - Y_{n+1}(Y_{n+1}^{-1}(it) + 1) \big) \\ &\quad + \lim_{t \to +\infty} \big(Y_{n+1}(Y_{n+1}^{-1}(it) + 1) - Y_{n+1}(Y_{n+1}^{-1}(it)) \big) - 1. \end{aligned}$$

By an inductive argument, one may see that for every $j \geq 1$, $\lim_{t \to +\infty}(u_n^j(it) - u_n^{j-1}(it))$ exists and is finite. Indeed, by (6.4), the absolute value of this limit is bounded from above by $(\delta_2)^{j-1} 2(C_3+1)/\delta_1$. It follows that $\lim_{t \to +\infty}(u_n^j(it) - (1+it))$ exists and is finite. Since u_n^j converges to u_n uniformly on $i[0, +\infty)$, we conclude the proposition. □

Remark 6.10. It should be clear from the argument in this section that the limiting curves u_n and their parametrisation do not depend on the particular choice of u_n^0. Any other choice for u_n^0 which lies within some uniform distance from u_n^0 leads the same limiting curve u_n. For this reason, one may see that when $\alpha \in \mathscr{B}$, the intersection of $\Phi_n^{-1}(u_n)$ and the Siegel Siegel of f_n coincides with an internal ray of the Siegel disk of f_n.

6.4. **An equivariant extension of the critical curve.** We need to extend the domain of definition of each u_n to $i[-1, +\infty)$, so that the extended maps collectively satisfy the functional relation in Prop 6.6. There are many choices for such extensions, as we present the details below.

Let us define the numbers t_n^j, for $n \geq -1$ and $j \geq 0$ according to

$$t_n^0 = -1, \text{ for } n \geq -1, \qquad \text{and} \qquad t_n^j = \operatorname{Im} Y_{n+1}(it_{n+1}^{j-1}), \text{ for } n \geq -1 \text{ and } j \geq 1.$$

Lemma 6.11. *For every $n \geq -1$, we have $t_n^0 < t_n^1 < t_n^2 < \cdots < 0$ with $t_n^j \to 0$ as $j \to +\infty$.*

Proof. It follows from Lem 3.1 and the definition of Y_n in (3.4) that for every $n \geq -1$, $\operatorname{Im} Y_{n+1}(-i) > -1$. This implies that $t_n^0 < t_n^1$, for all $n \geq -1$. Since each Y_n is injective and maps $i[-1,+\infty)$ into itself, the map $t \to \operatorname{Im} Y_n(it)$ is order preserving, for all $n \geq 0$. This implies that for all $n \geq -1$ and $j \geq 0$, $t_n^j < t_n^{j+1}$.

Finally, by (3.8), $|t_n^j| \leq (9/10)^j$, which implies the latter part of the lemma. □

Recall the set $\Pi_n = \Pi_{f_n}$ defined in Section 6.2.

Lemma 6.12. *For each $n \geq -1$, there is a continuous and injective map*[6]

$$u_n^0 : i[t_n^0, t_n^1] \to \Pi \setminus \operatorname{int}\,(\Upsilon_{n+1}(\Pi_{n+1}))$$

such that

(i) $u_n^0(it_n^0) = 1 - 2i$ *and* $u_n^0(it_n^1) = \Upsilon_{n+1}(1-2i)$,
(ii) $\sup\{\operatorname{Im} u_n^0(is) \mid n \geq -1, t_n^0 \leq s \leq t_n^1\} < +\infty$,
(iii) $u_n^0(i(t_n^0, t_n^1)) \subset \operatorname{int}\,(\Pi \setminus \Upsilon_{n+1}(\Pi_{n+1})$.

Proof. Recall that $\mathbb{E}\mathrm{xp}(1-2i) = -4e^{4\pi}/27$, and $\mathbb{E}\mathrm{xp}(\Upsilon_{n+1}(1-2i))$ is either equal to $\Phi_{n+1}^{-1}(1-2i)$ or $s \circ \Phi_{n+1}^{-1}(1-2i)$ depending on ε_{n+1}. Also, recall that $\Phi_{n+1}^{-1}(\Pi_{n+1})$ is a finite union of sectors bounded by analytic curves landing at 0. Moreover, this set contains a punctured neighbourhood of 0, is compactly contained in $B(0, 4e^{4\pi}/27)$, and $\Phi_{n+1}^{-1}(1-2i)$ lies on its boundary. Let us assume that $\varepsilon_{n+1} = -1$. There is a continuous curve $\gamma : [0,1] \to B(0, -4e^{4\pi}/27)$ such that $\gamma(0) = -4e^{4\pi}/27$, $\gamma(1) = \Phi_{n+1}^{-1}(1-2i)$, and $\gamma((0,1))$ does not meet the sets $\Phi_{n+1}^{-1}(\Pi_{n+1})$ and $[0,+\infty)$. We may choose this curve to be uniformly away from 0. One may lift the curve γ via $\mathbb{E}\mathrm{xp}$ to define the desired curve u_n^0, which may be re-parametrised on $i[t_n^0, t_n^1]$.

When $\varepsilon_{n+1} = +1$, one only needs to insert the complex conjugation map s in the appropriate places in the above argument. □

By induction on $j \geq 0$, we define the maps u_n^j on $i[t_n^j, t_n^{j+1}]$, for all $n \geq -1$. For $j = 0$ we let u_n^0 denote the map introduced in Lem 6.12. Assume that for some $j \geq 0$ and all $n \geq -1$, u_n^j is defined on $i[t_n^j, t_n^{j+1}]$. For all $n \geq -1$, we define u_n^{j+1} on $i[t_n^{j+1}, t_n^{j+2}]$ as

$$u_n^{j+1}(it) = \Upsilon_{n+1} \circ u_{n+1}^j \circ Y_{n+1}^{-1}(it). \tag{6.6}$$

Note that, by Lem 6.12-(i),

$$u_n^1(it_n^1) = \Upsilon_{n+1} \circ u_{n+1}^0 \circ Y_{n+1}^{-1}(it_n^1) = \Upsilon_{n+1} \circ u_{n+1}^0(i6_n^0) = \Upsilon_{n+1}(1-2i) = u_n^0(it_n^1).$$

In other words, the maps u_n^0 and u_n^1 match at the boundary of their respective domains of definitions. Repeating the above argument inductively, one may see that for all $n \geq -1$ and $j \geq 0$, $u_n^{j+1}(it_n^{j+1}) = u_n^j(it_n^{j+1})$. Thus we may define the map u_n on $i[-1,0)$ as

$$u_n(it) = u_n^j(it), \text{ for } t \in [t_n^j, t_n^{j+1}].$$

We set $u_n(0) = +1$, for each $n \geq -1$.

Lemma 6.13. *For every $n \geq -1$, $u_n : i[-1,0] \to \Pi$ is continuous. Moreover, there is $C_5 > 0$ such that for all $n \geq -1$ and all $t \in [-1,0]$ we have $|u_n(it) - (1+it)| \leq C_5$.*

Proof. Fix an arbitrary $n \geq -1$. By Lem 6.12 and (6.6), the restriction of u_n to each closed interval $i[t_n^j, t_n^{j+1}]$ is continuous, for $j \geq 0$. Hence, u_n is continuous on $i[-1,0)$.

Fix an arbitrary $n \geq -1$ and an arbitrary $j \geq 1$. By Lem 6.12-(ii), the Euclidean diameter of the curve $u_{n+j}(i[t_{n+j}^0, t_{n+j}^1])$ is uniformly bounded from above. By the pre-compactness of the class of maps IS, it follows that the Euclidean diameter of the curve $\Upsilon_{n+j}(u_{n+j}(i[t_{n+j}^0, t_{n+j}^1]))$ is uniformly bounded from above, independent of $n+j$. On the other hand, this curve also lies in $\Upsilon_{n+j}(\Pi)$, which is contained well inside the set Π, by Lem 6.2. These imply that the hyperbolic diameter of

[6] The operator int returns the topological interior of a given set.

the curve $\Upsilon_{n+j}(u_{n+j}(i[t^0_{n+j}, t^1_{n+j}]))$ in Π is uniformly bounded from above, independent of $n+j$. Then, we employ Prop 7.7, to conclude that there is a constant C, independent of n and j, such that the hyperbolic diameter of the curve $\Upsilon_{n+1} \circ \cdots \circ \Upsilon_{n+j}(u_{n+j}(i[t^0_{n+j}, t^1_{n+j}]))$ is bounded from above by $C(\delta_2)^{j-1}$. Similarly, by making C larger if necessary, we also conclude that the hyperbolic distance from $\Upsilon_{n+1} \circ \cdots \circ \Upsilon_{n+j}(u_{n+j}(it^1_{n+j}))$ to $+1$ is bounded from above by $C(\delta_2)^{j-1}$.

By (6.6), $\Upsilon_{n+1} \circ \cdots \circ \Upsilon_{n+j}(u_{n+j}(i[-1, t^1_{n+j}])) = u_n(i[t^j_n, t^{j+1}_n])$, and $\Upsilon_{n+1} \circ \cdots \circ \Upsilon_{n+j}(u_{n+j}(it^1_{n+j})) = u_n(it^{j+1}_n)$. Thus, by the above paragraph, the hyperbolic diameter of $u_n(i[t^j_n, t^{j+1}_n])$ is bounded from above by $C(\delta_2)^{j-1}$, and the hyperbolic distance from $u_n(it^{j+1}_n)$ to $+1$ is bounded from above by $C(\delta_2)^{j-1}$. These imply that the map u_n is continuous at 0. Moreover, the hyperbolic diameter of the curve $u_n(i[t^1_n, 0])$ is bounded from above by $C\sum_{j=1}^{\infty}(\delta_2)^{j-1} = C/(1-\delta_2)$. In particular, combining with Lem 6.12-(ii), we conclude that the Euclidean diameter of the curve $u_n(i[-1, 0])$ is uniformly bounded from above, independent of n. This implies the latter part of the lemma. □

Proposition 6.14. *For every $n \geq -1$, $u_n : i[-1, +\infty) \to \Pi$ is injective.*

Proof. Fix an arbitrary $n \geq -1$. We already proved in Prop 6.8 that u_n is injective on $i[0, +\infty)$.

Let us fix arbitrary points $y_n < x_n$ in $[-1, +\infty)$ with $y_n < 0$. We aim to show that $u_n(iy_n) \neq u_n(ix_n)$. First assume that there is $j \geq 0$ such that both x_n and y_n belong to the same interval $[t^j_n, t^{j+1}_n]$. Then, u_n on $i[t^j_n, t^{j+1}_n]$ is given by

$$u_n = \Upsilon_{n+1} \circ \cdots \circ \Upsilon_{n+j} \circ u_{n+j} \circ Y^{-1}_{n+j} \circ \cdots \circ Y^{-1}_{n+1},$$

while

$$Y^{-1}_{n+j} \circ \cdots \circ Y^{-1}_{n+1}(i[t^j_n, t^{j+1}_n]) = i[t^0_{n+j}, t^1_{n+j}].$$

However, each Υ_l is injective on Π, Y_l is injective on its domain, and by Lem 6.12, u_n is injective on $i[t^0_{n+j}, t^1_{n+j}]$. In particular, $u_n(iy_n) \neq u_n(ix_n)$.

Now assume that both x_n and y_n do not belong to one interval $[t^j_n, t^{j+1}_n]$. By Lem 6.11, there is $j \geq 0$ such that $t^j_n \leq y_n < t^{j+1}_n < x_n$. Let $y_{l+1} = \operatorname{Im} Y^{-1}_{l+1}(iy_l)$ and $x_{l+1} = \operatorname{Im} Y^{-1}_{l+1}(ix_l)$, for $n \leq l \leq n+j-1$. By the choice of j, we have $-1 \leq y_{n+j} < t^1_{n+j} < x_{n+j}$. On the other hand, by Lem 6.2 (and $y_{n+j} < t^1_{n+j}$),

$$u_{n+j}(iy_{n+j}) \in \Pi \setminus \Upsilon_{n+j+1}(\Pi_{n+j+1}),$$

while

$$u_{n+j}(ix_{n+j}) = \Upsilon_{n+j+1} \circ u_{n+j+1} \circ Y^{-1}_{n+j+1}(ix_{n+j}) \subset \Upsilon_{n+j+1}(\Pi) \subset \Upsilon_{n+j+1}(\Pi_{n+j+1}).$$

In particular, $u_{n+j}(ix_{n+j}) \neq u_{n+j}(iy_{n+j})$. Now, inductively one uses the commutative relation in (6.6), and the injectivity of the maps Y_l on its domain and Υ_l on Π, to conclude that $u_n(ix_n) \neq u_n(iy_n)$. □

The particular choice of the above curve and its parametrisation does not play any role in the sequel. But the feature we shall use is the commutative property

$$\Upsilon_{n+1} \circ u_{n+1}(it) = u_n \circ Y_{n+1}(it), \ t \in [-1, 0]. \tag{6.7}$$

7. Marked dynamical partitions for the post-critical set

In this section we define a nest of sets in the phase space, in analogy with the nest of sets used to define the arithmetic model in Sec. 3. Each element of the nest consists of a finite number of Jordan domains. Each piece comes with a marking of its boundary, and the markings match where the boundaries meet. This nest is similar to the nest of partitions Ω^j_n introduced in the papers [Che13, Che19, AC18]. However, the nest we introduce here has a simpler combinatorial and geometric features in terms of the number of pieces and overlaps.

7.1. **Marked curves $w_n^\pm$ and $v_n^\pm$.** Recall the curves $u_n : i[-1,+\infty) \to \Pi$, for $n \geq -1$, defined in Sections 6.3 and 6.4, as well as the sets Π_n and the maps $\Upsilon_n : \Pi_n \to \mathbb{C}$ defined in Sec. 6.2.

For the map Υ_n, the curve $u_n(i[-1,+\infty))$ plays the role of the vertical line $i[-1,+\infty)$ for the map Y_n. Here we define two other curves for Υ_n which play the analogous role of the vertical lines $1/\alpha_n + i[-1,+\infty)$ and $1/\alpha_n - 1 + i[-1,+\infty)$ for Y_n (see (3.6) and (3.7)). However, due to the presence of a critical point for Υ_n, as opposed to the injectivity of Y_n, we need to consider a pair of curves for the role of $1/\alpha_n - 1 + i[-1,+\infty)$. These curves are denoted by w_n^+ and $v_n^\pm$, respectively. For the sake of simplifying the arguments, we parametrise w_n^+ on $1/\alpha_n + i[-1,+\infty)$ and the pair of curves $v_n^\pm$ on $1/\alpha_n - 1 + i[-1,+\infty)$.

Proposition 7.1. *For every $n \geq 0$, there are continuous and injective maps*

$$w_n^+ : 1/\alpha_n + i[-1,+\infty) \to \Pi_n, \qquad w_n^- : 1/\alpha_n + i[-1,+\infty) \to \Pi_n$$

such that

(i) *for all $t \in [-1,+\infty)$ we have*

$$\Upsilon_n \circ w_n^+(1/\alpha_n + it) = \Upsilon_n \circ w_n^-(1/\alpha_n + it) = \Upsilon_n \circ u_n(it) - \varepsilon_n,$$

(ii) *on $1/\alpha_n + i(0,+\infty)$, $w_n^+ = w_n^-$,*
(iii) *$w_n^+(1/\alpha_n + i[-1,0)) \cap w_n^-(1/\alpha_n + i[-1,0)) = \emptyset$,*
(iv) *Υ_n has a critical point at $w_n^+(1/\alpha_n) = w_n^-(1/\alpha_n)$.*

Proof. From Sec. 4.2, recall the sets $A_n = A_{f_n}$, $B_n = B_{f_n}$, and $S_n = S_{f_n}$, as well as the integer $k_n = k_{f_n}$, associated to the map $h = f_n$. Also, recall from Sec. 6 that $u_n(i[-1,+\infty))$ is contained in $\Pi = \Phi_n(A_n \cup B_n) \subset \Pi_n$. Then, $\Phi_n^{-1} \circ u_n(i[-1,+\infty))$ is contained in $A_n \cup B_n$.

The map $\Phi_n^{-1} : \Pi_n \to \mathbb{C}$ covers the set $A_n \cup B_n$ several times. Its restriction $\Phi_n^{-1} : \Pi \to A_n \cup B_n$ is univalent. But its restriction $\Phi_n^{-1} : \Phi_n(S_n) + k_n \to A_n \cup B_n$ has a specific covering structure; it covers B_n in a one-to-one fashion, and covers A_n by a two-to-one proper branched covering map. The branch point is mapped to $-4/27 = \Phi_n^{-1}(+1)$; the critical value of f_n. See the discussion in Sec. 4.2. In particular, there is a unique continuous curve $w_n^+ : 1/\alpha_n + i[0,+\infty) \to \Phi_n(S_n) + k_n$, such that

$$\Phi_n^{-1}(w_n^+(1/\alpha_n + it)) = \Phi_n^{-1}(u_n(it)), \tag{7.1}$$

This defines w_n^+ on $1/\alpha_n + i[0,+\infty)$. We let $w_n^- = w_n^+$ on $1/\alpha_n + i[0,+\infty)$.

There are two ways to extend the map w_n^+ on $1/\alpha_n + i[-1,0)$ so that the above equation holds. These come from the double covering structure of the map Φ_n^{-1} from $\Phi_n(A_n^{-k_n}) + k_n$ onto A_n. Let us denote these maps by w_n^+ and w_n^-. The three curves $w_n^+(1/\alpha_n + i[-1,0])$, $w_n^-(1/\alpha_n + i[-1,0])$, and $w_n^+(1/\alpha_n + i[0,+\infty))$ land at $w_n^+(1/\alpha_n)$. There is a cyclic order on these curves consistent with the positive orientation on an infinitesimal circle at $w_n^+(1/\alpha_n)$. We relabel these curves so that $w_n^+(1/\alpha_n + i[0,+\infty)) < w_n^+(1/\alpha_n + i[-1,0]) < w_n^-(1/\alpha_n + i[-1,0])$. See Fig 7. Evidently, Υ_n has a critical point at $w_n^+(1/\alpha_n)$, and $w_n^+(1/\alpha_n + i[0,+\infty)) \cap w_n^-(1/\alpha_n + i[0,+\infty)) = \emptyset$.

