SATELLITE RENORMALISATION OF QUADRATIC POLYNOMIALS

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ABSTRACT. We prove the uniform *hyperbolicity* of the *near-parabolic renormalisation* operators acting on an infinite-dimensional space of holomorphic transformations. This implies the *universality of the scaling laws*, conjectured by physicists in the 70's, for a combinatorial class of bifurcations. Through near-parabolic renormalisations the *polynomial-like* renormalisations of *satellite* type are successfully studied here for the first time, and new techniques are introduced to analyze the fine-scale dynamical features of maps with such *infinite renormalisation structures*. In particular, we confirm the *rigidity conjecture* under a *quadratic growth* condition on the combinatorics. The class of maps addressed in the paper includes infinitely-renormalisable maps with degenerating geometries at small scales (lack of *a priori* bounds).

1. INTRODUCTION

1.1. **renormalisation conjecture.** In the 1970's, physicists Feigenbaum [Fei78] and independently Coullet-Tresser [TC78], working numerically, observed **universal scaling laws** in the cascades of *doubling bifurcations* in generic families of one-dimensional real analytic transformations. To explain this phenomena, they conjectured that a *renormalisation operator* acting on an infinite-dimensional function space is *hyperbolic* with a one-dimensional unstable direction and a co-dimension-one stable direction. Subsequently, this remarkable feature was observed in other *bifurcation combinatorics* (besides the doubling one) in generic families of real and complex analytic transformations [DGP79]. A conceptual explanation for this phenomena has been the focus of research ever since.

By the seminal works of Sullivan, McMullen, and Lyubich in the 90's, there is a proof of the renormalisation conjecture for combinatorial types arising for real and some complex analytic transformations, [Sul92, McM96, Lyu02], see also Avila-Lyubich [AL11]. A central concept in these works is the **pre-compactness** of the **polynomial-like renormalisation**; a nonlinear operator introduced by Douady and Hubbard in the 80's [DH84]. While this provides the first conceptual proof of the renormalisation conjecture for a class of combinatorial types, lack of the pre-compactness of this renormalisation operator with arbitrary combinatorics is a major obstacle to establishing the renormalisation conjecture for arbitrary combinatorics.

Inou and Shishikura in 2006 [IS06] introduced a sophisticated pair of renormalisations, called **near-parabolic renormalisations**, acting on an infinite-dimensional class of complex analytic transformations near **parabolic maps**. Using a new analytic technique introduced by the first author [Che19, Che13] to control the dependence of these nonlinear operators on the data, we prove the hyperbolicity of these renormalisation operators in this paper, Theorem B. This implies the universality of the scaling laws for a (combinatorial)

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class of bifurcations. Our result covers some combinatorial types where the polynomial-like renormalisations are not pre-compact.

Hyperbolicity versus rigidity. The proof of the expansion part of the hyperbolicity by Lyubich relies on a major result on the **combinatorial rigidity** of the underlying maps. The latter involves a detailed combinatorial and analytic study of the dynamics of the underlying maps, successfully accomplished through a decade of intense studies [Lyu97, GŚ97], see also [Hub93, McM94, LY97, LvS98, Hin00] and the references therein. As a result of this, it is slightly short of providing the rates of expansions. In contrast, the expansion part of the hyperbolicity stated here comes from the relations between the conformal data on the large-scale and the small-scale, related via the renormalisations, see Theorem A. This provides basic formulas for the rates of expansions, and in turn yields an elementary proof of the rigidity conjecture for a class of combinatorial types, see Theorem D.

Tame and wild dynamics. ***A priori* bounds**, a notion of pre-compactness on the non-linearities of long return maps to small scales, is a key concept that has been widely used since the 90's to analyze the fine-scale structure of the dynamics of real and complex analytic transformations (tame dynamics). This has also been at the center of the arguments by Sullivan-McMullen-Lyubich. The hyperbolicity result in this paper applies to classes of transformations that do not enjoy the *a priori* bounds. It also treats maps (of bounded type) that are conjectured to enjoy the *a priori* bounds, but remained mysterious to date. Our approach provides a strong set of tools to describe the fine-scale dynamics of these maps using **towers** of near-parabolic renormalisations, see Theorem C. In particular, in forthcoming papers we shall construct the first examples of analytic transformations with some pathological phenomena.

Below, we state the above notions and results more precisely.

1.2. **Near-parabolic renormalisation operators.** Renormalisation is a sophisticated tool to study fine-scale structures in low-dimensional dynamics. It is a procedure to control the divergence of large iterates of a map through **regularizations**. Starting with a class of maps, to each f in the class, one often identifies an appropriate iterate of f on a region in its domain of definition, which, once viewed in a suitable coordinate on the region (the regularization), belongs to the same class of maps. Remarkably, iterating a renormalisation operator on a class of maps provides significant information about the behaviour of individual maps in the class.

Inou and Shishikura in [IS06] introduced a renormalisation scheme to study the local dynamics of **near-parabolic** holomorphic transformations. More precisely, there is an infinite-dimensional class of maps $\mathcal{F}_0$ (**the non-linearities**), consisting of holomorphic maps h defined on a neighbourhood of 0, with $h(0) = 0$, $h'(0) = 1$, and h has a particular covering property from its domain onto its range. For $\rho > 0$, let (**the set of linearities**)

$$A(\rho) = \{\alpha \in \mathbb{C} \mid 0 < |\alpha|\rho, |\operatorname{Re}\alpha| \geq |\operatorname{Im}\alpha|\}.$$

Define the class of maps

$$A(\rho) \ltimes \mathcal{F}_0 = \{h(e^{2\pi \mathbf{i}\alpha} z) \mid \alpha \in A(\rho), h \in \mathcal{F}_0\}.$$

For ρ small enough, every $\alpha \ltimes h$ in the above class has two distinct fixed points 0 and $\sigma = \sigma(\alpha \ltimes h)$, with derivatives $(\alpha \ltimes h)'(0) = e^{2\pi \mathbf{i}\alpha}$ and $(\alpha \ltimes h)'(\sigma) = e^{2\pi \mathbf{i}\beta}$, where $\beta =$

$\beta(\alpha \ltimes h) \in \mathbb{C}$ and $-1/2 < \operatorname{Re} \beta \leq 1/2$. There are two renormalisation operators, called the **top near-parabolic renormalisation** and the **bottom near-parabolic renormalisation** acting on the class $A(\rho) \ltimes \mathcal{F}_0$. They are defined as some sophisticated notions of return maps of $\alpha \ltimes h$ near 0 and σ, respectively, viewed in some canonically defined coordinates. We denote these by $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$, respectively, and refer to them as NP-renormalisations. According to Inou and Shishikura, the non-linearities of $\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h)$ and $\mathcal{R}_{\text{NP-b}}(\alpha \ltimes h)$ belong to the same class $\mathcal{F}_0$, that is,

$$\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h) = (\hat{\alpha}(\alpha, h) \ltimes \hat{h}(\alpha, h)), \quad \mathcal{R}_{\text{NP-b}}(\alpha \ltimes h) = (\check{\alpha}(\alpha, h) \ltimes \check{h}(\alpha, h)),$$

where $\hat{h}(\alpha, h)$ and $\check{h}(\alpha, h)$ belong to $\mathcal{F}_0$. It follows from the construction that $\hat{\alpha}(\alpha, h) = -1/\alpha \mod \mathbb{Z}$ and $\check{\alpha}(\alpha, h) = -1/\beta \mod \mathbb{Z}$ (so $\hat{\alpha}$ and $\check{\alpha}$ are not necessarily in $A(\rho)$).

A crucial step here is to understand the dependence of these renormalisation operators on the data. In [IS06] $\mathcal{F}_0$ is identified with a Teichmüller metric in order to establish the contractions of the maps $h \mapsto \hat{h}(\alpha \ltimes h)$ and $h \mapsto \check{h}(\alpha \ltimes h)$ on $\mathcal{F}_0$, for each fixed α. On the other hand, to control the maps $\alpha \mapsto \hat{h}(\alpha \ltimes h)$ and $\alpha \mapsto \check{h}(\alpha \ltimes h)$, from $A(\rho)$ to $\mathcal{F}_0$, one faces the canonic transcendental mappings with highly distorting nature that appear as the regularizations in the definitions of these renormalisation operators.

An analytic approach has been introduced by the first author in [Che19, Che13] to control the geometric quantities, and their dependence on the data, involved in these renormalisation schemes. That is, to discard the distortions via certain model maps, and study the differences in the framework of nonlinear elliptic partial differential equations. We extend this approach here to prove an upper bound on the dependence of these regularizations (and the renormalisations) on the data. A key step here is to study the variations of (the hyperbolic norm of) the **Schwarzian derivatives** of $\hat{h}(\alpha \ltimes h)$ and $\check{h}(\alpha \ltimes h)$ as a function of α.

Theorem A. *There exists a constant L such that for every $h \in \mathcal{F}_0$, the maps $\alpha \mapsto \hat{h}(\alpha, h)$, and $\alpha \mapsto \check{h}(\alpha, h)$ are L-Lipschitz with respect to the Euclidean metric on $A(\rho)$ and the Teichmüller metric on $\mathcal{F}_0$.*

The Gauss maps $\hat{\alpha}(\alpha, h) = -1/\alpha \mod \mathbb{Z}$ and $\check{\alpha}(\alpha, h) = -1/\beta \mod \mathbb{Z}$ make $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$ expanding in the first coordinates. Combining these bounds, we build **cone fields** in $A(\rho) \ltimes \mathcal{F}_0$ that are respected by the maps $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$.

Theorem B. *The renormalisations operators $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$ are uniformly hyperbolic on $A(\rho) \ltimes \mathcal{F}_0$. Moreover, the derivatives of these operators at each point in $A(\rho) \ltimes \mathcal{F}_0$ have an invariant one-dimensional expanding direction and an invariant uniformly contracting co-dimension-one direction.*

In the above theorem, the rates of expansions along unstable directions are given in terms of the Gauss map and a holomorphic index formula.

1.3. **Polynomial-like renormalisable versus near-parabolic renormalisable.** To repeat applying $\mathcal{R}_{\text{NP-t}}$ or $\mathcal{R}_{\text{NP-b}}$ to the map $\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h) = \hat{\alpha} \ltimes \hat{h}$ one requires the complex rotation $\hat{\alpha} = -1/\alpha \mod \mathbb{Z}$ belong to $A(\rho)$. Similarly, to apply them to $\mathcal{R}_{\text{NP-b}}(\alpha \ltimes h) = \check{\alpha} \ltimes \check{h}$ one requires $\check{\alpha} = -1/\beta \mod \mathbb{Z}$ belong to $A(\rho)$. In general, to iterate some arbitrary composition of these operators at some $\alpha \ltimes h$, one needs the inductively defined **complex rotations** at 0 belong to $A(\rho)$. For instance, to apply $\mathcal{R}_{\text{NP-t}}$ infinitely often, we require α be real and the continued fraction expansion of $\alpha = [a_1, a_2, a_3, \dots]$ consist of entries $a_i \geq 1/\rho$. It follows from Theorem B that the set of $\alpha \ltimes h$, where an infinite mix of NP-renormalisations may be applied at, consists of a bundle over a Cantor set in $A(\rho)$, with fibers isomorphic to the class $\mathcal{F}_0$, see Theorem 5.1.

To employ the theory, one faces the problem of whether a given map lies on the (implicitly defined) set of infinitely NP-renormalisable maps. We discuss two strategies here: one is based on successive perturbations, and the other is based on somehow knowing the complex rotations of a sequence of periodic points of the given map beforehand. Below we discuss an instance of the first strategy and in Section 1.4 we discuss an instance of the second strategy.

Let $P_c(z) = z^2 + c$, $c \in \mathbb{C}$. The Mandelbrot set

$$M = \{c \in \mathbb{C} \mid \text{the orbit } \langle P_c^{\circ n}(0)\rangle_{n=0}^{\infty} \text{ remains bounded}\}$$

is the set of parameters $c \in \mathbb{C}$ where P_c has a connected Julia set. To explain the appearance of many homeomorphic copies of M within M, Douady and Hubbard in the 80's [DH85] introduced the foundational notion of **polynomial-like** (abbreviated by PL) **renormalisation**. A map $P_c : \mathbb{C} \to \mathbb{C}$ is PL-renormalisable if there exist an integer $q \geq 2$ and simply connected domains $0 \in U \Subset V \subset \mathbb{C}$ such that $P_c^{\circ q} : U \to V$ is a proper branched covering of degree two and the orbit of 0 under the map $P_c^{\circ q} : U \to V$ remains in U. Moreover, when there is a fixed point of P_c in all the domains $P_c^{\circ i}(U)$, for $0 \leq i \leq q-1$, the PL-renormalisation is said of **satellite** type. In turn, if $P_c^{\circ q} : U \to V$ is PL-renormalisable of satellite type, P_c is called two times PL-renormalisable of satellite type. **Infinitely PL-renormalisable of satellite type** is naturally defined as when this scenario occurs infinitely often. These are complex analogues of the period doubling bifurcations (Feigenbaum phenomena), which were successfully studied in the 90's, while these complex analogues remained widely out of reach.

When a P_c is PL-renormalisable of satellite type, the permutation of the domains $P_c^{\circ i}(U)$ about the fixed point, for $0 \leq i \leq q-1$, under the action of P_c may be described by a non-zero rational number $p/q \in (-1/2, 1/2]$. Naturally, an infinitely PL-renormalisable of satellite type gives rise to a sequence of non-zero rational numbers $\langle p_i/q_i\rangle_{i=1}^{\infty}$ in the interval $(-1/2, 1/2]$. This describes the **combinatorial behavior** of the map. We use the notation

$$\langle m_i : b_{i,j} : \varepsilon_{i,j}\rangle_{i=1}^{\infty},$$

with integers $m_i \geq 1$, $b_{i,j} \geq 2$, and $\varepsilon_{i,j} = \pm 1$, for $i \geq 1$ and $1 \leq j \leq m_i$, to denote the sequence of rational numbers defined by the (modified) continued fractions

$$\frac{p_i}{q_i} = \cfrac{\varepsilon_{i,1}}{b_{i,1} + \cfrac{\varepsilon_{i,2}}{\ddots + \cfrac{\varepsilon_{i,m_i}}{b_{i,m_i}}}}, i \geq 1.$$

Theorem C. *There exists $N \geq 2$ such that for every sequence of rational numbers $\langle m_i : b_{i,j} : \varepsilon_{i,j}\rangle_{i=1}^{\infty}$ with all $b_{i,j} \geq N$ there is $c \in M$ such that P_c is infinitely PL-renormalisable of satellite type with combinatorics $\langle m_i : b_{i,j} : \varepsilon_{i,j}\rangle_{i=1}^{\infty}$ and it is also infinitely NP-renormalisable. Moreover, the successive types of NP-renormalisations is given by*

$$\cdots \circ (\mathcal{R}_{\text{NP-t}}^{\circ(\text{m}_\text{n}\text{-1})} \circ \mathcal{R}_{\text{NP-b}}) \circ \cdots \circ (\mathcal{R}_{\text{NP-t}}^{\circ(\text{m}_2\text{-1})} \circ \mathcal{R}_{\text{NP-b}}) \circ (\mathcal{R}_{\text{NP-t}}^{\circ \text{m}_1\text{-1}} \circ \mathcal{R}_{\text{NP-b}}).$$

The parameter c in the above theorem is obtained by an infinite perturbation procedure. That is, by successively following the boundaries of the hyperbolic components of M bifurcating one from the previous one. To this end we introduce a continued fraction type of algorithm (with correction terms satisfying universal laws) that produces the successive complex rotations at 0 along the infinite NP renormalisations of P_c.

Successively applying NP-renormalisations produces a chain of maps linked via the regularizations, that is, the renormalisarion tower. This allows one to study fine-scale dynamical features of the original map. For instance, being infinitely $\mathcal{R}_{\text{NP-t}}$ renormalisable has already led to a breakthrough on the dynamics of maps tangent to irrational rotations. It was used by Buff and Chéritat [BC12] to complete a remarkable program to construct quadratic polynomials with Julia sets of positive area, see also [Yam08]. It is used in a series of papers [Che19, BBCO10, Che13, AC18, CC15] to confirm a number of conjectures on the dynamics of those maps, and is still being harvested. When $\mathcal{R}_{\text{NP-b}}$ appears infinitely often in the chain of NP-renormalisations, we are dealing with the complex analogues of the real Feigenbaum maps. We shall use Theorem C to describe fine dynamical features of these maps in a series of papers to appear in future.

1.4. **Rigidity conjecture.** The **combinatorial rigidity** conjecture in the quadratic family suggests that the "combinatorial behavior" of a quadratic polynomial P_c uniquely determines c, provided $c \in M$ and all periodic points of P_c are repelling. This remarkable feature is equivalent to the **local connectivity of the Mandelbrot set** and implies the **density of hyperbolic maps** within this family; a special case of a conjecture attributed to Fatou [Fat20].

Yoccoz in the 80's proved the rigidity conjecture for quadratic polynomials that are not PL-renormalisable, see [Hub93]. In [Sul92], Sullivan proposed a program, based on *a priori* bounds and pull-back methods, to study the rigidity conjecture for infinitely renormalisable quadratic polynomials. This has been the subject of intense studies for real values of c in the 90's, [GŚ97, Lyu97, LY97, LvS98, McM98]. The symmetry of the map with respect to the real line plays an important role in these studies. When PL-renormalisations are not of satellite type (called primitive type), the pre-compactness is established for a wide class of combinatorial types [Lyu97, Kah06, KL08]. However, when all PL-renormalisations are of satellite type, there is not a single combinatorial class for which the *a priori* bounds is known. But, it is known that for some combinatorial types this property may not hold [Sr00].

We study the rigidity problem for PL-renormalisations of satellite type via their near-parabolic renormalisations. To this end, we need to know when all PL-renormalisable maps of a given combinatorial type are infinitely NP-renormalisable. That is, to know, beforehand, the location of the complex rotations of the cycles associated with the PL-renormalisations. We gain this information using a control on the geometry of the boundaries of the hyperbolic components of M that is proved in this paper, as well as the combinatorial-analytic multiplier inequality of Pommerenke-Levin-Yoccoz [Hub93].

For $N \geq 1$, define the class of sequences of rational numbers

$$\mathcal{QG}_N = \Big\{ \langle \frac{p_i}{q_i} \rangle_{i=1}^{\infty} = \langle m_i : b_{i,j} : \varepsilon_{i,j} \rangle_{i=1}^{\infty} \Big| \begin{array}{l} b_{1,1} \geq N, b_{i,j+1} \geq b_{i,j}^2, b_{i+1,1} \geq q_i^2 \\ \forall i \geq 1, 1 \leq j \leq m_i - 1. \end{array} \Big\}.$$

For a sequence of non-zero rational numbers $\langle p_i/q_i \rangle_{i=1}^{n}$ in $(-1/2, 1/2]$, let $M(\langle p_i/q_i \rangle_{i=1}^{n})$ denote the set of c in M such that P_c is n times PL-renormalisable of satellite type $\langle p_i/q_i \rangle_{i=1}^{n}$.

Theorem D. *There are constants N, C, and $\lambda \in (0,1)$ such that for every $\langle p_i/q_i\rangle_{i=1}^\infty$ in $\mathcal{QG}_N$, we have*

$$\operatorname{diam} M(\langle p_i/q_i\rangle_{i=1}^n) \leq C\lambda^n.$$

In particular, if P_c is infinitely PL-renormalisable of satellite type $\langle p_i/q_i\rangle_{i=1}^\infty$ in $\mathcal{QG}_N$, it is combinatorially rigid, and the Mandelbrot set is locally connected at c.

If one chooses $m_i = 1$, for all $i \geq 1$, then a sequence $\langle p_i/q_i\rangle_{i=1}^\infty$ belongs to $\mathcal{QG}_N$ provided $q_1 \geq N$, $p_i = \pm 1$, and $q_{i+1} \geq q_i^2$, for all $i \geq 1$. Choosing a larger value for some m_i allows us to have a rational number p_i/q_i of mixed type, but this requires the later denominators become large because of the condition $b_{i+1,1} \geq q_i$. G. Levin had already proved the combinatorial rigidity under the relative growth conditions $\limsup_n q_{n+1}/(q_1 q_2 \dots q_n)^2 > 0$ and $\sup |p_n/q_n| q_0 q_1 \dots q_{n-1} < \infty$, [Lev11, Lev14]. This is a faster growth condition on the denominators, but it covers rational numbers with certain numerators that are not covered in the above theorem. For parameters satisfying these growth conditions, he controls the location of the sequence of periodic cycles that consecutively bifurcate one from another; quantifying a construction due to Douady and Hubbard [Sr00], to obtain non-locally connected Julia sets. His approach is different from the one presented in this paper.

We note that the **post-critical** set (i.e. the closure of the orbit of the critical point) of the maps in the above theorem do not enjoy the *a priori* bounds (bounded geometry) proposed in the program of Sullivan. The geometry of the post-critical set highly depends on the successive complex rotations at 0 produced by successive NP-renormalisations. Moreover, the Pommerenke-Levin-Yoccoz inequality does not provide the kind of estimates to deduce that the post-critical sets of all maps with the same combinatorial behavior have comparable geometries. Besides overcoming this issue, our approach allows us to treat all types of geometries at once, rather than dealing with fine geometric considerations dependent on the combinatorics, investigated in part two of [Lyu97] and in [Che10].

The combinatorial rigidity conjecture is meaningful for higher degree maps, and indeed it has been successfully established for a number of classes of maps through the program of Sullivan. Rational maps with all critical points periodic or pre-periodic are studied in [DH85]. See [LvS98, KSvS07] for real infinitely PL-renormalisable polynomials of higher degree, and [CvST17, CvS18] (and the references therein) for a broader result for real maps. The papers [AKLS09, KvS09, PT15] treat higher degree complex polynomials that are not PL-renormalisable. On the other hand, a near-parabolic renormalisation scheme for uni-critical maps (i.e. maps with a single critical point of higher degree), similar to the one studied here, has been announced by Chéritat in [Ché14]. The analysis of this paper may be carried out in that setting to obtain the corresponding results for the higher degree uni-critical maps.

One may refer to the book by de Melo-van Strien [dMvS93] for historical notes on early stages of the developments, and also for the real analysis tools that played a crucial role in the pioneering work of Sullivan on the subject. An alternative approach to the existence of the fixed point of doubling renormalisation was given by Martens in [Mar98]. A unified approach to the uniform contraction of polynomial-like renormalisation for uni-singular maps is presented in [AL11]. A number of renormalisation schemes in low-dimensional dynamics have been studied in parallel to the one for holomorphic maps on plane-domains discussed here. However, the issue of the pre-compactness addressed here does not arise in those cases. One may refer to [Lan82, EE86, dF92, Yam02, dFdM00, LŚ05, GdM17, KK14] for critical circle maps; [McM98] for linearizable maps of bounded type; [LŚ02, LŚ12] for critical circle covers; [GvST89, DCLM05] for Henon maps; [dFdMP06] for C^r uni-modal maps.

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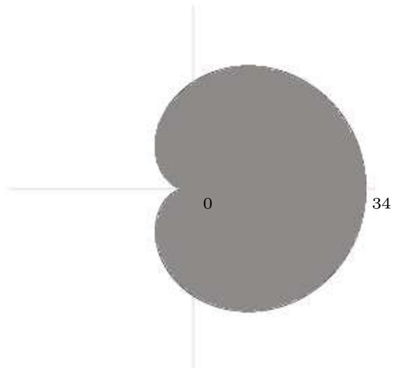

FIGURE 1. The domain V, in gray, contains 0 and $-1/3$, but not -1.

2. Near-parabolic renormalisation scheme

In this section we introduce the class of maps $\mathcal{F}_0$, and define the top and bottom near-parabolic renormalisations $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$.

2.1. **The class of maps $\mathcal{F}_0$ and their perturbations.** Consider the ellipse

$$E = \left\{x + iy \in \mathbb{C} \,\middle|\, \left(\frac{x+0.18}{1.24}\right)^2 + \left(\frac{y}{1.04}\right)^2 \leq 1\right\}$$

and define the domain

$$V = g(\hat{\mathbb{C}} \setminus E), \text{ where } g(z) = \frac{-4z}{(1+z)^2}.$$

The ellipse E is contained in the ball $|z| < 2$, and thus, the ball $|z| < 8/9$ is contained in V. Consider the cubic polynomial

$$P(z) = z(1+z)^2.$$

The polynomial P has a fixed point at 0 with multiplier $P'(0) = 1$, and it has two critical points $-1 \in \mathbb{C} \setminus V$ and $-1/3 \in V$, where $P(-1) = 0$ and $P(-1/3) = -4/27$. See Figures 1 and 5.

Following [IS06] we consider the class of maps

$$\mathcal{F}_0 = \left\{ f = P \circ \varphi^{-1} : \varphi(V) \to \mathbb{C} \,\middle|\, \begin{array}{l} \varphi : V \to \mathbb{C} \text{ is univalent}^1, \varphi(0) = 0, \varphi'(0) = 1, \text{ and} \\ \varphi \text{ has quasi-conformal extension onto } \mathbb{C}. \end{array} \right\}.$$

Every map in $\mathcal{F}_0$ has a parabolic fixed point at 0, and a unique critical point at $\varphi^{-1}(-1/3)$ which is mapped to $-4/27$. Indeed, every element of $\mathcal{F}_0$ has the same covering structure from its domain onto its range as the one of $P : V \to P(V)$. See Figure 5.

For $h \in \mathcal{F}_0$ and $\alpha \in \mathbb{C}$ we use the notation $\alpha \ltimes h$ to denote the map [2]

$$(\alpha \ltimes h)(z) = h(e^{2\pi i\alpha} z), \qquad z \in e^{-2\pi i\alpha}\, \mathrm{Dom}\,(h). \tag{2.1}$$

[1]Univalent is a standard terminology used for one-to-one holomorphic maps.

[2]$\mathrm{Dom}\,(f)$ denotes the domain of definition of a given map f, and is always assumed to be an open set.

In the same fashion, for a set $A \subseteq \mathbb{C}$, define the class of maps

$$A \ltimes \mathcal{F}_0 = \{\alpha \ltimes h \mid h \in \mathcal{F}_0, \alpha \in A\}.$$

For $r > 0$, we define

$$\begin{aligned} A^+(r) &= \{\alpha \in \mathbb{C} \mid 0 < |\alpha| \le r, \operatorname{Re}\alpha \ge |\operatorname{Im}\alpha|\}, \\ A^-(r) &= \{\alpha \in \mathbb{C} \mid 0 < |\alpha| \le r, \operatorname{Re}\alpha \le -|\operatorname{Im}\alpha|\}, \\ A(r) &= A^+(r) \cup A^-(r). \end{aligned}$$

We shall work on the class of maps $A(r) \ltimes \mathcal{F}_0$, for an appropriate constant r which will be determined in Section 2.4.

Every $f \in A(\infty) \ltimes \mathcal{F}_0$ has a unique critical point, denoted by cp_f. That is,

$$f'(\mathrm{cp}_f) = 0, \quad f(\mathrm{cp}_f) = -4/27 = \mathrm{cv}_f.$$

For our convenience, we normalise the quadratic polynomials into the form

$$Q_\alpha(z) = e^{2\pi i\alpha} z + \frac{27}{16} e^{4\pi i\alpha} z^2,$$

so that their critical values lie at $-4/27$ and $\alpha \ltimes Q_0 = Q_\alpha$.

We consider the topology of uniform convergence on compact sets on the space of holomorphic maps $g : \mathrm{Dom}\,(g) \to \mathbb{C}$, where $\mathrm{Dom}\,(g)$ is an open subset of $\mathbb{C}$. A basis for this topology is defined by

$$N(h; K, \varepsilon) = \Big\{ g : \mathrm{Dom}\,(g) \to \mathbb{C} \,\Big|\, K \subset \mathrm{Dom}\,(g) \text{ and } \sup_{z \in K} |g(z) - h(z)| < \varepsilon \Big\},$$

where $h : \mathrm{Dom}\,(h) \mapsto \mathbb{C}$ is a holomorphic map, $K \subset \mathrm{Dom}\,(h)$ is compact, and $\varepsilon > 0$. In this topology, a sequence $h_n : \mathrm{Dom}\,(h_n) \mapsto \mathbb{C}$ converges to h provided h_n is contained in any given neighbourhood of h defined as above, for large enough n. Note that the maps h_n are not necessarily defined on the same domain.

The class $\mathcal{F}_0$ naturally embeds into the space of univalent maps on the unit disk with a neutral fixed point at 0. Therefore, by the Koebe distortion Theorem [Leh87], $\mathcal{F}_0$ is a pre-compact class in the compact-open topology.

2.2. **Teichmuller metric and the holomorphic dependence.** Using the one-to-one correspondence between the class $\mathcal{F}_0$ and the quasi-conformal mappings on $\mathbb{C} \setminus \overline{V}$ one may define a metric on $\mathcal{F}_0$. This corresponds to the Teichmüller metric on the Teichmüller space of $\mathbb{C} \setminus \overline{V}$. One may refer to [Leh76] for the definition of quasi-conformal mappings, and to [GL00], [IT92], or [Leh87] for the theory of Teichmüller spaces. Recall that the dilatation quotient of a quasi-conformal mapping h is defined as

$$\mathrm{Dil}(h) = \sup_{z \in \mathrm{Dom}\ h} \frac{|h_z| + |h_{\overline{z}}|}{|h_z| - |h_{\overline{z}}|}.$$

The Teichmüller distance between any two elements $f = P \circ \varphi_f^{-1}$ and $g = P \circ \varphi_g^{-1}$ in $\mathcal{F}_0$ is defined as

$$\mathrm{d}_{\mathrm{Teich}}(f, g) = \inf \Big\{ \log \mathrm{Dil}(\hat\varphi_g \circ \hat\varphi_f^{-1}) \,\Big|\, \begin{array}{l} \hat\varphi_f \text{ and } \hat\varphi_g \text{ are quasi-conformal extensions} \\ \text{of } \varphi_f \text{ and } \varphi_g \text{ onto } \mathbb{C}, \text{ respectively.} \end{array} \Big\}.$$

It is known that the Teichmüller space of $\mathbb{C} \setminus \overline{V}$ equipped with the Teichmüller distance is a complete metric space, and so is $\mathcal{F}_0$ equipped with $\mathrm{d}_{\mathrm{Teich}}$. The convergence with respect to $\mathrm{d}_{\mathrm{Teich}}$ on $\mathcal{F}_0$ implies the uniform convergence on compact sets.

Let $f_\lambda : \mathrm{Dom}\,(f_\lambda) \to \mathbb{C}$ be a family of holomorphic maps parameterised by λ in a finite dimensional complex manifold Λ, such that for every $\lambda \in \Lambda$, $\mathrm{Dom}\,(f_\lambda) \subset \mathbb{C}$. We say that the family f_λ is a **holomorphic family** of maps, if for every $\lambda_0 \in \Lambda$ and every $z_0 \in \mathrm{Dom}\, f_{\lambda_0}$, the map $(z, \lambda) \mapsto f_\lambda(z)$ is defined and holomorphic in z and λ, for z sufficiently close to z_0 and λ sufficiently close to λ_0. Let $\Upsilon : X \to Y$ be a mapping where X and Y are some classes of holomorphic maps. We say that the mapping $f \mapsto \Upsilon(f)$ has **holomorphic dependence** on f, if for every holomorphic family of maps f_λ in X, the family $\Upsilon(f_\lambda)$ is a holomorphic family of maps.

2.3. **Fatou coordinates.** As we shall see in Lemma 3.10, the set of $f''(0)$ over all $f \in \mathcal{F}_0$ is compactly contained in $\mathbb{C}$. Thus every $f \in \mathcal{F}_0$ has a non-degenerate parabolic fixed point at 0. For $\alpha \in \mathbb{C}$ sufficiently close to 0 and $f \in \mathcal{F}_0$, the parabolic fixed point of $0 \ltimes f$ at 0 bifurcates into two distinct nearby fixed points for $\alpha \ltimes f$. These two fixed points play a central role in this paper. Below we introduce these formally, as they are needed for the definition of the near-parabolic renormalisations, and postpone the proofs to Section 3.

Proposition 2.1. *There exist a simply connected neighbourhood W of 0, bounded by a smooth curve, and a constant $r_1 > 0$ such that every map in $A(r_1) \ltimes \mathcal{F}_0$ has exactly two distinct fixed points in the closure of W.*

The proof of the above proposition appears in Section 3.4.

The non-zero fixed point of $f \in A(r_1) \ltimes \mathcal{F}_0$ contained in W is denoted by σ_f. There are complex numbers $\alpha(f)$ and $\beta(f)$ with their real parts in $(-1/2, 1/2]$ such that

$$f'(0) = e^{2\pi i \alpha(f)} \text{ and } f'(\sigma_f) = e^{2\pi i \beta(f)}.$$

These values are related by the holomorphic index formula

$$\frac{1}{2\pi i} \int_{\partial W} \frac{1}{z - f(z)}\, \mathrm{d}z = \frac{1}{1 - e^{2\pi i \alpha(f)}} + \frac{1}{1 - e^{2\pi i \beta(f)}}. \tag{2.2}$$

Proposition 2.2. *There exists $r_2 > 0$ such that for every f in $A^+(r_2) \ltimes \mathcal{F}_0$ there exist a domain $\mathcal{P}_f \subset \mathrm{Dom}\,(f)$ and a univalent map $\Phi_f : \mathcal{P}_f \to \mathbb{C}$ satisfying the following properties:*

- *a) $\mathcal{P}_f$ is bounded by piecewise smooth curves, the closure of $\mathcal{P}_f$ is contained in $\mathrm{Dom}\,(f)$, and the points cp_f, 0, and σ_f belong to $\partial \mathcal{P}_f$;*
- *b) $\mathrm{Im}\, \Phi_f(z) \to +\infty$ when $z \to 0$ in $\mathcal{P}_f$, and $\mathrm{Im}\, \Phi_f(z) \to -\infty$ when $z \to \sigma_f$ in $\mathcal{P}_f$;*
- *c) $\Phi_f(\mathcal{P}_f)$ contains the vertical strip $\mathrm{Re}\, w \in (0, 2)$;*
- *d) Φ_f satisfies*
  $$\Phi_f(f(z)) = \Phi_f(z) + 1,$$
  whenever z and $f(z)$ belong to $\mathcal{P}_f$;
- *e) Φ_f is uniquely determined by the above conditions and the normalisation $\Phi_f(\mathrm{cp}_f) = 0$. Moreover, with this normalisation, the map $f \mapsto \Phi_f$ has holomorphic dependence on f.*

When $f = Q_\alpha : \mathbb{C} \to \mathbb{C}$, with $\alpha \in A^+(r_2)$, the existence of a domain $\mathcal{P}_f$ and a coordinate $\Phi_f : \mathcal{P}_f \to \mathbb{C}$ satisfying the properties in the above proposition is rather classical. Indeed, these are among the basic tools in complex dynamics for over a century now. However, their existence for the maps in $A(r) \ltimes \mathcal{F}_0$ is highly non-trivial and is proved in [IS06]. The univalent map Φ_f with the above properties is called the **Fatou coordinate** of f on $\mathcal{P}_f$. See Figure 2.

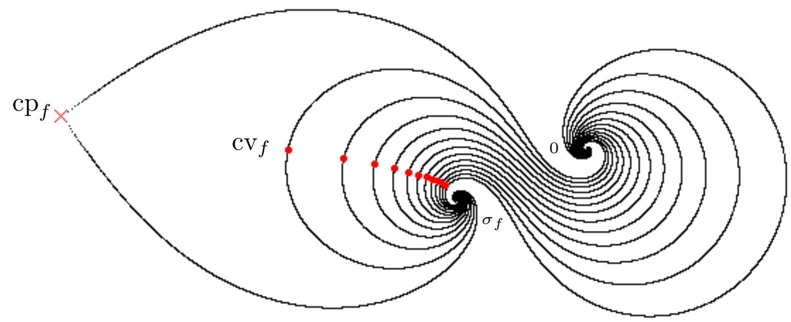

FIGURE 2. The pre-images of the vertical lines, with integer real parts, under ϕ_f. These curves land at 0 and σ_f, and spiral about 0 and σ_f at well-defined speeds when $\operatorname{Im}\alpha \neq 0$. The cross "×" in red is the location of cp_f, and the dots in red are the first few iterates of cp_f.

Proposition 2.3. *There are constants $r_3 \in (0, r_2)$ and $\mathbf{k}$ such that for all $f \in A^+(r_3) \ltimes \mathcal{F}_0$, or $f = Q_\alpha$ with $\alpha \in A^+(r_3)$, the domain $\mathcal{P}_f$ in Proposition 2.2 may be chosen (wide enough) to satisfy the additional property*

$$\Phi_f(\mathcal{P}_f) = \left\{ w \in \mathbb{C} \;\middle|\; 0 < \operatorname{Re}(w) < \operatorname{Re}\frac{1}{\alpha(f)} - \mathbf{k} \right\}.$$

The above proposition is proved in Sections 3.7.

In [Che19] it is proved that when α is real, Φ_f^{-1} of every vertical line in the image of Φ_f is a curve which lands at 0 and σ_f at some well-defined angles. That is, there are tangent lines to these curves at 0 and σ_f. However, this is not the case when $\operatorname{Im}\alpha \neq 0$. The pre-images of the vertical lines spiral about 0 and σ_f, and the corresponding speeds of spirals depend on $\operatorname{Im}\alpha$ and $\operatorname{Im}\beta$, respectively. This is stated in the next proposition.

Proposition 2.4. *There exists a constant k' such that for all $f \in A^+(r_3) \ltimes \mathcal{F}_0$, or $f = Q_\alpha$ with $\alpha \in A^+(r_3)$, there exists a continuous branch of argument defined on $\mathcal{P}_f$ satisfying the following properties.*

a) For all ξ_1 in $(0, \operatorname{Re}\frac{1}{\alpha(f)} - \mathbf{k})$ and $\xi_2 \geq 0$, we have

$$\lim_{\xi_2 \to +\infty} \left(\arg \Phi_f^{-1}(\xi_1 + i\xi_2) + 2\pi\xi_2 \operatorname{Im}\alpha \right) = \arg \sigma_f + 2\pi\xi_1 \operatorname{Re}\alpha + c_f,$$

where c_f is a real constant which depends only on f and $|c_f| \leq k'(1 - \log|\alpha|)$.

b) For all ξ_1 in $(0, \operatorname{Re}\frac{1}{\alpha(f)} - \mathbf{k})$ and $\xi_2 \leq 0$, we have

$$\lim_{\xi_2 \to -\infty} \left(\arg(\Phi_f^{-1}(\xi_1 + i\xi_2) - \sigma_f) - 2\pi\xi_2 \operatorname{Im}\beta \right) = \arg \sigma_f - 2\pi\xi_1 \operatorname{Re}\beta + c'_f,$$

where c'_f is a real constant which only depends on f and $|c'_f| \leq k'(1 - \log|\alpha|)$.

Part (a) of Proposition 2.4 is proved in Section 3.7, and its part (b) is proved in Section 3.9.

Remark 2.5. When $f \in A(r) \ltimes \mathcal{F}_0$ tends to a map $f_0 \in \mathcal{F}_0$, the fixed point σ_f tends to 0, and becomes a parabolic fixed point. Although it is not used in this paper, it may be useful to note that as f tends to $f_0 \in \mathcal{F}_0$, appropriately normalised Fatou coordinates Φ_f tend to some conformal mappings, called attracting and repelling Fatou coordinates, that still satisfy the relation in Proposition 2.2-(d). One may refer to [Shi00] for further details on this.

2.4. **Top and bottom near-parabolic renormalisations.** Let f either be in $A^+(r_2) \ltimes \mathcal{F}_0$ or be the quadratic polynomial Q_α with $\alpha \in A^+(r_2)$. Let $\Phi_f : \mathcal{P}_f \to \mathbb{C}$ be the Fatou coordinate of f introduced in the previous section. Define the sets

$$\begin{aligned} A_f &= \{z \in \mathcal{P}_f : 1/2 \leq \mathrm{Re}(\Phi_f(z)) \leq 3/2\,,\ 2 \leq \mathrm{Im}\,\Phi_f(z)\}, \\ C_f &= \{z \in \mathcal{P}_f : 1/2 \leq \mathrm{Re}(\Phi_f(z)) \leq 3/2\,,\ -2 \leq \mathrm{Im}\,\Phi_f(z) \leq 2\}, \\ B_f &= \{z \in \mathcal{P}_f : 1/2 \leq \mathrm{Re}(\Phi_f(z)) \leq 3/2\,,\ \mathrm{Im}\,\Phi_f(z) \leq -2\}. \end{aligned} \tag{2.3}$$

By Proposition 2.2, $\Phi_f(\mathrm{cv}_f) = +1$, and hence $\mathrm{cv}_f \in \mathrm{int}\,(C_f)$ [3]. Moreover, $0 \in \partial A_f$ and $\sigma_f \in \partial B_f$.

See Figures 3 and 4 for an illustration of the following two propositions.

Proposition 2.6. *For every $f \in A^+(r_3) \ltimes \mathcal{F}_0$, or $f = Q_\alpha$ with $\alpha \in A^+(r_3)$, there is a positive integer k_f^t satisfying the following properties.*

a) For every integer k, with $0 \leq k \leq k_f^t$, there exists a unique connected component of $f^{-k}(A_f)$ which is compactly contained in $\mathrm{Dom}\,(f)$ *and contains* 0 *on its boundary. We denote this component by A_f^{-k}.*

b) For every integer k, with $0 \leq k \leq k_f^t$, there exists a unique connected component of $f^{-k}(C_f)$ which has non-empty intersection with A_f^{-k}, and is compactly contained in $\mathrm{Dom}\,(f)$. *This component is denoted by $C_{f,t}^{-k}$.*

c) We have

$$A_f^{-k_f^t}, C_{f,t}^{-k_f^t} \subseteq \left\{ z \in \mathcal{P}_f \mid 1/2 < \mathrm{Re}\,\Phi_f(z) < \mathrm{Re}\,\frac{1}{\alpha(f)} - \mathbf{k} \right\}.$$

d) The map $f : C_{f,t}^{-k} \to C_{f,t}^{-k+1}$, for $2 \leq k \leq k_f^t$, and $f : A_f^{-k} \to A_f^{-k+1}$, for $1 \leq k \leq k_f^t$, are univalent. On the other hand, the map $f : C_{f,t}^{-1} \to C_f$ is a proper branched covering of degree two.

Proposition 2.7. *For every $f \in A^+(r_3) \ltimes \mathcal{F}_0$, or $f = Q_\alpha$ with $\alpha \in A^+(r_3)$, there is a positive integer k_f^b satisfying the following properties.*

a) For every integer k, with $0 \leq k \leq k_f^b$, there exists a unique connected component of $f^{-k}(B_f)$ which is compactly contained in $\mathrm{Dom}\,(f)$, *and contains σ_f on its boundary. We denote this component by B_f^{-k}.*

b) For every integer k, with $0 \leq k \leq k_f^b$, there exists a unique connected component of $f^{-k}(C_f)$ which has non-empty intersection with B_f^{-k}, and is compactly contained in $\mathrm{Dom}\,(f)$. *This component is denoted by $C_{f,b}^{-k}$.*

[3] The notation $\mathrm{int}\,(C)$ denotes the (topological) interior of a given set C.

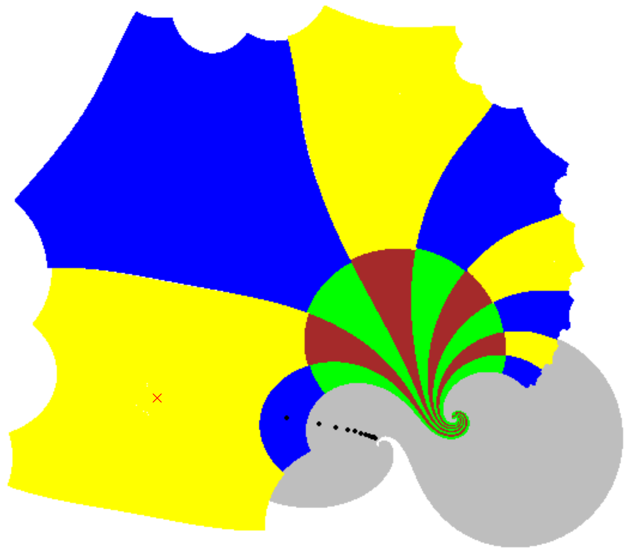

FIGURE 3. A presentation of the regions associated to the top renormalisation of f. The alternating green and brown shades denote the sets A_f^{-j}, and the alternating blue and yellow shades are the sets C_f^{-j}. The grey region is the petal $\mathcal{P}_f$. Here, $f = Q_\alpha$ and $\alpha = 0.01 - 0.02i$.

c) We have

$$B_f^{-k_f^b}, C_{f,b}^{-k_f^b} \subseteq \left\{ z \in \mathcal{P}_f \mid 1/2 < \operatorname{Re} \Phi_f(z) < \frac{1}{\operatorname{Re} \alpha(f)} - \mathbf{k} \right\}.$$

d) The map $f : C_{f,b}^{-k} \to C_{f,b}^{-k+1}$, *for* $2 \leq k \leq k_f^b$, *and* $f : B_f^{-k} \to B_f^{-k+1}$, *for* $1 \leq k \leq k_f^b$, *are univalent. On the other hand, the map* $f : C_{f,b}^{-1} \to C_f$ *is a proper branched covering of degree two.*

The above two propositions are appropriately adjusted and reformulated versions of several statements which appear in Section 5.A in [IS06]. See in particular Propositions 5.6 and 5.7 in that paper. Note that here we are working with the value $+2$ in place of the parameter η in that paper.

Let k_f^t and k_f^b be the smallest positive integers satisfying the above propositions.

Proposition 2.8. *There exists a constant* $\mathbf{k}''$ *such that for every* $f \in A^+(r_3) \ltimes \mathcal{F}_0$, *or* $f = Q_\alpha$ *with* $\alpha \in A^+(r_3)$, *we have a)* $k_f^t \leq \mathbf{k}''$ *and b)* $k_f^b \leq \mathbf{k}''$.

Part (a) of the above proposition is proved in Section 3.7 and its part (b) is proved in Section 3.9.

Define the set

$$S_f^t = A_f^{-k_f^t} \cup C_{f,t}^{-k_f^t}, \qquad S_f^b = B_f^{-k_f^b} \cup C_{f,b}^{-k_f^b}.$$

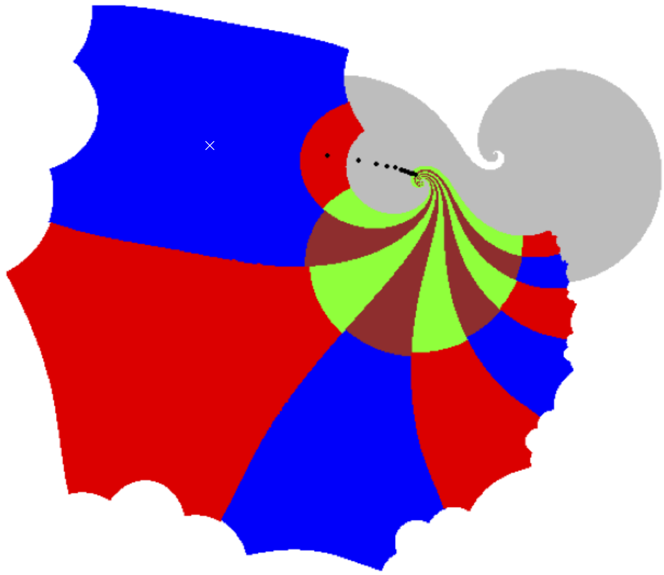

FIGURE 4. A presentation of the regions associated to the bottom renormalisation of f. The alternating green and brown shades show the sets B_f^{-j}, and the alternating red and blue shades show the sets C_f^{-j}.

Consider the induced maps

$$E_f^t = \Phi_f \circ f^{\circ k_f^t} \circ \Phi_f^{-1} : \Phi_f(S_f^t) \to \Phi_f(\mathcal{P}_f), \quad E_f^b = \Phi_f \circ f^{\circ k_f^b} \circ \Phi_f^{-1} : \Phi_f(S_f^b) \to \Phi_f(\mathcal{P}_f). \tag{2.4}$$

By the functional relation in 2.2-(d), we have $E_f^t(w+1) = E_f^t(w)+1$ whenever both w and $w+1$ are in $\Phi_f(S_f^t)$. Similarly, E_f^b commutes with the translation by $+1$ on the boundary of $\Phi_f(S_f^b)$.

Let us define the covering maps

$$\mathbb{E}\mathrm{xp}^t(w) = \frac{-4}{27} e^{2\pi i w}, \qquad \mathbb{E}\mathrm{xp}^b(w) = \frac{-4}{27} e^{-2\pi i w}. \tag{2.5}$$

The map $E_f^t : \Phi_f(S_f^t) \to \Phi_f(\mathcal{P}_f)$ projects via $\mathbb{E}\mathrm{xp}^t$ to a well-defined holomorphic map defined on a set containing a punctured neighbourhood of 0. We denote this map by $\mathcal{R}_{\text{NP-t}}(f)$. Similarly, $E_f^b : \Phi_f(S_f^b) \to \Phi_f(\mathcal{P}_f)$ projects via $\mathbb{E}\mathrm{xp}^b$ to a well-defined holomorphic map defined on a set containing a punctured neighbourhood of 0. This map is denoted by $\mathcal{R}_{\text{NP-b}}(f)$. Both of these maps have a removable singularity at 0 with asymptotic expansions

$$\mathcal{R}_{\text{NP-t}}(f)(z) = e^{-2\pi i/\alpha(f)} z + O(z^2), \quad \mathcal{R}_{\text{NP-b}}(f)(z) = e^{-2\pi i/\beta(f)} z + O(z^2),$$

near 0, where $f'(0) = e^{2\pi i \alpha(f)}$ and $f'(\sigma_f) = e^{2\pi i \beta(f)}$. The above asymptotic expansions are obtained from comparing f near 0 and σ_f to the linear maps $z \mapsto e^{2\pi i \alpha(f)} z$ and $z \mapsto \sigma_f + e^{2\pi i \beta(f)}(z - \sigma_f)$, respectively.

By Propositions 2.6 and 2.7, each of E^t_f and E^b_f has a unique critical point. As $\Phi_f(\mathrm{cv}_f) = 1$, the critical values of E^t_f and E^b_f lie at $+1$. On the other hand, as $\mathbb{E}\mathrm{xp}^t(+1) = \mathbb{E}\mathrm{xp}^b(+1) = -4/27$, each of $\mathcal{R}_{\text{NP-t}}(f)$ and $\mathcal{R}_{\text{NP-b}}(f)$ must have a unique critical value at $-4/27$.

The main result of [IS06] is formulated in the next theorem.

Theorem 2.9 (Inou-Shishikura)**.** *There exists a Jordan domain $U \supset \overline{V}$ satisfying the following. For all $f \in A^+(r_3) \ltimes \mathcal{F}_0$, or $f = Q_\alpha$ with $\alpha \in A^+(r_3)$, there are appropriate restrictions (to smaller domains about 0) of the maps $\mathcal{R}_{\text{NP-t}}(f)$ and $\mathcal{R}_{\text{NP-b}}(f)$ which belong to the classes $\{\frac{-1}{\alpha(f)}\} \ltimes \mathcal{F}_0$ and $\{\frac{-1}{\beta(f)}\} \ltimes \mathcal{F}_0$, respectively. That is, there exist quasi-conformal homeomorphisms $\psi, \varphi : \mathbb{C} \to \mathbb{C}$ which are holomorphic on V, $\psi(0) = \varphi(0) = 0$, $\psi'(0) = \varphi'(0) = 1$, and*

$$\mathcal{R}_{\text{NP-t}}(f)(z) = P \circ \psi^{-1}(e^{-2\pi i/\alpha(f)} z), \quad \forall z \in e^{2\pi i/\alpha(f)}\psi(V)$$
$$\mathcal{R}_{\text{NP-b}}(f)(z) = P \circ \varphi^{-1}(e^{-2\pi i/\beta(f)} z), \quad \forall z \in e^{2\pi i/\beta(f)}\varphi(V).$$

Moreover, when $f \in A^+(r_3) \ltimes \mathcal{F}_0$, $\psi : V \to \mathbb{C}$ and $\varphi : V \to \mathbb{C}$ extend to univalent maps on U.

See the figure below for the covering structure of the polynomial P on the set V.

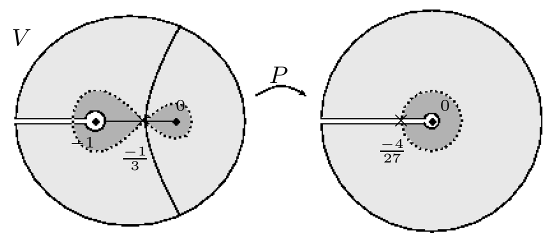

FIGURE 5. Illustration of the covering property of the polynomial P. Similar colors and line styles are mapped onto one another.

Remark 2.10. The renormalisations $\mathcal{R}_{\text{NP-t}}(f)$ and $\mathcal{R}_{\text{NP-b}}(f)$ are not obtained from the return maps (iterates of f) to a region, in contrast to other notions of renormalisation in holomorphic dynamics, such as the PL-renormalisation [DH85] or the sector renormalisation of Yoccoz [Yoc95]. Near 0 or σ_f, these renormalisations may be interpreted as return maps since E^t_f and all its integer translations project to the same map $\mathcal{R}_{\text{NP-t}}(f)$. That is, for $w \in \Phi_f(S^t_f)$ with $\operatorname{Im} w$ large enough, there is $i_w \in \mathbb{N}$ such that $E^t_f(w) + i_w \in \Phi_f(S^t_f)$; hence a return map. But, this may not happen for every w in $\Phi_f(S^t_f)$. For example, when $|\alpha|$ is small with $\arg \alpha = -\pi/4$, the σ_f is attracting and may attract the orbit of cp_f. Then, the orbit of cp_f may not visit S^t_f, (and then "go around" 0 to return back to $\mathcal{P}_f$). However, this does not contradict the above theorem and the top renormalisation is still defined. Here, $\mathcal{R}_{\text{NP-t}}(f)$ has a critical value, but it does not belong to the domain of $\mathcal{R}_{\text{NP-t}}(f)$. For such values of α, $|e^{-2\pi i/\alpha}|$ is large, and hence by Theorem 2.9, Dom $\mathcal{R}_{\text{NP-t}}(f)$ may be small and not contain the critical value at $-4/27$. As we shall see in Sections 3.10 and 6.3, in the interesting cases where both multipliers at 0 and σ_f are repelling, α belongs to a substantially smaller region and this scenario does not occur.

Definition 2.11. For every $f \in A^-(r_3) \ltimes \mathcal{F}_0$, the conjugate map $s \circ f \circ s$, where $s(z) = \overline{z}$ denotes the complex conjugation, belongs to $A^+(r_3) \ltimes \mathcal{F}_0$. We may extend the above definitions of renormalisations onto $A^-(r_3) \ltimes \mathcal{F}_0$ by letting

$$\mathcal{R}_{\text{NP-t}}(f) = \mathcal{R}_{\text{NP-t}}(s \circ f \circ s),\ \mathcal{R}_{\text{NP-b}}(f) = \mathcal{R}_{\text{NP-b}}(s \circ f \circ s), \quad \forall f \in A^-(r_3) \ltimes \mathcal{F}_0.$$

In particular, the Fatou coordinates are also defined for maps in $A^-(r_3) \ltimes \mathcal{F}_0$. Similarly, one defines the Fatou coordinates and the renormalisations for Q_α with $\alpha \in A^-(r_3)$.

The following proposition is a consequence of the holomorphic dependence of the Fatou coordinate on the map, Proposition 2.2-(e), and the definitions of the operators $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$.

Proposition 2.12. *The operators $f \mapsto \mathcal{R}_{\text{NP-t}}(f)$ and $f \mapsto \mathcal{R}_{\text{NP-b}}(f)$ have holomorphic dependence on $f \in A(r_3) \ltimes \mathcal{F}_0$. Similarly, $\alpha \mapsto \mathcal{R}_{\text{NP-t}}(Q_\alpha)$ and $\alpha \mapsto \mathcal{R}_{\text{NP-b}}(Q_\alpha)$ are holomorphic families of maps, parametrized on $A(r_3)$.*

Recall that $\beta(f)$ is selected to satisfy $\operatorname{Re}\beta(f) \in (-1/2, 1/2]$. *A priori*, this number, which is determined by Equation (2.2), may not be a continuous function of $\alpha(f)$ and f. However, when $\alpha(f)$ is close enough to 0, one can choose $\beta(f)$ close to 0, so that it continuously depends on $\alpha(f)$ and f. This condition is implicit in Proposition 2.12 and Theorem 2.9. That is, r_3 is small enough so that $\beta(f)$ continuously depends on f and satisfies $\operatorname{Re}\beta(f) \in (-1/2, 1/2]$.

The restrictions of the maps $\mathcal{R}_{\text{NP-t}}(f)$ and $\mathcal{R}_{\text{NP-b}}(f)$ to the smaller domain such that they belong to $A(\infty) \ltimes \mathcal{F}_0$, are called the **top** and **bottom near-parabolic renormalisation** of f, respectively. We use the notation $\mathcal{R}_{\text{NP-t}}(f)$ and $\mathcal{R}_{\text{NP-b}}(f)$ to denote these (domain restricted) maps. Note that in this definition, $\mathcal{R}_{\text{NP-t}}(f)$ and $\mathcal{R}_{\text{NP-b}}(f)$ have extension onto the larger domain U, by the above theorem. Also, note that although the renormalisation of a map has extension onto a larger domain (U), the renormalisations are defined only using the iterates of the map on the smaller domain (V).

3. Analytic properties of the near-parabolic renormalisations

3.1. **K-horizontal curves.** In this section we study the dependence of the operators $f \mapsto \mathcal{R}_{\text{NP-t}}(f)$ and $f \mapsto \mathcal{R}_{\text{NP-b}}(f)$ on the linearity of f (that is, $\alpha(f)$), and the non-linearity of f (that is, the higher order terms of f). Let us introduce some notations in order to simplify the statements that will follow.

By virtue of Theorem 2.9, using these notations, we may write

$$\begin{aligned} \mathcal{R}_{\text{NP-t}}(\alpha \ltimes h) &= \hat{\alpha}(\alpha \ltimes h) \ltimes \hat{h}(\alpha \ltimes h), \\ \mathcal{R}_{\text{NP-b}}(\alpha \ltimes h) &= \check{\alpha}(\alpha \ltimes h) \ltimes \check{h}(\alpha \ltimes h). \end{aligned} \tag{3.1}$$

Here, $\hat{\alpha}(\alpha \ltimes h)$ and $\check{\alpha}(\alpha \ltimes h)$ are complex numbers, which depend on α and h. Also, $\hat{h}(\alpha \ltimes h)$ and $\check{h}(\alpha \ltimes h)$ are elements of $\mathcal{F}_0$, which depend on α and h. Indeed, by definition, $\hat{\alpha}(\alpha \ltimes h) = -1/\alpha$ but $\check{\alpha}$, $\hat{h}$ and $\check{h}$ depend on both α and h. Recall from Section 2.3 that $\sigma_{\alpha \ltimes h}$ denotes the preferred non-zero fixed point of $\alpha \ltimes h$, and $\beta(\alpha \ltimes h)$ is a complex number satisfying

$$(\alpha \ltimes h)'(\sigma_{\alpha \ltimes h}) = e^{2\pi i \beta(\alpha \ltimes h)}.$$

We frequently consider maps $\Upsilon : \Delta \to \mathbb{C} \ltimes \mathcal{F}_0$ defined on a set $\Delta \subseteq \mathbb{C}$. We may write any such map as $\Upsilon(s) = \alpha(s) \ltimes h(s)$ where for all $s \in \Delta$, $\alpha(s) \in \mathbb{C}$ and $h(s) \in \mathcal{F}_0$. For $k > 0$, we say that Υ is k-**horizontal**, if Δ is connected, Υ is continuous on Δ, and for all $s_1, s_2 \in \Delta$ we have

$$\mathrm{d}_{\mathrm{Teich}}(h(s_1), h(s_2)) \leq k|\alpha(s_1) - \alpha(s_2)|.$$

We call the image of any such curve a k-**horizontal curve**.

We aim to show that there is $k > 0$ such that each of the operators $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$ map any k-horizontal curve to a $k/2$-horizontal curve. That means, these renormalisation operators map the "cone field" of k-horizontal curves passing through an arbitrary point $\alpha_0 \ltimes h_0$ well inside the cone field of k-horizontal curves passing through $\mathcal{R}_{\text{NP-t}}(\alpha_0 \ltimes h_0)$ and $\mathcal{R}_{\text{NP-b}}(\alpha_0 \ltimes h_0)$, respectively. We start with the following property.

Recall the constant r_3 introduced in Section 2.3.

Proposition 3.1. *There are constants $r_4 \in (0, r_3]$ and $k_1 > 0$ satisfying the following properties:*

a) for every k_1-horizontal curve Υ in $A(r_4) \ltimes \mathcal{F}_0$, $\mathcal{R}_{\text{NP-t}}(\Upsilon)$ and $\mathcal{R}_{\text{NP-b}}(\Upsilon)$ are k_1-horizontal curves in $\mathbb{C} \ltimes \mathcal{F}_0$;

b) the curves $\alpha \mapsto \mathcal{R}_{\text{NP-t}}(Q_\alpha)$ and $\alpha \mapsto \mathcal{R}_{\text{NP-b}}(Q_\alpha)$ are k_1-horizontal curves in $\mathbb{C} \ltimes \mathcal{F}_0$, for $\alpha \in A^+(r_4)$ and for $\alpha \in A^-(r_4)$.

In order to prove Proposition 3.1, we need to control the dependence of the linearities and non-linearities of $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$ on α and h. These are formulated in the following four propositions.

Proposition 3.2. *There exists a constant $c_{2,1}$ such that for all h in $\mathcal{F}_0 \cup \{Q_0\}$, and all α_1 and α_2 in $A^+(r_3)$ or all α_1 and α_2 in $A^-(r_3)$, we have*

$$\mathrm{d}_{\mathrm{Teich}}(\hat{h}(\alpha_1 \ltimes h), \hat{h}(\alpha_2 \ltimes h)) \leq c_{2,1}|\alpha_1 - \alpha_2|,$$
$$\mathrm{d}_{\mathrm{Teich}}(\check{h}(\alpha_1 \ltimes h), \check{h}(\alpha_2 \ltimes h)) \leq c_{2,1}|\alpha_1 - \alpha_2|.$$

The above proposition is stated in Section 1 as Theorem B. The Lipschitz property of $\alpha \mapsto \hat{h}(\alpha \ltimes h)$ on the real slice $(-r_3, 0) \cup (0, r_3)$ is proved in [CC15]. In that paper, it is crucial that $\mathcal{R}_{\text{NP-t}}$ is fibre preserving, that is, $\hat{\alpha}(\alpha \ltimes h)$ depends only on α. On the other hand, $\mathcal{R}_{\text{NP-b}}$ is not fibre preserving and the proof of the above proposition for $\check{h}(\alpha \ltimes h)$ involves further analysis carried out in this paper. Also, when α is real, $\partial \mathcal{P}_{\alpha \ltimes h}$ consists of curves landing at 0 and $\sigma_{\alpha \ltimes h}$ at well defined angles, and hence, we did not need to deal with the spiralling behaviour of the Fatou coordinates in Proposition 2.4.

Proposition 3.3. *There exists a constant $c_{2,2} \in (0, 1]$ such that for all α in $A(r_3)$, and all h_1 and h_2 in $\mathcal{F}_0$ we have*

$$\mathrm{d}_{\mathrm{Teich}}(\hat{h}(\alpha \ltimes h_1), \hat{h}(\alpha \ltimes h_2)) \leq c_{2,2}\, \mathrm{d}_{\mathrm{Teich}}(h_1, h_2),$$
$$\mathrm{d}_{\mathrm{Teich}}(\check{h}(\alpha \ltimes h_1), \check{h}(\alpha \ltimes h_2)) \leq c_{2,2}\, \mathrm{d}_{\mathrm{Teich}}(h_1, h_2).$$

Recall from Section 2.4 that $\check{\alpha}(\alpha \ltimes h) = -1/\beta(\alpha \ltimes h)$. By Proposition 2.12, for a fixed $h \in \mathcal{F}_0 \cup \{Q_0\}$, $\alpha \mapsto \check{\alpha}(\alpha \ltimes h)$ is a holomorphic mapping from $A(r_3)$ into $\mathbb{C}$.