By the above argument, the images of the curves $w_n^\pm$ are contained in $\Phi_n(S_n) + k_n$, that is, for all $s \in \{+,-\}$,

$$w_n^s : 1/\alpha_n + i[-1,+\infty) \to \Phi_n(S_n) + k_n. \tag{7.2}$$

Recall that $\Upsilon_n = \mathbb{E}\mathrm{xp}^{-1} \circ s \circ \Phi_n^{-1}$ or $\Upsilon_n = \mathbb{E}\mathrm{xp}^{-1} \circ \Phi_n^{-1}$, depending on the sign ε_n. Therefore, by (7.1) and the continuity of u_n, w_n^+, and Υ_n, there is an integer i_n such that for $s \in \{+,-\}$ and all $t \geq -1$, $\Upsilon_n \circ w_n^s(1/\alpha_n + it) = \Upsilon_n \circ u_n(it) + i_n$. On the other hand, the region bounded by the curves u_n and w_n^+ near $+i\infty$ is mapped by Φ_n^{-1} to a slit neighbourhood of 0 in a one-to-one fashion. It follows that $i_n = -\varepsilon_n$. ∎

Proposition 7.2. *There exists a constant C_6 such that for every $n \geq 0$, and every $t \geq -1$,*

$$|w_n^+(it + 1/\alpha_n) - (u_n(it) + 1/\alpha_n)| \leq C_6.$$

Moreover, for every $\varepsilon > 0$ there is C_ε such that for every $n \geq 0$ and every $t \geq C_\varepsilon$, we have

$$|w_n^+(1/\alpha_n + it) - (u_n(it) + 1/\alpha_n)| \leq \varepsilon.$$

Proof. Consider the map $\mathcal{E}_n = \Phi_n \circ \Phi_n^{-1} : \Phi_n(S_n) + k_n \to \Pi$; compare with (4.3). By the functional relation for the Fatou coordinate in Prop 4.1, this map commutes with the translation by $+1$. Therefore, $\mathcal{E}_n$ induces a holomorphic map, say $\tilde{\mathcal{E}}_n$, from $\Phi_n(S_n)/\mathbb{Z} \subset \mathbb{C}/\mathbb{Z}$ onto $\{w \in \mathbb{C}/\mathbb{Z} \mid \operatorname{Im} w \geq 2\}$. Moreover, by (7.1), $\tilde{\mathcal{E}}_n$ maps $w_n^+/\mathbb{Z}$ to $u_n/\mathbb{Z}$.

The image of $\tilde{\mathcal{E}}_n$ covers the region above the circle $\operatorname{Im} = +2$ in a univalent fashion. Also, we have $\lim_{\operatorname{Im} w \to +\infty} \operatorname{Im} \tilde{\mathcal{E}}_n(w) = +\infty$. We may apply the Koebe distortion theorem to $\tilde{\mathcal{E}}_n^{-1}$ on the annulus $\{w \in \mathbb{C}/\mathbb{Z} \mid \operatorname{Im} w > 2\}$. It implies that there is a constant c_n such that $|u_n(it) + c_n - w_n^+(1/\alpha_n + it)|$ converges to 0, uniformly independent of n, as $t \to +\infty$. Moreover, $|u_n(it) + c_n - w_n^+(1/\alpha_n + it)|$ is uniformly bounded from above when $\operatorname{Im} u_n(it) \geq 3$. By (5.2), $c_n = 1/\alpha_n$.

By Prop 6.7, if $t \geq C_2/(1-\delta_2) + 3$, we get $\operatorname{Im} u_n(it) \geq 3$. On the other hand, for $-1 \leq t \leq C_2/(1-\delta_2) + 3$ we employ the pre-compactness of the class $\mathcal{QIS}$, and the continuous dependence of the map Φ_h on $h \in \mathcal{QIS}$, to conclude that $u_n(it) + 1/\alpha_n - w_n^+(1/\alpha_n + it)$ is uniformly bounded from above. This proves the existence of the constant C_6. ∎

For $n \geq 0$, we consider the maps

$$v_n^+ : 1/\alpha_n - 1 + i[-1, +\infty) \to \Pi_n, \quad v_n^- : 1/\alpha_n - 1 + i[-1, +\infty) \to \Pi_n$$

defined as

$$v_n^+(1/\alpha_n - 1 + it) = w_n^+(1/\alpha_n + it) - 1, \quad v_n^-(1/\alpha_n - 1 + it) = w_n^-(1/\alpha_n + it) - 1. \tag{7.3}$$

Proposition 7.3. *For every $n \geq 0$ and every $t \geq -1$ the following hold:*

(i) *if $\varepsilon_{n+1} = -1$, then*

$$\Upsilon_{n+1} \circ v_{n+1}^+(1/\alpha_{n+1} - 1 + it) + a_n + \varepsilon_{n+1} = w_n^+(Y_{n+1}(it) + 1/\alpha_n),$$

$$\Upsilon_{n+1} \circ v_{n+1}^-(1/\alpha_{n+1} - 1 + it) + a_n + \varepsilon_{n+1} = w_n^-(Y_{n+1}(it) + 1/\alpha_n),$$

(ii) *if $\varepsilon_{n+1} = +1$, then*

$$\Upsilon_{n+1} \circ v_{n+1}^+(1/\alpha_{n+1} - 1 + it) + a_n + \varepsilon_{n+1} = w_n^-(Y_{n+1}(it) + 1/\alpha_n),$$

$$\Upsilon_{n+1} \circ v_{n+1}^-(1/\alpha_{n+1} - 1 + it) + a_n + \varepsilon_{n+1} = w_n^+(Y_{n+1}(it) + 1/\alpha_n).$$

Proof. Fix an arbitrary $n \geq 0$ and $s \in \{+, -\}$. Let us first assume that $\varepsilon_{n+1} = -1$ so that $f_{n+1} = \mathcal{R}(f_n)$.

By (7.2), for all $t' \geq -1$, $w_n^s(1/\alpha_n + it') - k_n \in \Phi_n(S_n)$. By Prop 4.1-(e), and (7.1),

$$f_n^{k_n}\left(\Phi_n^{-1}\left(w_n^s(1/\alpha_n + it') - k_n\right)\right) = \Phi_n^{-1}(w_n^s(1/\alpha_n + it')) = \Phi_n^{-1}(u_n(it')).$$

Hence, by the definition of renormalisation $\mathcal{R}(f_n) = f_{n+1}$, see (4.3), the above relation implies that

$$\begin{aligned} f_{n+1}\left(\mathbb{E}\mathrm{xp}\left(w_n^s(1/\alpha_n + it')\right)\right) &= f_{n+1}\left(\mathbb{E}\mathrm{xp}\left(w_n^s(1/\alpha_n + it') - k_n\right)\right) \\ &= \mathbb{E}\mathrm{xp}(u_n(it')). \end{aligned} \tag{7.4}$$

Let $it' = Y_{n+1}(it)$. The right hand side of the above equation becomes

$$\begin{aligned} \mathbb{E}\mathrm{xp}(u_n(it')) = \mathbb{E}\mathrm{xp}(u_n(Y_{n+1}(it))) &= \mathbb{E}\mathrm{xp} \circ \Upsilon_{n+1} \circ u_{n+1}(it) && \text{(Prop 6.6)} \\ &= \Phi_{n+1}^{-1}(u_{n+1}(it)) && \text{((6.1))} \\ &= \Phi_{n+1}^{-1}(w_{n+1}^s(1/\alpha_{n+1} + it)) && \text{((7.1))} \\ &= f_{n+1} \circ \Phi_{n+1}^{-1}(w_{n+1}^s(1/\alpha_{n+1} + it) - 1) && \text{(Prop 4.1-e)} \\ &= f_{n+1} \circ \Phi_{n+1}^{-1}(v_{n+1}^s(1/\alpha_{n+1} - 1 + it)). && \text{((7.3))} \end{aligned}$$

Combining the above equations, we conclude that

$$f_{n+1}\left(\mathbb{E}\mathrm{xp}\left(w_n^s(1/\alpha_n + Y_{n+1}(it))\right)\right) = f_{n+1}\left(\Phi_{n+1}^{-1}(v_{n+1}^s(1/\alpha_{n+1} - 1 + it))\right).$$

The above equation implies that

$$\begin{aligned}\mathbb{E}\mathrm{xp}\left(w_n^s(1/\alpha_n + Y_{n+1}(it))\right) &= \Phi_{n+1}^{-1}(v_{n+1}^s(1/\alpha_{n+1} - 1 + it)) \\ &= \mathbb{E}\mathrm{xp} \circ \Upsilon_{n+1}(v_{n+1}^s(1/\alpha_{n+1} - 1 + it)) \qquad ((6.1))\end{aligned}$$

As $\mathbb{E}\mathrm{xp}(w) = \mathbb{E}\mathrm{xp}(w')$ iff $w - w' \in \mathbb{Z}$, the above relation implies that for any $t \geq -1$, there must be an integer l_t, such that

$$w_n^s(1/\alpha_n + Y_{n+1}(it)) = \Upsilon_{n+1}(v_{n+1}^s(1/\alpha_{n+1} - 1 + it)) + l_t.$$

However, since $\Upsilon_{n+1}(v_{n+1}^s(1/\alpha_{n+1} - 1 + it))$ and $w_n^s(1/\alpha_n + Y_{n+1}(it))$ depend continuously on t, l_t must be independent of t. In order to identify the value of l_t we look at the limiting behaviour of the relation as $t \to +\infty$.

By Prop 7.2,

$$\begin{aligned}&\lim_{t\to+\infty}\left(\mathrm{Re}\left(v_{n+1}^s(1/\alpha_{n+1} - 1 + it) - u_{n+1}(it)\right)\right) \\ &\qquad = \lim_{t\to+\infty}\left(\mathrm{Re}(w_{n+1}^s(1/\alpha_{n+1} + it) - u_{n+1}(it))\right) - 1 = 1/\alpha_{n+1} - 1.\end{aligned}$$

Applying Υ_{n+1} and using (5.2),

$$\lim_{t\to+\infty}\left(\mathrm{Re}\left(\Upsilon_{n+1}(v_{n+1}^s(1/\alpha_{n+1} - 1 + it)) - \Upsilon_{n+1}(u_{n+1}(it))\right)\right) = \alpha_{n+1}(1/\alpha_{n+1} - 1) = 1 - \alpha_{n+1}.$$

On the other hand, by Propositions 6.6 and 7.2, we have

$$\begin{aligned}&\lim_{t\to+\infty}\mathrm{Re}(w_n^s(1/\alpha_n + Y_{n+1}(it)) - \Upsilon_{n+1}(u_{n+1}(it)) \\ &\qquad = \lim_{t\to+\infty}\mathrm{Re}(w_n^s(1/\alpha_n + Y_{n+1}(it)) - u_n(Y_{n+1}(it))) = 1/\alpha_n.\end{aligned}$$

Hence, we must have $1 - \alpha_{n+1} + l_t = 1/\alpha_n$, which by (2.2) and $\varepsilon_{n+1} = -1$, implies that $l_t = a_n - 1 = a_n + \varepsilon_{n+1}$. This completes the proof of the desired relation in the proposition when $\varepsilon_{n+1} = -1$.

Now assume that $\varepsilon_{n+1} = +1$. The argument is similar in this case, so we only look at the relation for v_n^+, and emphasis the differences with the above argument. As in the above argument, we have

$$f_n^{k_n}\left(\Phi_n^{-1}\left(w_n^-(1/\alpha_n + it') - k_n\right)\right) = \Phi_n^{-1}(u_n(it')).$$

When $\varepsilon_{n+1} = +1$, $\mathcal{R}(f_n) = s \circ f_{n+1} \circ s$. Therefore,

$$s \circ f_{n+1} \circ s\left(\mathbb{E}\mathrm{xp}\left(w_n^-(1/\alpha_n + it')\right)\right) = \mathbb{E}\mathrm{xp}(u_n(it')).$$

Let $it' = Y_{n+1}(it)$. The right hand side of the above equation becomes

$$\begin{aligned}\mathbb{E}\mathrm{xp}(u_n(it')) = \mathbb{E}\mathrm{xp}(u_n(Y_{n+1}(it))) &= \mathbb{E}\mathrm{xp} \circ \Upsilon_{n+1} \circ u_{n+1}(it) && \text{(Prop 6.6)} \\ &= s \circ \Phi_{n+1}^{-1}(u_{n+1}(it)) && ((6.1)) \\ &= s \circ f_{n+1} \circ \Phi_{n+1}^{-1}(v_{n+1}^-(1/\alpha_{n+1} - 1 + it)). && ((7.3))\end{aligned}$$

Combining the above equations, we conclude that

$$s \circ f_{n+1} \circ s\left(\mathbb{E}\mathrm{xp}\left(w_n^-(1/\alpha_n + Y_{n+1}(it))\right)\right) = s \circ f_{n+1}\left(\Phi_{n+1}^{-1}(v_{n+1}^-(1/\alpha_{n+1} - 1 + it))\right).$$

The above equation implies that

$$\begin{aligned}s \circ \mathbb{E}\mathrm{xp}\left(w_n^-(1/\alpha_n + Y_{n+1}(it))\right) &= \Phi_{n+1}^{-1}(v_{n+1}^+(1/\alpha_{n+1} - 1 + it)) \\ &= s \circ \mathbb{E}\mathrm{xp} \circ \Upsilon_{n+1}(v_{n+1}^+(1/\alpha_{n+1} - 1 + it)) \qquad ((6.1))\end{aligned}$$

Note that the change from v_{n+1}^- to v_{n+1}^+ in the above relation is due to the orientation reversing effect of $s \circ \mathbb{E}\mathrm{xp}$ on the left-hand side as opposed to the orientation preserving effect of $s \circ \mathbb{E}\mathrm{xp} \circ \Upsilon_{n+1}$ on the right hand side of the equation.

As in the previous case, there is an integer l_t, independent of t, such that

$$w_n^+(1/\alpha_n + Y_{n+1}(it)) = \Upsilon_{n+1}(v_{n+1}^-(1/\alpha_{n+1} - 1 + it)) + l_t.$$

Since Υ'_{n+1} is asymptotically equal to $-\alpha_{n+1}$ near $+i\infty$, in this case we obtain

$$\lim_{t\to+\infty} \big(\mathrm{Re}\,\big(\Upsilon_{n+1}(v_{n+1}^-(1/\alpha_{n+1} - 1 + it)) - \Upsilon_{n+1}(u_{n+1}(it))\big)\big) = -\alpha_{n+1}(1/\alpha_{n+1} - 1) = \alpha_{n+1} - 1.$$

Hence, $\alpha_{n+1} - 1 + l_t = 1/\alpha_n$, which by (2.2) and $\varepsilon_{n+1} = +1$, implies that $l_t = a_n + \varepsilon_{n+1}$. □

Recall the numbers $t_n^1 = \mathrm{Im}\, Y_{n+1}(-i) \in (-1, 0)$, for $n \geq -1$.

Proposition 7.4. *For all $n \geq 0$ and $s \in \{+, -\}$, we have*

$$w_n^+(1/\alpha_n + i[t_n^1, +\infty)) \cap (u_n(i[t_n^1, +\infty) + \mathbb{Z}) = \emptyset,$$
$$v_n^s(1/\alpha_n - 1 + i[t_n^1, +\infty)) \cap (u_n(i[t_n^1, +\infty) + \mathbb{Z}) = \emptyset,$$
$$v_n^s(1/\alpha_n - 1 + i[-1, +\infty)) \cap w_n^+(1/\alpha_n + i[-1, +\infty)) = \emptyset.$$

Proof. First assume that $\varepsilon_n = -1$ so that $\mathcal{R}(f_n) = f_{n+1}$. Recall from Sec. 6 that $u_{n+1}(i(-1, +\infty))$ lies in $\mathrm{int}\,(\Pi)$. Moreover, by Prop 6.6 and (6.7), we have

$$\mathbb{E}\mathrm{xp} \circ u_n(i(t_n^1, +\infty)) = \Phi_{n+1}^{-1}(u_{n+1}(i(-1, +\infty)).$$

Thus, $\mathbb{E}\mathrm{xp} \circ u_n(i(t_n^1, +\infty))$ is contained in the interior of $A_{n+1} \cup B_{n+1} = \Phi_{n+1}^{-1}(\Pi)$. It follows from (7.4) that $\mathbb{E}\mathrm{xp} \circ w_n^\pm(1/\alpha_n + i(t_n^1, +\infty))$ is contained in the interior of $A_{n+1}^{-1} \cup B_{n+1}^{-1}$. As the interiors of $A_{n+1} \cup B_{n+1}$ and $A_{n+1}^{-1} \cup B_{n+1}^{-1}$ are disjoint, we conclude that

$$\mathbb{E}\mathrm{xp} \circ u_n(i(t_n^1, +\infty)) \cap \mathbb{E}\mathrm{xp} \circ w_n^\pm(1/\alpha_n + i(t_n^1, +\infty)) = \emptyset.$$

Moreover, by Lem 6.12-(i) and (7.4),

$$\mathbb{E}\mathrm{xp} \circ u_n(it_n^1) = \Phi_{n+1}^{-1}(u_{n+1}(-i)) = \Phi_{n+1}^{-1}(1 - 2i) \neq \Phi_{n+1}^{-1}(-2i) \ni \mathbb{E}\mathrm{xp} \circ w_n^\pm(1/\alpha_n + it_n^1).$$

Combining the above equations, we have

$$\big(\mathbb{E}\mathrm{xp} \circ u_n(i[t_n^1, +\infty)\big) \cap \big(\mathbb{E}\mathrm{xp} \circ w_n^\pm(1/\alpha_n + i[t_n^1, +\infty))\big) = \emptyset.$$

By the definition of $v_n^\pm$ in terms of $w_n^\pm$, the above equation implies the first two properties in the proposition.

Since $u_n(i(-1, +\infty))$ lies in the interior of Π, $w_n^\pm(1/\alpha_n + i(-1, +\infty))$ must lie in the interior of $\Phi_n(S_n) + k_n$. The sets $\mathrm{int}\,(\Phi_n(S_n) + k_n)$ and $\mathrm{int}\,(\Phi_n(S_n) + k_n - 1)$ do not meet. Hence, $v_n^\pm(1/\alpha_n - 1 + i(-1, +\infty))$ are disjoint from $w_n^+(1/\alpha_n + i(-1, +\infty))$. Evidently, $v_n^\pm(1/\alpha_n - 1 - i) \neq w_n^+(1/\alpha_n - i)$. These imply the last property in the proposition.