Proposition 3.4. *There exists a constant $c_{1,1} > 0$ such that for all h in $\mathcal{F}_0 \cup \{Q_0\}$, and all α_1 and α_2 in $A^+(r_3)$ or all α_1 and α_2 in $A^-(r_3)$, we have*

$$\frac{|\alpha_1 - \alpha_2|}{c_{1,1}|\alpha_1 \alpha_2|} \leq |\check{\alpha}(\alpha_1 \ltimes h) - \check{\alpha}(\alpha_2 \ltimes h)| \leq \frac{c_{1,1}|\alpha_1 - \alpha_2|}{|\alpha_1 \alpha_2|}.$$

Proposition 3.5. *There exists a constant $c_{1,2} > 0$ such that for all α in $A(r_3)$ as well as all h_1 and h_2 in $\mathcal{F}_0$, we have*

$$|\check{\alpha}(\alpha \ltimes h_1) - \check{\alpha}(\alpha \ltimes h_2)| \leq c_{1,2}\, \mathrm{d}_{\mathrm{Teich}}(h_1, h_2).$$

Remark 3.6. By the definitions of $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$, we only need to prove the Propositions 3.2-3.5 for $\alpha \in A^+(r_3)$. The statements for $\alpha \in A^-(r_3)$ follow from the ones for $\alpha \in A^+(r_3)$. Thus, within the rest of this section we assume that $\operatorname{Re}\alpha > 0$, unless otherwise stated.

Proof of Proposition 3.1 assuming Propositions 3.2, 3.3, 3.4, and 3.5.
Define $k_1 = c_{2,1}/3$, and choose $r_4 > 0$ such that

$$4 + \frac{c_{1,2}c_{2,1}}{3} \leq \frac{1}{c_{1,1}r_4^2}. \tag{3.2}$$

Let $\Upsilon : \Delta \to A(r_4) \ltimes \mathcal{F}_0$ be a k_1-horizontal curve defined on a connected set $\Delta \subset \mathbb{C}$. Since $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$ are continuous operators, $\mathcal{R}_{\text{NP-t}} \circ \Upsilon$ and $\mathcal{R}_{\text{NP-b}} \circ \Upsilon$ are continuous maps, parametrised on the connected set Δ. Fix two (distinct) points $\alpha_1 \ltimes h_1$ and $\alpha_2 \ltimes h_2$ on $\Upsilon(\Delta)$, and consider the third point $\alpha_1 \ltimes h_2$ in $A(r_3) \ltimes \mathcal{F}_0$. We denoted the images of these points under $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$ by

$$\begin{aligned} \hat{\alpha}_1 \ltimes \hat{h}_1 &= \mathcal{R}_{\text{NP-t}}(\alpha_1 \ltimes h_1), & \check{\alpha}_1 \ltimes \check{h}_1 &= \mathcal{R}_{\text{NP-b}}(\alpha_1 \ltimes h_1), \\ \hat{\alpha}_2 \ltimes \hat{h}_2 &= \mathcal{R}_{\text{NP-t}}(\alpha_2 \ltimes h_2), & \check{\alpha}_2 \ltimes \check{h}_2 &= \mathcal{R}_{\text{NP-b}}(\alpha_2 \ltimes h_2), \\ \hat{\alpha}_3 \ltimes \hat{h}_3 &= \mathcal{R}_{\text{NP-t}}(\alpha_1 \ltimes h_2), & \check{\alpha}_3 \ltimes \check{h}_3 &= \mathcal{R}_{\text{NP-b}}(\alpha_1 \ltimes h_2). \end{aligned}$$

First we deal with $\mathcal{R}_{\text{NP-t}}$. We have

$$\begin{aligned} \mathrm{d}_{\mathrm{Teich}}(\hat{h}_1, \hat{h}_2) &\leq \mathrm{d}_{\mathrm{Teich}}(\hat{h}_1, \hat{h}_3) + \mathrm{d}_{\mathrm{Teich}}(\hat{h}_3, \hat{h}_2) && \text{(Triangle Inequality)} \\ &\leq c_{2,2}\, \mathrm{d}_{\mathrm{Teich}}(h_1, h_2) + c_{2,1}|\alpha_1 - \alpha_2| && \text{(Propositions 3.3 and 3.2)} \\ &\leq c_{2,2}k_1|\alpha_1 - \alpha_2| + c_{2,1}|\alpha_1 - \alpha_2| && (\Upsilon \text{ is } k_1\text{-horizontal}) \\ &= (c_{2,2}k_1 + c_{2,1})|\alpha_1 - \alpha_2|. \end{aligned} \tag{3.3}$$

On the other hand, as $\hat{\alpha}_1 = -1/\alpha_1$ and $\hat{\alpha}_2 = -1/\alpha_2$, we obtain

$$|\alpha_1 - \alpha_2| = |\alpha_1\alpha_2||\hat{\alpha}_1 - \hat{\alpha}_2|.$$

Combining the above two equations, and using $c_{2,2} \leq 1$, $3k_1 = c_{2,1}$, $|\alpha_1| \leq 1/2$, and $|\alpha_2| \leq 1/2$, we obtain

$$\mathrm{d}_{\mathrm{Teich}}(\hat{h}_1, \hat{h}_2) \leq (c_{2,2}k_1 + c_{2,1})|\alpha_1 - \alpha_2| = 4k_1|\alpha_1\alpha_2||\hat{\alpha}_1 - \hat{\alpha}_2| \leq k_1|\hat{\alpha}_1 - \hat{\alpha}_2|.$$

The above inequality implies that $\mathcal{R}_{\text{NP-t}}(\Upsilon)$ is a k_1-horizontal curve, because $\hat{\alpha}_1 \ltimes \hat{h}_1$ and $\hat{\alpha}_2 \ltimes \hat{h}_2$ are arbitrary points on $\mathcal{R}_{\text{NP-t}}(\Upsilon)$.

Now we deal with $\mathcal{R}_{\text{NP-b}}$. Repeating Equation (3.3) for $\mathcal{R}_{\text{NP-b}}$, we obtain

$$\mathrm{d}_{\mathrm{Teich}}(\check{h}_1, \check{h}_2) \leq (c_{2,2}k_1 + c_{2,1})|\alpha_1 - \alpha_2|.$$

On the other hand, we have

$$\begin{aligned} |\check{\alpha}_1 - \check{\alpha}_2| &\geq |\check{\alpha}_2 - \check{\alpha}_3| - |\check{\alpha}_1 - \check{\alpha}_3| && \text{(Triangle Inequality)} \\ &\geq \frac{|\alpha_1 - \alpha_2|}{c_{1,1}|\alpha_1\alpha_2|} - c_{1,2}\, \mathrm{d}_{\mathrm{Teich}}(h_1, h_2) && \text{(Propositions 3.4 and 3.5)} \\ &\geq \frac{|\alpha_1 - \alpha_2|}{c_{1,1}|\alpha_1\alpha_2|} - c_{1,2}k_1|\alpha_1 - \alpha_2| && (\Upsilon \text{ is } k_1\text{-horizontal}) \\ &\geq \Big(\frac{1}{c_{1,1}r_4^2} - \frac{c_{1,2}c_{2,1}}{3}\Big)|\alpha_1 - \alpha_2| && (|\alpha_1\alpha_2| \leq 1/4,\ 3k_1 = c_{2,1}) \\ &\geq 4|\alpha_1 - \alpha_2| && \text{(Equation (3.2))}. \end{aligned}$$

Combining the above two equations, and using $3k_1 = c_{2,1}$, $c_{2,2} \leq 1$, and 3.2 we obtain,

$$\mathrm{d}_{\mathrm{Teich}}(\check{h}_1, \check{h}_2) \leq (c_{2,2}k_1 + c_{2,1})|\alpha_1 - \alpha_2| \leq 4k_1|\alpha_1 - \alpha_2| \leq k_1|\check{\alpha}_1 - \check{\alpha}_2|.$$

This shows that $\mathcal{R}_{\text{NP-b}}(\Upsilon)$ is k_1-horizontal.

The proof of the second part of the proposition is a special case of the above argument. □

The remaining of Section 3 is denoted to the proofs of Propositions 3.2, 3.3, 3.4, and 3.5. In Section 3.2, we use the notion of Schwarzian derivative to reduce Proposition 3.2 to the Euclidean variation of the functions, stated in Proposition 3.8. The proof of Proposition 3.8 requires a rather long series of calculations. To make the main idea clear, we first analyse the top renormalisation, in Sections 3.4 to 3.8. Then, we prove the second part of Proposition 3.2 in Section 3.9. Proposition 3.3 follows from Theorem 2.9 and an infinite dimensional Schwartz lemma of Royden-Gardiner (further details appear later). The proofs of Propositions 3.4 and 3.5 appear in Section 3.10.

3.2. **Schwarzian derivative.** The Schwarzian derivative of a univalent map f is defined as

$$\mathrm{D_S}\, f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f}\right)^2,$$

where " $'$ " denotes the complex differentiation of an analytic map. This measures the deviation of a univalent map from the Möbius transformations. We shall use this notion to measure the Teichmüller distance between univalent maps. See [Leh87] for general properties of this derivative.

When the maps $\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h)$ and $\mathcal{R}_{\text{NP-b}}(\alpha \ltimes h)$ belong to $\mathbb{C} \ltimes \mathcal{F}_0$, there are univalent maps $\hat{\psi}_{\alpha\ltimes h} : V \to \mathbb{C}$ and $\check{\psi}_{\alpha\ltimes h} : V \to \mathbb{C}$, with $\hat{\psi}_{\alpha\ltimes h}(0) = \check{\psi}_{\alpha\ltimes h}(0) = 0$ and $\hat{\psi}'_{\alpha\ltimes h}(0) = \check{\psi}'_{\alpha\ltimes h}(0) = 1$, such that

$$\begin{aligned} \mathcal{R}_{\text{NP-t}}(\alpha \ltimes h)(z) &= P \circ \hat{\psi}^{-1}_{\alpha\ltimes h}(e^{2\pi i\hat{\alpha}} z),\ \forall z \in e^{-2\pi i\hat{\alpha}}\hat{\psi}_{\alpha\ltimes h}(V), \\ \mathcal{R}_{\text{NP-b}}(\alpha \ltimes h)(z) &= P \circ \check{\psi}^{-1}_{\alpha\ltimes h}(e^{2\pi i\check{\alpha}} z),\ \forall z \in e^{-2\pi i\check{\alpha}}\check{\psi}_{\alpha\ltimes h}(V). \end{aligned} \tag{3.4}$$

In terms of our earlier notations $\hat{h}$ and $\check{h}$ in Equation (3.1),

$$\hat{h} = P \circ \hat{\psi}^{-1}_{\alpha\ltimes h}, \quad \check{h} = P \circ \check{\psi}^{-1}_{\alpha\ltimes h}. \tag{3.5}$$

For α and α' in $A(r_3)$ and $h \in \mathcal{F}_0$, consider the maps

$$\hat{\Omega}_{\alpha,\alpha',h} = \hat{\psi}_{\alpha\ltimes h} \circ \hat{\psi}^{-1}_{\alpha'\ltimes h} : \hat{\psi}_{\alpha'\ltimes h}(V) \to \hat{\psi}_{\alpha\ltimes h}(V),$$
$$\check{\Omega}_{\alpha,\alpha',h} = \check{\psi}_{\alpha\ltimes h} \circ \check{\psi}^{-1}_{\alpha'\ltimes h} : \check{\psi}_{\alpha'\ltimes h}(V) \to \check{\psi}_{\alpha\ltimes h}(V).$$

Studying the Schwarzian derivatives of the above maps allows us to control the distances $\mathrm{d}_{\mathrm{Teich}}(\hat{h}(\alpha \ltimes h), \hat{h}(\alpha' \ltimes h))$ and $\mathrm{d}_{\mathrm{Teich}}(\check{h}(\alpha \ltimes h), \check{h}(\alpha' \ltimes h))$. Let $\hat{\eta}_{\alpha\ltimes h}|dz|$ and $\check{\eta}_{\alpha\ltimes h}|dz|$ denote the hyperbolic metrics of constant curvature -1 on $\hat{\psi}_{\alpha\ltimes h}(V)$ and $\check{\psi}_{\alpha\ltimes h}(V)$, respectively. Then the hyperbolic norms of the Schwarzian derivatives $\mathrm{D_S}\, \hat{\Omega}_{\alpha,\alpha',h}$ and $\mathrm{D_S}\, \check{\Omega}_{\alpha,\alpha',h}$ are defined as

$$\|\mathrm{D_S}\, \hat{\Omega}_{\alpha,\alpha',h}\|_{\hat{\psi}_{\alpha'\ltimes h}(V)} = \sup_{z\in\hat{\psi}_{\alpha'\ltimes h}(V)} \frac{|\mathrm{D_S}\, \hat{\Omega}_{\alpha,\alpha',h}(z)|}{|\hat{\eta}_{\alpha\ltimes h}(z)|^2},$$
$$\|\mathrm{D_S}\, \check{\Omega}_{\alpha,\alpha',h}\|_{\check{\psi}_{\alpha'\ltimes h}(V)} = \sup_{z\in\check{\psi}_{\alpha'\ltimes h}(V)} \frac{|\mathrm{D_S}\, \check{\Omega}_{\alpha,\alpha',h}(z)|}{|\check{\eta}_{\alpha\ltimes h}(z)|^2}.$$

Proposition 3.7. *There exists a constant D_1 such that for all h in $\mathcal{F}_0 \cup \{Q_0\}$, and all α, α' in $A^+(r_3)$, we have*

$$\|\mathrm{D_S}\, \hat{\Omega}_{\alpha,\alpha',h}\|_{\hat{\psi}_{\alpha'\ltimes h}(V)} \leq D_1|\alpha-\alpha'|,\ \|\mathrm{D_S}\, \check{\Omega}_{\alpha,\alpha',h}\|_{\check{\psi}_{\alpha'\ltimes h}(V)} \leq D_1|\alpha-\alpha'|.$$

Proof of Prop. 3.2 assuming Prop. 3.7. The domain V is bounded by a smooth curve, and hence is a quasi-circle. On the other hand, by Theorem 2.9, $\hat{\psi}_{\alpha\ltimes h} : V \to \mathbb{C}$ and $\check{\psi}_{\alpha\ltimes h} : V \to \mathbb{C}$ have univalent extensions onto the domain U, which contains the closure of V in its interior. This implies that there exists a constant K, depending only on V and $\mod (U \setminus V)$, such that the boundaries of $\hat{\psi}_{\alpha\ltimes h}(V)$ and $\check{\psi}_{\alpha\ltimes h}(V)$ are K-quasi-circles.

By a classical result on the relation between Schwarzian derivative and the quasi-conformal extension, see [Leh87, Chapter 2, Thm 4.1] or [Ahl63], there exists a constant $\varepsilon(K)$ such that $\hat{\Omega}_{\alpha,\alpha',h}$ and $\check{\Omega}_{\alpha,\alpha',h}$ can be extended to quasi-conformal maps of $\mathbb{C}$ whose complex dilatations $\hat{\mu}$ and $\check{\mu}$, respectively, satisfy

$$\|\hat{\mu}\|_\infty \leq \frac{\|\mathrm{D_S}\, \hat{\Omega}_{\alpha,\alpha',h}\|_{\hat{\psi}_{\alpha'\ltimes h}(V)}}{\varepsilon(K)}, \quad \|\check{\mu}\|_\infty \leq \frac{\|\mathrm{D_S}\, \check{\Omega}_{\alpha,\alpha',h}\|_{\check{\psi}_{\alpha'\ltimes h}(V)}}{\varepsilon(K)}.$$

By the definition of $\mathrm{d}_{\mathrm{Teich}}$ on $\mathcal{F}_0$, and Proposition 3.7, we conclude that Proposition 3.2 holds with the constant $c_{2,1} = D_1/\varepsilon(K)$. □

The Schwarzian derivative satisfies the chain rule

$$\left\|\mathrm{D_S}(\hat{\psi}_{\alpha\ltimes h} \circ \hat{\psi}^{-1}_{\alpha'\ltimes h})\right\|_{\hat{\psi}_{\alpha'\ltimes h}(V)} = \left\|\mathrm{D_S}\, \hat{\psi}_{\alpha\ltimes h} - \mathrm{D_S}\, \hat{\psi}_{\alpha'\ltimes h}\right\|_V,$$
$$\left\|\mathrm{D_S}(\check{\psi}_{\alpha\ltimes h} \circ \check{\psi}^{-1}_{\alpha'\ltimes h})\right\|_{\check{\psi}_{\alpha'\ltimes h}(V)} = \left\|\mathrm{D_S}\, \check{\psi}_{\alpha\ltimes h} - \mathrm{D_S}\, \check{\psi}_{\alpha'\ltimes h}\right\|_V. \tag{3.6}$$

By virtue of these relations, we may boil down Proposition 3.7 to the following statement.

Proposition 3.8. *For every Jordan domain V' with $V \Subset V' \Subset U$, there exists a constant D_2 such that for all $\alpha \in A(r_3)$, all $h \in \mathcal{F}_0 \cup \{Q_0\}$, and all $z \in V'$, we have*

a) $|\partial\hat{\psi}_{\alpha\ltimes h}(z)/\partial\alpha| \leq D_2$,
b) $|\partial\check{\psi}_{\alpha\ltimes h}(z)/\partial\alpha| \leq D_2$.

Proof of Prop. 3.7 assuming Prop. 3.8. By the estimates in Proposition 3.8, and the Cauchy integral formula, there is a constant D_2' such that for all $z \in V$ we have

$$\Big|\frac{\partial}{\partial \alpha}\hat{\psi}'_{\alpha \ltimes h}(z)\Big| \leq D_2', \quad \Big|\frac{\partial}{\partial \alpha}\hat{\psi}''_{\alpha \ltimes h}(z)\Big| \leq D_2', \quad \Big|\frac{\partial}{\partial \alpha}\hat{\psi}'''_{\alpha \ltimes h}(z)\Big| \leq D_2',$$

and

$$\Big|\frac{\partial}{\partial \alpha}\check{\psi}'_{\alpha \ltimes h}(z)\Big| \leq D_2', \quad \Big|\frac{\partial}{\partial \alpha}\check{\psi}''_{\alpha \ltimes h}(z)\Big| \leq D_2', \quad \Big|\frac{\partial}{\partial \alpha}\check{\psi}'''_{\alpha \ltimes h}(z)\Big| \leq D_2'.$$

Also, by the Koebe distortion theorem, $|\hat{\psi}'_{\alpha \ltimes h}|$, $|\hat{\psi}'_{\alpha' \ltimes h}|$, $|\check{\psi}'_{\alpha \ltimes h}|$ and $|\check{\psi}'_{\alpha' \ltimes h}|$ are uniformly bounded from above and away from zero, on V, with bounds depending only on mod $U \setminus V$. Combining these bounds together, and using Equation (3.6), one obtains the uniform bounds in Proposition 3.7. □

The proof of Proposition 3.8 constitutes a series of calculations that will be presented in Sections 3.4 to 3.8.

3.3. **Preliminary estimates for the maps in $\mathcal{F}_0$.**

Lemma 3.9. *For every $\alpha \in A(1/2)$, the following hold:*

a) for every $h \in \mathcal{F}_0$, $\alpha \ltimes h$ is defined on the ball $B(0, 2e^{-\pi/\sqrt{2}}/9)$ and is univalent on the ball $B(0, 4e^{-\pi/\sqrt{2}}/27)$;

b) for every $h \in \mathcal{F}_0$, the critical point of $\alpha \ltimes h$, $\mathrm{cp}_{\alpha \ltimes h}$, satisfies

$$4e^{-\pi/\sqrt{2}}/27 \leq |\mathrm{cp}_{\alpha \ltimes h}| \leq 4e^{\pi/\sqrt{2}}/3;$$

c) Q_α is univalent on $B(0, 8e^{-\pi/\sqrt{2}}/27)$, and its critical point cp_α satisfies

$$8e^{-\pi/\sqrt{2}}/27 \leq |\mathrm{cp}_\alpha| \leq 8e^{\pi/\sqrt{2}}/27.$$

Proof. a) Let $h = P \circ \varphi^{-1}$, as in the definition of the class $\mathcal{F}_0$. Because $E \subset B(0,2)$, $B(0, 8/9) \subset V$. Applying the Koebe 1/4-Theorem to the map $z \mapsto 9\varphi(8z/9)/8$, one concludes that $B(0, 2/9) \subseteq \varphi(V)$. Thus, $B(0, 2/9) \subset \mathrm{Dom}\, h$.

The polynomial P is univalent on the ball $B(0, 1/3)$ (which can be seen from the argument principle, for instance). By the above paragraph, φ is defined on the ball $B(0, 2/3) \subset B(0, 8/9)$. Thus, we may apply the classical Koebe distortion theorem to the map $z \mapsto 3\varphi(2z/3)/2$, to conclude that $\varphi(B(0, 1/3))$ contains $B(0, 4/27)$. (For the simplicity of calculations we have applied the Koebe Theorem to $z \mapsto 3\varphi(2z/3)/2$ instead of $z \mapsto 9\varphi(8z/9)/8$.) This means that every map $h \in \mathcal{F}_0$ is univalent on the ball $B(0, 4/27)$.

For α in $A(1/2)$, $|\operatorname{Im} \alpha| \leq \sqrt{2}/4$. Composing with the rescalings at 0, the above paragraphs imply that $\alpha \ltimes h$ must be defined on the ball $(2/9)e^{-\pi/\sqrt{2}}$, and must be univalent on the ball $(4/27)e^{-\pi/\sqrt{2}}$.

b) The polynomial P has a unique critical point at $-1/3$ within V. By the Koebe distortion theorem, applied to the map $z \mapsto 3\varphi(2z/3)/2$, we conclude that $|\varphi(-1/3)| \in [4/27, 4/3]$. Recall that the critical point of $h \in \mathcal{F}_0$ is equal to $\varphi(-1/3)$. Composing with the complex rotations $z \mapsto e^{2\pi i \alpha} z$, we conclude the bounds in Part b).

c) The unique critical point of Q_α lies at $-8e^{2\pi i \alpha}/27$. Further details are left to the reader. □

Lemma 3.10. *For every $h \in \mathcal{F}_0$, $h^{\circ n}(\mathrm{cp}_h)$ tends to 0 as n tends to $+\infty$, and $2 \leq |h''(0)| \leq 7$.*

The above lemma is stated in [IS06, main theorem 1]. The uniform bound in the latter part of the lemma also follows from the Area Theorem, since the conformal radius of the set V is strictly larger than $+1$.

Proof of Proposition 2.1. Although the class of maps $\mathcal{F}_0$ is not compact (their domain of definitions are quasi-circles), every sequence of maps in $\mathcal{F}_0$ has a sub-sequence which converges, in the compact-open topology, to a holomorphic map with a non-degenerate parabolic fixed point at 0. Indeed, as we saw in the proof of Lemma 3.9, the limiting map is defined on $B(0, 2/9)$, and by Lemma 3.10 the modulus of its second derivative at 0 belongs to the interval $[2, 7]$.

Every map h in the closure of $\mathcal{F}_0$, $\overline{\mathcal{F}_0}$, has a non-degenerate parabolic fixed point at 0. That is, a fixed point of order two. Every such h has an attracting and a repelling petal covering a punctured neighbourhood of 0. Hence, h may not have any fixed point on the union of the petals. Using Lemmas 3.9-a and 3.10, one may find a neighbourhood W of 0, bounded by a smooth curve, such that every $h \in \overline{\mathcal{F}_0}$ has a unique fixed point on the closure of W.

By the Argument Principle, there is $r_1 \in (0, 1/2)$ such that for all $\alpha \in A(r_1)$ and all $h \in \mathcal{F}_0$, $\alpha \ltimes h$ has two fixed points in W, counted with multiplicity. As $\alpha \neq 0$, 0 is a simple fixed point of $\alpha \ltimes h$, and hence, there must be another simple fixed point of $\alpha \ltimes h$ within W. □

In order to analyse the dependence of $\hat{h}$ on α we need to study the definitions of $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$ in detail. For $h \in \mathcal{F}_0$ and $\alpha \in A^+(+\infty)$, it is convenient to denote $(\alpha \ltimes h)$ by h_α, that is,

$$h_\alpha(z) = h(e^{2\pi i\alpha} z),\ z \in e^{-2\pi i\alpha}\,\mathrm{Dom}\,(h).$$

This is consistent with the notation $Q_\alpha(z) = Q_0(e^{2\pi i\alpha} z)$. For h in $\mathcal{F}_0 \cup \{Q_0\}$ and α in $A^+(r_2)$, Proposition 2.2 guarantees the existence of a Jordan domain $\mathcal{P}_{\alpha\ltimes h}$, and a conformal change of coordinate

$$\Phi_{\alpha\ltimes h} : \mathcal{P}_{\alpha\ltimes h} \to \mathbb{C},$$

which conjugates h_α to the translation by $+1$. We aim to study the dependence of $\Phi_{\alpha\ltimes h}$ on α. It is convenient to do this in a certain coordinate called the pre-Fatou coordinate, as we discuss in the next section.

3.4. **The top pre-Fatou coordinate.** Every map h_α in $A^+(r_1) \ltimes \mathcal{F}_0$ or in $A^+(r_1) \ltimes \{Q_0\}$, may be written of the form

$$h_\alpha(z) = z + z(z - \sigma_{\alpha\ltimes h}) u_{\alpha\ltimes h}(z), \tag{3.7}$$

where $u_{\alpha\ltimes h}$ is a holomorphic function defined on Dom h_α which is non-zero at 0 and $\sigma_{\alpha\ltimes h}$. As $\alpha \to 0$, $\sigma_{\alpha\ltimes h} \to 0$, and we may identify a holomorphic function $u_{0\ltimes h}$ such that

$$h_0(z) = z + z^2 u_{0\ltimes h}(z), \tag{3.8}$$

with $u_{0\ltimes h}(0) \neq 0$. By pre-compactness of the class $\mathcal{F}_0$, and the uniform bound in Lemma 3.10, $|u_{\alpha\ltimes h}(0)|$ is uniformly bounded from above and away from 0. More precisely, there is a constant D_3, independent of α in $A^+(r_1) \cup \{0\}$ and h in $\mathcal{F}_0 \cup \{Q_0\}$, such that

$$D_3^{-1} \leq u_{\alpha\ltimes h}(0) \leq D_3. \tag{3.9}$$

Differentiating Equation (3.8) at 0 and $\sigma_{\alpha\ltimes h}$ provides us with the formulas:

$$\sigma_{\alpha\ltimes h} = (1-e^{2\pi i\alpha})/u_{\alpha\ltimes h}(0),\ h'_\alpha(\sigma_{\alpha\ltimes h}) = 1+\sigma_{\alpha\ltimes h}u_{\alpha\ltimes h}(\sigma_{\alpha\ltimes h}). \tag{3.10}$$

In particular, there is a constant D_4 such that for all $\alpha \in A^+(r_1)$ and $h \in \mathcal{F}_0 \cup \{Q_0\}$, we have

$$\frac{1}{D_4}|\alpha| \leq |\sigma_{\alpha\ltimes h}| \leq D_4|\alpha|. \tag{3.11}$$

Following [Shi98], we consider the covering map $\tau_{\alpha\ltimes h} : \mathbb{C} \to \hat{\mathbb{C}} \setminus \{0, \sigma_{\alpha\ltimes h}\}$, defined as

$$\tau_{\alpha\ltimes h}(w) = \frac{\sigma_{\alpha\ltimes h}}{1-e^{-2\pi i\alpha w}}, \tag{3.12}$$

where $\hat{\mathbb{C}}$ denotes the Riemann sphere. We have,

$$\tau_{\alpha\ltimes h}(w+\alpha^{-1}) = \tau_{\alpha\ltimes h}(w),\ \lim_{\operatorname{Im}(\alpha w)\to+\infty} \tau_{\alpha\ltimes h}(w) = 0,\ \lim_{\operatorname{Im}(\alpha w)\to-\infty} \tau_{\alpha\ltimes h}(w) = \sigma_{\alpha\ltimes h}.$$

Also, $\tau_{\alpha\ltimes h}$ maps $\mathbb{Z}/\alpha$ to the point at infinity in $\hat{\mathbb{C}}$.

Lemma 3.11. *For all $h \in \mathcal{F}_0 \cup \{Q_0\}$ and all α in $A^+(r_1)$, we have the following estimates*

a) if $\operatorname{Im}(\alpha w) > 0$, then

$$|\tau_{\alpha\ltimes h}(w)| \leq D_4 \frac{|\alpha|}{e^{2\pi \operatorname{Im}(\alpha w)} - 1};$$

b) if $\operatorname{Im}(\alpha w) < 0$, then

$$|\tau_{\alpha\ltimes h}(w) - \sigma_{\alpha\ltimes h}| \leq D_4 \frac{|\alpha| e^{2\pi \operatorname{Im}(\alpha w)}}{1 - e^{2\pi \operatorname{Im}(\alpha w)}}.$$

Proof. By Equation (3.11), for w with $\operatorname{Im}(\alpha w) > 0$,

$$|\tau_{\alpha\ltimes h}(w)| \leq D_4|\alpha| \frac{1}{|1-e^{-2\pi i\alpha w}|} \leq D_4|\alpha| \frac{1}{e^{2\pi \operatorname{Im}(\alpha w)} - 1}.$$

Similarly, for w with $\operatorname{Im}(\alpha w) < 0$,

$$|\tau_{\alpha\ltimes h}(w) - \sigma_{\alpha\ltimes h}| \leq |\sigma_{\alpha\ltimes h}| \frac{|e^{-2\pi i\alpha w}|}{|1-e^{-2\pi i\alpha w}|} \leq D_4|\alpha| \frac{e^{2\pi \operatorname{Im}(\alpha w)}}{1-e^{2\pi \operatorname{Im}(\alpha w)}}. \qquad \square$$

We may lift $h_\alpha : \mathcal{P}_{\alpha\ltimes h} \to \mathbb{C}$ via $\tau_{\alpha\ltimes h}$ to a holomorphic map

$$F_{\alpha\ltimes h} : \tau_{\alpha\ltimes h}^{-1}(\mathcal{P}_{\alpha\ltimes h}) \to \mathbb{C},$$

which is determined upto an additive constant in $\mathbb{Z}/\alpha$. Since there are no pre-images of 0 and $\sigma_{\alpha\ltimes h}$ in $\mathcal{P}_{\alpha\ltimes h}$, $F_{\alpha\ltimes h}$ takes a finite value at every point in $\tau_{\alpha\ltimes h}^{-1}(\mathcal{P}_{\alpha\ltimes h})$. Being a lift, for any choice of the additive constant in $\mathbb{Z}/\alpha$, we must have

$$h_\alpha \circ \tau_{\alpha\ltimes h}(w) = \tau_{\alpha\ltimes h} \circ F_{\alpha\ltimes h}(w),\ F_{\alpha\ltimes h}(w+1/\alpha) = F_{\alpha\ltimes h}(w)+1/\alpha,\ w \in \tau_{\alpha\ltimes h}^{-1}(\mathcal{P}_{\alpha\ltimes h}). \tag{3.13}$$

Indeed, we have a formula for $F_{\alpha\ltimes h}$, in terms of $u_{\alpha\ltimes h}$ in (3.7),

$$F_{\alpha\ltimes h}(w) = w + \frac{1}{2\pi i\alpha} \log\Big(1 - \frac{\sigma_{\alpha\ltimes h}u_{\alpha\ltimes h}(z)}{1+zu_{\alpha\ltimes h}(z)}\Big),\ \text{with } z = \tau_{\alpha\ltimes h}(w). \tag{3.14}$$

A choice of the branch of log in the above formula corresponds to a choice of the additive constant in $\mathbb{Z}/\alpha$. In this paper, we work with the branch satisfying $\operatorname{Im}\log(\cdot) \subseteq (-\pi, +\pi)$, in order to guarantee

$$\lim_{\operatorname{Im}(\alpha w)\to+\infty} |F_{\alpha\ltimes h}(w) - (w+1)| = 0. \tag{3.15}$$

This can be verified using Equation (3.10). Let $\hat{\mathcal{P}}_{\alpha\ltimes h}$ denote the connected component of $\tau_{\alpha\ltimes h}^{-1}(\mathcal{P}_{\alpha\ltimes h})$ which separates 0 and $1/\alpha$. The unique critical point of h_α, which lies on the boundary of $\mathcal{P}_{\alpha\ltimes h}$, lifts under $\tau_{\alpha\ltimes h}$ to a $1/\alpha$-periodic set of points. There is a unique point in this set which lies on the boundary of $\hat{\mathcal{P}}_{\alpha\ltimes h}$. This is denoted by $\hat{\mathrm{cp}}_{\alpha\ltimes h}$.

For $r \in (0, +\infty)$, define the set

$$\Theta_\alpha(r) = \operatorname{int}\big(\mathbb{C} \setminus \cup_{n\in\mathbb{Z}} B(n/\alpha, r)\big).$$

Lemma 3.12. *There are constants $r_3' > 0$, D_5, and D_6 such that for all $h \in \mathcal{F}_0 \cup \{Q_0\}$ and $\alpha \in A^+(r_3')$, $F_{\alpha\ltimes h}$ is defined and univalent on $\Theta_\alpha(D_5)$ and satisfies the following properties:*

a) for all $w \in \Theta_\alpha(D_5)$,

$$|F_{\alpha\ltimes h}(w) - (w+1)| \le 1/4, \qquad |F'_{\alpha\ltimes h}(w) - 1| \le 1/4;$$

b) for all $w \in \Theta_\alpha(D_5)$ with $\operatorname{Im}(\alpha w) > 0$, we have

$$|F_{\alpha\ltimes h}(w) - (w+1)| \le D_6|\tau_{\alpha\ltimes h}(w)|, \qquad |F'_{\alpha\ltimes h}(w) - 1| \le D_6|\tau_{\alpha\ltimes h}(w)|;$$

c) for all $w \in \Theta_\alpha(D_5)$ with $\operatorname{Im}(\alpha w) < 0$, we have

$$|F_{\alpha\ltimes h}(w) - w + \frac{1}{2\pi i\alpha}\log h'_\alpha(\sigma_{\alpha\ltimes h})| \le D_6|\tau_{\alpha\ltimes h}(w) - \sigma_{\alpha\ltimes h}|,$$
$$|F'_{\alpha\ltimes h}(w) - 1| \le D_6|\tau_{\alpha\ltimes h}(w) - \sigma_{\alpha\ltimes h}|.$$

Proof. By Equation (3.11), there is $\delta_0 \in (0, r_1]$ such that for all $\alpha \in A^+(\delta_0)$ and every h in $\mathcal{F}_0 \cup \{Q_0\}$, $\sigma_{\alpha\ltimes h} \in B(0, 2e^{-\pi/\sqrt{2}}/27)$. Then, there is a constant $C_1 > 0$ such that for all α in $A^+(\delta_0)$ and h in $\mathcal{F}_0 \cup \{Q_0\}$, $\tau_{\alpha\ltimes h}(\Theta_\alpha(C_1))$ is contained in $B(0, 4e^{-\pi/\sqrt{2}}/27)$. By Lemma 3.9, h_α is univalent on $B(0, 4e^{-\pi/\sqrt{2}}/27)$. Thus, there are no pre-image of 0 and $\sigma_{\alpha\ltimes h}$ within $B(0, 4e^{-\pi/\sqrt{2}}/27)$, except 0 and $\sigma_{\alpha\ltimes h}$. This implies that there is a lift of h_α defined on $\Theta_\alpha(C_1)$, which is holomorphic and one-to-one. Moreover, we may choose the lift which agrees with the branch in Equation (3.14), on $\hat{\mathcal{P}}_{\alpha\ltimes h} \cap \Theta_\alpha(C_1)$.

We need to repeat the previous paragraph, with more explicit constants, so that the inequalities in part (a) hold. By the pre-compactness of the class $\mathcal{F}_0$, there is a constant C_2, independent of α and h, such that for all $z \in \tau_{\alpha\ltimes h}(\Theta_\alpha(C_1))$, $|u_{\alpha\ltimes h}(z)| \le C_2$. Choose $\delta_1 \in (0, \delta_0]$ such that $4\delta_1 D_4 C_2 < 1$, where D_4 is the constant in Equation (3.11). This implies that for all $\alpha \in A^+(\delta_1)$ and all $h \in \mathcal{F}_0 \cup \{Q_0\}$, $|\sigma_{\alpha\ltimes h}| \le \delta_1 D_4 < 1/(4C_2)$. Now, there is $C_3 \ge C_1$ such that for all $\alpha \in A^+(\delta_1)$, all $h \in \mathcal{F}_0 \cup \{Q_0\}$, and all $w \in \Theta_\alpha(C_3)$, $|\tau_{\alpha\ltimes h}(w)| \le 1/(2C_2)$. Putting these together, we have

$$\Big|\frac{\sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(z)}{1 + z u_{\alpha\ltimes h}(z)}\Big| \le \frac{D_4|\alpha|C_2}{1/2} \le 2D_4\delta_1 C_2 < 1/2, \quad \forall z \in \tau_{\alpha\ltimes h}(\Theta_\alpha(C_3)). \tag{3.16}$$

This guarantees that $1 - \sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(z)/(1 + z u_{\alpha\ltimes h}(z))$ is away from the negative real axis $(-\infty, 0]$. In particular, the branch of log with $\operatorname{Im}\log(\cdot) \subseteq (-\pi, \pi)$ is defined in formula (3.14).

Similarly, we have

$$|\sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(\sigma_{\alpha\ltimes h})| \le D_4 \delta_1 C_2 < 1/2. \tag{3.17}$$

Thus, by Equation (3.10), $h'_\alpha(\sigma_{\alpha\ltimes h})$ is away from the negative real axis as well. In particular, the same branch of log is defined at $h'_\alpha(\sigma_{\alpha\ltimes h})$.

We may now use the formula in (3.14) for $F_{\alpha\ltimes h}$ in order to prove the inequalities in the lemma. With $z = \tau_{\alpha\ltimes h}(w)$, we have

$$\begin{aligned} |F_{\alpha\ltimes h}(w) - w - 1| &= \Big|\frac{1}{2\pi i\alpha} \log\Big(1 - \frac{\sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(z)}{1 + z u_{\alpha\ltimes h}(z)}\Big) - 1\Big| \\ &= \frac{1}{2\pi|\alpha|}\Big| \log\Big(1 - \frac{\sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(z)}{1 + z u_{\alpha\ltimes h}(z)}\Big) - \log e^{2\pi i\alpha}\Big| \end{aligned} \tag{3.18}$$

Let us define C_4 as the maximum of the function $x \mapsto |\log'(x)| = |1/x|$, for x in the set

$$B(1, 2D_4C_2\delta_1) \cup \{1/x \mid x \in B(1, C_2D_4\delta_1) \cup \{e^{2\pi\alpha i} \mid \alpha \in A^+(\delta_1)\}.$$

Using $e^{2\pi i\alpha} = 1 - \sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(0)$ from (3.10), and (3.16), we have

$$\begin{aligned} \Big|\log\Big(1 - \frac{\sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(z)}{1 + z u_{\alpha\ltimes h}(z)}\Big) - \log e^{2\pi i\alpha}\Big| &\le C_4\Big|\big(1 - \frac{\sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(z)}{1 + z u_{\alpha\ltimes h}(z)}\big) - e^{2\pi i\alpha}\Big| \\ &= C_4\Big|\big(1 - \frac{\sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(z)}{1 + z u_{\alpha\ltimes h}(z)}\big) - (1 - \sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(0))\Big| \\ &= C_4|\sigma_{\alpha\ltimes h}|\Big|\frac{u_{\alpha\ltimes h}(z)}{(1 + z u_{\alpha\ltimes h}(z))} - u_{\alpha\ltimes h}(0)\Big| \\ &\le C_4 D_4 |\alpha| C_5 |z|. \end{aligned} \tag{3.19}$$

A uniform constant C_5 in the above equation exists because of the pre-compactness of $\mathcal{F}_0$.

Now, we may choose $r'_3 \in (0, \delta_1]$, and then choose $C_6 \ge C_3$ such that for all $\alpha \in A^+(r'_3)$, all h in $\mathcal{F}_0 \cup \{Q_0\}$, and all $z \in \tau_{\alpha\ltimes h}(\Theta_\alpha(C_6))$, we have $C_4D_4C_5|z|/(2\pi) \le 1/4$. Then, combining (3.18) and (3.19), we conclude the first inequalities of Parts a and b.

Using $z = \tau_{\alpha\ltimes h}(w)$, and $h'_\alpha(\sigma_{\alpha\ltimes h}) = 1 + \sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(\sigma_{\alpha\ltimes h})$ from (3.10), we have

$$\begin{aligned} &\Big|F_{\alpha\ltimes h}(w) - w + \frac{1}{2\pi\alpha i} \log h'_\alpha(\sigma_{\alpha\ltimes h})\Big| \\ &\qquad = \Big|\frac{1}{2\pi\alpha i} \log\Big(1 - \frac{\sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(z)}{1 + z u_{\alpha\ltimes h}(z)}\Big) - \frac{1}{2\pi i\alpha} \log \frac{1}{1 + \sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(\sigma_{\alpha\ltimes h}))}\Big| \\ &\qquad \le \frac{C_4}{2\pi|\alpha|}\Big| \Big(1 - \frac{\sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(z)}{1 + z u_{\alpha\ltimes h}(z)}\Big) - \frac{1}{1 + \sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(\sigma_{\alpha\ltimes h})}\Big|. \end{aligned} \tag{3.20}$$

On the other hand,
(3.21)

$$
\begin{aligned}
&\Big|\Big(1-\frac{\sigma_{\alpha\ltimes h}u_{\alpha\ltimes h}(z)}{1+zu_{\alpha\ltimes h}(z)}\Big)-\frac{1}{1+\sigma_{\alpha\ltimes h}u_{\alpha\ltimes h}(\sigma_{\alpha\ltimes h})}\Big|\\
&\le |z-\sigma_{\alpha\ltimes h}|\sup\Big\{\Big|\frac{d}{dy}\Big(\frac{1+(y-\sigma_{\alpha,h})u_{\alpha\ltimes h}(y)}{1+yu_{\alpha\ltimes h}(y)}\Big)\Big|; y=t\sigma_{\alpha\ltimes h}+(1-t)z, t\in(0,1)\Big\}\\
&=|z-\sigma_{\alpha\ltimes h}||\sigma_{\alpha\ltimes h}|\sup\Big\{\Big|\frac{u_{\alpha\ltimes h}(y)^2-u'_{\alpha\ltimes h}(y)}{(1+yu_{\alpha\ltimes h}(y))^2}\Big|; y=t\sigma_{\alpha\ltimes h}+(1-t)z, t\in(0,1)\Big\}\\
&\le |z-\sigma_{\alpha\ltimes h}|D_4|\alpha|C'_5.
\end{aligned}
$$

The constant C'_5 in the last inequality depends only on the class $\mathcal{F}_0$. Combining (3.20) and (3.21), we obtain the first inequality in Part c.

To prove the uniform bounds for the derivatives in Parts a, b, and c, one may use the Cauchy Integral formula for the first derivatives, at points in $\Theta_\alpha(C_6+1)$. This finishes the proof of the proposition by introducing $D_5=C_6+1$ and D_6 as the maximum of $C_4D_4C_5/(2\pi)$ and $C_4D_4C'_5/(2\pi)$. □

Lemma 3.13. *There exists a constant D_7 such that for all $\alpha,\alpha'\in A^+(r'_3)$ and all $h\in\mathcal{F}_0\cup\{Q_0\}$ we have the following inequalities:*

a) for all $w\in\Theta_\alpha(D_5)$ with $\operatorname{Im}(\alpha w)>0$,

$$|F_{\alpha\ltimes h}(w)-F_{\alpha'\ltimes h}(w)|\le D_7|\alpha-\alpha'||\tau_{\alpha\ltimes h}(w)|;$$

b) for all $w\in\Theta_\alpha(D_5)$ with $\operatorname{Im}(\alpha w)<0$,

$$|F_{\alpha\ltimes h}(w)-F_{\alpha'\ltimes h}(w)|\le D_7|\alpha-\alpha'|.$$

Proof. Consider the continuous function $B_1:\mathbb{C}\setminus(-\infty,-1]\to\mathbb{C}$ defined through $\log(1+x)=xB_1(x)$, and $B_2(\alpha,w)$, for $\alpha\in A^+(r'_3)$ and $w\in\Theta_\alpha(D_5)$, by the formula

$$B_2(\alpha,w)=\frac{(1-e^{-2\pi\alpha i})}{\alpha}\Big(\frac{u_{\alpha\ltimes h}(z)}{u_{\alpha\ltimes h}(0)(1+zu_{\alpha\ltimes h}(z))}-1\Big),\qquad z=\tau_{\alpha\ltimes h}(w).$$

In the proof of Lemma 3.12 we chose r'_3 and D_5 so that for $z\in\tau_{\alpha\ltimes h}(\Theta_\alpha(D_5))$, $1+zu_{\alpha\ltimes h}(z)$ is uniformly away from 0. Combining this with the pre-compactness of the class $\mathcal{F}_0$, we have

$$|B_2(\alpha,w)|=O(|z|)=O(|\tau_{\alpha\ltimes h}(w)|),\quad \Big|\frac{B_2(\alpha,w)}{\tau_{\alpha\ltimes h}(w)}-\frac{B_2(\alpha',w)}{\tau_{\alpha'\ltimes h}(w)}\Big|=O(|\alpha-\alpha'|),$$

for some uniform constants in O.

Let $w \in \Theta_\alpha(D_5)$. Using the formulas (3.10) and (3.14),

$$\begin{aligned}
F_{\alpha \ltimes h}(w) - w - 1 &= \frac{1}{2\pi\alpha i} \log\left(1 - \frac{\sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(z)}{1 + z u_{\alpha\ltimes h}(z)}\right) - \frac{1}{2\pi\alpha i} \log e^{2\pi\alpha i} \\
&= \frac{1}{2\pi\alpha i} \log\left(\left(1 - \left(\frac{1 - e^{2\pi i\alpha}}{u_{\alpha\ltimes h}(0)}\right) \frac{u_{\alpha\ltimes h}(z)}{1 + z u_{\alpha\ltimes h}(z)}\right) e^{-2\pi\alpha i}\right) \\
&= \frac{1}{2\pi\alpha i} \log\left(e^{-2\pi i\alpha} + \left(\frac{1 - e^{-2\pi i\alpha}}{u_{\alpha\ltimes h}(0)}\right) \frac{u_{\alpha\ltimes h}(z)}{1 + z u_{\alpha\ltimes h}(z)}\right) \\
&= \frac{1}{2\pi\alpha i} \log\left(1 + (1 - e^{-2\pi\alpha i})\left(\frac{u_{\alpha\ltimes h}(z)}{u_{\alpha\ltimes h}(0)(1 + z u_{\alpha\ltimes h}(z))} - 1\right)\right). \\
&= \frac{1}{2\pi i} B_2(\alpha, w) B_1(\alpha B_2(\alpha, w)).
\end{aligned}$$

Define the set

$$O = \partial\Theta_\alpha(D_5) \cup \{w \in \Theta_\alpha(D_5) \mid \mathrm{Im}(\alpha w) = 0\}.$$

For α' sufficiently close to α, $|\tau_{\alpha\ltimes h}(w)|/|\tau_{\alpha'\ltimes h}(w)|$ is uniformly bounded from above and away from 0, independent of $w \in O$ and $h \in \mathcal{F}_0$. For $w \in O$,

$$\begin{aligned}
|F_{\alpha\ltimes h}(w) - F_{\alpha'\ltimes h}(w)| &= \frac{1}{2\pi} \big|B_2(\alpha, w) B_1(\alpha B_2(\alpha, w)) - B_2(\alpha', w) B_1(\alpha' B_2(\alpha', w))\big| \\
&\leq \frac{1}{2\pi} |B_2(\alpha, w)| \big|B_1(\alpha B_2(\alpha, w)) - B_1(\alpha' B_2(\alpha', w))\big| \\
&\qquad + \frac{1}{2\pi} |B_1(\alpha' B_2(\alpha', w))| \big|B_2(\alpha, w) - B_2(\alpha', w)\big| \\
&\leq C|\tau_{\alpha\ltimes h}(w)||\alpha - \alpha'| + C'|\alpha - \alpha'||\tau_{\alpha\ltimes h}(w)|.
\end{aligned}$$

The constants C and C' depend only on the class $\mathcal{F}_0$. Below we use the maximum principle in the w variable in order to prove the estimates in (a) and (b).

By the above equation, the estimate in (a) holds on $\partial\{w \in \Theta_\alpha(D_5) \mid \mathrm{Im}(\alpha w) > 0\}$. It also holds as $\mathrm{Im}(\alpha w) \to +\infty$, since $|F_{\alpha\ltimes h}(w) - F_{\alpha'\ltimes h}(w)| \to 0$ by Equation (3.15). This implies that the uniform bound must hold for all w in $\Theta_\alpha(D_5)$ with $\mathrm{Im}(\alpha w) > 0$.

On $\{w \in O \mid \mathrm{Im}(\alpha w) \leq 0\}$, $|\tau_{\alpha\ltimes h}(w)|$ is uniformly bounded from above. Hence, by the above equation, $|F_{\alpha\ltimes h}(w) - F_{\alpha'\ltimes h}(w)|$ is bounded by a uniform constant times $|\alpha - \alpha'|$. We need to look at the asymptotic behaviour of this difference as $\mathrm{Im}(\alpha w) \to -\infty$. That is,

$$\begin{aligned}
&\lim_{\mathrm{Im}(\alpha w) \to -\infty} |F_{\alpha\ltimes h}(w) - F_{\alpha'\ltimes h}(w)| \\
&\quad = \left|\frac{1}{2\pi\alpha i} \log\left(1 - \frac{\sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(\sigma_{\alpha\ltimes h})}{1 + \sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(\sigma_{\alpha\ltimes h})}\right) - \frac{1}{2\pi\alpha' i} \log\left(1 - \frac{\sigma_{\alpha'\ltimes h} u_{\alpha'\ltimes h}(\sigma_{\alpha'\ltimes h})}{1 + \sigma_{\alpha'\ltimes h} u_{\alpha'\ltimes h}(\sigma_{\alpha'\ltimes h})}\right)\right| \\
&\quad = \left|\frac{1}{2\pi\alpha i} \log(1 + \sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(\sigma_{\alpha\ltimes h})) - \frac{1}{2\pi\alpha' i} \log(1 + \sigma_{\alpha'\ltimes h} u_{\alpha'\ltimes h}(\sigma_{\alpha'\ltimes h}))\right| \\
&\quad \leq C''|\alpha - \alpha'|,
\end{aligned}$$

for some constant C'' depending only on the class $\mathcal{F}_0$. By the maximum principle, the uniform bound in (b) must hold for all $w \in \Theta_\alpha(D_5)$ with $\mathrm{Im}(\alpha w) < 0$. □

Recall that the map $\alpha \ltimes h$ has a unique critical point in its domain of definition. This point may be lifted by $\tau_{\alpha\ltimes h}$ to a critical point for $F_{\alpha\ltimes h}$ which lies on $\partial\hat{\mathcal{P}}_{\alpha\ltimes h}$. We denote this point by $\hat{\mathrm{cp}}_{\alpha\ltimes h}$. By Proposition 2.2, several iterates of $\hat{\mathrm{cp}}_{\alpha\ltimes h}$ under $F_{\alpha\ltimes h}$ remain in $\hat{\mathcal{P}}_{\alpha\ltimes h}$, and there is the first moment when an iterate exits $\hat{\mathcal{P}}_{\alpha\ltimes h}$. Once the orbit exits this set, it falls in the connected component of $\mathbb{C}\setminus\hat{\mathcal{P}}_{\alpha\ltimes h}$ containing $1/\alpha$. In particular, by Lemma 3.12, there is the smallest $i_{\alpha\ltimes h}\in\mathbb{N}$ such that

$$\operatorname{Re} F_{\alpha\ltimes h}^{\circ i_{\alpha\ltimes h}}(\hat{\mathrm{cp}}_{\alpha\ltimes h})\in(\sqrt{2}D_5,\sqrt{2}D_5+2).$$

Here we are assuming that $\operatorname{Re}\alpha^{-1}\geq 2\sqrt{2}D_5+2$ so that such a point exists. This can be guaranteed by assuming that r_3' is small enough.

By the pre-compactness of $A^+(r_3')\ltimes\mathcal{F}_0$, $i_{\alpha\ltimes h}$ is uniformly bounded from above independent of α and h. The integer $i_{\alpha\ltimes h}$ may be chosen so that it is locally constant near a given $\alpha\in A^+(r_3')$ and $h\in\mathcal{F}_0$. We set the notation

$$v_{\alpha\ltimes h}=F_{\alpha\ltimes h}^{\circ i_{\alpha\ltimes h}}(\hat{\mathrm{cp}}_{\alpha\ltimes h}).$$

This point shall be used as a reference point for the normalisation of a number of maps we aim to introduce in order to study the dependence of $\Phi_{\alpha\ltimes h}$ on α.

Lemma 3.14. *There exists a constant D_8 such that for all $\alpha,\alpha'\in A^+(r_3')$ and all $h\in\mathcal{F}_0\cup\{Q_0\}$, we have*

$$|v_{\alpha\ltimes h}-v_{\alpha'\ltimes h}|\leq D_8|\alpha-\alpha'|.$$

Proof. By the definition of $F_{\alpha\ltimes h}$, $v_{\alpha\ltimes h}$ is the pre-image of $h_\alpha^{\circ i_{\alpha\ltimes h}}(\mathrm{cp}_{\alpha\ltimes h})=h_\alpha^{\circ(i_{\alpha\ltimes h}-1)}(-4/27)$. Also, by the pre-compactness of the class $\mathcal{F}_0$, $|h_\alpha^{\circ(i_{\alpha\ltimes h}-1)}(-4/27)|$ is uniformly bounded from above and away from 0. Then, the uniform bound in the lemma may be obtained by a argument similar to the one in the proof of Lemma 3.13. □

3.5. **Quasi-conformal Fatou coordinates.** Consider the univalent map

$$L_{\alpha\ltimes h}=\tau_{\alpha\ltimes h}^{-1}\circ\Phi_{\alpha\ltimes h}^{-1}:\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h})\to\hat{\mathcal{P}}_{\alpha\ltimes h},\tag{3.22}$$

where $\tau_{\alpha\ltimes h}^{-1}$ is the inverse of the map $\tau_{\alpha\ltimes h}:\hat{\mathcal{P}}_{\alpha\ltimes h}\to\mathcal{P}_{\alpha\ltimes h}$. Since $\Phi_{\alpha\ltimes h}$ conjugates $\alpha\ltimes h$ to the translation by $+1$, for every $\xi\in\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h})$ with $\xi+1\in\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h})$, we have

$$L_{\alpha\ltimes h}(\xi+1)=F_{\alpha\ltimes h}(L_{\alpha\ltimes h}(\xi)).\tag{3.23}$$

The map $L_{\alpha\ltimes h}$ may be extended onto the boundary of $\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h})$. It follows that $L_{\alpha\ltimes h}(0)=\hat{\mathrm{cp}}_{\alpha\ltimes h}$. Then, by the above functional equation, $L_{\alpha\ltimes h}(i_{\alpha\ltimes h})=v_{\alpha\ltimes h}$, where $i_{\alpha\ltimes h}$ and $v_{\alpha\ltimes h}$ are defined at the end of Section 3.4. Let Σ_2 denote the set of $w\in\mathbb{C}$ such that either

$$\arg(w-\sqrt{2}D_5)\in[-3\pi/4,3\pi/4]+2\pi\mathbb{Z},$$

or

$$\arg(w-1/\alpha+\sqrt{2}D_5)\in[\pi/4,7\pi/4]+2\pi\mathbb{Z}.$$

Using (3.23) and Lemma 3.12-(a), one may extend $L_{\alpha\ltimes h}^{-1}$ to a univalent map

$$L_{\alpha\ltimes h}^{-1}:\Sigma_2\to\mathbb{C}.$$

One may refer to [Che19] for a more detailed discussion on extending the domain of $L_{\alpha\ltimes h}^{-1}$.

Lemma 3.15. *There is a constant D_9, independent of α and h, such that for all w in Σ_2 we have*

$$1/D_9 \leq |(L_{\alpha\ltimes h}^{-1})'(w)| \leq D_9.$$

Moreover, as $\operatorname{Im} w \to +\infty$ within Σ_2, $(L_{\alpha\ltimes h}^{-1})'(w) \to +1$.

Proof. First assume $w \in \Sigma_2$ so that $B(w, 3/2) \subseteq \Sigma_2$. By the Koebe distortion theorem applied to $L_{\alpha\ltimes h}^{-1}$ on $B(w, 3/2)$ we know that $L_{\alpha\ltimes h}^{-1}$ is uniformly close to a (complex) linear map on the strictly smaller ball $B(w, 5/4)$. By Lemma 3.12-a, $|F_{\alpha\ltimes h}(w) - w - 1| \leq 1/4$. Hence w and $F_{\alpha\ltimes h}(w)$ lie in $B(w, 5/4)$, and are mapped by $L_{\alpha\ltimes h}^{-1}$ to a pair of points apart by $+1$. This implies that $(L_{\alpha\ltimes h}^{-1})'(w)$ must be uniformly bounded from above and away from zero.

For w near the vertical line $\operatorname{Re} w = \operatorname{Re}(1/(2\alpha))$ and with $\operatorname{Im} w$ large, there is a ball of radius comparable to $\operatorname{Im} w$, centred at w, which is contained in Σ_2. By the Koebe distortion theorem, $L_{\alpha\ltimes h}^{-1}$ tends to a (complex) linear map on $B(w, 3/2)$, as $\operatorname{Im} w \to +\infty$. Meanwhile, the points w and $F_{\alpha\ltimes h}(w)$ that are nearly apart by one, are mapped to two points exactly apart by one. This implies that $(L_{\alpha\ltimes h}^{-1})'(w) \to 1$ as $\operatorname{Im} w \to +\infty$.

An arbitrary $w \in \Sigma_2$ marches under the iterates of $F_{\alpha\ltimes h}$ to a point near the vertical line $\operatorname{Re} w = \operatorname{Re}(1/(2\alpha))$, where $(L_{\alpha\ltimes h}^{-1})'$ is uniformly bounded from above and away from zero. Moreover, this derivative tends to $+1$ as $\operatorname{Im} w \to +\infty$. By the uniform estimate in Lemma 3.12-a, the number of forward or backward iterates required to reach the proximity of the vertical line is linear in $\operatorname{Im} w$. Moreover, $|F'_{\alpha\ltimes h}|$ is uniformly bounded from above and away from 0 on Σ_2, and also $F'_{\alpha\ltimes h}$ tends to $+1$ exponentially fast as $\operatorname{Im} w \to +\infty$. Using this and the functional equation (3.22), we conclude that $(L_{\alpha\ltimes h}^{-1})'(w)$ is uniformly bounded from above and away from zero, and must tend to $+1$ as $\operatorname{Im} w \to +\infty$ within Σ_2. □

Remark 3.16. The proof of the above lemma provides us with a uniform bound on $|(L_{\alpha\ltimes h}^{-1})' - 1|$ of order $1/\operatorname{Im} w$. An exponentially decaying bound on $|(L_{\alpha\ltimes h}^{-1})' - 1|$ is obtained in [Che13], which requires more involved analysis. We do not need that finer estimate here.

We shall analyse the dependence of $L_{\alpha\ltimes h}$ on α by comparing it to two quasi-conformal changes of coordinates denoted by $H_{\alpha\ltimes h}^1$ and $H_{\alpha\ltimes h}^2$. For $\alpha \in A^+(r_3')$ and h in $\mathcal{F}_0 \cup \{Q_0\}$, define the map

$$H_{\alpha\ltimes h}^1 : \{\zeta \in \mathbb{C} \mid \operatorname{Re} \zeta \in [0, 1]\} \to \Theta_\alpha(D_5) \subseteq \operatorname{Dom} F_{\alpha\ltimes h}$$

as

$$H_{\alpha\ltimes h}^1(\zeta) = (1 - \operatorname{Re} \zeta)(v_{\alpha\ltimes h} + i \operatorname{Im} \zeta) + (\operatorname{Re} \zeta) F_{\alpha\ltimes h}(v_{\alpha\ltimes h} + i \operatorname{Im} \zeta).$$

When $\operatorname{Re} \zeta = 0$, we have

$$H_{\alpha\ltimes h}^1(\zeta + 1) = F_{\alpha\ltimes h}(H_{\alpha\ltimes h}^1(\zeta)).$$

We have normalised $H_{\alpha\ltimes h}^1$ by $H_{\alpha\ltimes h}^1(0) = v_{\alpha\ltimes h}$.

Lemma 3.17. *The map $H_{\alpha\ltimes h}^1$ is quasi-conformal and its complex dilatation satisfies*

$$\left| \partial_{\overline{\zeta}} H_{\alpha\ltimes h}^1(\zeta) / \partial_\zeta H_{\alpha\ltimes h}^1(\zeta) \right| \leq 1/3, \quad \forall \zeta \in \operatorname{Dom} H_{\alpha\ltimes h}^1.$$

Proof. It follows Lemma 3.12-a that $H^1_{\alpha\ltimes h}$ is a homeomorphism. The first partial derivatives of $H^1_{\alpha\ltimes h}$ exist and are given by

$$\begin{aligned}
\partial_\zeta H^1_{\alpha\ltimes h}(\zeta) &= \big(F_{\alpha\ltimes h}(v_{\alpha\ltimes h}+i\operatorname{Im}\zeta)-(v_{\alpha\ltimes h}+i\operatorname{Im}\zeta)+1+\operatorname{Re}\zeta(F'_{\alpha\ltimes h}(v_{\alpha\ltimes h}+i\operatorname{Im}\zeta)-1)\big)/2\\
\partial_{\overline{\zeta}} H^1_{\alpha\ltimes h}(\zeta) &= \big(F_{\alpha\ltimes h}(v_{\alpha\ltimes h}+i\operatorname{Im}\zeta)-(v_{\alpha\ltimes h}+i\operatorname{Im}\zeta)-1-\operatorname{Re}\zeta(F'_{\alpha\ltimes h}(v_{\alpha\ltimes h}+i\operatorname{Im}\zeta)-1)\big)/2.
\end{aligned} \tag{3.24}$$

By the estimates in Lemma 3.12, the complex dilatation of $H^1_{\alpha\ltimes h}$ satisfies the bound in the lemma. Evidently, $H^1_{\alpha\ltimes h}$ is absolutely continuous on lines. □

By the formulas in (3.24), and Lemma 3.12-(b)-(c), we have

– if $\operatorname{Im} H^1_{\alpha\ltimes h}(\zeta)\geq 0$, then

$$\begin{aligned}
|\partial_{\overline{\zeta}} H^1_{\alpha\ltimes h}(\zeta)| &\leq D_6|\tau_{\alpha\ltimes h}(v_{\alpha\ltimes h}+i\operatorname{Im}\zeta)|,\\
|\partial_\zeta H^1_{\alpha\ltimes h}(\zeta)-1| &\leq D_6|\tau_{\alpha\ltimes h}(v_{\alpha\ltimes h}+i\operatorname{Im}\zeta)|,
\end{aligned} \tag{3.25}$$

– if $\operatorname{Im} H^1_{\alpha\ltimes h}(\zeta)\leq 0$, then

$$\begin{aligned}
\Big|\partial_{\overline{\zeta}} H^1_{\alpha\ltimes h}(\zeta)+\frac{1}{2}+\frac{1}{4\pi\alpha i}\log h'_\alpha(\sigma_{\alpha\ltimes h})\Big| &\leq D_6|\tau_{\alpha\ltimes h}(v_{\alpha\ltimes h}+i\operatorname{Im}\zeta)-\sigma_{\alpha\ltimes h}|,\\
\Big|\partial_\zeta H^1_{\alpha\ltimes h}(\zeta)-\frac{1}{2}+\frac{1}{4\pi\alpha i}\log h'_\alpha(\sigma_{\alpha\ltimes h})\Big| &\leq D_6|\tau_{\alpha\ltimes h}(v_{\alpha\ltimes h}+i\operatorname{Im}\zeta)-\sigma_{\alpha\ltimes h}|.
\end{aligned} \tag{3.26}$$

Define the homeomorphism

$$G^1_{\alpha\ltimes h}=L^{-1}_{\alpha\ltimes h}\circ H^1_{\alpha\ltimes h}:\{\zeta\in\mathbb{C}\mid \operatorname{Re}\zeta\in[0,1]\}\to\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h}).$$

Using the complex chain rule, Lemma 3.15, and (3.25)-(3.26), we obtain:

– if $\operatorname{Im} H^1_{\alpha\ltimes h}(\zeta)\geq 0$, then

$$|\partial_{\overline{\zeta}} G^1_{\alpha\ltimes h}(\zeta)|\leq D_6D_9|\tau_{\alpha\ltimes h}(v_{\alpha\ltimes h}+i\operatorname{Im}\zeta)|, \tag{3.27}$$

– if $\operatorname{Im} H^1_{\alpha\ltimes h}(\zeta)\leq 0$, then

$$\Big|\partial_{\overline{\zeta}} G^1_{\alpha\ltimes h}(\zeta)+\frac{1}{2}+\frac{1}{4\pi\alpha i}\log h'_\alpha(\sigma_{\alpha\ltimes h})\Big|\leq D_6D_9|\tau_{\alpha\ltimes h}(v_{\alpha\ltimes h}+i\operatorname{Im}\zeta)-\sigma_{\alpha\ltimes h}|. \tag{3.28}$$

Note that the chain rule does not automatically give us a similar upper bound on $|\partial_\zeta G^1_{\alpha\ltimes h}-1|$. However, it implies that

$$|(G^1_{\alpha\ltimes h})'|=O(1),\ \text{and}\ \lim_{\operatorname{Im}\zeta\to+\infty}(G^1_{\alpha\ltimes h})'(\zeta)=1, \tag{3.29}$$

with a uniform constant in O independent of α and h.