The proof for $\varepsilon_n = +1$ is similar. □

7.2. **Elements of the dynamical partition.** Fix an arbitrary $n \geq 0$. By Prop 7.4, the curves $u_n(i(-1, +\infty))$, $v_n^\pm(1/\alpha_n - 1 + i(-1, +\infty))$ and $w_n^+(1/\alpha_n + i(-1, +\infty))$ are disjoint. These curves lie in the interior of Π_n, with their end points, $u_n(-i)$, $v_n^\pm(1/\alpha_n - 1 - i)$ and $w_n^+(1/\alpha_n - i)$, on the boundary of Π_n. Therefore, the curves u_n and w_n^+ cut off a region of Π_n which contains $v_n^\pm$. Let us denote the closure of that region with Δ_n^0. The curves $v_n^+(1/\alpha_n - 1 + i[-1, 0])$ and $v_n^-(1/\alpha_n - 1 + i[-1, 0])$ divide the set Δ_n^0 into two components. We denote the closure of the component containing $v_n^\pm(1/\alpha_n - 1 + i(0, +\infty))$ with $\mathcal{M}_n^0$.

The curve $v_n^+(1/\alpha_n - 1 + i[0, +\infty))$ divides $\mathcal{M}_n^0$ into two disjoint sets. Let $\mathcal{J}_n^0$ be the closure of the component of $\mathcal{M}_n^0 \setminus v_n^+(1/\alpha_n - 1 + i[0, +\infty))$ which contains w_n^+, and let $\mathcal{K}_n^0$ denote the closure of the connected component of $\mathcal{M}_n^0 \setminus v_n^+(1/\alpha_n - 1 + i[0, +\infty))$ which contains u_n. We have $\mathcal{M}_n^0 = \mathcal{K}_n^0 \cup \mathcal{J}_n^0$. These are in analogy with the sets M_n^0, J_n^0, and K_n^0 defined in Sec. 3.2. See Fig 7 for an illustration of the sets $\mathcal{K}_n^0$ and $\mathcal{J}_n^0$.

For every $n \geq 0$ the map Υ_n is defined on $\mathcal{M}_n^0$, and gives values in $\{w \in \mathbb{C} \mid \mathrm{Im}\, w \geq -2\}$. Each Υ_n is either holomorphic or anti-holomorphic, depending on the sign of ε_n, and $\Upsilon_n(+1) = +1$.

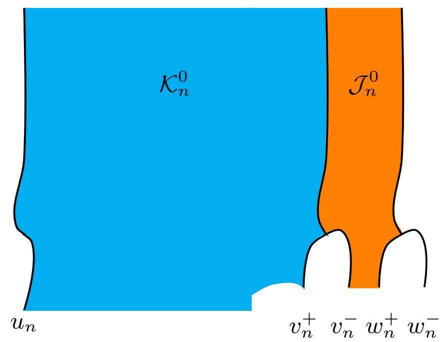

FIGURE 7. Illustration of the curves u_n, $v_n^{\pm}$ and $w_n^{\pm}$. The set $\mathcal{K}_n^0$ is coloured in blue and the set $\mathcal{J}_n^0$ is coloured in orange.

Lemma 7.5. *For every $n \geq 0$, Υ_n is injective on $\mathcal{M}_n^0$.*

Proof. Fix an arbitrary $n \geq 0$. By the definition of the curve w_n^+, the map Φ_n^{-1} is injective on the set $\mathcal{M}_n^0 \setminus (u_n \cup w_n^+)$; turn to the proof of Prop 7.1. This implies that Υ_n, which is either equal to $\mathbb{E}\mathrm{xp}^{-1} \circ \Phi_n^{-1}$ or $\mathbb{E}\mathrm{xp}^{-1} \circ s \circ \Phi_n^{-1}$, is injective on that set. On the other hand, by Prop 7.1, the curves u_n and w_n^+ are mapped to disjoint curves under Υ_n, lying on the boundary of $\Upsilon_n(\operatorname{int} \mathcal{M}_n^0)$. Therefore, Υ_n is injective on $\mathcal{M}_n^0$. ☐

Below we define the connected sets $\mathcal{M}_n^j$, $\mathcal{J}_n^j$, and $\mathcal{K}_n^j$, for $n \geq 0$ and $j \geq 0$, in analogy with the sets M_n^j, J_n^j and K_n^j in Section 3.2. Assume that $\mathcal{M}_n^j$, $\mathcal{J}_n^j$, and $\mathcal{K}_n^j$ are defined for some $j \geq 0$ and all $n \geq 0$. We define these sets for $j+1$ and all $n \geq 0$ as follows.

When $\varepsilon_{n+1} = -1$, by Prop 7.1, the set $\Upsilon_{n+1}(\mathcal{M}_{n+1}^{j-1})$ is bounded by the curves u_n and $u_n + 1$. Also, in this case, by Prop 7.3, $\Upsilon_{n+1}(\mathcal{K}_{n+1}^j) + a_n - 1$ is bounded by the curves $u_n + a_n - 1$ and w_n^+. So, for $\varepsilon_{n+1} = -1$, we let

$$\mathcal{M}_n^{j+1} = \bigcup_{l=0}^{a_n - 2} \big(\Upsilon_{n+1}(\mathcal{M}_{n+1}^j) + l\big) \bigcup \big(\Upsilon_{n+1}(\mathcal{K}_{n+1}^j) + a_n - 1\big). \tag{7.5}$$

On the other hand, when $\varepsilon_{n+1} = +1$, by Prop 7.1, the set $\Upsilon_{n+1}(\mathcal{M}_{n+1}^{j-1})$ is bounded by the curves $u_n - 1$ and u_n. Also, in this case, by Prop 7.3, $\Upsilon_{n+1}(\mathcal{J}_{n+1}^j) + a_n + 1$ is bounded by the curves $u_n + a_n$ and w_n^+. So, for $\varepsilon_{n+1} = +1$, we let

$$\mathcal{M}_n^{j+1} = \bigcup_{l=1}^{a_n} \big(\Upsilon_{n+1}(\mathcal{M}_{n+1}^j) + l\big) \bigcup \big(\Upsilon_{n+1}(\mathcal{J}_{n+1}^j) + a_n + 1\big). \tag{7.6}$$

The set $\mathcal{M}_n^{j+1}$ is closed, and bounded by piece-wise analytic curves. The interior of this set is connected. To see more details, recall the numbers $t_n^j = \operatorname{Im} Y_{n+1}(it_{n+1}^{j-1})$, with $t_n^0 = -1$, for $n \geq -1$ and $j \geq 0$. One may see that when $\varepsilon_{n+1} = -1$,

$$u_n\left(i[t_n^j, +\infty)\right) \subset \partial\mathcal{M}_n^j,\ \Upsilon_{n+1}\left(v_{n+1}^+\left(1/\alpha_{n+1} - 1 + i[t_{n+1}^{j-1}, \infty)\right)\right) + a_n - 1 \subset \partial\mathcal{M}_n^j.$$

When $\varepsilon_{n+1} = +1$,

$$\Upsilon_{n+1}\left(w_{n+1}^+\left(1/\alpha_{n+1} + i[t_{n+1}^{j-1}, +\infty)\right)\right) + 1 \subset \partial\mathcal{M}_n^j,$$

and

$$\Upsilon_{n+1}\left(v_{n+1}^-\left(1/\alpha_{n+1} - 1 + i[t_{n+1}^{j-1}, \infty)\right)\right) + a_n + 1 \subset \partial\mathcal{M}_n^j.$$

By Prop 7.3,

$$\Upsilon_{n+1}(v_{n+1}^+(1/\alpha_{n+1} - 1 + i[0, +\infty)) + a_n + \varepsilon_{n+1} - 1 = v_n^+(1/\alpha_n - 1 + i[0, +\infty)).$$

In particular, $v_n^+(1/\alpha_n - 1 + i[0,+\infty))$ divides $\mathcal{M}_n^{j+1}$ into two connected components. Let $\mathcal{J}_n^{j+1}$ denote the closure of the connected component of $\mathcal{M}_n^{j+1} \setminus v_n^+(1/\alpha_n - 1 + i[0,+\infty))$ which meets w_n^+, and let $\mathcal{K}_n^{j+1}$ denote the closure of the connected component of $\mathcal{M}_n^{j+1} \setminus v_n^+(1/\alpha_n - 1 + i[0,+\infty))$ which intersects u_n. This completes the induction step to define the sets $\mathcal{M}_n^j$, $\mathcal{K}_n^j$, and $\mathcal{J}_n^k$, for $n \geq 0$ and $j \geq 0$.

For $n = -1$, we may let

$$\mathcal{M}_{-1}^j = \Upsilon_0(\mathcal{M}_0^{j+1}) + (\varepsilon_0 + 1)/2.$$

For each $n \geq -1$, the sets $\mathcal{M}_n^j$, for $j \geq 0$, form a nest of closed sets, that is, $\mathcal{M}_n^{j+1} \subset \mathcal{M}_n^j$. Let

$$\mathcal{M}_n = \bigcap_{j=0}^{+\infty} \mathcal{M}_n^j. \tag{7.7}$$

7.3. **Uniform contraction of the change of coordinates in renormalisation.** In this section we establish a uniform contraction property of the maps Υ_n on $\mathcal{M}_n^0$, with respect to suitable hyperbolic metrics on the domain and range. This will be carried out in the fashion of Lem 6.2 and Prop 6.3.

Lemma 7.6. *There is $\delta_3 > 0$ satisfying the following property. For every $n \geq 0$, there are open sets $\tilde{\mathcal{M}}_n^0$, $\tilde{\mathcal{K}}_n^0$ and $\tilde{\mathcal{J}}_n^0$, with*

$$\mathcal{M}_n^0 \subset \tilde{\mathcal{M}}_n^0, \qquad \mathcal{K}_n^0 \subset \tilde{\mathcal{K}}_n^0 \subset \tilde{\mathcal{M}}_n^0, \qquad \mathcal{J}_n^0 \subset \tilde{\mathcal{J}}_n^0 \subset \tilde{\mathcal{M}}_n^0,$$

such that Υ_n is defined on $\tilde{\mathcal{M}}_n^0$, and the following hold

(i) *for all integers l with $(\varepsilon_{n+1}+1)/2 \leq l \leq a_n + \varepsilon_{n+1} - 1$, $B_{\delta_3}(\Upsilon_{n+1}(\tilde{\mathcal{M}}_{n+1}^0) + l) \subset \tilde{\mathcal{M}}_n^0$;*
(ii) *if $\varepsilon_{n+1} = -1$, $B_{\delta_3}(\Upsilon_{n+1}(\tilde{\mathcal{K}}_{n+1}^0) + a_n - 1) \subset \tilde{\mathcal{M}}_n^0$;*
(iii) *if $\varepsilon_{n+1} = +1$, $B_{\delta_3}(\Upsilon_{n+1}(\tilde{\mathcal{J}}_{n+1}^0) + a_n + 1) \subset \tilde{\mathcal{M}}_n^0$.*

Proof. Recall from Prop 6.3 that $\Upsilon_n : \Pi \to \Pi$ is uniformly contracting with respect to the hyperbolic metric ρ on Π. The uniform contraction factor is δ_2. Fix arbitrary constants $R > 0$ and R' such that $\delta_2 R < R' < R$. There are continuous and injective maps γ_n^+ and γ_n^- from $i[-1,+\infty)$ to Π such that $\gamma_n^+(-i) = \gamma_n^-(-i) = u_n(-i)$, and for all $t > -1$, $R' \leq d_\rho(\gamma_n^+(it), u_n(it)) \leq R$, and $R' \leq d_\rho(\gamma_n^-(it), u_n(it)) \leq R$. Moreover, we may choose γ_n^- lying on the left hand side of u_n and γ_n^+ lying on the right hand side of u_n. We consider these curves for all $n \geq 0$, but using the same constants R and R'.

As in the proof of Prop 7.1, we may use the map $\Phi_n \circ \Phi_n^{-1}$ to pull-back the curves γ_n^- and γ_n^+ to define the curves $\eta_n^\pm : 1/\alpha_n + i[-1,+\infty) \to \Pi_n$ such that

$$\Phi_n^{-1}(\gamma_n^+(it)) = \Phi_n^{-1}(\eta_n^+)(1/\alpha_n + it), \quad \Phi_n^{-1}(\gamma_n^-(t)) = \Phi_n^{-1}(\eta_n^-(1/\alpha_n + it))$$

so that η_n^+ lies to the right hand side of w_n^- with $\eta_n^+(1/\alpha_n - i) = w_n^-(1/\alpha_n - i)$ and η_n^- lies to the left hand side of w_n^+ with $\eta_n^-(1/\alpha_n - i) = w_n^+(1/\alpha_n - i)$. Similarly, we define

$$\nu_n^+(1/\alpha_n - 1 + it) = \eta_n^+(1/\alpha_n + it) - 1, \quad \nu_n^-(1/\alpha_n - 1 + it) = \eta_n^-(1/\alpha_n + it).$$

Then, ν_n^+ lies on the right hand side of v_n^- with $\nu_n^+(1/\alpha_n - 1 - i) = v_n^-(1/\alpha_n - 1 - i)$, and ν_n^- lies on the left hand side of v_n^+ with $\nu_n^-(1/\alpha_n - 1 - i) = v_n^+(1/\alpha_n - 1 - i)$. See Fig 8 for an illustration of these curves.

We define the domain $\tilde{\mathcal{M}}_n^0$ as the region inside Π_n cut off by the curves γ_n^- and η_n^+. Also, $\tilde{\mathcal{K}}_n^0$ denotes the region inside Π_n cut off by the curves γ_n^- and ν_n^+. Similarly, $\tilde{\mathcal{J}}_n^0$ denotes the region inside Π_n cut off by the curves ν_n^- and η_n^+. We have $\tilde{\mathcal{M}}_n^0 = \tilde{\mathcal{K}}_n^0 \cup \tilde{\mathcal{J}}_n^0$.

Let V_n denote the region in Π enclosed by $\gamma_n^- \cup \gamma_n^+$. Since Υ_{n+1} sends u_{n+1} to u_n, and is uniformly contracting with respect to ρ, it maps V_{n+1} inside V_n. Indeed, it follows from Lem 6.2 that ρ is uniformly bounded from above and below on $\Upsilon_{n+1}(V_{n+1})$. This implies that there is $\delta > 0$, independent of n, such that $B_\delta(\Upsilon_{n+1}(V_{n+1})) \subset V_n$. It follows that when $\varepsilon_{n+1} = -1$, the

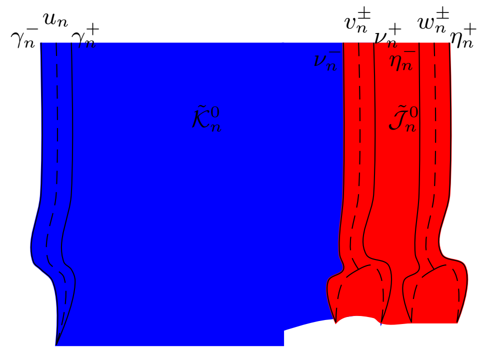

FIGURE 8. Illustration of the curves $\gamma_n^\pm$, $\nu_n^\pm$ and $\eta_n^\pm$ in the proof of Lem 7.6. The set $\tilde{\mathcal{K}}_n^0$ is drawn in blue and the set $\tilde{\mathcal{J}}_n^0$ is drawn in red. These two sets overlap on the darker red set.

δ-neighbourhood of the curves $\Upsilon_{n+1}(\gamma_{n+1}^-)$ and $\Upsilon_{n+1}(\nu_{n+1}^+)+a_n-1$ are contained in $\mathcal{M}_n^0$. Similarly, when $\varepsilon_{n+1}=+1$, the δ-neighbourhood of the curves $\Upsilon_{n+1}(\eta_{n+1}^-)+1$ and $\Upsilon_{n+1}(\nu_{n+1}^+)+a_n+1$ are contained in $\mathcal{M}_n^0$.

The set $\cup_{i=0}^{k_n} f_n^{\circ i}(S_n)$ is compactly contained in the domain of f_n and also compactly contained in the image of f_n. Note that by Prop 4.3, k_n is uniformly bounded from above, independent of n. By the pre-compactness of the class $\mathcal{QIS}$, there is $\delta'>0$, independent of n, such that δ'-neighbourhood of $\cup_{i=0}^{k_n} f_n^{\circ i}(S_n)$ is contained in the domain and also in the image of f_n. This implies that there is $\delta'>0$, independent of n, such that δ'-neighbourhood of $\Upsilon_n(\partial\tilde{\mathcal{M}}_n^0\setminus(\gamma_n^-\cup\eta_n^+))$ is contained in $\mathcal{M}_{n-1}^0$.

We define $\delta_3=\min\{\delta,\delta'\}$. □

Let $\varrho_n|dz|$ denote the hyperbolic metric of constant curvature -1 on $\tilde{\mathcal{M}}_n^0$, for $n\geq 0$.

Proposition 7.7. *There is a constant $\delta_5\in(0,1)$ such that for all $n\geq 0$ we have,*

(i) *for all integers l with $(\varepsilon_{n+1}+1)/2\leq l\leq a_n+\varepsilon_{n+1}-1$, and all $z\in\mathcal{M}_{n+1}^0$,*
$$(\Upsilon_{n+1}+l)^*\varrho_n(z)\leq\delta_5\varrho_{n+1}(z);$$
(ii) *if $\varepsilon_{n+1}=-1$, for all $z\in\mathcal{K}_{n+1}^0$, $(\Upsilon_{n+1}+a_n-1)^*\varrho_n(z)\leq\delta_5\varrho_{n+1}(z)$;*
(iii) *if $\varepsilon_{n+1}=+1$, for all $z\in\mathcal{J}_{n+1}^0$, $(\Upsilon_{n+1}+a_n+1)^*\varrho_n(z)\leq\delta_5\varrho_{n+1}(z)$.*

Proof. One may repeat the proof of Prop 6.3; replacing Lem 6.2 by Lem 7.6. We obtain the uniform contractions with respect to the corresponding hyperbolic metrics on $\tilde{\mathcal{M}}_{n+1}^0$ and $\tilde{\mathcal{M}}_n^0$. In particular, the uniform contractions also hold on $\mathcal{M}_{n+1}^0$, $\mathcal{K}_{n+1}^0$, and $\mathcal{J}_{n+1}^0$. □

7.4. **Iterates, shifts, and lifts.** Here we related the iterates of the map f to the translations by integers and lifts in the renormalisation tower of f. To do that we need to define the notion of trajectory for a given point in $\mathcal{M}_{-1}$. This is in analogy with the corresponding notion for points in the arithmetic model M_{-1}, presented in Sec. 3.5.

Given $z_{-1}\in\mathcal{M}_{-1}$, we inductively define $z_i\in\mathcal{M}_i$ and $l_i\in\mathbb{Z}$, for $i\geq 0$, as follows. There is a unique $z_0\in\mathcal{M}_0$ such that $z_{-1}=\Upsilon_0(z_0)+(\varepsilon_0+1)/2$. Then, inductively identify the integer l_i and the point $z_{i+1}\in\mathcal{M}_{i+1}$, for $i\geq 0$, so that

$$z_i-l_i\in\Upsilon_{i+1}(\mathcal{M}_{i+1}),\qquad\Upsilon_{i+1}(z_{i+1})+l_i=z_i. \tag{7.8}$$

It follows that for all $n\geq 0$, we have

$$z_{-1}=(\Upsilon_0+(\varepsilon_0+1)/2)\circ(\Upsilon_1+l_0)\circ\cdots\circ(\Upsilon_n+l_{n-1})(z_n), \tag{7.9}$$

and by (7.5) and (7.6), for all $i \geq 0$,

$$(1+\varepsilon_{i+1})/2 \leq l_i \leq a_i + \varepsilon_{i+1}. \tag{7.10}$$

We refer to the sequence $(z_i; l_i)_{i\geq 0}$ as the **trajectory** of z_{-1}, in the renormalisation tower of f, or simply, as the trajectory of z_{-1}, when it is clear what map is involved. This algorithm might not associate a unique sequence $(z_i; l_i)_{i\geq 0}$ to some z_{-1}. That is because, for some z_i there might be two integers l_i satisfying (7.8). By the trajectory of z_{-1}, we mean any sequence $(z_i; l_i)_{i\geq 0}$ which satisfies both (7.9) and (7.10).

Lemma 7.8. *Let $p \geq 0$, and assume that we have $w_1 \in \mathcal{J}_p^0$, $w_2 \in \Upsilon_{p+1}(\mathcal{M}_{p+1}^0) + (\varepsilon_p + 1)/2$, and $\mathcal{R}(f_p)(\mathbb{E}\mathrm{xp}(w_1)) = \mathbb{E}\mathrm{xp}(w_2)$. Then, $f_p(\Phi_p^{-1}(w_1)) = \Phi_p^{-1}(w_2)$.*

Proof. Recall the set $S_p = S_{f_p}$ from Sec. 4.2, and that $\mathbb{E}\mathrm{xp}(\Phi_p(S_p)) = \mathrm{Dom}\,\mathcal{R}(f_p)$. Since $\mathbb{E}\mathrm{xp}(w_1)$ belongs to $\mathrm{Dom}\,\mathcal{R}(f_p)$, there is an integer l_1 such that $w_1 - l_1 \in \Phi_p(S_p)$. By the definition of renormalisation, see Sec. 4.2,

$$\mathbb{E}\mathrm{xp} \circ \Phi_p \circ f_p^{\circ k_p} \circ \Phi_p^{-1}(w_1 - l_1) = \mathcal{R}(f_p)(\mathbb{E}\mathrm{xp}(w_1)).$$

Therefore, by the hypothesis in the lemma, we must have

$$\mathbb{E}\mathrm{xp} \circ \Phi_p \circ f_p^{\circ k_p} \circ \Phi_p^{-1}(w_1 - l_1) = \mathbb{E}\mathrm{xp}(w_2).$$

This implies that there is $l_2 \in \mathbb{Z}$ such that

$$\Phi_p \circ f_p^{\circ k_p} \circ \Phi_p^{-1}(w_1 - l_1) + l_2 = w_2.$$

Now we look at the relation between l_1 and l_2. Recall that $\mathcal{J}_p^0$ is bounded by the curves w_p^+, $v_p^- = w_p^- - 1$, and part of the boundary of $\Phi_p(S_p) + \mathbb{Z}$. By (7.2), w_p^+ is contained in $\Phi_p(S_p) + k_p$ and v_p^- is contained in $\Phi_p(S_p) + k_p - 1$. Thus, $\mathcal{J}_p^0 \subset (\Phi_p(S_p) + k_p - 1) \cup (\Phi_p(S_p) + k_p)$. Since $w_1 - l_1 \in \Phi_p(S_p)$, we conclude that either $l_1 = k_p$ or $l_1 = k_p - 1$.

Recall that the curves u_p and $u_p + 1$ lie on the boundary of $\Upsilon_{p+1}(\mathcal{M}_{p+1}^0) + (\varepsilon_p + 1)/2$, and u_p is contained in $\Pi = \Phi_p(A_{f_p} \cup B_{f_p})$. In particular, $l_2 \in \{0, 1\}$. However, using (7.1), we note that when $l_1 = k_p$, we must have $l_2 = 1$. Similarly, when $l_1 = k_p - 1$, we must have $l_2 = 0$. In each of these cases we look back at the above equation. In the former case, we obtain $\Phi_p \circ f_p^{\circ k_p} \circ \Phi_p^{-1}(w_1 - k_p) + 1 = w_2$. Applying Φ_p^{-1} and using 4.1-(e), we conclude that

$$\Phi_p^{-1}(w_2) = \Phi_p^{-1}\left(\Phi_p \circ f_p^{\circ k_p} \circ \Phi_p^{-1}(w_1 - k_p) + 1\right) = f \circ \left(f_p^{\circ k_p} \circ \Phi_p^{-1}(w_1 - k_p)\right) = f_p(\Phi_p^{-1}(w_1)).$$

The latter case directly gives the desired relation in the proposition. ∎

Recall that for each $n \geq 0$, $\mathcal{M}_n \subset \mathcal{M}_n^0 = \mathcal{K}_n^0 \cup \mathcal{J}_n^0$. For $n \geq 0$, we consider the map

$$\mathcal{E}_n : J_n^0 \to \mathcal{M}_n^0, \quad \mathcal{E}_n(z) = \Phi_n \circ f_n \circ \Phi_n^{-1}(z). \tag{7.11}$$

Compare the above map to the one in (4.3).

Proposition 7.9. *Assume that $\alpha \in \mathrm{HT}_N$, $f \in \mathcal{QIS}_\alpha$, and $z_{-1} \in \mathcal{M}_{-1}$ is an arbitrary point with trajectory $(z_i; l_i)_{i\geq 0}$. The following hold:*

(i) *if there is $n \geq 0$ such that $z_n \in \mathcal{K}_n$ and for all $0 \leq i \leq n-1$, $z_i \in \mathcal{M}_i \setminus \mathcal{K}_i$, then*

$$s \circ f \circ s(\mathbb{E}\mathrm{xp}(z_{-1})) = \mathbb{E}\mathrm{xp}\left(\Upsilon_0 + \frac{\varepsilon_0 + 1}{2}\right) \circ \left(\Upsilon_1 + \frac{\varepsilon_1 + 1}{2}\right) \circ \cdots \circ \left(\Upsilon_n + \frac{\varepsilon_n + 1}{2}\right)(z_n + 1).$$

(ii) *if for all $i \geq 0$, $z_i \in \mathcal{M}_i \setminus \mathcal{K}_i$, then for all $n \geq 0$,*

$$s \circ f \circ s(\mathbb{E}\mathrm{xp}(z_{-1})) = \mathbb{E}\mathrm{xp}\left(\Upsilon_0 + \frac{\varepsilon_0 + 1}{2}\right) \circ \left(\Upsilon_1 + \frac{\varepsilon_1 + 1}{2}\right) \circ \cdots \circ \left(\Upsilon_n + \frac{\varepsilon_n + 1}{2}\right)(\mathcal{E}_n(z_n)).$$

Proof. Part (i). Since $z_n \in \mathcal{K}_n^0$, $z_n + 1 \in \mathcal{M}_n^0$ and $\Upsilon_n(z_n+1) + (\varepsilon_n+1)/2$ is defined and belongs to $\mathcal{M}_{n-1}^0$.

By the functional equation for the Fatou coordinates in Prop 4.1-(e), we have

$$f_n \circ \Phi_n^{-1}(z_n) = \Phi_n^{-1}(z_n+1). \tag{7.12}$$

Now we consider two cases, based on whether $n = 0$ or $n \geq 1$.

First assume that $n = 0$. If $\varepsilon_0 = -1$, then, by Prop 6.1, $f_0 = s \circ f \circ s$, and by (6.1), $\mathbb{E}\mathrm{xp} \circ \Upsilon_0 = \Phi_0^{-1}$. Moreover, by the definition of trajectory, $\mathbb{E}\mathrm{xp}(z_{-1}) = \Phi_0^{-1}(z_0)$. Thus, using (7.12) with $n = 0$, and Prop 4.1-(e), we get

$$\begin{aligned} s \circ f \circ s(\mathbb{E}\mathrm{xp}(z_{-1})) = f_0(\Phi_0^{-1}(z_0)) &= \Phi_0^{-1}(z_0+1) \\ &= \mathbb{E}\mathrm{xp} \circ \Upsilon_0(z_0+1) = \mathbb{E}\mathrm{xp}(\Upsilon_0 + (\varepsilon_0+1)/2)(z_0+1). \end{aligned}$$

Similarly, if $\varepsilon_0 = +1$, $f_0 = f$ and $\mathbb{E}\mathrm{xp} \circ \Upsilon_0 = s \circ \Phi_0^{-1}$. Then, $\mathbb{E}\mathrm{xp}(z_{-1}) = s \circ \Phi_0^{-1}(z_0)$. Therefore,

$$\begin{aligned} s \circ f \circ s(\mathbb{E}\mathrm{xp}(z_{-1})) = s \circ f(\Phi_0^{-1}(z_0)) &= s \circ f_0(\Phi_0^{-1}(z_0)) \\ &= s \circ \Phi_0^{-1}(z_0+1) = \mathbb{E}\mathrm{xp}(\Upsilon_0 + (\varepsilon_0+1)/2)(z_0+1). \end{aligned}$$

This completes the proof when $n = 0$.

Now assume that $n \geq 1$. By considering two cases based on $\varepsilon_n = \pm 1$, as in the previous case, one may see that (7.12) and (7.8), imply that

$$\mathcal{R}(f_{n-1})(\mathbb{E}\mathrm{xp}(z_{n-1})) = \mathbb{E}\mathrm{xp}\left(\Upsilon_n + \frac{\varepsilon_n+1}{2}\right)(z_n+1). \tag{7.13}$$

Then, we apply Lem 7.8, with $p = n-1$, $w_2 = (\Upsilon_n + (\varepsilon_n+1)/2)(z_n+1)$, and $w_1 = z_{n-1}$ to obtain

$$f_{n-1} \circ \Phi_{n-1}^{-1}(z_{n-1}) = \Phi_{n-1}^{-1}(\Upsilon_n + (\varepsilon_n+1)/2)(z_n+1). \tag{7.14}$$

Compare the above relation to the one in (7.12).

We repeat the above process, replacing the relation in (7.12) with the one in (7.14). If $n-1 = 0$, we conclude that

$$s \circ f \circ s(\mathbb{E}\mathrm{xp}(z_{n-2})) = \mathbb{E}\mathrm{xp}\left(\Upsilon_{n-1} + \frac{\varepsilon_{n-1}+1}{2}\right) \circ \left(\Upsilon_n + \frac{\varepsilon_n+1}{2}\right)(z_n+1).$$

This is the desired relation in Part (i), when $n-1 = 0$. If $n-1 \geq 1$, (7.14) implies that

$$\mathcal{R}(f_{n-2})(\mathbb{E}\mathrm{xp}(z_{n-2})) = \mathbb{E}\mathrm{xp}\left(\Upsilon_{n-1} + \frac{\varepsilon_{n-1}+1}{2}\right) \circ \left(\Upsilon_n + \frac{\varepsilon_n+1}{2}\right)(z_n+1).$$

Repeating the above process, until we reach level 0, leads to the desired relation in Part (i).

Part(ii). By the definition of renormalisation in Sec. 4.2, the map $\mathcal{E}_n$ induces the relation

$$\mathcal{R}(f_n)(\mathbb{E}\mathrm{xp}(z_n)) = \mathbb{E}\mathrm{xp}(\mathcal{E}_n(z_n)).$$

By Lem 7.8, the above relation implies that

$$f_n \circ \Phi_n^{-1}(z_n) = \Phi_n^{-1}(\mathcal{E}_n(z_n)). \tag{7.15}$$

Now one may repeat the argument in Part (i); replacing (7.12) with (7.15). □

Proposition 7.10. *Assume that $\alpha \in \mathrm{HT}_N$, $f \in \mathcal{QIS}_\alpha$, and $z_{-1} \in \mathcal{M}_{-1}$ with trajectory $(z_i; l_i)_{i \geq 0}$. For every $m \geq 1$, there is a sequence of integers $(p_i)_{i=0}^{v}$ such that either*

$$\begin{aligned} &s \circ f^{\circ m} \circ s(\mathbb{E}\mathrm{xp}(z_{-1})) \\ &\quad = \mathbb{E}\mathrm{xp} \circ (\Upsilon_0 + (\varepsilon_0+1)/2) \circ (\Upsilon_1 + p_0) \circ (\Upsilon_2 + p_1) \circ \cdots \circ (\Upsilon_v + p_{v-1})(z_v + p_v), \end{aligned} \tag{7.16}$$

or

$$s \circ f^{\circ m} \circ s(\mathbb{E}\mathrm{xp}(z_{-1})) = \mathbb{E}\mathrm{xp} \circ (\Upsilon_0 + (\varepsilon_0+1)/2) \circ (\Upsilon_1 + p_0) \circ (\Upsilon_2 + p_1) \circ \cdots \circ (\Upsilon_v + p_{v-1})(\mathcal{E}_v(z_v)). \tag{7.17}$$

Each map Υ_j in the above proposition is only considered on the set $\mathcal{M}_j$.

Proof. We prove the proposition by induction on m. The statement for $m=1$ follows directly from Prop 7.9. Assume that the statement holds for some $m-1 \geq 1$. We aim to prove it for m.

By the induction hypothesis, there is a finite sequence of integers $(j_i)_{i=0}^n$ such that either

$$s \circ f^{\circ m-1} \circ s(\mathbb{E}\mathrm{xp}(z_{-1})) = \mathbb{E}\mathrm{xp} \circ (\Upsilon_0 + (\varepsilon_0+1)/2) \circ (\Upsilon_1 + j_0) \circ (\Upsilon_2 + j_1) \circ \cdots \circ (\Upsilon_n + j_{n-1})(z_n + j_n). \tag{7.18}$$

or

$$s \circ f^{\circ m-1} \circ s(\mathbb{E}\mathrm{xp}(z_{-1})) = \mathbb{E}\mathrm{xp} \circ (\Upsilon_0 + (\varepsilon_0+1)/2) \circ (\Upsilon_1 + j_0) \circ (\Upsilon_2 + j_1) \circ \cdots \circ (\Upsilon_n + j_{n-1})(\mathcal{E}_n(z_n)). \tag{7.19}$$

Let us first assume that (7.18) holds. Define

$$w_{-1} = (\Upsilon_0 + (\varepsilon_0+1)/2) \circ (\Upsilon_1 + j_0) \circ (\Upsilon_2 + j_1) \circ \cdots \circ (\Upsilon_n + j_{n-1})(z_n + j_n).$$

The point w_{-1} has a trajectory $(w_i; s_i)_{i \geq 0}$ which satisfies the relations

$$w_i = (\Upsilon_{i+1} + j_i) \circ \cdots \circ (\Upsilon_n + j_{n-1})(z_n + j_n), \quad \text{for } 0 \leq i \leq n-1, \tag{7.20}$$

$$w_n = z_n + j_n, \tag{7.21}$$

$$w_i = z_i, \quad \text{for } i \geq n+1. \tag{7.22}$$

By (7.18),

$$s \circ f^{\circ m} \circ s(\mathbb{E}\mathrm{xp}(z_{-1})) = (s \circ f \circ s) \circ s \circ f^{\circ m-1} \circ s(\mathbb{E}\mathrm{xp}(z_{-1})) = s \circ f \circ s(\mathbb{E}\mathrm{xp}(w_{-1})). \tag{7.23}$$

Now, we consider two cases:

Case 1: There is $m \geq 0$ such that $w_m \in \mathcal{K}_m^0$, and for all $0 \leq i \leq m-1$, $w_i \in \mathcal{M}_i \setminus \mathcal{K}_i$.

In this case, we may employ Prop 7.9, to obtain

$$s \circ f \circ s(\mathbb{E}\mathrm{xp}(w_{-1})) = \mathbb{E}\mathrm{xp}\left(\Upsilon_0 + \frac{\varepsilon_0+1}{2}\right) \circ \left(\Upsilon_1 + \frac{\varepsilon_1+1}{2}\right) \circ \cdots \circ \left(\Upsilon_m + \frac{\varepsilon_m+1}{2}\right)(w_m+1). \tag{7.24}$$

There are three scenarios based on the value of m relative n.