We do not *a priori* know the image of the map $G^1_{\alpha\ltimes h}$, except that when $\operatorname{Re}\zeta=0$, we must have

$$G^1_{\alpha\ltimes h}(\zeta+1)=G^1_{\alpha\ltimes h}(\zeta)+1. \tag{3.30}$$

However, (3.27), (3.28), and (3.30) make this map to be almost an affine one. The precise statement is formulated in the next lemma.

Lemma 3.18. *There exists a constant D_{10}, independent of α and h, such that for all ζ with $\operatorname{Re}\zeta \in [0,1]$ and $\operatorname{Im}\zeta \geq -2$,*

$$|G^1_{\alpha\ltimes h}(\zeta) - \zeta| \leq D_{10}(1 - \log|\alpha|).$$

Moreover, $\lim_{\operatorname{Im}\zeta\to+\infty}(G^1_{\alpha\ltimes h}(\zeta) - \zeta)$ exists and is a finite number.

Proof. Fix real numbers δ_1 and δ_2 such that $\delta_2 > \delta_1 + 1 \geq 1$, and define the set

$$A = \{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta \in [0,1], \operatorname{Im}\zeta \in [\delta_1, \delta_2]\}.$$

By the uniform bound in (3.27), we have

$$\Big|\iint_A \partial_{\overline{\zeta}} G^1_{\alpha\ltimes h}(\zeta)\, d\zeta d\overline{\zeta}\Big| \leq D_6 D_9 \iint_A |\tau_{\alpha\ltimes h}(v_{\alpha\ltimes h} + i\operatorname{Im}\zeta)|\, d\zeta d\overline{\zeta}. \tag{3.31}$$

By the pre-compactness of $\mathcal{F}_0$, $|v_{\alpha\ltimes h}|$ is uniformly bounded from above, independent of α and h. By Lemma 3.11, there is a constant C, independent of α and h, such that the right hand side of the above inequality is bounded by

$$\begin{aligned}
&C\left(1 + |\alpha| \int_{\delta_1+1}^{\delta_2} \frac{1}{e^{2\pi t|\alpha|\cos(\arg\alpha)} - 1}\, dt\right)\\
&\leq C\left(1 + |\alpha| \frac{1}{2\pi|\alpha|\cos(\arg\alpha)}\left(\left(\log(1 - e^{-2\pi t|\alpha|\cos(\arg\alpha)}\right)\Big|_{t=\delta_1+1}^{t=\delta_2}\right)\right)\\
&\leq C\left(1 + \frac{1}{\pi\sqrt{2}}\left(\left(\log(1 - e^{-2\pi t|\alpha|\cos(\arg\alpha)}\right)\Big|_{t=\delta_1+1}^{t=+\infty}\right)\right)\\
&\leq C\left(1 - \frac{1}{\pi\sqrt{2}}\log(1 - e^{-2\pi(\delta_1+1)|\alpha|\cos(\arg\alpha)})\right).
\end{aligned} \tag{3.32}$$

In the above inequalities we have used $\arg\alpha \in [-\pi/4, \pi/4]$ and so $\cos(\arg\alpha) \leq \sqrt{2}/2$. Since $r_1 \geq 0$, we have

$$1 - \frac{1}{\pi\sqrt{2}}\log\left(1 - e^{-2\pi(\delta_1+1)|\alpha|\cos(\arg\alpha)}\right) \leq 1 - \frac{1}{\pi\sqrt{2}}\log\left(1 - e^{-2\pi|\alpha|\cos(\arg\alpha)}\right) \leq C'(1 - \log|\alpha|),$$

for some explicit constant C' independent of α. Combining the above inequalities we conclude that the right hand side of Equation (3.31) is bounded from above by a uniform constant times $1 - \log|\alpha|$.

On the other hand, by the Green's integral formula, and using the relation in Equation (3.30) on the vertical side of A where $\operatorname{Re}\zeta = 0$, the integral in the left hand side of (3.31) is equal to

$$\oint_{\partial A} G^1_{\alpha\ltimes h}(\zeta)\, d\zeta = (\delta_2 - \delta_1)i - \int_0^1 G^1_{\alpha\ltimes h}(\delta_2 i + t)\, dt + \int_0^1 G^1_{\alpha\ltimes h}(\delta_1 i + t)\, dt. \tag{3.33}$$

The derivative $|\partial_\zeta G^1_{\alpha\ltimes h}(\zeta)|$ is uniformly bounded from above independent of α and h, because of the bounds in Equations (3.25)-(3.26) and Lemma 3.15. This implies that

$$\Big|\int_0^1 G^1_{\alpha\ltimes h}(\delta_2 i + t)\, dt - G^1_{\alpha\ltimes h}(\delta_2 i)\Big|, \quad \Big|\int_0^1 G^1_{\alpha\ltimes h}(\delta_1 i + t)\, dt - G^1_{\alpha\ltimes h}(\delta_1 i)\Big|$$

are uniformly bounded from above, independent of α and h.

Let us choose $\delta_1 = 0$ and a point $\zeta \in A$ with $\operatorname{Re}\zeta = 0$. By the above arguments we conclude that $|G^1_{\alpha\ltimes h}(\zeta) - \zeta| \leq C'(1 - \log|\alpha|) + |G^1_{\alpha\ltimes h}(0)|$. However, $|G^1_{\alpha\ltimes h}(0)| = |i_{\alpha\ltimes h}|$ is uniformly bounded from above, independent of α and h. This implies the desired upper bound in the first part of the lemma at ζ. Now, the uniform bound in (3.29) may be used to establish the first part of the lemma at other points $\zeta \in \text{Dom } G^1_{\alpha\ltimes h}$ with $\operatorname{Im}\zeta \geq -2$.

Let ζ_1 and ζ_2 be two points with $\operatorname{Re}\zeta_1 = \operatorname{Re}\zeta_2 = 0$ and $\delta_2 = \operatorname{Im}\zeta_2 > \delta_1 = \operatorname{Im}\zeta_1$. By the asymptotic estimate in Equation (3.25) and in Lemma 3.15, $\partial_\zeta G^1_{\alpha\ltimes h}(\zeta) \to 1$ as $\operatorname{Im}\zeta \to +\infty$. This implies that the integral in (3.33) tends to $(\zeta_2 - G^1_{\alpha\ltimes h}(\zeta_2)) - (\zeta_1 - G^1_{\alpha\ltimes h}(\zeta_1))$ as $\operatorname{Im}\zeta_2$ and $\operatorname{Im}\zeta_1$ tend to $+\infty$. On the other hand, as δ_2 and δ_1 tend to $+\infty$, the right hand side of the first " $\leq$" in (3.32) tends to 0. This means that the difference $(\zeta_2 - G^1_{\alpha\ltimes h}(\zeta_2)) - (\zeta_1 - G^1_{\alpha\ltimes h}(\zeta_1))$ satisfies the Cauchy's criterion, and hence the limit of $(\zeta - G^1_{\alpha\ltimes h}(\zeta))$ exists as $\operatorname{Im}\zeta \to +\infty$ along the line $\operatorname{Re}\zeta = 0$. Finally, since $\partial_\zeta G^1_{\alpha\ltimes h}(\zeta) \to 1$ as $\operatorname{Im}\zeta \to +\infty$ within $\text{Dom } G^1_{\alpha\ltimes h}$, the limit must exist as $\operatorname{Im}\zeta \to +\infty$ within $\text{Dom } G^1_{\alpha\ltimes h}$. □

Proposition 3.19. *There exists a constant D_{11} such that for all $\alpha \in A^+(r_3')$, $h \in \mathcal{F}_0 \cup \{Q_0\}$, and all $\xi \in \text{Dom } L_{\alpha\ltimes h}$ with $\operatorname{Im}\xi \geq -2$ we have*

$$|L_{\alpha\ltimes h}(\xi) - \xi| \leq D_{11}(1 - \log|\alpha|).$$

Moreover, the limit $\lim_{\operatorname{Im}\xi\to+\infty} L_{\alpha\ltimes h}(\xi) - \xi$ exists.

Proof. Recall that $H^1_{\alpha\ltimes h}(0) = v_{\alpha\ltimes h}$, and $|v_{\alpha\ltimes h}|$ is uniformly bounded from above independent of α and h. This implies that $|H^1_{\alpha\ltimes h}(\zeta) - \zeta|$ is uniformly bounded from above, independent of α, h, and $\zeta \in \text{Dom } H^1_{\alpha\ltimes h}$. Indeed, $H^1_{\alpha\ltimes h}(\zeta) - \zeta$ converges to $v_{\alpha\ltimes h}$ as $\operatorname{Im}\zeta \to +\infty$. Combining with Lemma 3.18, we obtain the uniform bound on $|L_{\alpha\ltimes h}(\xi) - \xi|$ on the image of $G^1_{\alpha\ltimes h}$. Finally, the functional relation in (3.22) and the estimates on $F_{\alpha\ltimes h}$ in Lemma 3.12 may be used to prove the proposition at points $\xi \in \text{Dom } L_{\alpha\ltimes h}$ with $\operatorname{Im}\xi \geq -2$. In the same fashion, the latter part of the proposition follows from the latter part of Lemma 3.18. □

The quasi-conformal change of coordinate $H^1_{\alpha\ltimes h}$ allows us to analyze the behavior of $L_{\alpha\ltimes h}$ near the left hand side of $\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h})$. For example, it shall allow us to study the spiraling behavior of $\Phi^{-1}_{\alpha\ltimes h}(x + iy)$, as y tends to $+\infty$ or to $-\infty$, for small positive values of x. On the other hand, we do not *a priori* know the size of $\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h})$, say, its vertical width in terms of $1/\alpha$. We also need to understand the behavior of $L_{\alpha\ltimes h}$ near the right side of $\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h})$. However, a large number of iterates of $F_{\alpha\ltimes h}$, (about $\operatorname{Re}(1/\alpha)$ near $+i\infty$), on the image of $H^1_{\alpha\ltimes h}$ are needed to cover $\hat{\mathcal{P}}_{\alpha\ltimes h}$. For this reason, it is not possible to use the functional equation (3.22) and the uniform estimates on $F_{\alpha\ltimes h}$ in Lemma 3.12 to derive estimates on $L_{\alpha\ltimes h}$ near the right hand side of $\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h})$. For this purpose we need to define an alternative quasi-conformal change of coordinate for the right-hand side. But, the issue here is that there is no reference point similar to $v_{\alpha\ltimes h}$ near the right hand side of $\hat{\mathcal{P}}_{\alpha\ltimes h}$. However, as $F_{\alpha\ltimes h}$ commutes with the translation by $1/\alpha$, we expect that the behavior of $L_{\alpha\ltimes h}$ near the left hand side and the right hand side of $\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h})$ should be similar in nature. This is investigated through the quasi-conformal change of coordinate

$$H^2_{\alpha\ltimes h} : \{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta \in [0,1]\} \to \Theta_\alpha(D_5) \subseteq \text{Dom } F_{\alpha\ltimes h}$$

defined below.

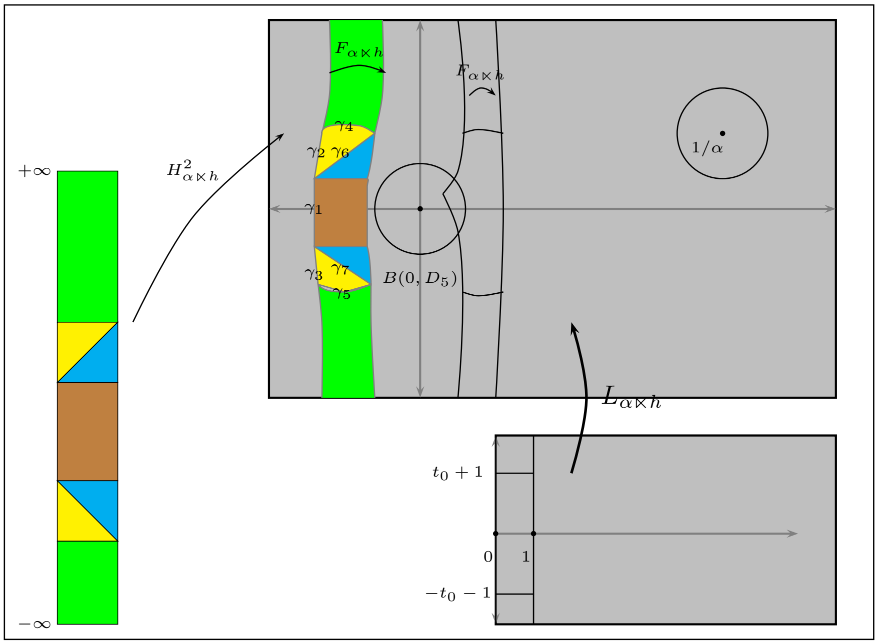

FIGURE 6. Schematic presentation of the quasi-conformal change of coordinates $H^2_{\alpha\ltimes h}$. It is defined as interpolations of some analytic and quasi-conformal mappings.

Recall that $H^1_{\alpha\ltimes h}(0) = v_{\alpha\ltimes h}$ and $|v_{\alpha\ltimes h}|$ is uniformly bounded from above. Since $F_{\alpha\ltimes h}$ is uniformly close to the translation by one, and $G^1_{\alpha\ltimes h}$ is quasi-conformal with a uniform bound on its dilatation, independent of α and h, there is a constant $t_0 > 0$ and a positive integer i such that

$$\forall \xi \in \{t_0 i, 1 + t_0 i, -t_0 i, 1 - t_0 i\}, \operatorname{Re} F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(\xi)) \in [-\operatorname{Re}(1/\alpha) + D_5, -D_5],$$

and for all ξ with $\operatorname{Re}\xi \in [0, 1]$ and $|\operatorname{Im}\xi| \geq t_0$, $F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(\xi))$ is defined. Indeed, by making t_0 large enough, the four points $F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(t_0 i))$, $F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(t_0 i + 1))$, $F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(-t_0 i))$, and $F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(-t_0 i + 1))$ become arbitrarily close to the vertices of a parallelogram whose two sides converge to two horizontal segments of length one.

Let us define the curves (see Figure 6)

$$\gamma_1(t) = (\frac{t_0 - t}{2t_0})F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(-t_0 i)) + (\frac{t_0 + t}{2t_0})F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(t_0 i)), \text{for } t \in [-t_0, t_0];$$
$$\gamma_2(t) = (1-t)F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(t_0 i)) + tF^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}((t_0+1)i)), \text{for } t \in [0,1]$$
$$\gamma_3(t) = (1-t)F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(-t_0 i)) + tF^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}((-t_0-1)i)), \text{for } t \in [0,1]$$
$$\gamma_4(s) = F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(s + (t_0+1)i)), \text{for } s \in [0,1];$$
$$\gamma_5(s) = F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(s + (-t_0-1)i)), \text{for } s \in [0,1];$$
$$\gamma_6(s) = (1-s)F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(t_0 i)) + sF^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(1 + (t_0+1)i)), \text{for } s \in [0,1];$$
$$\gamma_7(s) = (1-s)F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(-t_0 i)) + sF^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(1 + (-t_0-1)i)), \text{for } s \in [0,1].$$

For t_0 large enough, independent of α and h, the curves γ_4, γ_5, $F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}([0,1] + t_0 i))$ and $F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}([0,1] - t_0 i))$ are nearly horizontal. The curve γ_6 has slope close to 1, and the curve γ_7 has slope close to -1. In particular, for large enough t_0, the curves γ_4 and γ_6 intersect only at their end points $F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(1 + (t_0+1)i))$, wile the curves γ_5 and γ_7 intersect only at the point $F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(1 - (t_0+1)i))$. The curves γ_6 and $F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}([0,1] + t_0 i))$ only intersect at their staring points, and similarly the curves γ_7 and $F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}([0,1] - t_0 i))$ intersect only at their starting points.

Define the map $H^2_{\alpha\ltimes h}(s + ti)$, for $s \in [0,1]$ and $t \in \mathbb{R}$, as follows:

$$\begin{cases} F^{-i}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(s+ti)), & \text{if } |t| \geq t_0 + 1 \\ (1-s)\gamma_1(t) + sF_{\alpha\ltimes h}(\gamma_1(t)), & \text{if } |t| \leq t_0 \\ (\frac{t-t_0-s}{1-s})\gamma_4(s) + \frac{t_0-t+1}{1-s}\gamma_6(s), & \text{if } s \in [0,1], t \in [t_0+s, t_0+1] \\ (\frac{s-1}{t-t_0-1})\gamma_6(t-t_0) + \frac{t-t_0-s}{t-t_0-1}F_{\alpha\ltimes h}(\gamma_2(t)), & \text{if } t \in [t_0, t_0+1], s \in [t-t_0, 1] \\ (\frac{-t-t_0-s}{1-s})\gamma_5(s) + \frac{t_0+t+1}{1-s}\gamma_7(s), & \text{if } s \in [0,1], t \in [-t_0-s, -t_0-1] \\ (\frac{s-1}{t-t_0-1})\gamma_7(t-t_0) + \frac{t-t_0-s}{t-t_0-1}F_{\alpha\ltimes h}(\gamma_3(t)), & \text{if } t \in [-t_0, -t_0-1], s \in [t-t_0, 1]. \end{cases}$$

It follows from the above definition that

$$H^2_{\alpha\ltimes h}(1 + ti) = F_{\alpha\ltimes h}(H^2_{\alpha\ltimes h}(ti)), \forall t \in \mathbb{R}.$$

Moreover, from the construction, one can see that the following lemma holds.

Lemma 3.20. *The map $H^2_{\alpha\ltimes h}$ is quasi-conformal on the strip $\operatorname{Re}\zeta \in [0,1]$ and the size of its dilatation $|\partial_{\overline{\zeta}} H^2_{\alpha\ltimes h}/\partial_\zeta H^2_{\alpha\ltimes h}|$ is uniformly bounded from above at almost every point on this strip. Moreover, there is a constant D_{12} such that for all α, β in $A(r_3')$, all h in $\mathcal{F}_0 \cup \{Q_\alpha\}$, and all ζ with $\operatorname{Re}\zeta \in [0,1]$ we have*

$$\big|H^2_{\alpha\ltimes h}(\zeta) - H^2_{\beta,h}(\zeta)\big| \leq D_{12}|\alpha - \beta|.$$

For $c \in \mathbb{C}$, we denote the translation by c on the complex plane with

$$T_c(w) = w + c.$$

Since $F_{\alpha\ltimes h}$ is periodic of period $1/\alpha$, the quasi-conformal mapping $T_{1/\alpha} \circ H^2_{\alpha\ltimes h}$ also conjugates the translation by one to the action of $F_{\alpha\ltimes h}$. We shall compare the map $L_{\alpha\ltimes h}$ to this coordinate by studying the map

$$G^2_{\alpha\ltimes h} = T_{-1/\alpha} \circ L^{-1}_{\alpha\ltimes h} \circ T_{1/\alpha} \circ H^2_{\alpha\ltimes h} : \{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta \in [0,1]\} \to \mathbb{C}.$$

By the periodicity of $F_{\alpha\ltimes h}$ and the functional equations for $L_{\alpha\ltimes h}$ and $H^2_{\alpha\ltimes h}$, we must have $G^2_{\alpha\ltimes h}(\zeta+1) = G^2_{\alpha\ltimes h}(\zeta)+1$, whenever $\operatorname{Re}\zeta = 0$. We shall use this map to analyze the conformal change of coordinate $L_{\alpha\ltimes h}$ near the right hand side of its domain of definition. By the definition of this map, $G^2_{\alpha\ltimes h}$ is quasi-conformal with $|\partial_{\bar\zeta} G^2_{\alpha\ltimes h}/\partial_\zeta G^2_{\alpha\ltimes h}|$ uniformly bounded from above. Moreover, it is normalised by making

$$\lim_{\operatorname{Im}\zeta\to+\infty} |\operatorname{Im}(G^2_{\alpha\ltimes h}(\zeta)-\zeta)| = 0. \tag{3.34}$$

The above normalisation, and the uniform bound on the dilatation of the map allows us to prove a uniform bound on the dependence of this map on α.

Lemma 3.21. *For every $D'_{13} > 0$ there is a constant D_{13} such that for all α, β in $A(r'_3)$ and all ζ with $\operatorname{Re}\zeta \in [0,1]$ and $|\operatorname{Im}\zeta| \le D'_{13}$ we have*

a)

$$|G^1_{\alpha\ltimes h} \circ (G^1_{\beta,h})^{-1}(\xi) - \xi| \le D_{13}|\alpha-\beta|.$$

b)

$$|G^2_{\alpha\ltimes h} \circ (G^2_{\beta,h})^{-1}(\xi) - \xi| \le D_{13}|\alpha-\beta|.$$

Proof. Part a) Let $\mu_{\alpha\ltimes h}$ denote the complex dilatation of the map $G^1_{\alpha\ltimes h}$. By Lemma 3.17, $|\mu_{\alpha\ltimes h}|$ is uniformly bounded from above, independent of α and h, by a constant < 1. The complex dilation of the composition $G^1_{\alpha\ltimes h} \circ (G^1_{\beta,h})^{-1}$ at ζ is given by the formula

$$\frac{\mu_{\alpha\ltimes h} - \mu_{\beta,h}}{1-\mu_{\beta,h}\overline{\mu_{\alpha\ltimes h}}}\Big(\frac{\partial_\zeta G^1_{\beta,h}(\zeta)}{|\partial_\zeta G^1_{\beta,h}(\zeta)|}\Big)^2.$$

On the other hand, as $G^1_{\alpha\ltimes h} = L^{-1}_{\alpha\ltimes h} \circ H^1_{\alpha\ltimes h}$ and $L_{\alpha\ltimes h}$ is holomorphic, by the chain rule, the complex dilation of $G^1_{\alpha\ltimes h}$ is equal to the complex dilatation of $H^1_{\alpha\ltimes h}$. Combining with the above equation, and the formulas in Equation (3.24), we conclude that the size of the complex dilation of $G^1_{\alpha\ltimes h} \circ (G^1_{\beta,h})^{-1}$ is bounded from above by a uniform constant times $|\alpha-\beta|$.

Recall that $G^1_{\alpha\ltimes h}(0) = i_{\alpha\ltimes h}$, and $i_{\alpha\ltimes h}$ is locally constant. Thus, for α and β sufficiently close, $G^1_{\alpha\ltimes h} \circ (G^1_{\beta,h})^{-1}(i_{\alpha\ltimes h}) = i_{\alpha\ltimes h}$. By the classical results on the dependence of the solution of the Beltrami equation on the Beltrami coefficient, see [AB60, Section 5.1], $|G^1_{\alpha\ltimes h} \circ (G^1_{\beta,h})^{-1}(\xi) - \xi|$, for ξ in a compact set, is bounded from above by a uniform constant times $|\alpha-\beta|$. This finishes the proof of the first part.

Part b) The proof is similar to the one given for Part a, except that we use the normalisation of the maps at infinity instead; Equation (3.34). □

3.6. Dependence of the Fatou coordinate on the linearity.

Proposition 3.22. *For all $D'_{14} > 0$ there exists a constant D_{14}, independent of α and h, such that*

a) for all ξ in $\operatorname{Dom} L_{\alpha\ltimes h} \cap B(0, D'_{14})$

$$|\frac{\partial}{\partial\alpha} L_{\alpha\ltimes h}(\xi)| \le D_{14},$$

b) for all w in $\mathrm{Dom}\, L^{-1}_{\alpha\ltimes h} \cap B(0, D'_{14})$,

$$|\frac{\partial}{\partial\alpha} L^{-1}_{\alpha\ltimes h}(w)| \leq D_{14}.$$

c) for all ξ in $(\mathrm{Dom}\,(L_{\alpha\ltimes h}) - 1/\alpha) \cap B(0, D'_{14})$,

$$|\frac{\partial}{\partial\alpha}(T_{-1/\alpha} \circ L_{\alpha\ltimes h} \circ T_{1/\alpha})(\xi)| \leq D_{14}.$$

Proof. Part a) We have

$$\begin{aligned}|L_{\alpha\ltimes h}(\xi) - L_{\beta,h}(\xi)| &\leq |H^1_{\alpha\ltimes h} \circ (G^1_{\alpha\ltimes h})^{-1}(\xi) - H^1_{\alpha\ltimes h} \circ (G^1_{\beta,h})^{-1}(\xi)| \\ &\quad + |H^1_{\alpha\ltimes h} \circ (G^1_{\beta,h})^{-1}(\xi) - H^1_{\beta,h} \circ (G^1_{\beta,h})^{-1}(\xi)| \\ &\leq \sup_z |DH^1_{\alpha\ltimes h}| \sup_\xi |D(G^1_{\alpha\ltimes h})^{-1}(\xi)||\xi - G^1_{\alpha\ltimes h}(G^1_{\beta,h})^{-1}(\xi)| \\ &\quad + |H^1_{\alpha\ltimes h}(z) - H^1_{\beta,h}(z)| \\ &\leq C|\alpha - \beta|.\end{aligned}$$

In the last line of the above equation we have used the uniform bound in Lemma 3.21, and a uniform bound on the dependence of $H^1_{\alpha\ltimes h}$ on α. The latter bound is obtained from the definition of the map $H^1_{\alpha\ltimes h}$ and the uniform bound on the dependence of the map $F_{\alpha\ltimes h}$ and the point $v_{\alpha\ltimes h}$ on α obtained in Lemmas 3.13 and 3.14.

Part b) With $w = L_{\beta,h}(\xi)$, we have

$$\begin{aligned}|L^{-1}_{\alpha\ltimes h}(w) - L^{-1}_{\beta,h}(w)| &= |L^{-1}_{\alpha\ltimes h}(L_{\beta,h}(\xi)) - L^{-1}_{\alpha\ltimes h}(L_{\alpha\ltimes h}(\xi))| \\ &\leq \sup_w |(L^{-1}_{\alpha\ltimes h})'(w)||L_{\alpha\ltimes h}(\xi) - L_{\beta,h}(\xi)| \\ &\leq C'C|\alpha - \beta|.\end{aligned}$$

In the above inequalities, the uniform bound on $|(L^{-1}_{\alpha\ltimes h})'(w)|$, when w is restricted to $B(0, D'_{14})$, may be obtained from the pre-compactness of the class of maps $\mathcal{F}_0$, and the continuous dependence of $L_{\alpha\ltimes h}$ on α and h. The constant C is the one introduced in the proof of Part a).

Part c) The argument here is similar to the one in part a) withe difference that we use the maps $H^2_{\alpha\ltimes h}$ and $G^2_{\alpha\ltimes h}$. That is, with $w = H^2_{\alpha\ltimes h}(\zeta)$, we have

$$\begin{aligned}&|(T_{-1/\alpha} \circ L_{\alpha\ltimes h} \circ T_{1/\alpha})(\xi) - (T_{-1/\beta} \circ L_{\beta,h} \circ T_{1/\beta})(\xi)| \\ &\qquad = |H^2_{\alpha\ltimes h} \circ (G^2_{\alpha\ltimes h})^{-1}(\xi) - H^2_{\beta,h} \circ (G^2_{\beta,h})^{-1}(\xi)| \\ &\qquad \leq |H^2_{\alpha\ltimes h} \circ (G^2_{\alpha\ltimes h})^{-1}(\xi) - H^2_{\alpha\ltimes h} \circ (G^2_{\beta,h})^{-1}(\xi)| \\ &\qquad\quad + |H^2_{\alpha\ltimes h} \circ (G^2_{\beta,h})^{-1}(\xi) - H^2_{\beta,h} \circ (G^2_{\beta,h})^{-1}(\xi)| \\ &\qquad \leq \sup_z |DH^2_{\alpha\ltimes h}| \sup_\xi |D(G^2_{\alpha\ltimes h})^{-1}(\xi)||\xi - G^2_{\alpha\ltimes h}(G^2_{\beta,h})^{-1}(\xi)| \\ &\qquad\quad + |H^2_{\alpha\ltimes h}(z) - H^2_{\beta,h}(z)| \\ &\qquad \leq C|\alpha - \beta|,\end{aligned}$$

for some constant C, independent of α and h. In the last line of the above inequalities we have used the uniform bounds in Lemmas 3.15, 3.21. □

3.7. **Geometry of the petals.**

Proof of Proposition 2.3. Let r_3 be the constant r_3' introduced in Lemma 3.12. We shall continue to use the notation $\alpha \ltimes h$ for the maps f in the class $A^+(r_3') \ltimes \mathcal{F}_0$, introduced in Equation (2.1). Recall the covering map $\tau_{\alpha\ltimes h}$ defined in (3.12), the lift $F_{\alpha\ltimes h}$ of $\alpha \ltimes h$ defined in Equation (3.14), and the univalent map $L_{\alpha\ltimes h}$ which conjugates the translation by one with $F_{\alpha\ltimes h}$. Then, $\Phi_{\alpha\ltimes h}^{-1}$ is the same as the composition $\tau_{\alpha\ltimes h} \circ L_{\alpha\ltimes h}$.

Let us define $x_{\alpha\ltimes h}$ as the supremum of the set of $x \geq 0$ such that $L_{\alpha\ltimes h}$ has a univalent extension onto the set $(0, x) + i\mathbb{R}$, and $L_{\alpha\ltimes h}$ maps this infinite strip into $\Sigma_2 \cup \hat{\mathcal{P}}_{\alpha\ltimes h}$, where Σ_2 is defined before Lemma 3.15 and $\hat{\mathcal{P}}_{\alpha\ltimes h}$ is the lift of $\mathcal{P}_{\alpha\ltimes h}$ separating 0 from $1/\alpha$. By Proposition 2.2, $x_{\alpha\ltimes h} \geq 2$. Also, $L_{\alpha\ltimes h}(x_{\alpha\ltimes h} + i\mathbb{R})$ intersects the right hand side boundary of Σ_2 at some point whose imaginary part is uniformly close to $\mathrm{Im}(1/\alpha)$.

Consider the sets

$$B_1 = \{\xi \in \mathbb{C} \mid \mathrm{Re}\,\xi \in [0, 1]\}, B_2 = \{\xi \in \mathbb{C} \mid \mathrm{Re}\,\xi \in [x_{\alpha\ltimes h} - 1, x_{\alpha\ltimes h}]\}.$$

We aim to show that the curve $L_{\alpha\ltimes h}(x_{\alpha\ltimes h} + i\mathbb{R})$ is within a uniformly bounded distance from a translation of the curve $L_{\alpha\ltimes h}(i\mathbb{R})$. Next we show that the translation constant is $\mathrm{Re}(1/\alpha)$. As $F_{\alpha\ltimes h}$ tends to the translation by one near $+i\infty$, the functional equation (3.22) implies that $x_{\alpha\ltimes h}$ is uniformly close to $\mathrm{Re}(1/\alpha)$.

There is a constant $\eta > 0$ such that every $\xi \in B_1$ with $|\xi| \geq \eta$, $L_{\alpha\ltimes h}(\xi)$ belongs to $\Sigma_2 - 1/\alpha$. Indeed, by the pre-compactness of the class $\mathcal{F}_0$, η may be chosen independent of α and h.