If $m \leq n-1$, by (7.20),

$$\begin{aligned} &\mathbb{E}\mathrm{xp}\left(\Upsilon_0 + \frac{\varepsilon_0+1}{2}\right) \circ \left(\Upsilon_1 + \frac{\varepsilon_1+1}{2}\right) \circ \cdots \circ \left(\Upsilon_m + \frac{\varepsilon_m+1}{2}\right)(w_m+1) \\ &\quad = \mathbb{E}\mathrm{xp}\left(\Upsilon_0 + \frac{\varepsilon_0+1}{2}\right) \circ \left(\Upsilon_1 + \frac{\varepsilon_1+1}{2}\right) \circ \cdots \circ \left(\Upsilon_m + \frac{\varepsilon_m+1}{2}\right) \\ &\qquad \circ (\Upsilon_{m+1} + j_m + 1) \circ \cdots \circ (\Upsilon_n + j_{n-1})(z_n + j_n). \end{aligned}$$

If $m = n$, by (7.21),

$$\begin{aligned} &\mathbb{E}\mathrm{xp}\left(\Upsilon_0 + \frac{\varepsilon_0+1}{2}\right) \circ \left(\Upsilon_1 + \frac{\varepsilon_1+1}{2}\right) \circ \cdots \circ \left(\Upsilon_m + \frac{\varepsilon_m+1}{2}\right)(w_m+1) \\ &\quad = \mathbb{E}\mathrm{xp}\left(\Upsilon_0 + \frac{\varepsilon_0+1}{2}\right) \circ \left(\Upsilon_1 + \frac{\varepsilon_1+1}{2}\right) \circ \cdots \circ \left(\Upsilon_n + \frac{\varepsilon_n+1}{2}\right)(z_n + j_n + 1). \end{aligned}$$

If $m \geq n+1$, by (7.22),

$$\begin{aligned}\mathbb{E}\mathrm{xp}\left(\Upsilon_0+\frac{\varepsilon_0+1}{2}\right)\circ\left(\Upsilon_1+\frac{\varepsilon_1+1}{2}\right)\circ\cdots\circ\left(\Upsilon_m+\frac{\varepsilon_m+1}{2}\right)(w_m+1)\\ =\mathbb{E}\mathrm{xp}\left(\Upsilon_0+\frac{\varepsilon_0+1}{2}\right)\circ\left(\Upsilon_1+\frac{\varepsilon_1+1}{2}\right)\circ\cdots\circ\left(\Upsilon_m+\frac{\varepsilon_m+1}{2}\right)(z_m+1).\end{aligned}$$

Combining the above relations with (7.23) and (7.24), we conclude that (7.16) holds, for some integers j_i.

Case 2: There is no $m \geq 0$ with $w_m \in \mathcal{K}_m$.

By Prop 7.9, and (7.22), we obtain

$$\begin{aligned}&s\circ f\circ s(\mathbb{E}\mathrm{xp}(w_{-1}))\\ &=\mathbb{E}\mathrm{xp}\left(\Upsilon_0+\frac{\varepsilon_0+1}{2}\right)\circ\left(\Upsilon_1+\frac{\varepsilon_1+1}{2}\right)\circ\cdots\circ\left(\Upsilon_{n+1}+\frac{\varepsilon_{n+1}+1}{2}\right)(\mathcal{E}_{n+1}(w_{n+1}))\\ &=\mathbb{E}\mathrm{xp}\left(\Upsilon_0+\frac{\varepsilon_0+1}{2}\right)\circ\left(\Upsilon_1+\frac{\varepsilon_1+1}{2}\right)\circ\cdots\circ\left(\Upsilon_{n+1}+\frac{\varepsilon_{n+1}+1}{2}\right)(\mathcal{E}_{n+1}(z_{n+1})).\end{aligned}$$

This implies that (7.17) holds, for some integers j_i.

Now assume that (7.19) holds. Define

$$w_{-1}=(\Upsilon_0+(\varepsilon_0+1)/2)\circ(\Upsilon_1+j_0)\circ(\Upsilon_2+j_1)\circ\cdots\circ(\Upsilon_n+j_{n-1})(\mathcal{E}_n(z_n)).$$

The point w_{-1} has a trajectory $(w_i;s_i)_{i\geq 0}$ which satisfies

$$w_i=(\Upsilon_{i+1}+j_i)\circ\cdots\circ(\Upsilon_n+j_{n-1})(\mathcal{E}_n(z_n)),\quad \text{for } 0\leq i\leq n-1, \tag{7.25}$$

$$w_n=\mathcal{E}_n(z_n). \tag{7.26}$$

By (7.19),

$$s\circ f^{\circ m}\circ s(\mathbb{E}\mathrm{xp}(z_{-1}))=(s\circ f\circ s)\circ s\circ f^{\circ m-1}\circ s(\mathbb{E}\mathrm{xp}(z_{-1}))=s\circ f\circ s(\mathbb{E}\mathrm{xp}(w_{-1})). \tag{7.27}$$

Now we consider two cases.

Case 1: There is $m \leq n-1$, such that $w_m \in \mathcal{K}_m^0$, and for all $0 \leq i \leq m-1$, we have $w_i \in \mathcal{M}_i^0 \setminus \mathcal{K}_i^0$.

In this case, using Prop 7.9, and (7.25),

$$\begin{aligned}&s\circ f\circ s(\mathbb{E}\mathrm{xp}(w_{-1}))\\ &=\mathbb{E}\mathrm{xp}\left(\Upsilon_0+\frac{\varepsilon_0+1}{2}\right)\circ\left(\Upsilon_1+\frac{\varepsilon_1+1}{2}\right)\circ\cdots\circ\left(\Upsilon_m+\frac{\varepsilon_m+1}{2}\right)(w_m+1)\\ &=\mathbb{E}\mathrm{xp}\left(\Upsilon_0+\frac{\varepsilon_0+1}{2}\right)\circ\left(\Upsilon_1+\frac{\varepsilon_1+1}{2}\right)\circ\cdots\circ\left(\Upsilon_m+\frac{\varepsilon_m+1}{2}\right)\\ &\qquad\circ(\Upsilon_{m+1}+j_m+1)\circ\cdots\circ(\Upsilon_n+j_{n-1})(\mathcal{E}_n(z_n)).\end{aligned}$$

Case 2: For all i with $0 \leq i \leq n-1$, $w_i \in \mathcal{M}_i^0 \setminus \mathcal{K}_i^0$.

By Prop 7.9,

$$\begin{aligned}&s\circ f\circ s(\mathbb{E}\mathrm{xp}(w_{-1}))\\ &=\mathbb{E}\mathrm{xp}\left(\Upsilon_0+\frac{\varepsilon_0+1}{2}\right)\circ\left(\Upsilon_1+\frac{\varepsilon_1+1}{2}\right)\circ\cdots\circ\left(\Upsilon_n+\frac{\varepsilon_n+1}{2}\right)(w_n+1)\end{aligned} \tag{7.28}$$

There are two scenarios based on whether $z_{n+1} \in \mathcal{K}_{n+1}^0$, or not.

If $z_{n+1} \in \mathcal{K}_{n+1}$, then $z_{n+1}+1 \in \mathcal{M}_{n+1}$, and hence $f_{n+1} \circ \Phi_{n+1}^{-1}(z_{n+1}) = \Phi_{n+1}^{-1}(z_{n+1}+1)$. The latter relation implies that

$$\mathcal{R}(f_n)(\mathbb{E}\mathrm{xp}(z_n))=\mathbb{E}\mathrm{xp}\circ(\Upsilon_{n+1}+(\varepsilon_{n+1}+1)/2)(z_{n+1}+1).$$

We may apply Lem 7.8, to get

$$f_n \circ \Phi_n^{-1}(z_n) = \Phi_n^{-1} \circ (\Upsilon_{n+1} + (\varepsilon_{n+1}+1)/2)(z_{n+1}+1).$$

This implies that

$$\mathcal{E}_n(z_n) = \Phi_n \circ f_n \circ \Phi_n^{-1}(z_n) = (\Upsilon_{n+1} + (\varepsilon_{n+1}+1)/2)(z_{n+1}+1).$$

By (7.26), and the above relation, we obtain

$$\begin{aligned}
&\mathbb{E}\mathrm{xp}\left(\Upsilon_0 + \frac{\varepsilon_0+1}{2}\right) \circ \left(\Upsilon_1 + \frac{\varepsilon_1+1}{2}\right) \circ \cdots \circ \left(\Upsilon_n + \frac{\varepsilon_n+1}{2}\right)(w_n+1)\\
&\quad = \mathbb{E}\mathrm{xp}\left(\Upsilon_0 + \frac{\varepsilon_0+1}{2}\right) \circ \left(\Upsilon_1 + \frac{\varepsilon_1+1}{2}\right) \circ \cdots \circ \left(\Upsilon_n + \frac{\varepsilon_n+1}{2}\right)(\mathcal{E}_n(z_n)+1)\\
&\quad = \mathbb{E}\mathrm{xp}\left(\Upsilon_0 + \frac{\varepsilon_0+1}{2}\right) \circ \left(\Upsilon_1 + \frac{\varepsilon_1+1}{2}\right) \circ \cdots \circ \left(\Upsilon_n + \frac{\varepsilon_n+1}{2}\right)\\
&\qquad\qquad \circ \left(\Upsilon_{n+1} + \frac{\varepsilon_{n+1}+1}{2} + 1\right)(z_{n+1}+1).
\end{aligned}$$

If $z_{n+1} \in \mathcal{M}_{n+1} \setminus \mathcal{K}_{n+1}$, as in the previous case, one may see that

$$\mathcal{E}_n(z_n) = (\Upsilon_{n+1} + (\varepsilon_{n+1}+1)/2)(\mathcal{E}_{n+1}(z_{n+1})).$$

Therefore, by (7.26), and the above relation, we obtain

$$\begin{aligned}
&\mathbb{E}\mathrm{xp}\left(\Upsilon_0 + \frac{\varepsilon_0+1}{2}\right) \circ \left(\Upsilon_1 + \frac{\varepsilon_1+1}{2}\right) \circ \cdots \circ \left(\Upsilon_n + \frac{\varepsilon_n+1}{2}\right)(w_n+1)\\
&\quad = \mathbb{E}\mathrm{xp}\left(\Upsilon_0 + \frac{\varepsilon_0+1}{2}\right) \circ \left(\Upsilon_1 + \frac{\varepsilon_1+1}{2}\right) \circ \cdots \circ \left(\Upsilon_n + \frac{\varepsilon_n+1}{2}\right)(\mathcal{E}_n(z_n)+1)\\
&\quad = \mathbb{E}\mathrm{xp}\left(\Upsilon_0 + \frac{\varepsilon_0+1}{2}\right) \circ \left(\Upsilon_1 + \frac{\varepsilon_1+1}{2}\right) \circ \cdots \circ \left(\Upsilon_n + \frac{\varepsilon_n+1}{2}\right)\\
&\qquad\qquad \circ \left(\Upsilon_{n+1} + \frac{\varepsilon_{n+1}+1}{2} + 1\right)(\mathcal{E}_{n+1}(z_{n+1})).
\end{aligned}$$

This completes the proof of the proposition. □

Remark 7.11. Assume that a point z_{-1} in $\mathcal{M}_{-1}$ has a trajectory $(z_i; l_i)_{i\geq 0}$ such that there are infinitely many distinct $n \geq 0$ with $z_n \in \mathcal{K}_n^0$. It is evident from the proof of Prop 7.10 that for every $m \geq 1$, one may always have (7.16). For instance, when $\varepsilon_i = -1$, for all $i \geq 0$, one may employ the uniform contraction in Prop 7.7, to conclude that for every $z_{-1} \in \mathcal{M}_{-1}$, infinitely often $z_n \in \mathcal{K}_n^0$. This feature holds for many other values of α. However, when $\varepsilon_i = +1$, for all $i \geq 0$, there are $z_{-1} \in \mathcal{M}_{-1}$ such that for all $n \geq 0$, $z_n \in \mathcal{M}_n \setminus \mathcal{K}_n^0$. For such values of z_{-1}, it is not possible to have (7.16) for all $m \geq 1$. It may not be clear at this point that the set of such z_{-1} forms a countable union of continuous curves in $\mathcal{M}_{-1}$. This will become clear when we establish the relation between $\mathcal{M}_{-1}$ and M_{-1} in Sec. 8.

The inverse of the statement in Prop 7.10 is also true, which we state below.

Proposition 7.12. *Assume that $\alpha \in \mathrm{HT}_N$, $f \in \mathcal{QIS}_\alpha$, and $z_{-1} \in \mathcal{M}_{-1}$ is an arbitrary point with trajectory $(z_i; l_i)_{i\geq 0}$. Then, the following hold:*

- (i) *for every sequence of integers $(p_i)_{i=0}^{v}$ with $p_v \geq 1$, there is $m \geq 1$ such that* (7.16) *holds, provided each Υ_j in the right hand side of* (7.16) *is considered on $\mathcal{M}_j$.*
- (ii) *for every sequence of integers $(p_i)_{i=0}^{v-1}$, there is $m \geq 1$ such that* (7.17) *holds, provided each Υ_j in the right hand side of* (7.17) *is considered on $\mathcal{M}_j$.*

We will not use the above proposition in this paper, it is only stated for the record.

Proof. This follows from the definition of renormalisation. By Prop 4.1-(e), each translation in $\mathcal{M}_j$ corresponds to an iterate of f_j. Each iterate of f_j corresponds to an iterate of $\mathcal{R}(f_{j-1})$. Each iterate of $\mathcal{R}(f_{j-1})$ corresponds to a finite number of iterates by f_{j-1}. Combining these steps, one concludes that the translations and lifts correspond to some iterate of f under the changes of coordinates. The argument is similar to the proof of Lem 7.8, so we leave the details to the reader. One may consult the proofs of similar statements in [Che13, Che19, AC18]. ☐

7.5. **Capturing the post-critical set.** We define the set

$$\begin{aligned} \mathcal{A}_f &= \partial(s \circ \mathbb{E}\mathrm{xp}(\mathcal{M}_{-1})) \setminus \{0\}, \quad \text{if } \alpha \in \mathscr{B}, \\ \mathcal{A}_f &= s \circ \mathbb{E}\mathrm{xp}(\mathcal{M}_{-1}) \cup \{0\}\,, \quad \text{if } \alpha \notin \mathscr{B}. \end{aligned} \tag{7.29}$$

Recall from Sec. 1 that the post-critical set of a map f is denoted by $\Lambda(f)$. In this section we prove that $\Lambda(f) \subseteq \mathcal{A}_f$, and later in Sec. 8.4 we prove that $\Lambda(f) = \mathcal{A}_f$.

Lemma 7.13. *For every $n \geq 0$, and every integer l with $0 \leq l \leq a_n + \varepsilon_n$, we have*

$$(u_n(i[-1,0)) + l) \cap \mathcal{M}_n^0 = \emptyset,\ w_n^{\pm}(1/\alpha_n + i[-1,0)) \cap \mathcal{M}_n^0 = \emptyset,\ v_n^{\pm}(1/\alpha_n - 1 + i[-1,0)) \cap \mathcal{M}_n^0 = \emptyset.$$

Proof. Fix arbitrary $n \geq 0$ and $0 \leq l \leq a_n + \varepsilon_n$. Recall from Prop 7.1 and Prop 7.3 that for all $i \geq 0$, $w_i^+(1/\alpha_i) = w_i^-(1/\alpha_i)$ and $v_i^+(1/\alpha_i - 1) = v_i^-(1/\alpha_i - 1)$. By propositions 7.1 and 7.3, for every z_i in the set $\{u_i(0) + l, w_i^+(1/\alpha_i), v_i^+(1/\alpha_i - 1)\}$, there are a point z_{i+1} in $\{u_{i+1}(0), w_{i+1}^+(1/\alpha_i), v_{i+1}^+(1/\alpha_i - 1)\}$ and an integer j_i such that $\Upsilon_{i+1}(z_{i+1}) + j_i = z_i$. This implies that z_n has a trajectory $(z_i; j_i)_{i \geq n+1}$, such that for all $i \geq n+1$, z_i is one of the points $u_i(0)$, $w_i^+(1/\alpha_i)$, or $v_i^+(1/\alpha_i - 1)$.

Now assume that there is $t \in [-1, 0)$ such that $u_n(it) + l$ belongs to $\mathcal{M}_n$. Let $w_n = u_n(it) + l$ and $z_n = u_n(0) + l$. It follows from the propositions 7.1 and 7.3, that z_n and w_n have trajectories $(z_i; j_i)_{i \geq n+1}$ and $(w_i; j_i)_{i \geq n+1}$, respectively. That is, the integers j_i in the corresponding trajectories are identical. By the definition of trajectories, for all $m \geq n+2$, we have

$$\begin{aligned} z_n &= (\Upsilon_{n+1} + (\varepsilon_{n+1} + 1)/2 + l) \circ (\Upsilon_{n+2} + j_{n+1}) \circ \cdots \circ (\Upsilon_m + j_{m-1})(z_m), \\ w_n &= (\Upsilon_{n+1} + (\varepsilon_{n+1} + 1)/2 + l) \circ (\Upsilon_{n+2} + j_{n+1}) \circ \cdots \circ (\Upsilon_m + j_{m-1})(w_m). \end{aligned}$$

Recall the hyperbolic metric ϱ_m on $\tilde{\mathcal{M}}_m^0 \supset \mathcal{M}_m^0$, discussed in Sec. 7.3. Since z_{m+1} and w_{m+1} belong to $\mathcal{M}_{m+1} \subset \mathcal{M}_{m+1}^0 \subset \tilde{\mathcal{M}}_{m+1}^0$, it follows from Lem 7.6, that z_m and w_m are well-contained in $\tilde{\mathcal{M}}_m^0$. One infers that the distance between w_m and z_m with respect to ϱ_m is uniformly bounded from above, by a constant independent of m. Then, by the uniform contraction of the maps $\Upsilon_i + j_{i-1}$ in Prop 7.7, we must have $w_n = z_n$. That is, $u_n(0) = u_n(it)$. This contradicts the injectivity of u_n on $[-1, +\infty)$, proved in Prop 6.8.

The same argument applies to prove the latter two properties in the lemma, where one employs the injectivity of the curves $w_n^{\pm}$ and $v_n^{\pm}$; see Prop 7.1 and Prop 7.3. ☐

Proposition 7.14. *For all $\alpha \in \mathrm{HT}_N$ and $f \in \mathcal{QIS}_\alpha$, $\Lambda(f) \subseteq \mathcal{A}_f$.*

Proof. Recall that $u_i(0) = +1$, and $\mathbb{E}\mathrm{xp}(+1) = -4/27$, where $-4/27$ is the critical value of each map f_i, for $i \geq 0$. By the functional equations in Propositions 7.1 and 7.3, there is a trajectory $(z_i; l_i)_{i \geq 0}$ for $z_{-1} = 0$ so that for all $i \geq 0$,

$$z_i \in \{+1, v_i^+(1/\alpha_i - 1), w_i^+(1/\alpha_i)\}.$$

Note that

$$\{+1, v_i^+(1/\alpha_i - 1), w_i^+(1/\alpha_i)\} \subset \mathcal{M}_i.$$

On the other hand, by Prop 7.10, for every integer $m \geq 0$, there is a sequence of integers $(j_i)_{i=0}^n$ such that either, $z_n \in \mathcal{K}_n^0$ and

$$s \circ f^{\circ m} \circ s(-4/27) = \mathbb{E}\mathrm{xp} \circ (\Upsilon_0 + (\varepsilon_0 + 1)/2) \circ (\Upsilon_1 + j_0) \circ \cdots \circ (\Upsilon_n + j_{n-1})(z_n + j_n).$$

or, $z_n \in \mathcal{M}_n \setminus \mathcal{K}_n^0$ and

$$s \circ f^{\circ m} \circ s(-4/27) = \mathbb{E}\mathrm{xp} \circ (\Upsilon_0 + (\varepsilon_0+1)/2) \circ (\Upsilon_1 + j_0) \circ \cdots \circ (\Upsilon_n + j_{n-1})(\mathcal{E}_n(z_n)).$$

Recall that $v_n^+(1/\alpha_n - 1) + 1 = w_n^+(1/\alpha_n)$, and $\mathcal{E}_n(w_n^+(1/\alpha_n)) = 2$. It follows that $z_n + j_n$ and $\mathcal{E}_n(z_n)$, in the above equation, belong to $\mathcal{M}_n$. Therefore, the right hand side of the above equation belongs to $\mathbb{E}\mathrm{xp}(\mathcal{M}_{-1})$. However, since each map Υ_i is injective on $\mathcal{M}_i^0$, see Lem 7.5, and each Υ_i is an open mapping, one infers from Lem 7.13 that the right hand side of the above equation must be in $\partial\, \mathbb{E}\mathrm{xp}(\mathcal{M}_{-1})$. Thus, the left hand side of the above equation must belong to $\partial\, \mathbb{E}\mathrm{xp}(\mathcal{M}_{-1})$. Therefore, the orbit of the critical value of f remains in $s(\partial\, \mathbb{E}\mathrm{xp}(\mathcal{M}_{-1})) = \partial(s \circ \mathbb{E}\mathrm{xp}(\mathcal{M}_{-1})) = \mathcal{A}_f$. Since $\mathcal{A}_f$ is a compact set, it must contain $\Lambda(f)$. ☐

Proposition 7.15. *For all $\alpha \in \mathrm{HT}_N$ and $f \in \mathcal{QIS}_\alpha$, f is injective on $\mathcal{A}_f$.*

Proof. First we note that $f^{-1}(0) \cap \mathcal{A}_f = \{0\}$. Recall the sets $S_0 = S_{f_0}$ and $\mathcal{P}_0 = \mathcal{P}_{f_0}$, as well as the integer $k_0 = k_{f_0}$, defined in Sec. 4.2. It follows from the properties (i)-(iv) in Sec. 4.2 that there is no pre-image of 0 by f_0 within $\cup_{i=0}^{k_0} f_0^{\circ i}(S_0) \cup \mathcal{P}_0$. This implies that there is no pre-image of 0 by f_0 within $\Phi_0^{-1}(\mathcal{M}_0^0)$. Since f is conjugate to f_0, by s or the identity map, using (6.1), we conclude that there is no pre-image of 0 by f within $s \circ \mathbb{E}\mathrm{xp}(\mathcal{M}_{-1}^1) = s \circ \mathbb{E}\mathrm{xp} \circ (\Upsilon_0 + (\varepsilon_0+1)/2)(\mathcal{M}_0^0)$. Note that $\mathcal{A}_f$ is contained in $s \circ \mathbb{E}\mathrm{xp}(\mathcal{M}_{-1}^1) \cup \{0\}$ Thus, the only preimage of 0 by f, which is contained in $\mathcal{A}_f$, is 0.

Assume that there are ζ and ξ in $\mathcal{A}_f$ such that $f(\zeta) = f(\xi) \neq 0$. Let us choose ζ_{-1} and ξ_{-1} in $\mathcal{M}_{-1}$ such that $s \circ \mathbb{E}\mathrm{xp}(\zeta_{-1}) = \zeta$ and $s \circ \mathbb{E}\mathrm{xp}(\xi_{-1}) = \xi$. There are unique points ζ_0 and ξ_0 in $\mathcal{M}_0$ such that $\Upsilon_0(\zeta_0) + (\varepsilon_0+1)/2 = \zeta_{-1}$ and $\Upsilon_0(\xi_0) + (\varepsilon_0+1)/2 = \xi_{-1}$.

By Prop 7.9, one of the following holds:

(a1) $\zeta_0 \in \mathcal{K}_0^0$, and $s \circ f(\zeta) = \mathbb{E}\mathrm{xp}(\Upsilon_0 + (\varepsilon_0+1)/2)(\zeta_0+1)$,
(a2) $\zeta_0 \in \mathcal{M}_0 \setminus \mathcal{K}_0^0$, and $s \circ f(\zeta) = \mathbb{E}\mathrm{xp}(\Upsilon_0 + (\varepsilon_0+1)/2)(\mathcal{E}_0(\zeta_0))$.

Similarly, we have one of

(b1) $\xi_0 \in \mathcal{K}_0^0$, and $s \circ f(\xi) = \mathbb{E}\mathrm{xp}(\Upsilon_0 + (\varepsilon_0+1)/2)(\xi_0+1)$,
(b2) $\xi_0 \in \mathcal{M}_0 \setminus \mathcal{K}_0^0$, and $s \circ f(\xi) = \mathbb{E}\mathrm{xp}(\Upsilon_0 + (\varepsilon_0+1)/2)(\mathcal{E}_0(\xi_0))$.

Since $s \circ f(\zeta) = s \circ f(\xi)$, we conclude that at least one of the following four cases must occur:

(c1) $\zeta_0 \in \mathcal{K}_0^0$, $\xi_0 \in \mathcal{K}_0^0$, and $\mathbb{E}\mathrm{xp} \circ \Upsilon_0(\zeta_0+1) = \mathbb{E}\mathrm{xp} \circ \Upsilon_0(\xi_0+1)$;
(c2) $\zeta_0 \in \mathcal{K}_0^0$, $\xi_0 \in \mathcal{M}_0 \setminus \mathcal{K}_0^0$, and $\mathbb{E}\mathrm{xp} \circ \Upsilon_0(\zeta_0+1) = \mathbb{E}\mathrm{xp} \circ \Upsilon_0(\mathcal{E}_0(\xi_0))$;
(c3) $\zeta_0 \in \mathcal{M}_0 \setminus \mathcal{K}_0^0$, $\xi_0 \in \mathcal{K}_0^0$, and $\mathbb{E}\mathrm{xp} \circ \Upsilon_0(\mathcal{E}_0(\zeta_0)) = \mathbb{E}\mathrm{xp} \circ \Upsilon_0(\xi_0+1)$;
(c4) $\zeta_0 \in \mathcal{M}_0 \setminus \mathcal{K}_0^0$, $\xi_0 \in \mathcal{M}_0 \setminus \mathcal{K}_0^0$, and $\mathbb{E}\mathrm{xp} \circ \Upsilon_0(\mathcal{E}_0(\zeta_0)) = \mathbb{E}\mathrm{xp} \circ \Upsilon_0(\mathcal{E}_0(\xi_0))$.

It follows from Lem 7.5 and Prop 7.1 that $\mathbb{E}\mathrm{xp} \circ \Upsilon_0$ is injective on each of $\mathcal{K}_0^0$ and $\mathcal{K}_0^0+1$.

If (c1) occurs, since $\mathbb{E}\mathrm{xp} \circ \Upsilon_0$ is injective on $\mathcal{K}_0+1$, we must have $\zeta_0 = \xi_0$, and hence $\zeta = \xi$.

If (c2) occurs, there are two possibilities. We must have either

(c2-1) $\zeta_0 + 1 = \mathcal{E}_0(\xi_0)$, or
(c2-2) $\zeta_0 + 1 \in w_0^+$, $\mathcal{E}_0(\xi_0) \in u_0$, and $\Upsilon_0(\mathcal{E}_0(\xi_0)) = \Upsilon_0(\zeta_0+1) - \varepsilon_0$.

If (c2-1) holds, since $\mathcal{E}_0(J_0)$ and K_0^0+1 only meet on u_0+1, we must have $\zeta_0 \in u_0$ and $\mathcal{E}_0(\xi_0) \in u_0+1$. By the definition of $\mathcal{E}_0$ in (7.11), and (7.1), the latter relation implies that $\xi_0 \in w_0^+$. Thus, there are t_1 and t_2 in $[-1,+\infty)$ such that $\zeta_0 = u_0(it_1)$ and $\xi_0 = w_0^+(1/\alpha_0 + it_2)$. On the other hand, $\zeta_0 + 1 = \mathcal{E}_0(\xi_0)$ implies that $u_0(it_1) = \Phi_0 \circ f_0 \circ \Phi_0^{-1}(w_0^+(1/\alpha_0+it_2)) - 1$, and hence $\Phi_0^{-1}(u_0(it_1)) = \Phi_0^{-1}(w_0^+(1/\alpha_0+it_2))$. By (7.1), we conclude that $t_1 = t_2$. Then, by Prop 7.1, we must have $\Upsilon_0(\xi_0) = \Upsilon_0(\zeta_0) - \varepsilon_0$. Thus, $s(\zeta) = \mathbb{E}\mathrm{xp} \circ \Upsilon_0(\zeta_0) = \mathbb{E}\mathrm{xp} \circ \Upsilon_0(\xi_0) = s(\xi)$, and hence $\zeta = \xi$.

If (c2-2) holds, by Prop 7.1, there is $t \geq -1$ such that $\zeta_0 + 1 = w_0^+(1/\alpha_0 + it)$ and $\mathcal{E}_0(\xi_0) = u_0(it)$. By the definition of $\mathcal{E}_0$ in (7.11), and (7.1), the latter relation implies that $\xi_0 = w_0^-(1/\alpha_0+it) - 1$. Since $\zeta_0 = w_0^+(1/\alpha_0+it) - 1 = v_0^+(1/\alpha_0 - 1 + it)$ belongs to $\mathcal{M}_0$, by Lem 7.13, we must have $t \geq 0$. Then, by Prop 7.3, we obtain $\xi_0 = w_0^-(1/\alpha_0+it) - 1 = w_0^+(1/\alpha_0+it) - 1 = \zeta_0$. Hence, $\zeta = \xi$.

If (c3) occurs, as in (c2), or by symmetry, we conclude that $\zeta = \xi$.

Assume that (c4) occurs. Since $\mathcal{E}_0(\zeta_0)$ and $\mathcal{E}_0(\xi_0)$ belong to $\mathcal{K}_0$, and $\mathbb{E}\mathrm{xp}\circ\Upsilon_0$ is injective on $\mathcal{K}_0^0$, we must have $\mathcal{E}_0(\zeta_0)=\mathcal{E}_0(\xi_0)$. Since $\mathcal{E}_0$ is injective on $\mathcal{J}_0^0$, we conclude that $\xi_0=\zeta_0$, and then $\zeta=\xi$. □

8. Uniformisation of the post-critical set

8.1. **Summary of the markings.** Let the sets M_n^j and the maps Y_n be the objects for the topological model we introduced in Sec. 3. These correspond to the dynamical objects $\mathcal{M}_n^j$ and the changes of coordinates Υ_n we introduced in Sec. 7. In this section we build homeomorphisms from the sets $\mathcal{M}_n^j$ to the sets M_n^j which "respect" the transformations Y_n and Υ_n.

In Sections 6 and 7 we introduced the curves u_n, $v_n^\pm$, and $w_n^\pm$, for $n\geq 0$. These curves are contained in $\mathcal{M}_n^0$. Also, in those sections we established some remarkable equivariant properties of these curves. Those properties play a key role in this section. For the convenience of the reader, we collect (and reformulate some of) the required relations, and present them below.

For every $n\geq 0$, the following hold:

(i) $u_n: i[-1,+\infty)\to\partial\mathcal{M}_n^0$, and for all $t\geq -1$,

$$\Upsilon_n\circ u_n(it)=u_{n-1}\circ Y_n(it). \tag{8.1}$$

(ii) $w_n^+: 1/\alpha_n+i[-1,+\infty)\to\partial\mathcal{M}_n^0$, $w_n^-: 1/\alpha_n+i[-1,+\infty)\to\Pi_n$, and for all $t\geq -1$,

$$\Upsilon_n\circ w_n^+(1/\alpha_n+it)=\Upsilon_n\circ w_n^-(1/\alpha_n+it)=u_{n-1}\circ Y_n(it)-\varepsilon_n; \tag{8.2}$$

(iii) $v_n^\pm: (1/\alpha_n-1)+i[-1,\infty)\to\mathcal{M}_n$, and for all $t\geq -1$,

$$\begin{aligned}\Upsilon_n\circ v_n^\pm(1/\alpha_n-1+it)+a_{n-1}+\varepsilon_n&=w_{n-1}^\pm(1/\alpha_{n-1}+Y_n(it)),\quad \text{if } \varepsilon_n=-1,\\ \Upsilon_n\circ v_n^\pm(1/\alpha_n-1+it)+a_{n-1}+\varepsilon_n&=w_{n-1}^\mp(1/\alpha_{n-1}+Y_n(it)),\quad \text{if } \varepsilon_n=+1.\end{aligned} \tag{8.3}$$

See Fig 9 for an illustration of the above relations.

8.2. **Partial uniformisations matching the markings.** Let $n\geq 0$ and $j\geq 0$. We say that a map $\Omega:\mathcal{M}_n^j\to M_n^j$ **matches** $(u_n,v_n^\pm,w_n^+)$, if the following four properties hold:

(i) for every $z\in M_n^j$ with $\mathrm{Re}\,z=0$ we have $\Omega\circ u(z)=z$;

(ii) for every $z\in M_n^j$ with $\mathrm{Re}\,z=1/\alpha_n$ we have $\Omega\circ w_n^+(z)=z$;

(iii) for every $z\in M_n^j$ with $\mathrm{Re}\,z=1/\alpha_n-1$ we have $\Omega\circ v_n^+(z)=z$;

(iv) for every $z\in M_n^j$ with $\mathrm{Re}\,z=1/\alpha_n-1$ we have $\Omega\circ v_n^-(z)=z$;

In other words, $\Omega:\mathcal{M}_n^j\to M_n^j$ matches $(u_n,v_n^\pm,w_n^+)$ if it is equal to the inverses of the maps u_n, v_n^+, v_n^-, and w_n^+, where they are defined.

Note that by Propositions 7.1 and 7.4, for all $n\geq 0$, when $t\geq 0$, $v_n^+(1/\alpha_n-1+it)=v_n^-(1/\alpha_n-1+it)$. These mean that items (iii) and (iv) in the above list do not contradict. On the other hand, $v_n^+(1/\alpha_n-1+i[-1,0))\cap v_n^-(1/\alpha_n-1+i[-1,0))=\emptyset$, which means that Ω_n may not be injective. For this reason, we are not able to work with maps from M_n^j to $\mathcal{M}_n^j$, as any such map must be multivalued; sending $w\in 1/\alpha_n-1+i[-1,0)$ to $v_n^+(w)$ and $v_n^-(w)$. As we shall see in a moment, this does not cause any problems, since the curves $v_n^+(1/\alpha_n-1+i[-1,0))$ and $v_n^-(1/\alpha_n-1+i[-1,0))$ are not in the post-critical set.

Proposition 8.1. *There is a constant C_8 such that for every $n\geq 0$ there exists a continuous and surjective map*

$$\Omega_n^0:\mathcal{M}_n^0\to M_n^0$$

which matches $(u_n,v_n^\pm,w_n^+)$, and for all $z\in\mathcal{M}_n^0$, $|\Omega_n^0(z)-z|\leq C_8$.

Proof. Recall that $\mathcal{M}_n^0$ is bounded by the curves u_n, w_n^+, and some continuous curve μ_n which connects $u_n(-i)$ to $w_n^+(1/\alpha_n - i)$. By Propositions 6.7 and 7.2 the maps u_n, $v_n^\pm$ and w_n^+ are uniformly close to the identity map. The map μ_n may also be re-parameterised so that it is uniformly close to the identity map. These imply that one can extend these maps from the boundary of M_n^0 to the boundary of $\mathcal{M}_n^0$ to a homeomorphism from M_n^0 to $\mathcal{M}_n^0$. This may be carried out by partitioning the sets M_n^0 and $\mathcal{M}_n^0$ into Jordan domains with uniformly bounded diameters, and then matching the corresponding pieces, while respecting the boundary maps, where they exist. The inverse of this extension is Ω_n^0. We shall present more details below.