The uniform estimate in Lemma 3.12 also hold on $\Sigma_2 - 1/\alpha$, since $F_{\alpha\ltimes h}$ is periodic of period $1/\alpha$. This implies that for every $L_{\alpha\ltimes h}(\xi)$ with $|\mathrm{Im}\,\xi| \geq \eta$ and $\mathrm{Re}\,\xi \in [0, 1]$, there is $j_\xi \in \mathbb{Z}$ with $F_{\alpha\ltimes h}^{\circ j_\xi}(L_{\alpha\ltimes h}(\xi)) \in L_{\alpha\ltimes h}(B_2) - 1/\alpha$. For $\xi \in B_1$ with $|\xi| \geq \eta$, define the map

$$H(\xi) = L_{\alpha\ltimes h}^{-1} \circ T_{1/\alpha} \circ F_{\alpha\ltimes h}^{\circ j_\xi} \circ L_{\alpha\ltimes h}(\xi).$$

The map H may have discontinuities on its domain of definition, but since it commutes with the translation by one, it induces a continuous map from the top and bottom ends of the cylinder $B_1/\mathbb{Z}$ to the cylinder $B_2/\mathbb{Z}$. By the pre-compactness of the class of maps $F_{\alpha\ltimes h}$, applied on a compact neighbourhood of 0, the map H may be extended to a quasi-conform mapping from $B_1/\mathbb{Z}$ to $B_2/\mathbb{Z}$, whose complex dilatation is uniformly bounded away from the unit circle. Comparing the asymptotic expansions of the maps, near the top end $\mathrm{Im}\,H$ is asymptotic to the translation by $\mathrm{Im}(1/\alpha)$. Note that since H is conformal near the two ends of the cylinder, it maps every vertical line in $B_1/\mathbb{Z}$ to a curve in $B_2/\mathbb{Z}$, going from one end to the other, and spirals around the cylinder by a uniformly bounded amount. This implies that the lift of H to a map from $\mathbb{C}$ to $\mathbb{C}$ is uniformly close to a translation.

By the above paragraph, the map $T_{1/\alpha} \circ F_{\alpha\ltimes h}^{\circ j_\xi}$ is uniformly close to a translation, as a map from $L_{\alpha\ltimes h}(B_1)$ to $L_{\alpha\ltimes h}(B_2)$. Thus, $F_{\alpha\ltimes h}^{\circ j_\xi}$ from $L_{\alpha\ltimes h}(B_1)$ to $T_{-1/\alpha} \circ L_{\alpha\ltimes h}(B_2)$ must be close to a translation. However, since $|j_\xi|$ is uniformly bounded form above for when $\mathrm{Im}\,\xi = \eta$, and each iterate of $F_{\alpha\ltimes h}$ is uniformly close to the translation by one, $F_{\alpha\ltimes h}^{\circ j_\xi}$ must be uniformly close to the identity map. This implies that, $L_{\alpha\ltimes h}(B_2)$ is uniformly close to $T_{1/\alpha} \circ L_{\alpha\ltimes h}(B_1)$.

By Proposition 3.19, the sets $L_{\alpha\ltimes h}(B_1)$ to $L_{\alpha\ltimes h}(B_2)$ are asymptotically vertical, and also $F_{\alpha\ltimes h}$ tends to the translation by one near $+i\infty$. Thus, the number iterates by $F_{\alpha\ltimes h}$ required to go from $L_{\alpha\ltimes h}(B_1)$ to $L_{\alpha\ltimes h}(B_2)$ must be uniformly close to $\mathrm{Re}(1/\alpha)$. By the functional equation (3.22), $x_{\alpha\ltimes h}$ is uniformly close to $\mathrm{Re}(1/\alpha)$.

Recall that $\tau_{\alpha\ltimes h}$ is periodic of period $1/\alpha$. Finally, since $L_{\alpha\ltimes h}(x_{\alpha\ltimes h}+i\mathbb{R})$ is uniformly close the $L_{\alpha\ltimes h}(i\mathbb{R})+\mathrm{Re}(1/\alpha)$, by subtracting a uniformly bounded number from $x_{\alpha\ltimes h}$, if necessary, we may assume that $\tau_{\alpha\ltimes h}$ is univalent on $L_{\alpha\ltimes h}((0,x_{\alpha\ltimes h})+i\mathbb{R})$. Thus, the composition $\tau_{\alpha\ltimes h}\circ L_{\alpha\ltimes h}$ is univalent on the set $(0,x_{\alpha\ltimes h})+i\mathbb{R})$. This finishes the proof of the proposition. □

Proof of Proposition 2.4 – Part a). We continue to use the notation $\alpha\ltimes h$ for the maps f in $A^+(r_3')\ltimes\mathcal{F}_0$. Recall that r_3 is the constant r_3' obtained in Lemma 3.12. We shall use the decomposition of $\Phi^{-1}_{\alpha\ltimes h}$ as $\tau_{\alpha\ltimes h}\circ L_{\alpha\ltimes h}$.

Let $\xi=\xi_1+i\xi_2$, and $w=w_1+iw_2=L_{\alpha\ltimes h}(\xi)$, with ξ_1, ξ_2, w_1, and w_2 in $\mathbb{R}$. By Proposition 3.19, for a fixed $\xi_1\in(0,\mathrm{Re}\,\frac{1}{\alpha}-\mathbf{k})$, as ξ_2 tends to $+\infty$, w_1 tends to a finite constant say w_1', and ξ_2-w_2 tends to a finite constant say w_2'. Indeed, we have

$$|\xi_1-w_1'|\leq D_{11}(1-\log|\alpha|),|w_2'|\leq D_{11}(1-\log|\alpha|),\tag{3.35}$$

where D_{11} is independent of α and h.

When $\xi_2\to+\infty$, $|e^{-2\pi i\alpha L_{\alpha\ltimes h}(\xi)}|\to+\infty$. Hence, $e^{-2\pi i\alpha L_{\alpha\ltimes h}(\xi)}$ and $-1+e^{-2\pi i\alpha L_{\alpha\ltimes h}(\xi)}$ have the same argument modulo 2π as $\xi_2\to+\infty$. Therefore,

$$\begin{aligned}
&\lim_{\xi_2\to+\infty}\big(\arg(\tau_{\alpha\ltimes h}\circ L_{\alpha\ltimes h}(\xi))+2\pi\xi_2\,\mathrm{Im}\,\alpha\big)\\
&\quad=\arg\sigma_{\alpha\ltimes h}+\lim_{\xi_2\to+\infty}\big(\arg\frac{1}{1-e^{-2\pi i\alpha L_{\alpha\ltimes h}(\xi)}}+2\pi w_2\,\mathrm{Im}\,\alpha\big)+\lim_{\xi_2\to+\infty}2\pi(\xi_2-w_2)\,\mathrm{Im}\,\alpha\\
&\quad=\arg\sigma_{\alpha\ltimes h}+\lim_{\xi_2\to+\infty}\big(\pi-2\pi w_2\,\mathrm{Im}\,\alpha+2\pi w_1\,\mathrm{Re}\,\alpha+2\pi w_2\,\mathrm{Im}\,\alpha\big)+2\pi w_2'\,\mathrm{Im}\,\alpha\\
&\quad=\arg\sigma_{\alpha\ltimes h}+\pi+2\pi w_1'\,\mathrm{Re}\,\alpha+2\pi w_2'\,\mathrm{Im}\,\alpha\\
&\quad=\arg\sigma_{\alpha\ltimes h}+2\pi\xi_1\,\mathrm{Re}\,\alpha+\pi+2\pi(w_1'-\xi_1)\,\mathrm{Re}\,\alpha+2\pi w_2'\,\mathrm{Im}\,\alpha.
\end{aligned}$$

We define $c_{\alpha\ltimes h}$ as $\pi+2\pi(w_1'-\xi_1)\,\mathrm{Re}\,\alpha+2\pi w_2'\,\mathrm{Im}\,\alpha$. Since $\mathrm{Re}\,\alpha\in(0,1/2)$ and $\mathrm{Im}\,\alpha\in[-\sqrt{2}/4,\sqrt{2}/4]$, by Equation (3.35), $|c_{\alpha\ltimes h}|$ is bounded from above by a uniform constant times $(1-\log|\alpha|)$. □

Proof of Proposition 2.8– Part a). Recall the set $S^t_{\alpha\ltimes h}$ defined in Section 2.4. By the definition, $\Phi_{\alpha\ltimes h}(S^t_{\alpha\ltimes h})$ is contained in the set $\{\xi\in\mathbb{C}\mid\mathrm{Re}\,\xi\in(0,\mathrm{Re}\,\frac{1}{\alpha}-\mathbf{k})\}$. First we note that the projection of this set onto the real line must have uniformly bounded diameter, independent of α and h. That is because, by Theorem 2.9 and the Koebe distortion theorem, the set of maps $\mathcal{R}_{\text{NP-t}}(\alpha\ltimes h)$, over all $\alpha\in A^+(r_3)$ and $h\in\mathcal{F}_0\cup\{Q_0\}$, forms a compact class of map. In particular, the pre-image of a straight ray landing at 0 under any of these maps, spirals at most a uniformly bounded number of times about 0. Lifting this property by $\mathbb{E}\mathrm{xp}^t$, we conclude that $\Phi_{\alpha\ltimes h}(S^t_{\alpha\ltimes h})$ must have a uniformly bounded horizontal width. As $k^t_{\alpha\ltimes h}$ is chosen as the smallest positive integer satisfying Proposition 2.6, $\Phi_{\alpha\ltimes h}(S^t_{\alpha\ltimes h})$ must be contained in the set

$$\{\xi\in\mathbb{C}\mid\mathrm{Re}\,\xi\in(\mathrm{Re}\,\frac{1}{\alpha}-\mathbf{k}-\delta,\mathrm{Re}\,\frac{1}{\alpha}-\mathbf{k})\},$$

for some δ independent of α and h.

On the other hand, $k^t_{\alpha\ltimes h}$ is the number of iterates by $F_{\alpha\ltimes h}$ required to go from $L_{\alpha\ltimes h}\circ\Phi_{\alpha\ltimes h}(S^t_{\alpha\ltimes h})$ to $L_{\alpha\ltimes h}(\{\xi\in\mathbb{C}\mid\mathrm{Re}\,\xi\in[1/2,3/2]\})+1/\alpha$. By Proposition 3.19, $L_{\alpha\ltimes h}\circ\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h})$ is bounded by two curves that are asymptotically vertical near the top end. Sine $L'_{\alpha\ltimes h}$ tends to $+1$ near the top, see Lemma 3.15, the width of the top end of $L_{\alpha\ltimes h}\circ$

$\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h})$ tends to $\operatorname{Re}\frac{1}{\alpha}-\mathbf{k}$. By the same lemma, $L_{\alpha\ltimes h}(\Phi_{\alpha\ltimes h}(S^t_{\alpha\ltimes h}))$ is contained within uniformly bounded distance from the left side of $L_{\alpha\ltimes h}\circ\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h})$. The lift $F_{\alpha\ltimes h}$ is uniformly close to the translation by one. Thus, the number of iterates by $F_{\alpha\ltimes h}$ required to go from $L_{\alpha\ltimes h}\circ\Phi_{\alpha\ltimes h}(S_{\alpha\ltimes h})$ to $L_{\alpha\ltimes h}(\{\xi\in\mathbb{C}\mid \operatorname{Re}\xi\in[1/2,3/2]\})+1/\alpha$ is uniformly bounded from above. That is, $k^t_{\alpha\ltimes h}$ is uniformly bounded from above, independent of α and h. □

We shall prove the other half of the proposition, the uniform bound on k^b_f at the end of Section 3.9.

3.8. **Ecalle map and its dependence on α.** Recall the notation $h_\alpha=\alpha\ltimes h$, as well as let $S^t_{\alpha\ltimes h}$ and $S^b_{\alpha\ltimes h}$ denote the sectors $S^t_{h_\alpha}$ and $S^b_{h_\alpha}$, respectively, defined in Section 2.4. Similarly, let $k^t_{\alpha\ltimes h}$ denote the positive integer introduced in Propositions 2.6, for the map $f=h_\alpha$. The map

$$E^t_{\alpha\ltimes h}=\Phi_{\alpha\ltimes h}\circ h_\alpha^{\circ k^t_{\alpha\ltimes h}}\circ\Phi^{-1}_{\alpha\ltimes h}:\Phi_{\alpha\ltimes h}(S^t_{\alpha\ltimes h})\to\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h}),$$

induces, via the projection $\mathbb{E}\mathrm{xp}^t(\xi)=(-4/27)e^{2\pi i\xi}$, the renormalisation $\mathcal{R}_{\text{NP-t}}(\alpha\ltimes h)$. Recall the domains $V\Subset U$ introduced in Section 2. By Theorem 2.9, $\mathcal{R}_{\text{NP-t}}(\alpha\ltimes h)$ has a restriction to a domain that belongs to the class $\{-1/\alpha\}\ltimes\mathcal{F}_0$. With the notations in Equation (3.4), this implies that

$$e^{2\pi i/\alpha}\hat{\psi}_{\alpha\ltimes h}(V)\subseteq\mathbb{E}\mathrm{xp}^t(\Phi_{\alpha\ltimes h}(S^t_{\alpha\ltimes h})).$$

By Theorem 2.9, $\hat{\psi}_{\alpha\ltimes h}$ has univalent extension onto U. Let V' be an arbitrary Jordan neighbourhood of 0, cf. Proposition 3.8, such that

$$V\Subset V'\Subset U. \tag{3.36}$$

The set $e^{2\pi i/\alpha}\hat{\psi}_{\alpha\ltimes h}(V')$ may, or may not, contain $\mathbb{E}\mathrm{xp}^t(\Phi_{\alpha\ltimes h}(S^t_{\alpha\ltimes h}))$.

By the above paragraph, there is a connected set $X_{\alpha\ltimes h}\subset\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h})$, that is equal to the set $\Phi_{\alpha\ltimes h}(S^t_{\alpha\ltimes h})$ above some horizontal line, and projects under $\mathbb{E}\mathrm{xp}^t$ onto $V'\setminus\{0\}$. Moreover, the map $E^t_{\alpha\ltimes h}$ has holomorphic extension onto $X_{\alpha\ltimes h}$ with $E_{\alpha\ltimes h}(X_{\alpha\ltimes h})\subset\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h})$.

In the pre-Fatou coordinate, $E_{\alpha\ltimes h}:X_{\alpha\ltimes h}\to\mathbb{C}$ corresponds to the map

$$I^t_{\alpha\ltimes h}=L_{\alpha\ltimes h}\circ E^t_{\alpha\ltimes h}\circ L^{-1}_{\alpha\ltimes h}:L_{\alpha\ltimes h}(X_{\alpha\ltimes h})\to\operatorname{Dom}F_{\alpha\ltimes h}.$$

A key point here is that $I_{\alpha\ltimes h}$ is given by a uniformly bounded number of iterates of $F_{\alpha\ltimes h}$ plus a translation. This is stated in the next lemma.

Lemma 3.23. *There exists a constant $D_{15}>0$ such that for all $\alpha\in A^+(r_3')$ and all $h\in\mathcal{F}_0\cup\{Q_0\}$, we have the following:*

a) for all $w\in L_{\alpha\ltimes h}(X_{\alpha\ltimes h})$,

$$I^t_{\alpha\ltimes h}(w)=F_{\alpha\ltimes h}^{\circ k^t_{\alpha\ltimes h}}(w)-\frac{1}{\alpha}.$$

b) for all $w\in L_{\alpha\ltimes h}(X_{\alpha\ltimes h})$,

$$\left|\frac{\partial F_{\alpha\ltimes h}^{\circ k^t_{\alpha\ltimes h}}}{\partial\alpha}(w)\right|\le D_{15}.$$

Proof. Recall the covering map $\tau_{\alpha\ltimes h} = \Phi^{-1}_{\alpha\ltimes h} \circ L^{-1}_{\alpha\ltimes h}$. For all $w \in L_{\alpha\ltimes h}(X_{\alpha\ltimes h})$, we have

$$\begin{aligned}\tau_{\alpha\ltimes h} \circ I^t_{\alpha\ltimes h}(w) &= h_\alpha^{\circ k^t_{\alpha\ltimes h}} \circ \Phi^{-1}_{\alpha\ltimes h} \circ L^{-1}_{\alpha\ltimes h}(w)\\ &= h_\alpha^{\circ k^t_{\alpha\ltimes h}} \circ \tau_{\alpha\ltimes h}(w)\\ &= \tau_{\alpha\ltimes h} \circ F^{\circ k^t_{\alpha\ltimes h}}_{\alpha\ltimes h}(w).\end{aligned}$$

Since $\tau_{\alpha\ltimes h}$ is $1/\alpha$-periodic, the above equality implies that the difference between $I^t_{\alpha\ltimes h}(w)$ and $F^{\circ k^t_{\alpha\ltimes h}}_{\alpha\ltimes h}(w)$ is equal to a constant in $\mathbb{Z}/\alpha$, and the value of the constant is independent of w. However, since $F_{\alpha\ltimes h}$ is asymptotically equal to the translation by one near $+i\infty$, $F^{\circ k^t_{\alpha\ltimes h}}_{\alpha\ltimes h}(w)$ belongs to $\hat{\mathcal{P}}_{\alpha\ltimes h} + 1/\alpha$. Thus, the difference is equal to $1/\alpha$. This finishes the proof of the first part of the lemma.

The positive integer $k^t_{\alpha\ltimes h}$ is uniformly bounded from above independent of α and h. This part of the Proposition 2.8 is proved earlier. On the other hand, by Lemma 3.13, $\partial_\alpha F_{\alpha\ltimes h}$ is uniformly bounded from above on Dom $F_{\alpha\ltimes h}$, and by Lemma 3.12, $|\partial_w F_{\alpha\ltimes h}|$ is also uniformly bounded from above on $L_{\alpha\ltimes h}(X_{\alpha\ltimes h})$. This implies the second part of the lemma. □

Proof of Proposition 3.8- Part a. Let V' be a Jordan neighbourhood of 0 satisfying (3.36) and assume $X_{\alpha\ltimes h}$ is the lift of V' defined in the paragraph after Equation (3.36).

For w in $X_{\alpha\ltimes h} - 1/\alpha$ we have

$$\begin{aligned}E^t_{\alpha\ltimes h} \circ T_{1/\alpha} &= L^{-1}_{\alpha\ltimes h} \circ I^t_{\alpha\ltimes h} \circ L_{\alpha\ltimes h} \circ T_{1/\alpha}\\ &= L^{-1}_{\alpha\ltimes h} \circ T_{-1/\alpha} \circ F^{k^t_{\alpha\ltimes h}}_{\alpha\ltimes h} \circ L_{\alpha\ltimes h} \circ T_{1/\alpha}\\ &= L^{-1}_{\alpha\ltimes h} \circ F^{k^t_{\alpha\ltimes h}}_{\alpha\ltimes h} \circ T_{-1/\alpha} \circ L_{\alpha\ltimes h} \circ T_{1/\alpha}.\end{aligned}$$

Let us fix a constant $C > 0$ large enough such that $\mathbb{E}\mathrm{xp}^t\{w \in X_{\alpha\ltimes h} \mid |\operatorname{Im} w| \le C\}$ contains the annulus $V' \setminus V$. The existence of a uniform C, independent of α and h, is guaranteed by the Koebe distortion Theorem applied to the map $\hat{\psi}_{\alpha\ltimes h} : U \to \mathbb{C}$. Now, assume that w belongs to $X_{\alpha\ltimes h} - 1/\alpha$, and $|\operatorname{Im}(w - 1/\alpha)| \le C$. Then, by Proposition 3.22, there is a constant D_{14}, depending only on C, such that $|\partial(T_{-1/\alpha} \circ L_{\alpha\ltimes h} \circ T_{1/\alpha})(w)/\partial\alpha|$ is uniformly bounded from above. By Lemma 3.23, $|\partial F^{k^t_{\alpha\ltimes h}}_{\alpha\ltimes h}/\partial\alpha|$ is uniformly bounded from above. Also, since $k^t_{\alpha\ltimes h}$ is uniformly bounded from above, see Proposition 2.8, the iterates $F^{k^t_{\alpha\ltimes h}}_{\alpha\ltimes h}$ displace a point by a uniformly bounded amount. Thus, we may apply Proposition 3.22, with a constant D'_{14} depending only on C and the uniform bound on $k^t_{\alpha\ltimes h}$, to conclude that $|\partial L^{-1}_{\alpha\ltimes h}/\partial\alpha|$ is uniformly bounded from above at $F^{k^t_{\alpha\ltimes h}}_{\alpha\ltimes h} \circ T_{-1/\alpha} \circ L_{\alpha\ltimes h} \circ T_{1/\alpha}(w)$. Combining these argument we conclude that for every w in $X_{\alpha\ltimes h}$ with $|\operatorname{Im} w| \le C$,

$$\Big|\frac{\partial}{\partial\alpha}(E^t_{\alpha\ltimes h} \circ T_{\alpha\ltimes h}(w))\Big|$$

is uniformly bounded from above. The map $E_{\alpha\ltimes h} \circ T_{1/\alpha}$ projects via $\mathbb{E}\mathrm{xp}^t$ to the map $\hat{\psi}_{\alpha\ltimes h}$. Therefore, $|\partial\hat{\psi}_{\alpha\ltimes h}/\partial\alpha|$ must be uniformly bounded from above on $V' \setminus V$. By the maximum principle, this it must be uniformly bounded from above on V'. □

3.9. **Analysis of the bottom NP-renormalisation.** For $h \in \mathcal{F}_0 \cup \{Q_0\}$ and $\alpha \in A^+(r_3)$, we continue to use the notation $(\alpha \ltimes h)$ for the map h_α defined as $h_\alpha(z) = h(e^{2\pi i\alpha} z)$. By Proposition 2.2, there is a Jordan domain, $\mathcal{P}_{\alpha\ltimes h}$ and a conformal change of coordinate $\Phi_{\alpha\ltimes h} : \mathcal{P}_\alpha \to \mathbb{C}$ conjugating the dynamics of h_α on $\mathcal{P}_{\alpha\ltimes h}$ to the translation by one. Let $\sigma_{\alpha\ltimes h}$ denote the non-zero fixed point of h_α obtained in Proposition 2.1. Recall that the complex rotation of h_α at $\sigma_{\alpha\ltimes h}$ is denoted by $\beta = \beta(\alpha \ltimes h)$, that is, $h'_\alpha(\sigma_{\alpha\ltimes h}) = e^{2\pi i\beta}$.

Consider the covering

$$\check{\tau}_{\alpha\ltimes h}(w) = \frac{\sigma_{\alpha\ltimes h}}{1 - e^{-2\pi i\beta w}}.$$

This is periodic of period $1/\beta$, where $+i\infty$ corresponds to 0 and $-i\infty$ corresponds to $\sigma_{\alpha\ltimes h}$.

The petal $\mathcal{P}_{\alpha\ltimes h}$ lifts under $\check{\tau}_{\alpha\ltimes h}$ to a periodic set, one of its connected components separates 0 from $-1/\beta$. Note that when $\alpha \in A^+(r_3)$, by Formula (2.2), $\operatorname{Re}(-1/\beta) \geq 0$. We denote this component by $\check{\mathcal{P}}_{\alpha\ltimes h}$. The map h_α on $\mathcal{P}_{\alpha\ltimes h}$ lifts under $\check{\tau}_{\alpha\ltimes h}$ to a univalent map $\check{F}_{\alpha\ltimes h}$ defined on $\check{\tau}_{\alpha\ltimes h}^{-1}(\mathcal{P}_{\alpha\ltimes h})$. This lift satisfies,

$$h_\alpha \circ \check{\tau}_{\alpha\ltimes h}(w) = \check{\tau}_{\alpha\ltimes h} \circ \check{F}_{\alpha\ltimes h}(w), \quad \check{F}_{\alpha\ltimes h}(w + 1/\beta) = \check{F}_{\alpha\ltimes h}(w) + 1/\beta, \quad w \in \check{\tau}_{\alpha\ltimes h}^{-1}(\mathcal{P}_{\alpha\ltimes h}),$$

and is given by the formula,

$$\check{F}_{\alpha\ltimes h}(w) = w + \frac{1}{2\pi i\beta} \log\big(1 - \frac{\sigma_{\alpha\ltimes h} u_{\alpha\ltimes h}(z)}{1 + z u_{\alpha\ltimes h}(z)}\big), \text{with } z = \check{\tau}_{\alpha\ltimes h}(w).$$

As in the previous case, we work with the branch of log with $\operatorname{Im}\log(\cdot) \subseteq (-\pi, +\pi)$. With this choice, $\check{F}_{\alpha\ltimes h}$ is asymptotic to a translation by $+1$ near the lower end,

$$\lim_{\operatorname{Im}(\beta w)\to+\infty} |\check{F}_{\alpha\ltimes h}(w) - (w + 1)| = 0.$$

Note that $\operatorname{Im}(\beta w) \to +\infty$ corresponds to tending to the lower end of $\check{\mathcal{P}}_{\alpha\ltimes h}$, due to the sign of β.

The unique critical point of h_α lifts under $\check{\tau}_{\alpha\ltimes h}$ to a $1/\beta$-periodic set of points, one of which lies on $\check{\mathcal{P}}_{\alpha\ltimes h}$ and is denoted by $\check{\mathrm{cp}}_{\alpha\ltimes h}$.

One may repeat all the constructions and arguments in Sections 3.4 to 3.8, replacing α by $-\beta$. That is, the analysis is now carried out near the lower end of the domain $\check{\mathcal{P}}_{\alpha\ltimes h}$. This provides us with a proof for part b of Proposition 3.8, and a proof for part b of Proposition 2.8. Note that by the holomorphic index formula, and the pre-compactness of the class $\mathcal{F}_0$, $|1/\alpha + 1/\beta|$ is uniformly bounded from above, see Lemma 3.24. We give a proof of part b of Proposition 2.4, where there is a slight difference between the calculations.

Proof of Proposition 2.4– Part b). Let $\check{L}_{\alpha\ltimes h}$ be the univalent map (analogue of $L_{\alpha\ltimes h}$) that conjugates the translation by one to the map $\check{F}_{\alpha\ltimes h}$, which is normalised by mapping 0 to $\check{\mathrm{cp}}_{\alpha\ltimes h}$. We use the decomposition of the map $\Phi_{\alpha\ltimes h}^{-1}$ as $\check{\tau}_{\alpha\ltimes h} \circ \check{L}_{\alpha\ltimes h}$. where $\check{\tau}_{\alpha\ltimes h}$ is the covering map defined above.

Let $\xi = \xi_1 + i\xi_2$, and $w = w_1 + iw_2 = \check{L}_{\alpha\ltimes h}(\xi)$, where w_1, w_2, ξ_1, and ξ_2 are real numbers. As ξ_2 tends to $-\infty$, w_2 tends to $-\infty$. Let w'_1 denote the limit of w_1 as ξ_2 tends to $-\infty$, and let w'_2 denote the limit of $\xi_2 - w_2$, as ξ_2 tends to $-\infty$. By the analogue of Lemma 3.19 for $\check{L}_{\alpha\ltimes h}$ near the bottom end, we have

$$|w'_2| \leq D_{11}(1 - \log|\alpha|), \quad |\xi_1 - w'_1| \leq D_{11}(1 - \log|\alpha|).$$

When $\xi_2 \to -\infty$, $1 - e^{-2\pi i\beta \check{L}_{\alpha\ltimes h}(\xi)} \to 1$. Then, the limit in part b of the proposition may be calculates as,

$$\begin{aligned}
&\lim_{\xi_2\to-\infty} \big(\arg(\check{\tau}_{\alpha\ltimes h} \circ \check{L}_{\alpha\ltimes h}(\xi) - \sigma_{\alpha\ltimes h}) - 2\pi\xi_2 \operatorname{Im}\beta\big) \\
&\quad= \arg\sigma_{\alpha\ltimes h} + \lim_{\xi_2\to-\infty} \Big(\arg \frac{e^{-2\pi i\beta \check{L}_{\alpha\ltimes h}(\xi)}}{1-e^{-2\pi i\beta \check{L}_{\alpha\ltimes h}(\xi)}} - 2\pi w_2 \operatorname{Im}\beta\Big) + \lim_{\xi_2\to-\infty} 2\pi(w_2-\xi_2)\operatorname{Im}\beta \\
&\quad= \arg\sigma_{\alpha\ltimes h} + \lim_{\xi_2\to-\infty} \big(2\pi w_2 \operatorname{Im}\beta - 2\pi w_1 \operatorname{Re}\beta - 2\pi w_2 \operatorname{Im}\beta\big) - 2\pi w_2' \operatorname{Im}\beta \\
&\quad= \arg\sigma_{\alpha\ltimes h} - 2\pi w_1' \operatorname{Re}\beta - 2\pi w_2' \operatorname{Im}\beta \\
&\quad= \arg\sigma_{\alpha\ltimes h} - 2\pi \xi_1 \operatorname{Re}\beta + 2\pi(\xi_1 - w_1')\operatorname{Re}\beta - 2\pi w_2' \operatorname{Im}\beta.
\end{aligned}$$

□

3.10. **Pairs of complex rotations.** For $h \in \mathcal{F}_0 \cup \{Q_0\}$ and $\alpha \in A(r_1)$, the map $\alpha \ltimes h$ has a non-zero fixed point in W denoted by $\sigma(\alpha \ltimes h)$; see Proposition 2.1. This fixed point has holomorphic dependence on h and α. Moreover, when $\alpha \in A(r_3)$, $\alpha \ltimes h$ is renormalisable. It follows from the definition of renormalisation that $\arg(\alpha \ltimes h)'(\sigma(\alpha \ltimes h)) \neq 0$.

Hence, there is a choice of $\beta(\alpha \ltimes h)$, with $\operatorname{Re}\beta(\alpha \ltimes h) \in (-1, 0)$ and $\beta(\alpha \ltimes h)$ holomorphic in h and α, such that $(\alpha \ltimes h)'(\sigma(\alpha \ltimes h)) = e^{2\pi i\beta(\alpha\ltimes h)}$. Moreover, as α tends to zero in $A(r_3)$, $\beta(\alpha \ltimes h)$ tends to 0 in a sector. See Lemma 3.25 for further details.

The function

$$I(\alpha \ltimes h) = \frac{1}{2\pi i}\int_{\partial W} \frac{1}{z - (\alpha \ltimes h)(z)} \mathrm{d}z, \quad \alpha \in A(r_1), h \in \mathcal{F}_0 \cup \{Q_0\},$$

is holomorphic in h and α.

Lemma 3.24. *There exist positive constants B_1, B_2, B_3 such that for all $h_1, h_2 \in \mathcal{F}_0$ and α in $A(r_1)$ we have*

a) $|I(\alpha \ltimes h_1)| \leq B_1$;
b) $|\frac{\partial}{\partial\alpha} I(\alpha \ltimes h_1)| \leq B_2$;
c) $|I(\alpha \ltimes h_1) - I(\alpha \ltimes h_2)| \leq B_3 \, \mathrm{d}_{\mathrm{Teich}}(h_1, h_2)$.

Proof. Part a) By Proposition 2.1, h_α has no fixed point on ∂W. Thus, by the pre-compactness of the class of maps $\mathcal{F}_0$, there is $\delta > 0$ such that for all $z \in \partial W$, $|z - h_\alpha(z)| \geq \delta$. Hence, $|I(\alpha \ltimes h)(z)| \leq \ell(W)/(2\pi\delta)$, where $\ell(W)$ denotes the circumference of W.

Part b) First note that by the Koebe distortion Theorem, $|h'|$ is uniformly bounded from above on $e^{2\pi i\alpha}\overline{W}$. Thus,

$$\begin{aligned}
|\frac{\partial}{\partial\alpha} I(\alpha \ltimes h)| &\leq \frac{1}{2\pi}\int_{\partial W} \frac{|\frac{\partial}{\partial\alpha} h_\alpha(z)|}{|z - h_\alpha(z)|^2}\, dz \\
&\leq \frac{\ell(W)}{2\pi\delta^2} \sup_{z\in W} \Big(|h'(e^{2\pi i\alpha} z)| 2\pi |e^{2\pi i\alpha}||z|\Big).
\end{aligned}$$

Part c) For this part of the lemma we use the majorant principle. Fix h_1 and h_2 in $\mathcal{F}_0$ and let $R = \mathrm{d}_{\mathrm{Teich}}(h_1, h_2)$. For $i = 1, 2$, let $h_i = P \circ \psi_i^{-1}$, where $\psi_i : V \to \mathbb{C}$ is a univalent map with $\psi_i(0) = 0$, $\psi_i'(0) = 1$, and ψ_i has a quasi-conformal extension onto $\mathbb{C}$. Then, by the definition of $\mathrm{d}_{\mathrm{Teich}}$, and the compactness of the class of normalised quasi-conformal mappings with dilatation bounded from above by a constant, there is a quasi-conformal mapping $\psi : \mathbb{C} \to \mathbb{C}$, which is identical to $\psi_1 \circ \psi_2^{-1}$ on $\psi_2(V)$, and $\log \operatorname{Dil}(\psi) = R$.