The curves $v_n^\pm$ lie in $\mathcal{M}_n^0$, and $v_n^\pm(1/\alpha_n - 1 - i)$ belong to μ_n. By Prop 7.4, $v_n^\pm$ are disjoint from w_n^+ and u_n. Moreover, by the same proposition, all translations of u_n by integers are disjoint from $v_n^\pm$ and w_n^+. Recall that $\{w \in \mathbb{C} \mid 0 \leq \operatorname{Re} w \leq 1/\alpha_n - c_1\}$ is contained in $\mathcal{M}_n^0$, where c_1 is the constant in Prop 4.2. Thus, for integers j with $0 \leq j \leq 1/\alpha_n - c_1 - 1$, the curves $u_n + j$ are contained in $\mathcal{M}_n^0$ and lie on the left hand side of v_n^+. Moreover, by the definition of μ_n, all those curves $u_n + j$ meet the curve μ_n. Let us define l_n as the largest integer less than or equal to $1/\alpha_n - c_1 - 1$. The curves $u_n + j$, for $0 \leq j \leq l_n$, v_n^+, and w_n^+ partition $\mathcal{M}_n^0$ into $l_n + 2$ pieces, say $A_{n,k}^0$, for $1 \leq k \leq l + 2$. Note that each $A_{n,k}^0$ is uniformly close to a half-infinite vertical strip of width one. The next step is to divide each $A_{n,k}^0$ into infinitely many nearly-square Jordan domains with uniformly bounded diameters. To see this, we note that for each $A_{n,k}^0$, with "vertical" boundary curves, say $u_n + j$ and $u_n + j + 1$, and any parameter $t \in [-1, +\infty)$, the points $u_n(it) + j$ and $u_n(it) + j + 1$ may be connected by a path in $A_{n,k}^0$, which has a uniformly bounded diameter. We may use countably many such paths to partition each $A_{n,k}^0$ into Jordan domains with uniformly bounded diameters. □

Proposition 8.2. *Let $n \geq 1$ and $j \geq 0$. Assume that $\Omega_n^j : \mathcal{M}_n^j \to M_n^j$ is a continuous and surjective map which matches $(u_n, v_n^\pm, w_n^+)$. Then, there exists a continuous and surjective map*

$$\Omega_{n-1}^{j+1} : \mathcal{M}_{n-1}^{j+1} \to M_{n-1}^{j+1}$$

which matches $(u_{n-1}, v_{n-1}^\pm, w_{n-1}^+)$, and for all integers l satisfying $(\varepsilon_n+1)/2 \leq l \leq a_{n-1} + (\varepsilon_n - 1)/2$,

$$\Omega_{n-1}^{j+1} \circ (\Upsilon_n(z) + l) = Y_n \circ \Omega_n^j(z) + l,$$

whenever both sides of the equation are defined.

See Fig 9 for an illustration of the proof of Prop 8.2.

Proof. Fix an arbitrary $n \geq 1$ and $j \geq 0$. Let us first assume that $\varepsilon_n = -1$. Recall from (3.10) and (7.5) that

$$M_{n-1}^{j+1} = \bigcup_{l=0}^{a_{n-1}-2} \big(Y_n(M_n^j) + l\big) \bigcup \big(Y_n(K_n^j) + a_{n-1} - 1\big).$$

and

$$\mathcal{M}_{n-1}^{j+1} = \bigcup_{l=0}^{a_{n-1}-2} \big(\Upsilon_n(\mathcal{M}_n^j) + l\big) \bigcup \big(\Upsilon_n(\mathcal{K}_n^j) + a_{n-1} - 1\big).$$

For each $0 \leq l \leq a_{n-1} - 2$, define $\Omega_{n-1}^{j+1} : \Upsilon_n(\mathcal{M}_n^j) + l \to Y_n(M_n^j) + l$ as

$$\Omega_{n-1}^{j+1}(z) = Y_n \circ \Omega_n^j \circ \Upsilon_n^{-1}(z - l) + l.$$

Since Y_n, Ω_n^j, and Υ_n are continuous, the above map is continuous. As Ω_n^j is surjective, Ω_{n-1}^{j+1} covers $Y_n(M_n^j) + l$. Similarly, we define $\Omega_{n-1}^{j+1} : \Upsilon_n(\mathcal{K}_n^j) + a_{n-1} - 1 \to Y_n(K_n^j) + a_{n-1} - 1$ as

$$\Omega_{n-1}^{j+1}(z) = Y_n \circ \Omega_n^j \circ \Upsilon_n^{-1}(z - a_{n-1} + 1) + a_{n-1} - 1.$$

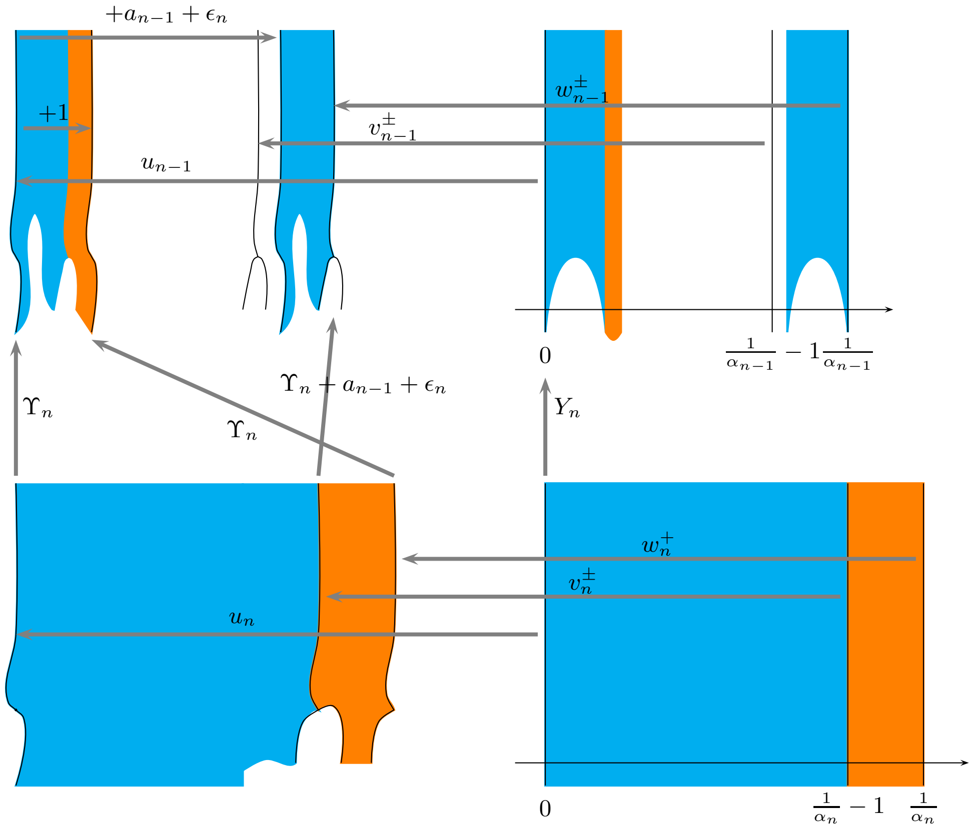

FIGURE 9. Illustration of the system of parameterised curves and commutative relations, when $\epsilon_n=-1$.

Since $\Omega_n^j:\mathcal{M}_n^j\to M_n^j$ matches u_n and v_n^+, it follows that Ω_n^j maps $\mathcal{K}_n^j$ to K_n^j. In particular, the restriction of Ω_n^j to $\mathcal{K}_n^j$ is continuous and covers K_n^j.

We need to show that the map Ω_{n-1}^{j+1}, which is defined in a piece-wise fashion, is well-defined on the common boundaries of $\Upsilon_n(\mathcal{M}_n^j)+l$ and $\Upsilon_n(\mathcal{M}_n^j)+l+1$, for integers l with $0\le l\le a_{n-1}-2$. Fix an arbitrary z on the common boundary. There is $t'\ge \operatorname{Im} Y_n(-i)$ such that $z=u_{n-1}(it')+l+1$. Let $it'=Y_n(it)$, for some $t\ge -1$, (see (3.5)). The map induced from the left-hand component gives us

$$\begin{aligned}\Omega_{n-1}^{j+1}(z)&=Y_n\circ\Omega_n^j\circ\Upsilon_n^{-1}(u_{n-1}(it')+l+1-l)+l\\&=Y_n\circ\Omega_n^j\circ w_n^+(1/\alpha_n+Y_n^{-1}(it'))+l &&((8.2))\\&=Y_n(1/\alpha_n+Y_n^{-1}(it'))+l &&(\Omega_n^j \text{ matches } w_n^+)\\&=it'+1+l. &&((3.6))\end{aligned}$$

The map induced from the right-hand component gives us

$$\begin{aligned}\Omega_{n-1}^{j+1}(z)&=Y_n\circ\Omega_n^j\circ\Upsilon_n^{-1}(u_{n-1}(it')+l+1-l-1)+l+1\\&=Y_n\circ\Omega_n^j\circ u_n(Y_n^{-1}(it'))+l+1 &&((8.1))\\&=Y_n(Y_n^{-1}(it'))+l+1 &&(\Omega_n^j \text{ matches } u_n)\\&=it'+l+1.\end{aligned}$$

Thus, the two induced maps are identical on the common boundaries.

By the definition of Ω_{n-1}^{j+1}, the functional equations in the proposition hold. There remains to show that Ω_{n-1}^{j+1} matches $(u_{n-1}, v_{n-1}^{\pm}, w_{n-1}^{+})$.

To see that Ω_{n-1}^{j+1} matches u_{n-1}, let $z = u_{n-1}(it')$ for some $t' \geq \operatorname{Im} Y_n(-i)$. Let $it' = Y_n(it)$, for some $t \geq -1$. By the definition of Ω_{n-1}^{j+1} (use $l = 0$), we have

$$\Omega_{n-1}^{j+1}(u_{n-1}(it')) = Y_n \circ \Omega_n^j \circ \Upsilon_n^{-1}(u_{n-1}(it')) = Y_n \circ \Omega_n^j \circ u_n(Y_n^{-1}(it')) = Y_n(Y_n^{-1}(it')) = it'.$$

In the above equation, the second "=" comes from (8.1) and the third "=" is because Ω_n^j matches u_n. Thus, Ω_{n-1}^{j+1} matches u_{n-1}.

To see that Ω_{n-1}^{j+1} matches w_{n-1}^{+}, let $z = w_{n-1}^{+}(it' + 1/\alpha_{n-1})$ for some $t' \geq -1$. Note that $w_{n-1}^{+}(it' + 1/\alpha_{n-1}) \in \Upsilon_n(\mathcal{K}_n^j) + a_{n-1} - 1$. Then, by the definition of Ω_{n-1}^{j+1},

$$\begin{aligned}
\Omega_{n-1}^{j+1}(z) &= Y_n \circ \Omega_n^j \circ \Upsilon_n^{-1}(w_{n-1}^{+}(it' + 1/\alpha_{n-1}) - (a_{n-1} - 1)) + a_{n-1} - 1 \\
&= Y_n \circ \Omega_n^j \circ v_n^{+}(1/\alpha_n - 1 + Y_n^{-1}(it')) + a_{n-1} - 1 && ((8.3)) \\
&= Y_n(1/\alpha_n - 1 + Y_n^{-1}(it')) + a_{n-1} - 1 && (\Omega_n^j \text{ matches } v_n^{+}) \\
&= Y_n(Y_n^{-1}(it') + 1 - \alpha_n + a_{n-1} - 1 && ((3.7)) \\
&= it' + 1/\alpha_{n-1}. && (1/\alpha_{n-1} = a_{n-1} - \alpha_n)
\end{aligned}$$

Thus, Ω_{n-1}^{j+1} matches w_{n-1}^{+}.

To see that Ω_{n-1}^{j+1} matches v_{n-1}^{+}, let $z = v_{n-1}^{+}(it' + 1/\alpha_{n-1} - 1)$ for some $t' \geq -1$. Using $\Omega_{n-1}^{j+1}(z+1) = \Omega_{n-1}^{j+1}(z) + 1$, we have

$$\begin{aligned}
\Omega_{n-1}^{j+1}(v_{n-1}^{+}(it' + 1/\alpha_{n-1} - 1)) &= \Omega_{n-1}^{j+1}(v_{n-1}^{+}(it' + 1/\alpha_{n-1} - 1) + 1) - 1 \\
&= \Omega_{n-1}^{j+1}(w_{n-1}^{+}(it' + 1/\alpha_{n-1})) - 1 \\
&= it' + 1/\alpha_{n-1} - 1. && (\Omega_n^j \text{ matches } w_n^{+})
\end{aligned}$$

Thus, Ω_{n-1}^{j+1} matches v_{n-1}^{+}.

To see that Ω_{n-1}^{j+1} matches v_{n-1}^{-}, let $z = v_{n-1}^{-}(it' + 1/\alpha_{n-1} - 1)$ for some $t' \geq \operatorname{Im} Y_n(-i)$. Let $it' = Y_n(it)$. Since $\varepsilon_n = -1$, $1/\alpha_{n-1} = a_{n-1} + \varepsilon_n \alpha_n = a_{n-1} - \alpha_n$. Thus, $1/\alpha_{n-1} - 1$ lies strictly between the integers $a_{n-1} - 2$ and $a_{n-1} - 1$. By the definition of Ω_{n-1}^{j+1} (here use $l = a_{n-1} - 2$), we have

$$\begin{aligned}
&\Omega_{n-1}^{j+1}(v_{n-1}^{-}(it' + 1/\alpha_{n-1} - 1)) \\
&\quad = Y_n \circ \Omega_n^j \circ \Upsilon_n^{-1}(v_{n-1}^{-}(it' + 1/\alpha_{n-1} - 1) - (a_{n-1} - 2)) + a_{n-1} - 2 \\
&\quad = Y_n \circ \Omega_n^j \circ \Upsilon_n^{-1}(w_{n-1}^{-}(it' + 1/\alpha_{n-1}) - (a_{n-1} - 1)) + a_{n-1} - 2 \\
&\quad = Y_n \circ \Omega_n^j \circ v_n^{-}(Y_n^{-1}(it') + 1/\alpha_n - 1)) + a_{n-1} - 2 && ((8.3)) \\
&\quad = Y_n(Y_n^{-1}(it') + 1/\alpha_n - 1)) + a_{n-1} - 2 && (\Omega_n^j \text{ matches } v_n^{-}) \\
&\quad = Y_n(Y_n^{-1}(it') + 1 - \alpha_n + a_{n-1} - 2 && ((3.7)) \\
&\quad = it' + 1/\alpha_{n-1} - 1. && (1/\alpha_{n-1} = a_{n-1} - \alpha_n)
\end{aligned}$$

Thus, Ω_{n-1}^{j+1} matches v_{n-1}^{-}. This completes the proof of the proposition when $\varepsilon_n = -1$.

Now assume that $\varepsilon_n = +1$. One may carry out the same argument to prove the proposition in this case as well. We briefly explain two parts of the argument which may require careful attention.

To see that Ω_{n-1}^{j+1} matches w_{n-1}^+, let $z = w_{n-1}^+(it' + 1/\alpha_{n-1})$ for some $t' \geq -1$. Note that $w_{n-1}^+(it' + 1/\alpha_{n-1}) \in \Upsilon_n(\mathcal{J}_n^j) + a_{n-1} + 1$. Then, by the definition of Ω_{n-1}^{j+1},

$$\begin{aligned}
&\Omega_{n-1}^{j+1}(w_{n-1}^+(it' + 1/\alpha_{n-1})) \\
&\quad = Y_n \circ \Omega_n^j \circ \Upsilon_n^{-1}(w_{n-1}^+(it' + 1/\alpha_{n-1}) - (a_{n-1} + 1)) + a_{n-1} + 1 \\
&\quad = Y_n \circ \Omega_n^j \circ v_n^-(Y_n^{-1}(it') + 1/\alpha_n - 1) + a_{n-1} + 1 && ((8.3)) \\
&\quad = Y_n(Y_n^{-1}(it') + 1/\alpha_n - 1) + a_{n-1} + 1 && (\Omega_n^j \text{ matches } v_n^-) \\
&\quad = Y_n(Y_n^{-1}(it') + \alpha_n - 1 + a_{n-1} + 1 && ((3.7)) \\
&\quad = it' + 1/\alpha_{n-1}. && (1/\alpha_{n-1} = a_{n-1} + \alpha_n)
\end{aligned}$$

Thus, Ω_{n-1}^{j+1} matches w_{n-1}^+.

To see that Ω_{n-1}^{j+1} matches v_{n-1}^-, let $z = v_{n-1}^-(it' + 1/\alpha_{n-1} - 1)$ for some $t' \geq \operatorname{Im} Y_n(-i)$. Let $it' = Y_n(it)$. Since $\varepsilon_n = +1$, $1/\alpha_{n-1} = a_{n-1} + \varepsilon_n\alpha_n = a_{n-1} + \alpha_n$. Thus, $1/\alpha_{n-1} - 1$ lies strictly between the integers $a_{n-1} - 1$ and a_{n-1}. By the definition of Ω_{n-1}^{j+1}, we have

$$\begin{aligned}
&\Omega_{n-1}^{j+1}(v_{n-1}^-(it' + 1/\alpha_{n-1} - 1)) \\
&\quad = Y_n \circ \Omega_n^j \circ \Upsilon_n^{-1}(v_{n-1}^-(it' + 1/\alpha_{n-1} - 1) - a_{n-1}) + a_{n-1} \\
&\quad = Y_n \circ \Omega_n^j \circ \Upsilon_n^{-1}(w_{n-1}^-(it' + 1/\alpha_{n-1}) - (a_{n-1} + 1)) + a_{n-1} \\
&\quad = Y_n \circ \Omega_n^j \circ v_n^+(Y_n^{-1}(it') + 1/\alpha_n - 1)) + a_{n-1} && ((8.3)) \\
&\quad = Y_n(Y_n^{-1}(it') + 1/\alpha_n - 1) + a_{n-1} && (\Omega_n^j \text{ matches } v_n^+) \\
&\quad = Y_n(Y_n^{-1}(it') + \alpha_n - 1 + a_{n-1} && ((3.7)) \\
&\quad = it' + 1/\alpha_{n-1} - 1. && (1/\alpha_{n-1} = a_{n-1} + \alpha_n)
\end{aligned}$$

Thus, Ω_{n-1}^{j+1} matches v_{n-1}^-. □

8.3. **Convergence of the partial uniformisations.** Fix an arbitrary $n \geq 0$. By Prop 8.1, for each integer $j \geq 0$, there is a continuous and surjective map $\Omega_{n+j}^0 : \mathcal{M}_{n+j}^0 \to M_{n+j}^0$ which matches $(u_{n+j}, v_{n+j}^\pm, w_{n+j}^+)$. Inductively applying Prop 8.2 several times, we obtain the continuous and surjective map

$$\Omega_n^j : \mathcal{M}_n^j \to M_n^j.$$

Proposition 8.3. *There is a constant C_9 such that for every $n \geq 0$, every $j \geq 0$, and every $z \in \mathcal{M}_n^{j+1}$ we have*

$$\left|\Omega_n^{j+1}(z) - \Omega_n^j(z)\right| \leq C_9(0.9)^j.$$

In particular, for each $n \geq 0$, as j tends to $+\infty$, the sequence of maps $\Omega_n^j : \mathcal{M}_n^j \to M_n^j$ converges to a continuous map

$$\Omega_n : \mathcal{M}_n \to M_n.$$

Proof. We prove the inequality in the proposition by induction on j. First we prove that there is a constant C_9 such that the inequality holds for all $n \geq 0$, $j = 0$, and $z \in \mathcal{M}_n^1$.

Fix an arbitrary $n \geq 0$, and let $z_n \in \mathcal{M}_n^1$. There is an integer l_n such that $z_n - l_n \in \Upsilon_{n+1}(\mathcal{M}_{n+1}^0)$ and $z_{n+1} = \Upsilon_{n+1}^{-1}(z_n - l_n)$ is defined. Note that $\Omega_n^0 : \mathcal{M}_n^0 \to M_n^0$ and $\Omega_n^1 : \mathcal{M}_n^1 \to M_n^1$, where $\mathcal{M}_n^1 \subset \mathcal{M}_n^0$. Thus, both maps are defined on $\mathcal{M}_n^1$. There is a uniform constant $C \geq 1$, independent of n, such that for all $z \in \mathcal{M}_n^0$, $\operatorname{Im} z + C \geq -1$. Let us consider the points

$$w_1 = \Upsilon_{n+1}^{-1}(z_n - l_n) + C, \qquad w_2 = \Upsilon_{n+1}^{-1}(z_n - l_n).$$

By Propositions 5.2 and 8.1, as well as Lem 3.3, we have

$$\begin{aligned}|\Omega_n^1(z_n) - z_n| &= \big|\big(Y_{n+1} \circ \Omega_{n+1}^0 \circ \Upsilon_{n+1}^{-1}(z_n - l_n) + l_n\big) - z_n\big| \\ &\leq \big|\big(Y_{n+1} \circ \Omega_{n+1}^0(w_2) + l_n\big) - (Y_{n+1}(w_1) + l_n)\big| \\ &\quad + |(Y_{n+1}(w_1) + l_n) - (\Upsilon_{n+1}(w_2) + l_n)| \\ &\leq 0.9 \cdot \big|\Omega_{n+1}^0(w_2) - w_1\big| + C_3 \cdot C. \\ &\leq 0.9 \cdot (C_8 + C) + C_3 \cdot C.\end{aligned}$$

Therefore,

$$|\Omega_n^1(z_n) - \Omega_n^0(z_n)| \leq |\Omega_n^1(z_n) - z_n| + |z_n - \Omega_n^0(z_n)| \leq 2C_8 + C(C_3 + 1).$$

Let us introduce C_9 as $2C_8 + C(C_3 + 1)$.

Now assume that the inequality holds for some $j - 1 \geq 0$, all $n \geq 0$, and all $z_n \in \mathcal{M}_n^j$. We aim to prove it for j. For $z_n \in \mathcal{M}_n^{j+1}$, we note that $\Upsilon_{n+1}^{-1}(z_n - l_n) \in \mathcal{M}_{n+1}^j$. Using the functional relation in Prop 8.2, the uniform contraction of Y_{n+1} in Lem 3.3, and the induction hypothesis, we obtain

$$\begin{aligned}\big|\Omega_n^{j+1}(z_n) - \Omega_n^j(z_n)\big| &= \big|Y_{n+1} \circ \Omega_{n+1}^j \circ \Upsilon_{n+1}^{-1}(z_n - l_n) - Y_{n+1} \circ \Omega_{n+1}^{j-1} \circ \Upsilon_{n+1}^{-1}(z_n - l_n)\big| \\ &\leq 0.9 \cdot C_9(0.9)^{j-1} = C_9(0.9)^j.\end{aligned}$$

This completes the induction step.

By the inequality in the proposition, the sequence of continuous maps Ω_n^j, for $j \geq 0$, is uniformly Cauchy on $\mathcal{M}_n \subset \mathcal{M}_n^j$. Hence, the sequence converges to a continuous map on $\mathcal{M}_n$. □

Corollary 8.4. *For every $n \geq 1$, and all integers l with $(\varepsilon_n + 1)/2 \leq l \leq a_{n-1} + (\varepsilon_n - 1)/2$, we have*

$$\Omega_{n-1} \circ (\Upsilon_n + l) = Y_n \circ \Omega_n + l,$$

whenever both sides of the equation are defined.

Proof. This follows from taking limits in the functional relations of Prop 8.2. □

We may define $\Omega_{-1} : \mathcal{M}_{-1} \to M_{-1}$, as

$$\Omega_{-1}(z) = Y_0 \circ \Omega_0 \circ \Upsilon_0^{-1}(z - (\varepsilon_0 + 1)/2) + (\varepsilon_0 + 1)/2).$$

By Prop 7.1, for $t \geq 0$,

$$\Omega_{-1}(it + 1) = \Omega_{-1}(it) + 1.$$

Proposition 8.5. *There is a constant C_{10} such that for every $n \geq -1$ and every $z \in \mathcal{M}_n$,*

$$|\Omega_n(z) - z| \leq C_{10}.$$

Proof. By Propositions 8.1 and 8.3, for all $n \geq -1$, all $j \geq 0$, and all $z \in \mathcal{M}_n$ we have

$$\begin{aligned}\big|\Omega_n^j(z) - z\big| &\leq \big|\Omega_n^j(z) - \Omega_n^0(z)\big| + \big|\Omega_n^0(z) - z\big| \\ &\leq \sum_{l=0}^{j-1} \big|\Omega_n^{l+1}(z) - \Omega_n^l(z)\big| + \big|\Omega_n^0(z) - z\big| \\ &\leq \sum_{l=0}^{+\infty} C_9(0.9)^l + C_8 = 10C_9 + C_8.\end{aligned}$$ □

Proposition 8.6. *For every $n \geq -1$, $\Omega_n : \mathcal{M}_n \to M_n$ is injective.*

Proof. Fix an arbitrary $n \geq -1$. Let $z_n \neq z'_n$ be arbitrary points in $\mathcal{M}_n$. Since $\mathcal{M}_n = \cap_{j\geq 0}\mathcal{M}_n^j$, we may inductively identify integers l_i and l'_i such that

$$z_i - l_i \in \Upsilon_{i+1}(\mathcal{M}_{i+1}) \setminus u_{i+1}(i[-1,+\infty)), \quad z'_i - l'_i \in \Upsilon_{i+1}(\mathcal{M}_{i+1}) \setminus u_{i+1}(i[-1,+\infty)),$$

with $z_{i+1} = \Upsilon_{i+1}^{-1}(z_i - l_i)$ and $z'_{i+1} = \Upsilon_{i+1}^{-1}(z'_i - l'_i)$ defined.

First assume that there is a smallest $i \geq n$ such that $l_i \neq l'_i$. This implies that $\Omega_i(z_i) \neq \Omega_i(z'_i)$. Then, one uses the commutative relations in Corollary 8.4 to conclude that $\Omega_n(z_n) \neq \Omega_n(z'_n)$.

On the other hand, assume that for all $i \geq n$, $l_i = l'_i$. Recall the sets $\tilde{\mathcal{M}}_n^0$ and the hyperbolic metrics ϱ_n defined in Sec. 7.3. By the uniform contraction of the maps $\Upsilon_i + l_{i-1}$ in Prop 7.7, we must have $d_{\varrho_i}(z_i, z'_i) \to +\infty$ as $i \to +\infty$. By Lem 7.6, $\mathcal{M}_{n+j}^1$ is well-contained in $\tilde{\mathcal{M}}_{n+1}^0$. This implies that $|z_i - z'_i| \to +\infty$, as $i \to +\infty$. In particular, there is $i \geq n$, such that $|z_i - z'_i| > 2C_{10}$. By virtue of Prop 8.5, we must have $\Omega_i(z_i) \neq \Omega_i(z'_i)$. Then we use the commutative diagrams in Corollary 8.4 as well as the injectivity of Y_l on M_l^0 and Υ_l on $\mathcal{M}_l^0$, to conclude that $\Omega_n(z_n) \neq \Omega_n(z'_n)$. □

Proposition 8.7. *For every $n \geq -1$, $\Omega_n : \mathcal{M}_n \to M_n$ is surjective.*

Proof. Fix an arbitrary $n \geq -1$, and $z \in M_n$. As $M_n = \cap_{j\geq 0} M_n^j$ and for each $j \geq 0$, $\Omega_n^j : \mathcal{M}_n^j \to M_n^j$ is surjective, there is $z_j \in \mathcal{M}_n^j$ with $\Omega_n^j(z_j) = z$. By Prop 8.5, z_j are contained in a bounded subset of $\mathcal{M}_n^0$. Thus, there is a sub-sequence of this sequence, say z_{j_k}, for $k \geq 1$, which converges to some $z' \in \mathcal{M}_n^0$. For every integer $l \geq 0$, z_{j_k} is contained in $\mathcal{M}_n^l$, for all $j_k \geq l$. Thus, z' belongs to $\mathcal{M}_n^l$. As l is arbitrary, $z' \in \mathcal{M}_n$. The uniform convergence of Ω_n^j to Ω_n implies that $\Omega_n(z') = z$. □

8.4. **The harvest.** Let N be the integer introduced in Prop 6.1. Assume that $\alpha \in \mathrm{HT}_N$ and $f \in \mathcal{QIS}_\alpha$. Recall the straight topological model A_α introduced in Section 3.2, and the closed set $\mathcal{A}_f$, which contains $\Lambda(f)$, introduced in Sec. 7.5. In this section we relate the system $T_\alpha : A_\alpha \to A_\alpha$ to $f : \Lambda(f) \to \Lambda(f)$, and transfer the features of the former system to the latter one.

Recall the map $\Omega_{-1} : \mathcal{M}_{-1} \to M_{-1}$ built in Sec. 8.3. Since $\Omega_{-1}(it+1) = \Omega_{-1}(it)+1$, for $t \geq 0$, by Propositions 8.3, 8.6 and 8.7, Ω_{-1} induces a homeomorphism

$$\Psi_f : \mathcal{A}_f \to A_\alpha. \tag{8.4}$$

Note that Ψ_f maps the critical value of f, which lies in $\mathcal{A}_f$, to $+1$ in A_α. Recall that by Prop 7.14, $\Lambda(f) \subseteq \mathcal{A}_f$.

We are now ready to prove that $f : \Lambda(f) \to \Lambda(f)$ is topologically conjugate to $T_\alpha : A_\alpha \to A_\alpha$.

Theorem 8.8. *Assume that $\alpha \in \mathrm{HT}_N$ and $f \in \mathcal{QIS}_\alpha$. Then, $\Lambda(f) = \mathcal{A}_f$ and hence $\Psi_f : \Lambda(f) \to A_\alpha$ is a homeomorphism. Moreover, $T_\alpha \circ \Psi_f = \Psi_f \circ f$ on $\Lambda(f)$.*

Proof. By Prop 7.9 and the definition of $\mathcal{A}_f$, f maps $\mathcal{A}_f$ into $\mathcal{A}_f$. Moreover, it follows from Prop 7.9, the definition of T_α in Sec. 3.5, and Corollary 8.4 that $T_\alpha \circ \Psi_f = \Psi_f \circ f$ on $\mathcal{A}_f$. Indeed, one only needs to verify the conjugacy relation on the set of points which satisfy item (i) in the definition of T_α, because the set of such points in dense in A_α.

Recall that Ψ_f maps the critical value of f in $\mathcal{A}_f$ to $+1 \in A_\alpha$. By Thm 3.7, $\omega(+1) = A_\alpha$, that is, the orbit of $+1$ by the iterates of T_α is dense in A_α. Since Ψ_f is a topological conjugacy of $f : \mathcal{A}_f \to \mathcal{A}_f$ and $T_\alpha : A_\alpha \to A_\alpha$, we conclude that the orbit of the critical value of f is dense in $\mathcal{A}_f$. By the definition of $\Lambda(f)$, we must have $\Lambda(f) = \mathcal{A}_f$. □

Thm A follows from Thm 8.8 and the trichotomy of A_α in Thm 3.6. Theorems B and C follow from Theorems 8.8 and 3.7. Our analysis of the post-critical set in this paper provides more information about the post-critical set than its topology. It also shows some geometric properties of the post-critical set. For example, we have the following result.

Corollary 8.9. *For every non-Brjuno number $\alpha \in \mathrm{HT}_N$ and every $f \in \mathcal{QIS}_\alpha$, every connected component of $\Lambda(f) \setminus \{0\}$ lands at 0 at a well-defined angle.*

Proof. By Prop 6.9, for every $n \geq -1$, $\lim_{t\to+\infty} \operatorname{Re} u_n(it)$ exists and is finite. This implies that the connected component of $\Lambda(f) \setminus \{0\}$ which contains the critical value of f lands at 0 at a well-defined angle. Any iterate of this curve by f lands at 0 at a well-defined angle. Since the set of the angles of all those rays is dense on $\mathbb{R}/\mathbb{Z}$, any other component of $\Lambda(f) \setminus \{0\}$ must also land at 0 at a well-defined angle. □

Given a connected set $X \subset \mathbb{C}$, a point $z \in X$ is called an **end point** of X if the set $X \setminus \{z\}$ is connected.

In a forthcoming paper [Che22], we employ the tools presented in this paper to show that when $\alpha \in \mathrm{HT}_N$ is not a Brjuno number, every component of $\Lambda(f) \setminus \{0\}$ is a C^1 curve, except possibly at its end point. Similarly, when $\alpha \in \mathrm{HT}_N$ is a Brjuno number but not a Herman number, every component of $\Lambda(f)$ minus the Siegel disk of f is a C^1 curve, except possibly at its end point.

Corollary 8.10. *For every $\alpha \in \mathrm{HT}_N \setminus \mathscr{H}$ and every $f \in \mathcal{QIS}_\alpha$, the critical point and the critical value of f are end points of $\Lambda(f)$.*

Proof. Recall that Ψ_f maps the critical value of f to $+1$ in A_α. By (3.19), $+1$ is an end point of A_α when α is an irrational number outside $\mathscr{H}$. Since $\Psi_f : \Lambda(f) \to A_\alpha$ is a homeomorphism, we conclude that the critical value of f is an end point of $\Lambda(f)$. On the other hand, by Propositions 7.14 and 7.15, $f : \Lambda(f) \to \Lambda(f)$ is injective. Thus, the critical point of f is also an end point of $\Lambda(f)$. □

As a corollary of Thm 8.8 we obtain the following statement.

Corollary 8.11. *For every $\alpha \in \mathrm{HT}_N$ and every f and g in $\mathcal{QIS}_\alpha$ with $f'(0) = g'(0) = e^{2\pi i\alpha}$, $f : \Lambda(f) \to \Lambda(f)$ and $g : \Lambda(g) \to \Lambda(g)$ are topologically conjugate, with a conjugacy which maps the critical point of f to the critical point of g.*

There is an essentially different proof of the above theorem using the holomorphic motion of the critical orbit of the maps, parametrised over the infinite dimensional complex manifold $\mathcal{IS}_\alpha$. However, the proof of Thm 8.8 only uses the pre-compactness of the class $\mathcal{IS}_\alpha$, and does not require any complex structure on $\mathcal{IS}_\alpha$. By classic results from complex analysis and Teichmüller theory, see [MSS83, Slo91], the conjugacy obtained from the holomorphic motion extends to a quasi-conformal mapping of the complex plane. However, the conjugacy may enjoy higher level of regularity due to general universality features in analytic dynamics.

It is proved in [AC18] that for maps in $\mathcal{QIS}_\alpha$ with $\alpha \in \mathrm{HT}_N \cap \mathscr{B}$, the Siegel disk has the small cycles property. That is, any neighbourhood of the closure of the Siegel disk contains infinitely many periodic cycles. This is a generalisation of Yoccoz's small-cycles property near 0 when Q_α is not linearisable, [Yoc95b]. If $\alpha \in \mathscr{B} \setminus \mathscr{H}$, those cycles stay uniformly away from the critical point. By a topological and combinatorial argument this property implies that the critical point of Q_α is not accessible through the external rays of the Julia set; see [Dou83, Kiw00] for details. In particular, we obtain the following corollary.

Corollary 8.12. *If $\alpha \in \mathrm{HT}_N$ is a Brjuno but not a Herman number, the Julia set of Q_α is not locally connected.*

In light of Thm A, it is likely that for irrational values of α, the Julia set of Q_α is locally connected if and only if $\alpha \in \mathscr{H}$. In [BBCO10], the authors make progress on describing the topology of the Julia set when it is not locally connected.

In [PM97] Perez-Marco introduced the notion of hedgehogs, or Siegel-compacta, for a general holomorphic map with an irrationally indifferent fixed point. That is, a local invariant compact set containing the fixed point. In general, a Siegel compacta may have a complicated topology, see for instance [Che11]. But, if the holomorphic map is a restriction of an element of $\mathcal{QIS}_\alpha$ to a neighbourhood of the fixed point, then the hedgehog may not have a complicated topology, provided

$\alpha \in \mathrm{HT}_N$. Indeed, using a general result of Mañé, and the lack of expansion along orbits in a Siegel compacta, any Siegel compacta must be contained in the post-critical set of the map. This is true for any element of $\mathcal{QIS}_\alpha$, [AC18, Section 4.3]. Thus, such Siegel compacta and hedgehogs must be one of the invariant sets presented in Thm C, and have zero area. Our work suggests that for an arbitrary rational function of the Riemann sphere, every hedgehog has a rather tame topology; either a Cantor bouquet or a hairy ball. In contrast, for an arbitrary holomorphic germ of diffeomorphism of $(\mathbb{C}, 0)$, this is far from true as it is shown in [Bis16].

Cantor bouquets are familiar objects in transcendental dynamics, [DK84, AO93, Rem06]. For instance, they appear as the closure of the set of escaping points of the exponential maps λe^z, for $0 < \lambda < 1/e$. Part (iii) of Thm A establishes a similarity between the post-critical set of Q_α and those objects, widely observed in computer simulations. Naively speaking, this is due to the behaviour of the changes of coordinates that appear in the renormalisation tower of a given map in $\mathcal{QIS}$. That is, below a certain horizontal line, the change of coordinate behaves like an exponential map.

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Email address, Davoud Cheraghi: `d.cheraghi@imperial.ac.uk`

(Davoud Cheraghi) Department of Mathematics, Imperial College London, London SW7 2AZ, UK