Let us define the dilatation quotient $\mu(z) = \partial_{\overline{z}}\psi/\partial_z\psi$, and let $r = \|\mu\|_\infty < 1$. Then, we have $(1+r)/(1-r) = e^R$. For each $\lambda \in B(0, 1/r)$, let $\psi^\lambda : \mathbb{C} \to \mathbb{C}$ denote the unique quasi-conformal mapping with dilatation quotient $\lambda\mu$ normalised with $\psi^\lambda(0) = 0$ and $(\psi^\lambda)'(0) = 1$. That is, ψ^λ is the unique solution of the Beltrami equation $\partial_{\overline{z}}\psi^\lambda = (\lambda\mu)\partial_z\psi^\lambda$ with the normalisation at 0. By the classical results on Beltrami equation, see for example [AB60], the map ψ^λ has holomorphic dependence on λ. For $\lambda = 1$, we have $\psi^1 = \psi_1 \circ \psi_2^{-1}$.

Define the holomorphic map $h^\lambda = P \circ \psi_1^{-1} \circ \psi^\lambda$. By definition, $h^0 = h_1$ and $h^1 = h_2$, and h^λ has holomorphic dependence on λ. Now consider the holomorphic map

$$G(\lambda) = I(\alpha \ltimes h_1) - I(\alpha \ltimes h^\lambda), \lambda \in B(0, 1/r).$$

We have $G(0) = 0$, and by part a of the lemma, $|G| \le 2B_1$, on $B(0, 1/r)$. Then, by the Schwarz lemma, $|G(1)| \le 2B_1 r$.

When $\mathrm{d}_{\mathrm{Teich}}(h_1, h_2) \ge 2B_1$, the left hand side of the inequality in Part c is bounded by $2B_1$. So, the inequality holds for $B_3 = 1$. On the other hand, when $\mathrm{d}_{\mathrm{Teich}}(h_1, h_2) = R \le 2B_1$, by the relation $(1+r)/(1-r) = e^R$, R and r are comparable. This finishes the proof of part c). □

Lemma 3.25. *There exist positive constants B_4, B_5, B_6 such that for all $h_1, h_2 \in \mathcal{F}_0$ and α in $A(r_3)$ we have*

1) $B_4^{-1}|\alpha| \le |\beta(\alpha \ltimes h_1)| \le B_4|\alpha|$
2) $B_5^{-1} \le |\frac{\partial\beta}{\partial\alpha}(\alpha \ltimes h_1)| \le B_5$
3) $|\beta(\alpha \times h_1) - \beta(\alpha \ltimes h_2)| \le B_6|\alpha|^2 \,\mathrm{d}_{\mathrm{Teich}}(h_1, h_2)$

Proof. Part 1) For $\alpha \in A(r_3)$, $|e^{2\pi i\alpha}|$ is uniformly bounded from above and away from 0. By the Koebe distortion theorem, for any univalent map $\varphi : V \to \mathbb{C}$, with $\varphi(0) = 0$ and $\varphi'(0) = 1$, $|\varphi'|$ is uniformly bounded from above and away from 0 on W. Combining the two statements we conclude that $|e^{2\pi i\beta(\alpha\ltimes h)}|$ must be uniformly bounded from above and away from 0. In particular, $\operatorname{Im}\beta(\alpha \ltimes h)$ is uniformly bounded from above and below.

Let us define the holomorphic map G according to $G(z)z = 1 - e^{2\pi iz}$, for $z \in \mathbb{C}$. By the above paragraph, $|G(\beta(\alpha \ltimes h))|$ and $|G(\alpha)|$ are uniformly bounded from above and below. Then, the holomorphic index formula may be written as

$$\alpha G(\alpha) + \beta(\alpha \ltimes h)G(\beta(\alpha \ltimes h)) = \alpha\beta(\alpha \ltimes h)G(\alpha)G(\beta(\alpha \ltimes h))I(\alpha \ltimes h).$$

By the uniform bound in Lemma 3.24-a, we conclude the two estimates in the first part.

Part 2) We differentiate the index formula in Equation (2.2) with respect to α to obtain

$$\frac{e^{2\pi i\alpha}}{(1 - e^{2\pi i\alpha})^2} + \frac{e^{2\pi i\beta(\alpha\ltimes h)}}{(1 - e^{2\pi i\beta(\alpha\ltimes h)})^2}\frac{\partial}{\partial\alpha}\beta(\alpha \ltimes h) = \frac{1}{2\pi i}\frac{\partial}{\partial\alpha}I(\alpha \ltimes h).$$

Using the formula $\sin z = (e^{iz} - e^{-iz})/(2i)$, this reduces to

$$\frac{1}{\sin^2(\pi\alpha)} + \frac{1}{\sin^2(\pi\beta(\alpha \ltimes h))}\frac{\partial}{\partial\alpha}\beta(\alpha \ltimes h) = -\frac{2}{\pi i}\frac{\partial}{\partial\alpha}I(\alpha \ltimes h).$$

Thus,

$$\frac{\partial}{\partial\alpha}\beta(\alpha \ltimes h) = \frac{-\sin^2(\pi\beta(\alpha \ltimes h))}{\sin^2(\pi\alpha)} - \frac{2}{\pi i}\sin^2(\pi\beta(\alpha \ltimes h))\frac{\partial}{\partial\alpha}I(\alpha \ltimes h)$$

The upper bound in part 2 follows from the upper bound in Lemma 3.24-2 and the uniform bounds in the previous part. To obtain the lower bound, we need to restrict α to a smaller region. First note that by the bounds in part 1, the first ration on the right-hand side of the above formula is compactly contained in $\mathbb{C} \setminus \{0\}$. The, by restricting $\alpha \in A(r_4)$, for some $r_4 \in (0, +\infty)$, we guarantee that the second ration is small enough. This implies the for $\alpha \in A(r_4)$, $|\frac{\partial}{\partial\alpha}\beta(\alpha \ltimes h)|$ is uniformly bounded away from 0.

Part 3) We subtract the holomorphic index formula for the maps $(\alpha \ltimes h_1)$ and $(\alpha \ltimes h_2)$ to get

$$\frac{1}{\beta(\alpha \ltimes h_1)G(\beta(\alpha \ltimes h_1))} - \frac{1}{\beta(\alpha \ltimes h_2)G(\beta(\alpha \ltimes h_2))} = I(\alpha \ltimes h_1) - I(\alpha \ltimes h_2),$$

which provides us with

$$\beta(\alpha \ltimes h_2)G(\beta(\alpha \ltimes h_2)) - \beta(\alpha \ltimes h_1)G(\beta(\alpha \ltimes h_1)) = \\ \beta(\alpha \ltimes h_1)\beta(\alpha \ltimes h_2)G(\beta(\alpha \ltimes h_1))G(\beta(\alpha \ltimes h_2))\big(I(\alpha \ltimes h_1) - I(\alpha \ltimes h_2)\big).$$

Since $\beta(\alpha \ltimes h)$ is uniformly bounded, $|G(\beta(\alpha \ltimes h_1)) - G(\beta(\alpha \ltimes h_2))| = O(|\beta(\alpha \ltimes h_1) - \beta(\alpha \ltimes h_2)|)$. Also, by the estimates in part 1, $|\beta(\alpha \ltimes h_1)|$ and $|\beta(\alpha \ltimes h_2)|$ are bounded by $B_4|\alpha|$. Combining these with the above formula, one obtains the desired inequality using the bound in part 3 of Lemma 3.24. □

Proof of proposition 3.3. This mainly follows from Theorem 2.9. For each fixed $\alpha \in A(r_3)$, the maps $h \mapsto \hat{h}(\alpha \ltimes h)$ and $h \mapsto \check{h}(\alpha \ltimes h)$ induce holomorphic mappings from the Teichmuler space of $\mathbb{C} \setminus \overline{V}$ to the Teichmuller space of $\mathbb{C} \setminus \overline{V}$. By the Royden-Gardiner theorem, any holomorphic map of Teichmuller spaces does not expand distances.

Indeed, by the holomorphic extension property in Theorem 2.9 the image of this map is a compact subset of the Teichmuller space of $\mathbb{C} \setminus \overline{V}$. It follows that this map is uniformly contracting, that is, $c_{2,2} < 1$, but we do not need this feature in this paper. □

Remark 3.26. By Theorem 2.9, the maps $\hat{h}(\alpha \ltimes h)$ and $\check{h}(\alpha \ltimes h)$ extend onto holomorphic maps on the strictly larger domain U, which contains the closure of V. This may be used to prove the existence of a constant $c_{2,2}$ which is strictly less than 1. However, we do not need this uniform contraction in this paper.

Proof of proposition 3.4. Recall the relation $\check{\alpha}(\alpha \ltimes h) = -1/\beta(\alpha \ltimes h)$. From the estimates in Lemma 3.25, we have

$$|\frac{\partial\check{\alpha}}{\partial\alpha}(\alpha \ltimes h)| = |\frac{1}{\beta(\alpha \ltimes h)}|^2|\frac{\partial\beta}{\partial\alpha}(\alpha \ltimes h)| \geq \frac{1}{B_4^2|\alpha|^2}B_5^{-1}.$$

□

Proof of Proposition 3.5. Here we use the estimates in Lemma 3.25.

$$\begin{aligned}|\check{\alpha}(\alpha \ltimes h_1) - \check{\alpha}(\alpha \ltimes h_2)| &= |\frac{-1}{\beta(\alpha \ltimes h_1)} + \frac{1}{\beta(\alpha \ltimes h_2)}| \\ &\leq \frac{B_4^2}{|\alpha|^2} |\beta(\alpha \ltimes h_1) - \beta(\alpha \ltimes h_2)| \\ &\leq \frac{B_4^2}{|\alpha|^2} B_6 |\alpha^2| \, \mathrm{d}_{\mathrm{Teich}}(h_1, h_2) = B_4^2 B_6 \, \mathrm{d}_{\mathrm{Teich}}(h_1, h_2).\end{aligned}$$

□

4. Polynomial-like renormalisations and their combinatorics

In this section we outline the basic properties of the dynamics of quadratic polynomials that we refer to in this paper. One may consult [Mil06] and [Bea91] for basic notions in complex dynamics. The material on the polynomial-like renormalisation presented here is mainly following the foundational work of Douady and Hubbard presented in [DH84, DH85]. One may also refer to [Mil00] [Sch04] for detailed discussions on the combinatorial aspects of the topic.

In this section, we assume that all quadratic polynomials are normalised so that they are monic and their critical points are at 0. That is, they are of the form $P_c(z) = z^2 + c$.

4.1. Combinatorial rotation of the dividing fixed point.

The **filled Julia** set $K(P_c)$ and the **Julia** set $J(P_c)$ of the quadratic polynomial P_c are defined as

$$K(P_c) = \{z \in \mathbb{C} \mid \sup_{n\in\mathbb{N}} |P_c^{\circ n}(z)| < +\infty\}, J(P_c) = \partial K(P_c).$$

These are compact subsets of $\mathbb{C}$. Each set $K(P_c)$ is connected if and only if the critical point 0 belongs to $K(P_c)$. In this paper we only work with quadratics with connected Julia sets.

By the maximum principle, $K(P_c)$ is full, i.e. its complement has no bounded connected component. The **Böttcher** coordinate of P_c is the conformal isomorphism

$$\varphi_c : \mathbb{C} \setminus K(P_c) \to \mathbb{C} \setminus \overline{\mathbb{D}}, \text{ where } \mathbb{D} = \{w \in \mathbb{C} \mid |w| < 1\},$$

that is tangent to the identity near infinity. It conjugates P_c on $\mathbb{C} \setminus K(P_c)$ to $w \mapsto w^2$ on $\mathbb{C} \setminus \overline{\mathbb{D}}$. By means of this isomorphism, the **external ray** of angle $\theta \in [0, 2\pi)$ and **equipotential** of radius $r \in (1, +\infty)$ are defined as

$$R_c^\theta = \{\varphi_c^{-1}(re^{i\theta}) \mid r \in (1, +\infty)\}, E_c^r = \{\varphi_c^{-1}(re^{i\theta}) \mid \theta \in [0, 2\pi]\}.$$

The map P_c send E_c^r to $E_c^{r^2}$ and send R_c^θ to $R_c^{2\theta}$. An external ray R_c^θ is called **periodic** under P_c, if there is $n \in \mathbb{N}$ with $P_c^{\circ n}(R_c^\theta) = R_c^\theta$. Equivalently, this occurs if $2^n\theta = \theta \mod 2\pi\mathbb{Z}$. An external ray R_c^θ is said to **land** at a point, if the limit of $\varphi_c^{-1}(re^{i\theta})$, as $r \to 1$, exists in $\mathbb{C}$. The landing point of a periodic ray R_c^θ is necessarily a periodic point of P_c.

Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be a holomorphic map with a periodic point $z \in U$. The **multiplier** of z is defined as $(f^{\circ n})'(z)$, where n is the smallest positive integer with $f^{\circ n}(z) = z$. The periodic point is called **parabolic**, if its multiplier is a root of unity.

We recall the Douady's theorem on the landing property of repelling and parabolic periodic points. One may refer to [DH84] for a proof of this result, and also [EL89, Mil06, Pet93, Pom86] for alternative proofs and generalizations of this result.

Proposition 4.1 (Douady, 1987)**.** *Let P_c be a quadratic polynomial with connected $K(P_c)$. Every repelling or parabolic periodic point of P_c is the landing point of at least one, but at most a finite number of, external rays.*

Let a_0 denote the landing point of the unique fixed ray of P_c; R_c^0. When P_c has only one fixed point, a_0 must be parabolic with multiplier 1. This occurs for $c = 1/4$. For $c \neq 1/4$, a_0 is repelling. The other fixed point of P_c, denoted by a_c, is **attracting** in a region called the **main hyperbolic component** of the Mandelbrot set. This is the large cardioid visible in the center of the Mandelbrot set.

For c outside the closure of the main hyperbolic component of M, a_c is **repelling**. By the above proposition there are at least one, but a finite number of, external rays landing on a_c. By a simple topological argument (since the map has only one critical point) the set of rays landing at a_c is formed of the orbit of a single periodic ray. Let $\theta_j \in [0, 2\pi)$, for $1 \leq j \leq q$ and $q \geq 2$, denote the angles of the external rays landing at a_c, labeled in increasing order. There is a non-zero integer $p \in (-q/2, q/2]$, with $(|p|, q) = 1$, such that $P_c(R_c^{\theta_j}) = R_c^{\theta_{j'}}$ where $j' = j + p \pmod q$. The rational number p/q is called the **combinatorial rotation number** of P_c at a_c. The fixed point a_c, when it is repelling or parabolic, is referred to as **the dividing fixed point** of P_c. It follows that for any rational number $p/q \in (-1/2, 1/2]$, there are parameters c in the Mandelbrot set where a_c has combinatorial rotation number p/q at a_c. We shall come back to this in a moment.

4.2. **Polynomial-like renormalisation.** Assume that the fixed point a_c is repelling. The closure of the q rays landing at a_c cut the complex plane into q (open) connected components which we denote by Y_j, for $0 \leq j \leq q-1$. By a simple topological consideration, P_c on these pieces has a simple covering property. One of these components, which we denote by Y_0, contains both critical point 0 and the pre-fixed point $-a_c$, while its image under P_c covers all Y_j, for $0 \leq j \leq q-1$. We may relabel the other components so that $P_c(Y_j) = Y_{j+1}$, for $1 \leq j \leq q-2$, and $P_c(Y_{q-1}) = Y_0$. Thus, the critical point is mapped into Y_1 in one iterate of P_c, and is mapped back into Y_0 under q iterates of P_c.

Fix $r > 1$. The equipotential E_c^r divides each piece Y_j, for $0 \leq j \leq q-1$, into two connected components. We denote by Y_j^1, for $0 \leq j \leq q-1$, the closure of the bounded connected component of $Y_j \setminus E_c^r$. These are called **puzzle pieces** of level 1. The closure of the connected components of $P_c^{-i}(\text{int } Y_j^1)$, for $i \in \mathbb{N}$ and $0 \leq j \leq q-1$, are called puzzle pieces of level $i+1$. They form nests of pieces breaking the Julia set into components. As $P_c(Y_0)$ covers $\cup_{j=0}^q - 1 \, \text{int } Y_j^1$, the connected components of $P_c^{-1}(\text{int } Y_j^1)$ contained in Y_0 divide Y_0 into q pieces. We denote the one containing 0 by Z_0^2 and the remaining ones by Z_j^2, for $1 \leq j \leq q-1$.

For c in the Mandelbrot set with a_c repelling and q rays landing at a_c, P_c is called **polynomial-like renormalisable** of **satellite** type, if $P_c^{\circ(jq)}(0) \in Z_0^2$, for all $j \in \mathbb{N}$. Using $|P_c'(a_c)| > 1$, one builds a simply connected domain $\tilde{Z}_0^2$ containing the closure of Z_0^2 such that $P_c^{\circ q}(\tilde{Z}_0^2)$ contains the closure of $\tilde{Z}_0^2$ and $P_c^{\circ q} : \tilde{Z}_0^q \to P_c^{\circ q}(\tilde{Z}_0^2)$ is a proper branched covering of degree two. We denote this map by $\tilde{\mathcal{R}}_{PL}(P_c)$, and note that the orbit of 0 under $\tilde{\mathcal{R}}_{PL}(P_c)$ remains in $\tilde{Z}_0^2$.

For c as in the above paragraph, if P_c is not polynomial-like renormalisable of satellite type, there is the smallest $n \in \mathbb{N}$ such that $P_c^{\circ(nq)}(0) \in Z_j^2$, for some $1 \leq j \leq q-1$.

Then, let V^1 denote the closure of the connected component of $P_c^{-nq}(\text{int } Z_j^2)$ containing 0 that is obtained by pulling back along the orbit $0, P_c(0), \dots, P_c^{\circ nq}(0)$. If the orbit of 0 under P_c never enters int V^1, then P_c is not polynomial-like renormalisable. Otherwise, let $j_1 \in \mathbb{N}$ be the smallest integer with $P_c^{\circ j_1}(0) \in \text{int } V^1$, and denote by V^2 the closure of the component of $P_c^{-j_1}(\text{int } V^1)$ that is obtained by pulling back along $0, P_c(0), \dots, P_c^{\circ j_1}(0)$. Here, $P_c^{\circ j_1} : \text{int } V^2 \to \text{int } V^1$ is a proper branched covering of degree two. Inductively, we define domains $V^1 \supset V^2 \supset V^3 \supset \dots$, and positive integers $j_1, j_2, j_3, \dots$ such that each j_m is the smallest positive integer with $P_c^{\circ j_m}(0) \in \text{int } V^m$ and V^{m+1} is the closure of the pull-back of int V^m along the orbit $0, P_c(0), \dots, P_c^{\circ j_m}(0)$. For all m, V^{m+1} is compactly contained in the interior of V^m, and $P_c^{\circ j_m} : V^{m+1} \to V^m$ is a proper branched covering of degree two. The quadratic P_c is **polynomial-like renormalisable** of **primitive type**, if there is $m \in \mathbb{N}$ such that the orbit of 0 under $P_c^{\circ j_m} : V^{m+1} \to V^m$ remains in V^{m+1}. Note that if this occurs, the sequence j_m is eventually constant. For the smallest positive integer m satisfying this property, we denoted $P_c^{\circ j_m} : V^{m+1} \to V^m$ by $\tilde{\mathcal{R}}_{PL}(P_c)$. If there is no m with this property, then P_c is not polynomial-like renormalisable.

A quadratic map is called **polynomial-like renormalisable**, if it is polynomial-like renormalisable of either satellite or primitive type.

A **polynomial-like mapping** of degree d is a proper branched covering holomorphic map $f : U \to V$ of degree d, where U and V are simply connected domains with U compactly contained in V. For example, the restriction of any polynomial P to $P^{-1}(B(0, R))$, for sufficiently large R, is a polynomial-like map. The successive renormalisations obtained above are non-trivial examples of polynomial-like maps of degree two. One may define the **filled Julia** set and the **Julia** set of a polynomial-like map in the same fashion;

$$K(f) = \{z \in U \mid f^{\circ j}(z) \in U, \forall j \in \mathbb{N}\}, J(f) = \partial K(f).$$

Similarly, $K(f)$ and $J(f)$ are connected if and only if the orbits of all branched points of f remain in U.

Two polynomial-like mappings $f : U \to V$ and $g : U' \to V'$ are **quasi-conformally** conjugate if there is a quasi-conformal map $h : V \to V'$ such that $g \circ h = h \circ f$ on U. They are called **hybrid** conjugate if they are quasi-conformally conjugate and the quasi-conformal conjugacy h between them may be chosen so that $\overline{\partial} h = 0$ on $K(f)$. A remarkable result of Douady and Hubbard is that the dynamics of a polynomial-like map is the same as the dynamics of some polynomial.

Theorem 4.2 (Straightening [DH85]). *Let $f : U \to V$ be a polynomial-like map of degree d with connected Julia set. Then, f is hybrid conjugate to an appropriate restriction of a polynomial of degree d. Moreover, the polynomial is unique up to an affine conjugacy.*

In the normalised quadratic family $z \mapsto z^2 + c$, $c \in \mathbb{C}$, each affine conjugacy class contains only one element. Thus, every polynomial-like map of degree two is hybrid conjugate to a unique element of this family. Also, although the hybrid conjugacy h in the above theorem is not unique, h is uniquely determined on $J(f)$ upto affine conjugacy.

Recall the polynomial-like renormalisation of satellite type $\tilde{\mathcal{R}}_{PL}(P_c) = P_c^{\circ q} : \tilde{Z}_0^2 \to P_c^{\circ q}(\tilde{Z}_0^2)$ obtained above. Let $M(p/q)$ denote the set of all $c \in M$ such that the dividing fixed point a_c of P_c has combinatorial rotation p/q and $P_c^{\circ jq}(0) \in Z_0^2$, for all $j \in \mathbb{N}$. By the straightening theorem, for all $c \in M(p/q)$, except at c where $P_c'(a_c) = e^{2\pi i p/q}$, $\tilde{\mathcal{R}}_{PL}(P_c)$ is hybrid conjugate

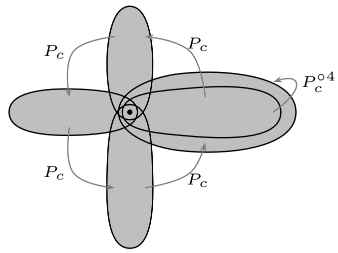

FIGURE 7. A PL-renormalisation of satellite type, $P_c^{\circ 4} : U \to V$. In this example, the combinatorial type of PL-renormalisation is 1/4.

to some quadratic polynomial denoted by $\mathcal{R}_{\text{PL}}(P_c)$. Indeed, through straightening theorem, Douady and Hubbard obtain a homeomorphism[4]

$$\Phi_{p/q} : M(p/q) \to M.$$

The set $M(p/q)$ is called the p/q-**satellite copy** of the Mandelbrot set.

Similarly, for polynomial-like renormalisation of primitive type $\tilde{\mathcal{R}}_{PL}(P_c) = P_c^{\circ j_m} : V^{m+1} \to V^m$, one may consider the connected component containing c of all parameters c where the isotopy type of V^{m+1} remains constant and the Julia set of $\tilde{\mathcal{R}}_{PL}(P_c) : V^{m+1} \to V^m$ is connected. Through straightening theorem, this gives rise to a homeomorphic copy of M within M, called a **primitive copy** of the Mandelbrot set.

The satellite and primitive copies of M obtained above are maximal in the sense that they are not contained in any other homeomorphic copy of M, except M itself. In this paper we will only work with the satellite copies of the Mandelbrot set.

The main hyperbolic component of M is the set of c where $|P_c'(a_c)| < 1$. For all rational numbers $p/q \in [-1/2, 1/2]$ with $(p, q) = 1$, there is c on the boundary of this component where $P_c'(a_c) = e^{2\pi i p/q}$. There are q rays landing at a_c with combinatorial rotation p/q, and the parameter c gives rise to the satellite copy $M(p/q)$ attached to the main hyperbolic component of M at c.

4.3. **Combinatorics of the PL-renormalisation.** Assume that P_c is polynomial-like renormalisable of satellite type $M(p_1/q_1)$, for some $p_1/q_1 \in (-1/2, 1/2] \cap \mathbb{Q}$. If the quadratic polynomial $\mathcal{R}_{\text{PL}}(P_c)$ is also polynomial-like renormalisable of satellite type, the corresponding parameter, $\mathcal{R}_{\text{PL}}(P_c)(0)$, belongs to $M(p_2/q_2)$, for some $p_2/q_2 \in (-1/2, 1/2] \cap \mathbb{Q}$. We have the return map $\tilde{\mathcal{R}}_{PL}(\mathcal{R}_{\text{PL}}(P_c))$ that is hybrid conjugate to $\mathcal{R}_{\text{PL}}^{\circ 2}(P_c)$. The parameter c belongs to the homeomorphic copy

$$M(p_1/q_1, p_2/q_2) = \Phi_{p_1/q_1}^{-1}(M(p_2/q_2))$$

of M contained in $M(p_1/q_1)$. Similarly, for an infinitely polynomial-like renormalisable of satellite type quadratic map we obtain a sequence of maximal satellite copies $M(p_j/q_j)$, for

[4]The homeomorphism sends the parameter c with $P_c'(a_c) = e^{2\pi i p/q}$ to the point $1/4 \in M$.

$j = 1, 2, \dots$, where $M(p_j/q_j)$ determining the type of renormalisation at level $j-1$. Then, we define the nest of Mandelbrot copies

$$M(p_1/q_1) \supset M(p_1/q_1, p_2/q_2) \supset M(p_1/q_1, p_2/q_2, p_3/q_3) \supset \dots,$$

where, for $n \geq 3$,

$$M(p_1/q_1, p_2/q_2, \dots, p_n/q_n) = \Phi^{-1}_{p_1/q_1} \circ \Phi^{-1}_{p_2/q_2} \circ \dots \circ \Phi^{-1}_{p_{n-1}/q_{n-1}}(M(p_n/q_n)).$$

For the simplicity of the notation we set

$$M_1(c) = M(p_1/q_1), M_2(c) = M(p_1/q_1, p_2/q_2), \dots,$$
$$M_n(c) = M(p_1/q_1, p_2/q_2, \dots, p_n/q_n).$$

The **root points** of these copies are defined as

$$c_0 = 1/4, c_1 = \Phi^{-1}_{p_1/q_1}(1/4) \in M_1(c), c_2 = \Phi^{-1}_{p_1/q_1} \circ \Phi^{-1}_{p_2/q_2}(1/4) \in M_2(c),$$
$$c_n = \Phi^{-1}_{p_1/q_1} \circ \Phi^{-1}_{p_2/q_2} \circ \dots \circ \Phi^{-1}_{p_n/q_n}(1/4) \in M_n(c), n \geq 3.$$

In other words, $c_n \in M_n$ is the unique parameter with $\mathcal{R}^{\circ n}_{\text{PL}}(P_{c_n})(z) = z^2 + 1/4$.

For an infinitely polynomial-like renormalisable P_c, we have the sequence of quadratic polynomials $\mathcal{R}^{\circ n}_{\text{PL}}(P_c)$, for $n \geq 0$, and the return maps $\tilde{\mathcal{R}}_{PL}(\mathcal{R}^{\circ n}_{\text{PL}}(P_c))$. By the straightening theorem, there are quasi-conformal maps S_n, for $n \geq 0$, that hybrid conjugate $\tilde{\mathcal{R}}_{\text{PL}}(\mathcal{R}^{\circ n}_{\text{PL}}(P_c))$ to $\mathcal{R}^{\circ(n+1)}_{\text{PL}}(P_c)$. Let $a(\mathcal{R}^{\circ n}_{\text{PL}}(P_c))$ denote the dividing fixed point of the quadratic map $\mathcal{R}^{\circ n}_{\text{PL}}(P_c)$, and set

$$a_1 = a(P_c) = a_c, a_2 = S_0^{-1}(a(\mathcal{R}_{\text{PL}}(P_c))), a_3 = S_0^{-1} \circ S_1^{-1}(a(\mathcal{R}^{\circ 2}_{\text{PL}}(P_c))),$$
$$a_{n+1} = S_0^{-1} \circ S_1^{-1} \circ \dots \circ S_{n-1}^{-1}(a(\mathcal{R}^{\circ n}_{\text{PL}}(P_c))), \text{for } n \geq 3.$$

Each a_n, for $n \geq 1$, is a dividing periodic point of P_c of minimal period $\Pi_{j=1}^{n-1} q_j$. The multipliers of these periodic points play a significant role in the remaining of this paper.

4.4. **The rigidity and MLC conjectures.** The **combinatorics** of a quadratic polynomial P_c is defined as an equivalence relation on the set of angles of external rays. That is, two angles θ_1 and θ_2 in $[0, 2\pi]$ are considered equivalent, if the external rays $R_c^{\theta_1}$ and $R_c^{\theta_2}$ land at the same point on $J(P_c)$. For an infinitely polynomial-like renormalisable quadratic polynomial P_c, it turns out that the combinatorics of the map is uniquely determined by the nest of relatively maximal Mandelbrot copies containing c.

Two quadratic polynomials are called **combinatorially equivalent**, if they induce the same equivalence relation on the circle. In the case of infinitely polynomial-like renormalisable maps, combinatorially equivalent maps fall into the same nest of relatively maximal Mandelbrot copies. We note that if two maps P_c and $P_{c'}$ are combinatorially equivalent and one of them is infinitely polynomial-like renormalisation, the other one must also be infinitely polynomial-like renormalisable.

The **rigidity conjecture** states that combinatorially equivalent quadratic polynomials with all their periodic points repelling are conformally conjugate. This conjecture is equivalent to the local connectivity of the Mandelbrot set [DH84] in its general form. However, in the case of infinitely polynomial-like renormalisable maps, there is a simple criterion that implies both conjectures at the corresponding parameter. If a nest of Mandelbrot copies shrinks to a single point c, then P_c is combinatorially rigid, c lies on the boundary of the Mandelbrot set, and P_c may be approximated by hyperbolic quadratic polynomials with connected Julia sets. This is because each Mandelbrot copy in M contains a parameter c' where $P_{c'}$ has a periodic critical point, and a root point c'' on ∂M where $P_{c''}$ has a parabolic periodic point with multiplier $+1$.

Proposition 4.3. *If a nest of Mandelbrot copies shrinks to a single point, then the Mandelbrot set is locally connected at the intersection.*

By the result of Douady and Hubbard on the equivalence of the two conjecture, the shrinking of a nest of Mandelbrot copies implies the local connectivity of the Mandelbrot set at the intersection. However, this has never been stated in the above form, although it may be proved following the standard techniques developed in 1980's. Here we briefly outline a proof, which requires some basic definitions presented below.

The definition of the Böttcher coordinate may be extended to quadratic polynomials with disconnected Julia sets. For $c \in \mathbb{C} \setminus M$, there is a connected domain U_c bounded by piece-wise analytic curves, a real number $r_c > 1$, and a conformal bijection $\varphi_c : U_c \to \{z \in \mathbb{C} \mid |z| > r_c\}$ that is tangent to the identity at infinity. Moreover, the critical value c belongs to U_c and $|\varphi_c(c)| = r_c^2$. Douady and Hubbard proved that the mapping defined as

$$c \mapsto \varphi_c(c)$$

provides a conformal bijection from $\mathbb{C} \setminus M$ to $\mathbb{C} \setminus \overline{\mathbb{D}}$. Through this map, one may define the external rays of the Mandelbrot set as the preimages of the straight rays $(1, +\infty)e^{i\theta}$, for $\theta \in [0, 2\pi]$.

A point $c \in M$ is called a **Misiurewicz parameter**, if there is an integer $n \geq 2$ such that $P_c^{\circ n}(0)$ is a repelling periodic point of P_c. Let us also say that $c \in M$ is a **parabolic parameter** if P_c has a parabolic periodic point. A key result regarding the landing property of the external rays of the Mandelbrot set is the following.

Proposition 4.4 (Douady-Hubbard [DH84])**.** *Every parabolic parameter $c \in M$ different from $1/4$ is the landing point of two distinct external rays of the Mandelbrot set. Every Misiurewicz parameter $c \in M$ is the landing point of at least one, but at most a finite number of, external rays of the Mandelbrot set.*

Let M' be a homeomorphic copy of M strictly contained in M. There is a unique parameter $c' \in M'$ that corresponds to the point $1/4$ under the homeomorphism mapping M' to M. The map $P_{c'}$ has a parabolic periodic point with multiplier $+1$. This is the root point of the copy M' defined above, and since $M' \neq M$, $c' \neq 1/4$. By the above proposition, there are two external rays of M landing at c'. The union of these rays and their landing point c' divide the complex plane into two connected components. We let $W(M')$ denote the closure of the connected component containing $M' \setminus \{c'\}$. In the literature, the interior of $W(M')$ (sometimes with the root point c' added to it) is called the **parabolic wake** containing the copy M'. Note that as the set M is connected, $W(M') \cap M$ must be connected.

If there are more than one external ray landing at a Misiurewicz parameter $m \in M$, the union of these rays and their landing point m divides the complex plane into a finite number of regions. One of these regions contains 0 in its interior. The remaining connected components are called **Misiurewicz wakes** at m. A Misiurewicz wake of the Mandelbrot set is, by definition, a Misiurewicz wake at some Misiurewicz parameter $m \in M$, and is assumed to be an open subsets of the plane. It follows that the intersection of any Misiurewicz wake with the Mandelbrot set is a connected subset of $\mathbb{C}$, and also the Mandelbrot set M minus any Misiurewicz wake is a connected subset of $\mathbb{C}$.

Proposition 4.5 ([Mil00]). *Let M' be a homeomorphic copy of the Mandelbrot set strictly contained in M. Then the set M' is obtained from removing a countable number of Misiurewicz wakes from the set $W(M') \cap M$.*

Now we are ready to prove the result we need.

Proof of Proposition 4.3. Assume that $M \supset M_1 \supset M_2 \supset \dots$ is a nest of Mandelbrot copies shrinking to a single point $c \in M$. Then, P_c is infinitely polynomial-like renormalisable, and in particular, c is not the root point of any of the copies M_i, $i \geq 1$.

Fix $\varepsilon > 0$ and choose $n \in \mathbb{N}$ such that the diameter of M_n is less than $\varepsilon/2$. By Proposition 4.5, M_n is equal to $W(M_n) \cap M$ minus a countable number of Misiurewicz wakes.

Let U be a ball of diameter ε containing M_n. The set $(\mathbb{C} \setminus U) \cap M$ is a compact subset of M. Thus, by the above paragraph, there are a finite number of Misiurewicz wakes $L_1, L_2, \dots L_n$ such that $(\mathbb{C} \setminus U) \cap M$ in contained in the union of these wakes. The set U minus the closure of $\cup_{i=1}^n L_i$, is an open subset of $\mathbb{C}$, and has a connected intersection with M. This set has diameter at most ε.

As ε was chosen arbitrary, this implies that there is a basis of neighbourhoods U_i containing c, $i \geq 1$, such that each $M \cap U_i$ is a connected set. □

4.5. **Bounds on multipliers.** We need a relation between the combinatorial rotation number of a repelling period cycle and the (analytic) multiplier of that cycle. A formula of this type is given by the so called Pommerenke-Levin-Yoccoz inequality, see [Hub93], [Pom86], [Lev91], [Pet93]. Indeed, the general form of the inequality applies to repelling periodic points of arbitrary degree polynomials, but here we only state the version for the quadratic polynomials.

Theorem 4.6 (PLY inequality). *Let P_c be a polynomial with a connected Julia set. Suppose that ζ is a repelling periodic point of P_c such that*

a) the minimal period of ζ is k;
b) there are q, $0 < q < +\infty$, external rays landing at ζ;
c) the external rays landing at ζ are cyclically permuted with combinatorial rotation number p/q.

Then, for a suitable branch of log*, the multiplier of ζ, denoted by ρ, satisfies*

$$\Big| \log \rho - \big(2\pi \text{ i } \frac{p}{q} + \frac{k}{q} \log 2\big) \Big| \leq \frac{k}{q} \log 2.$$

Let $\langle p_i/q_i \rangle_{i=1}^{\infty}$ be a sequence of rational numbers in $[-1/2, 1/2]$ and n be a positive integer. For $c \in M(\langle p_i/q_i \rangle_{i=1}^{n})$, P_c has dividing periodic points $a_1, \dots, a_n$, where each a_j has period k_j given by the formula

$$k_1 = 1, \quad k_j = \prod_{i=1}^{j-1} q_i, \text{for } 2 \leq j \leq n. \tag{4.1}$$

Moreover, the combinatorial rotation number of a_n is p_n/q_n and there are q_n external rays landing at a_n. By the above theorem, there is a branch of log such that the multiplier ρ_n of the n-th dividing periodic point a_n of P_c satisfies

$$\Big|\frac{1}{2\pi i}\log\rho_n - \big(\frac{p_n}{q_n} - i\frac{k_n}{q_n}\frac{\log 2}{2\pi}\big)\Big| \leq \frac{k_n}{q_n}\frac{\log 2}{2\pi}. \tag{4.2}$$

See Figure 8. We have $\frac{1}{2\pi}\log 2 = 0.110\ldots$.

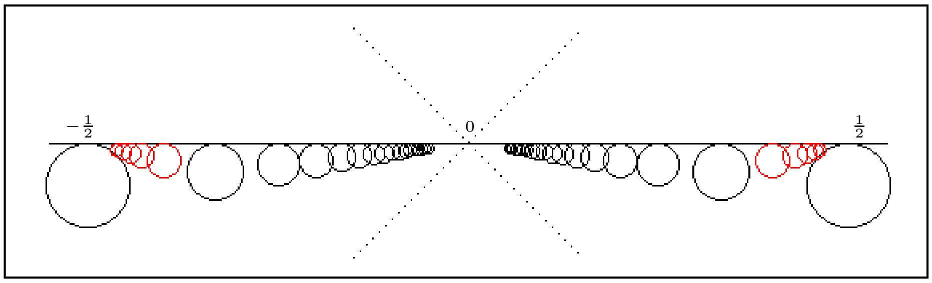

FIGURE 8. The black circles denote the locus of $\frac{1}{2\pi i}\log\rho_1$ for combinatorial rotations $\pm 1/i$, $2 \leq i \leq 20$. The red circles denote the locus of this quantity for combinatorial rotations $\langle(2,+1),(i,+1)\rangle)$, for $2 \leq i \leq 7$, as well as $\langle(2,-1),(i,+1)\rangle)$, for $2 \leq i \leq 7$.

5. Infinitely near-parabolic renormalisable maps

By Theorem 2.9 and Definition 2.11, for $f \in A(r_3) \ltimes \mathcal{F}_0$ and $f = Q_\alpha$ with $\alpha \in A(r_3)$, the top and bottom near-parabolic renormalisations $\mathcal{R}_{\text{NP-t}}(f)$ and $\mathcal{R}_{\text{NP-b}}(f)$ are defined. If either of these maps belongs to $A(r_3) \ltimes \mathcal{F}_0$, which depends on whether $\alpha(\mathcal{R}_{\text{NP-t}}(f))$ and $\alpha(\mathcal{R}_{\text{NP-b}}(f))$ belong to $A(r_3)$, we may define the top and bottom near-parabolic renormalisation of that map in order to obtain the second near-parabolic renormalisation of f. This successive near-parabolic renormalisation process may be carried out infinitely often for some maps f. One may associate a one sided infinite sequence of t's and b's to determine the type of the successive near-parabolic renormalisations, where "t" stands for "top" and "b" stands for "bottom". In other words, for any κ in the set

$$\mathcal{T} = \{t,b\}^{\mathbb{N}} = \{(\kappa_1,\kappa_2,\kappa_3,\ldots) \mid \forall i \geq 1, \kappa_i \in \{t,b\}\}, \tag{5.1}$$

we say that a map $f \in A(r_3) \ltimes \mathcal{F}_0$, or $f = Q_\alpha$ with $\alpha \in A(r_3)$, is **infinitely near-parabolic renormalisable of type** κ if the following infinite sequence of maps is defined

$$f_1 = f,\; f_{n+1} = \begin{cases} \mathcal{R}_{\text{NP-t}}(f_n), & \text{if } \kappa_n = t, \\ \mathcal{R}_{\text{NP-b}}(f_n), & \text{if } \kappa_n = b. \end{cases} \tag{5.2}$$

Then, there are $\alpha_n, \beta_n \in \mathbb{C}$, for $n \geq 1$, such that

$$f_n'(0) = e^{2\pi i\alpha_n}, \operatorname{Re}\alpha_n \in (-1/2, 1/2]$$
$$f_n'(\sigma_n) = e^{2\pi i\beta_n}, \operatorname{Re}\beta_n \in (-1/2, 1/2].$$

We shall use the notations

$$\alpha_n = \alpha(f_n) \in A(r_3), \beta_n = \beta(f_n), \sigma_n = \sigma(f_n),$$

throughout the rest of this paper.

The rotation numbers α_n and β_n, for $n \geq 1$, are related by the formulas

$$\frac{1}{1-e^{2\pi i \alpha_n}} + \frac{1}{1-e^{2\pi i \beta_n}} = \frac{1}{2\pi i} \int_{\partial W} \frac{1}{z - f_n(z)} \, \mathrm{d}z, \tag{5.3}$$

and

$$\alpha_{n+1} = \begin{cases} -1/\alpha_n - [\mathrm{Re}(-1/\alpha_n)] & \text{if } \kappa_n = t, \\ -1/\beta_n - [\mathrm{Re}(-1/\beta_n)] & \text{if } \kappa_n = b, \end{cases} \tag{5.4}$$

where $[\cdot]$ denotes the closest integer function. We proved in Lemma 3.24 that the absolute value of the right-hand side of Equation (5.3) is uniformly bounded from above. This implies that when some α_n is small (and non-zero), then β_n is small and the sign of $\mathrm{Re}\, \beta_n$ is equal to the sign of $\mathrm{Re}\, \alpha_n$ times -1.

We say that a map is **infinitely near-parabolic renormalisable** if there is $\kappa \in \mathcal{T}$ such that the map is infinitely near-parabolic renormalisable of type κ. By a continuity argument one may see that for any $\kappa \in \mathcal{T}$, the set of infinitely near-parabolic renormalisable maps of type κ is non-empty. See Figure 9.

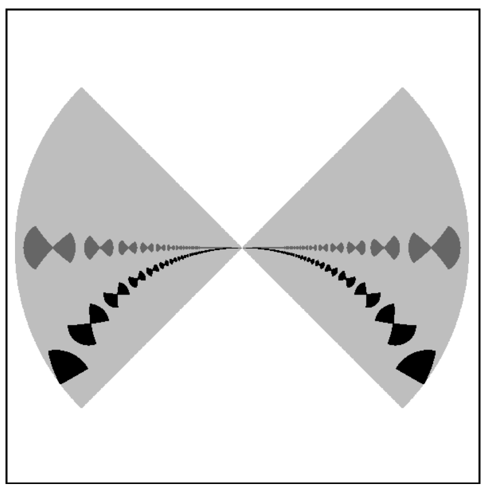

FIGURE 9. This is a schematic presentation of the values of the complex rotation α where one and two iterates of the top and bottom near-parabolic renormalisations are defined. The light gray region shows the set $A(r_3)$. The darker gray region is where $\mathcal{R}_{\text{NP-t}}^{\circ 2}$ and $\mathcal{R}_{\text{NP-b}} \circ \mathcal{R}_{\text{NP-t}}$ are defined. On the black region the iterates $\mathcal{R}_{\text{NP-b}}^{\circ 2}$ and $\mathcal{R}_{\text{NP-t}} \circ \mathcal{R}_{\text{NP-b}}$ are defined.

When a map is infinitely near-parabolic renormalisable, one may obtain fine scale understanding of the dynamics of that map. For example, for κ of constant type $\kappa_i = t$ for all $i \geq 1$, this has lead to important properties of the dynamics of quadratic polynomials Q_α, with irrational α of high type, in [BC12], [Che19], [Che13], [AC18], and [CC15]. Thus, it is a significant problem to understand when a map is infinitely near-parabolic renormalisable. By virtue of Theorem 2.9, the covering structure of the top and bottom renormalisations of a one time renormalisable map are determined. Thus, the answer to this question relies on controlling the multipliers of the successive renormalisations of the map at the origin.

5.1. **Cantor structure in the bifurcation loci.** Let $\kappa = (\kappa_1, \kappa_2, \kappa_3, \dots) \in \mathcal{T}$, and $r \in (0, r_3]$, where r_3 is the constant obtained in Proposition 2.6. Define $\Lambda_r(\kappa_1) = A(r)$. By Theorem 2.9, for every $f \in \mathcal{F}_0$ and $\alpha \in A(r)$, $\mathcal{R}_{\text{NP-t}}(\alpha \ltimes f)$ and $\mathcal{R}_{\text{NP-b}}(\alpha \ltimes f)$ are defined. For integers $n \geq 1$, we consider

$$\Lambda_r(\langle \kappa_i \rangle_{i=1}^n) = \left\{ \alpha \ltimes f \;\middle|\; \begin{array}{c} \mathcal{R}_{\text{NP-}\kappa_n} \circ \dots \circ \mathcal{R}_{\text{NP-}\kappa_1}(\alpha \ltimes f) \text{ is defined} \\ \text{and } \forall i \text{ with } 1 \leq i \leq n, \alpha_i \in A(r). \end{array} \right\}. \tag{5.5}$$

We recall that in the above definition, α_i is the rotation number of the map $\mathcal{R}_{\text{NP-}\kappa_{i-1}} \circ \dots \circ \mathcal{R}_{\text{NP-}\kappa_1}(\alpha \ltimes f)$ at 0. In other words, $\Lambda_r(\kappa_1, \dots, \kappa_n)$ is the set of maps $\alpha \ltimes f$ that are n times near parabolic renormalisable of type $\kappa_1, \dots, \kappa_n$ with the rotation number of all the successive renormalisations in the set $A(r)$. Given $\kappa \in \mathcal{T}$, one naturally defines

$$\Lambda_r(\kappa) = \bigcap_{n=1}^{\infty} \Lambda_r(\langle \kappa_i \rangle_{i=1}^n).$$

The main result of this paper is stated in the following theorem. We recall that k_1 and r_4 are the constant obtained in Proposition 3.1.

Theorem 5.1. *For all k_1-horizontal family of maps $\Upsilon : A(r_4) \to A(r_4) \ltimes \mathcal{F}_0$ and all $\kappa \in \mathcal{T}$, every connected component of the set $\Lambda_{r_4}(\kappa) \cap \Upsilon(A(r_4))$ is a single point.*

In particular, for all $f \in \mathcal{F}_0 \cup \{Q_0\}$, and all $\kappa \in \mathcal{T}$, every connected component of the set $\Lambda_{r_4}(\kappa) \cap (A(r_4) \ltimes f)$ is a single point.

Proof. Let A_1 be a connected component of $\Lambda_{r_4}(\kappa) \cap \Upsilon(A(r_4))$. By definition, for every $n \geq 1$, every point in A_1 is near parabolic renormalisable of type $\langle \kappa_i \rangle_{i=1}^n$. Inductively define the sets $A_{n+1} = \mathcal{R}_{\text{NP-}\kappa_n}(A_n)$, for $n \geq 1$. By Proposition 3.1, each A_i, $i \geq 1$, is a k_1-horizontal curve.

Let us define the numbers ϑ_i as the diameter of the projection of the set A_i onto the α coordinate. Then, since the diameter of each component of $A(r_4)$ is equal to $\sqrt{2} r_4$, we have $\vartheta_i \leq \sqrt{2} r_4$. On the other hand, by Proposition 3.4, $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$ are strictly expanding in the horizontal direction. We must have $\vartheta_i = 0$, for all $i \geq 1$. It follows from the definition of k_1- horizontal curves that the diameter of each A_i must be zero. Indeed, the uniform contraction implies that there are constants C and $\mu \in (0,1)$ such that for all k_1-horizontal family of maps $\Upsilon : A(r_4) \to A(r_4) \ltimes \mathcal{F}_0$ and all $\kappa \in \mathcal{T}$

$$\text{diam}\left(\Lambda_{r_4}(\langle \kappa_i \rangle_{i=1}^n) \cap \Upsilon(A(r_4))\right) \leq C\mu^n.$$

The latter part of the theorem follows from the first part as the family $\Upsilon(\alpha) = (\alpha \ltimes f)$ may be thought of a 0-horizontal family. □

The quadratic polynomial P_c, with $c \in \mathbb{C}$, is conformally conjugate to some $Q_{\alpha(c)}$, with $\operatorname{Re} \alpha(c) \in (-1/2, 1/2]$. Indeed, there are two choices for $\alpha(c)$ with this property. The choice does not make any difference for the sake of the next statement, although we will make one of these choices in the next section for our convenience. An immediate corollary of Theorem 5.1 applied to the quadratic family is formulated in the next corollary.

It follows from the proof of the above theorem that the set $\Lambda_{r_4}(\kappa)\cap\Upsilon(A(r_4))$ is isomorphic to a $\mathcal{F}_0$-bundle over a Cantor set. This is, formulated in the next corollary.

Corollary 5.2. *For all $\kappa\in\mathcal{T}$ the restriction of the map $\alpha \ltimes h\mapsto h$ to each connected component of the set $\Lambda_{r_4}(\kappa)\cap\Upsilon(A(r_4))$ is one-to-one and onto whose image is equal to $\mathcal{F}_0$.*

The operators $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$ map the $\mathcal{F}_0$-fibers of the set $\cup_{\kappa\in\mathcal{T}}\Lambda_{r_4}(\kappa)\cap\Upsilon(A(r_4))$ to the fibers. Combining with the uniform contraction in Theorem 2.9, we obtain the uniform contraction of the operators $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$ on the co-dimension one fibers.

Corollary 5.3. *Let $\langle p_i/q_i\rangle_{i=1}^{\infty}$ be a sequence of rational numbers in $(-1/2,1/2]$ and $\kappa\in\mathcal{T}$ be a type such that for all $c\in M(\langle p_i/q_i\rangle_{i=1}^{\infty})$, $Q_{\alpha(c)}$ is infinitely near parabolic renormalisable of type κ and for every $i\geq 1$ the rotation number α_i of the i-th renormalisation of $Q_{\alpha(c)}$ belongs to $A(r_4)$. Then, the nest of Mandelbrot copies $M(\langle p_i/q_i\rangle_{i=1}^{n})$ shrinks (geometrically) to a single point as n tends to infinity.*

Although in the above corollary the sequence of rational numbers and the type κ are not related *a priori*, in Section 6.3 we associate a canonic type κ to any given sequence of rational numbers in $(-1/2,1/2]$.

6. Application to the complex quadratic polynomials

6.1. **Modified continued fractions.**

We work with a modified notion of continued fractions that is more suitable in the study of the near-parabolic renormalisation.

For $x\in\mathbb{R}$, let $[x]$ denote the closest integer to x, where we use the convention

$$x\in([x]-1/2,[x]+1/2], \text{for } x>0,$$
$$x\in[[x]-1/2,[x]+1/2), \text{for } x<0.$$

We have adapted the above convention to obtain a $x\mapsto -x$ symmetry in the continued fraction expansion introduced below.

Let $x\in[-1/2,1/2]\setminus\{0\}$ be a rational number and set $x_1=x$. There is a positive integer n such that the numbers

$$x_{i+1}=\frac{-1}{x_i}-\left[\frac{-1}{x_i}\right], 1\leq i\leq n, \tag{6.1}$$

are defined and $x_{n+1}=0$. For $1\leq i\leq n$, we define the signs $\varepsilon'_i=+1$ if $x_i>0$ and $\varepsilon'_i=-1$ if $x_i<0$, and then define the integers $b_i\geq 2$ according to

$$b_i=\begin{cases}\left[\frac{-1}{x_i}\right] & \text{if } \varepsilon'_i=-1,\\ \left[\frac{1}{x_i}\right] & \text{if } \varepsilon'_i=+1.\end{cases}$$

It follows that $[-1/x_i]=-\varepsilon'_i b_i$, and hence $x_i^{-1}=\varepsilon'_i b_i-x_{i+1}$. Now let us define the signs $\varepsilon_1=\varepsilon'_1$, and $\varepsilon_i=(-1)\varepsilon'_{i-1}\varepsilon'_i$, for $2\leq i\leq n$. Then, one can see that x is given by the finite continued fraction

$$x=x_1=1/(\varepsilon'_1 b_1-x_2)=\varepsilon_1/(b_1-\varepsilon'_1 x_2),$$

$$=\cfrac{\varepsilon_1}{b_1-\varepsilon'_1\cfrac{1}{\varepsilon'_2 b_2-x_3}}=\cfrac{\varepsilon_1}{b_1+\cfrac{-1\varepsilon'_1\varepsilon'_2}{b_2-\varepsilon'_2 x_3}}=\cfrac{\varepsilon_1}{b_1+\cfrac{\varepsilon_2}{b_2-\varepsilon'_2 x_3}}.$$

Inductively, repeating the above process until $x_{n+1} = 0$, we obtain

$$x = \cfrac{\varepsilon_1}{b_1 + \cfrac{\varepsilon_2}{\ddots + \cfrac{\varepsilon_n}{b_n}}}.$$

Remark 6.1. The above continued fraction expansion is slightly different from the usual notion of modified (closest integer) continued fraction expansion in the literature, where we use $x - [x]$ instead of $d(x, \mathbb{Z})$ and allow x_i to be negative as well as positive. However, the only difference between the two expansions is in the signs ε_i. The reason for adapting to the above algorithm is that we shall later extend the map $x \mapsto x - [x]$ to a holomorphic map on a neighbourhood of the interval $[-1/2, 1/2]$. This allows us to study the maps $\alpha \mapsto \mathcal{R}_{\text{NP-t}}(\alpha \ltimes f)$ and $\alpha \mapsto \mathcal{R}_{\text{NP-b}}(\alpha \ltimes f)$ as holomorphic maps of α rather than anti-holomorphic maps of α.

Given $n \geq 1$ and a sequence of pairs $\langle b_i : \varepsilon_i \rangle_{i=1}^n$, where each $b_i \geq 2$ is an integer and $\varepsilon_i \in \{+1, -1\}$, we use the notation $[\langle b_i : \varepsilon_i \rangle_{i=1}^n]$ to denote the rational number generated by this sequence of pairs. That is,

$$[b_1 : \varepsilon_1] = \frac{\varepsilon_1}{b_1}, \quad [(b_1 : \varepsilon_1), (b_2 : \varepsilon_2)] = \cfrac{\varepsilon_1}{b_1 + \cfrac{\varepsilon_2}{b_2}},$$

and for $n \geq 3$,

$$[\langle b_i : \varepsilon_i \rangle_{i=1}^n] = \cfrac{\varepsilon_1}{b_1 + \cfrac{\varepsilon_2}{\ddots + \cfrac{\varepsilon_n}{b_n}}}.$$

However, note that a rational number of the above form is not necessarily in the interval $[-1/2, 1/2]$. But, it is fairly close. The only condition we need to impose to obtain a rational number in the interval $[-1/2, 1/2]$ is that when $b_1 = 2$ we must have $\varepsilon_1 \varepsilon_2 = +1$.

6.2. **Sequences of rational numbers.** Let $n \geq 1$ be an integer, and let $m_i \geq 1$ and $b_{i,j} \geq 2$ be integers, and $\varepsilon_{i,j} \in \{+1, -1\}$, for $1 \leq i \leq n$ and $1 \leq j \leq m_i$. We define the notation

$$\langle m_i : b_{i,j} : \varepsilon_{i,j} \rangle_{i=1}^n$$

to represent the finite sequence of rational numbers

$$\langle [b_{i,j} : \varepsilon_{i,j}]_{j=1}^{m_i} \rangle_{i=1}^n$$

For each i and l with $1 \leq i \leq n$ and $1 \leq l \leq m_i$ we let

$$\frac{p_{i,l}}{q_{i,l}} = [\langle b_{i,j} : \varepsilon_{i,j} \rangle_{j=1}^l].$$

Thus,

$$\langle m_i : b_{i,j} : \varepsilon_{i,j} \rangle_{i=1}^n = \langle \frac{p_{i,m_i}}{q_{i,m_i}} \rangle_{i=1}^n.$$

Although we may remove the second subscript m_i from the numerator and denominator of the above sequence to get the simpler notation $p_i/q_i = p_{i,m_i}/q_{i,m_i}$, these have been put there to avoid possible future confusions that the sequence p_i/q_i forms the fractions of a single number. That is, *a priori* there is no relation between these fractions. When we are interested in infinite sequences of rational numbers, we shortened the notation $\langle m_i : b_{i,j} : \varepsilon_{i,j}\rangle_{i=1}^{\infty}$ to $\langle m_i : b_{i,j} : \varepsilon_{i,j}\rangle$.

By setting the initial data $p_{i,-1} = q_{i,0} = 1$ and $p_{i,0} = q_{i,-1} = 0$, we have the usual recursive formulas for the continued fractions $p_{i,l}/q_{i,l}$, $0 \leq l \leq m_i$

$$q_{i,l+1} = b_{i,l+1}q_{i,l} + \varepsilon_{i,l+1}q_{i,l-1}, \quad p_{i,l+1} = b_{i,l+1}p_{i,l} + \varepsilon_{i,l+1}p_{i,l-1},$$

By an inductive process, the above formulas imply that for all $i \geq 1, 0 \leq l \leq m_i - 1$, we have

$$q_{i,l+1} > q_{i,l}. \tag{6.2}$$

Moreover, for every $i \geq 1$,

$$\frac{1}{q_{i,m_i}} = \Big|\prod_{k=1}^{m_i}[\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=k}^{m_i}]\Big|. \tag{6.3}$$

6.3. **Pairs of multipliers vs pairs of complex rotations.** Let $\kappa = (\kappa_1, \kappa_2, \kappa_3, \dots)$ be in $\mathcal{T}$, where the set of types $\mathcal{T}$ is defined in Equation (5.1). In analogy with the set $\Lambda_r(\kappa_1, \dots, \kappa_n)$ defined in Equation (5.5), we consider the sets

$$\Lambda_r^1(\langle\kappa_i\rangle_{i=1}^n) = \Big\{\alpha \in A(r_3) \Big| \begin{array}{c} \mathcal{R}_{\text{NP-}\kappa_n} \circ \dots \circ \mathcal{R}_{\text{NP-}\kappa_1}(Q_\alpha) \text{ is defined} \\ \text{and } \forall i \text{ with } 1 \leq i \leq n, \alpha_i \in A(r). \end{array} \Big\}$$

For example, by Theorem 2.9, $\mathcal{R}_{\text{NP-t}}(Q_\alpha)$ and $\mathcal{R}_{\text{NP-b}}(Q_\alpha)$ are defined for $\alpha \in \Lambda_{r_3}^1(\kappa_1)$.

Each Q_α with $\alpha \in \mathbb{C}$ is conformally conjugate to some quadratic polynomial P_c with a unique $c \in \mathbb{C}$. The connectedness locus of the family Q_α, that is, the set of α such that the Julia set of Q_α is connected, is $\mathbb{Z}$-periodic in α. However, this connectedness locus modulo $\mathbb{Z}$ forms a double cover of the Mandelbrot set, branched over $c = 1/4$ (which is only covered once). Here $\alpha = 0$ is mapped to $c = 1/4$. For each $c \in \mathbb{C} \setminus \{1/4\}$ there are two distinct parameters w_1 and w_2 in $\mathbb{C}$ such that for all α in $w_1 + \mathbb{Z}$ and $w_2 + \mathbb{Z}$, Q_α is conformally conjugate to P_c. See Figure 10.

For α in the upper half plane, 0 is an attracting fixed point of Q_α, while for α in $\mathbb{R}$ the multiplier of Q_α at 0 belongs to the unit circle. In analogy to the Mandelbrot set, the connectedness locus of Q_α minus $\mathbb{R}$ consists of the upper half plane and the connected components attached to the real line at rational values of α. There is a unique connected component attached to the real line at 0. We denote the closure of this component by M_α. Then, there is a one-to-one correspondence between the Mandelbrot set and M_α such that the corresponding quadratic polynomials are conformally conjugate. Let $c \mapsto \alpha(c)$, from M to M_α, denote this bijection. We define the sets

$$M_\alpha(\langle\frac{p_i}{q_i}\rangle_{i=1}^n) = \big\{\alpha(c) \mid c \in M(\langle\frac{p_i}{q_i}\rangle_{i=1}^n)\big\}.$$

As in Section 4, the notions of dividing periodic points and their combinatorial rotation numbers are defined on the above components.

Let Q_α be an infinitely polynomial-like renormalisable. We denote the sequence of the dividing periodic points of Q_α by ν_i, $i \geq 1$. In other words, ν_i in the periodic point of Q_α that is mapped to a_i by the conformal map conjugating $Q_{\alpha(c)}$ to P_c, for each $i \geq 1$. We recall

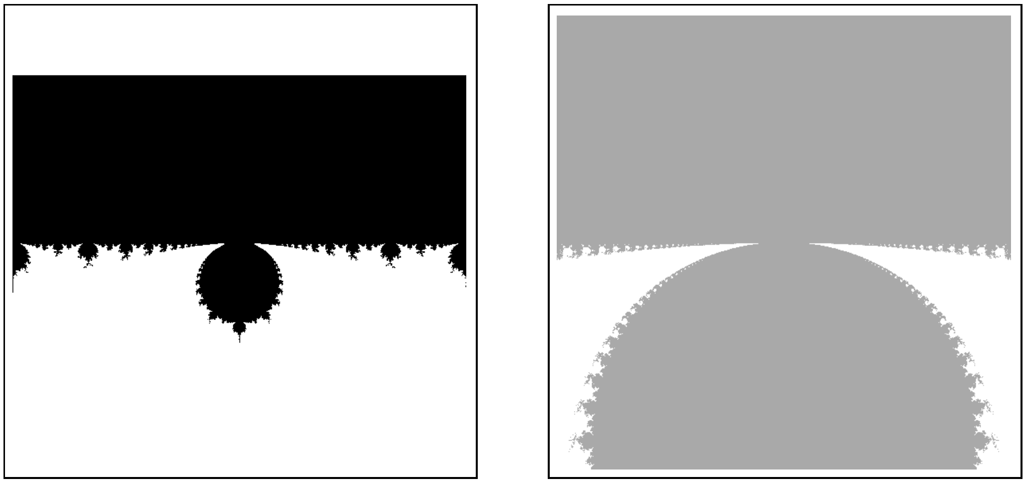

FIGURE 10. The connectedness locus of the family Q_α and a zoom into a neighbourhood of 0 on the right hand side. Need to produce better figures.

that ν_1 is a fixed point. Note that ν_i is not an arbitrary element in the cycle of ν_i. Rather, it is a particular point in this cycle. Let us denote the multiplier of ν_i by ρ_i, for $i \geq 1$. That is,

$$\rho_1 = Q'_\alpha(\nu_1),\ \rho_2 = (Q_\alpha^{\circ q_1})'(\nu_2),\ \rho_n = (Q_\alpha^{\circ(q_1 \cdots q_{n-1})})'(\nu_n), n \geq 3.$$

For $\alpha \in M_\alpha \setminus \{0\}$, 0 is a repelling fixed point of Q_α while $\sigma(Q_\alpha)$ may be either attracting or repelling, depending on $\mathrm{Im}(\beta(Q_\alpha))$. For $\alpha \in M_\alpha \setminus \{0\}$, $\nu_1 = \sigma_1 = \sigma(Q_\alpha)$ is the non-zero fixed point of Q_α, and we have

$$\beta_1 = \frac{1}{2\pi i} \log \rho_1. \tag{6.4}$$

Let p_i/q_i be a sequence of non-zero rational numbers in $(-1/2, 1/2]$. By the previous section, there is $\langle m_i : b_{i,j} : \varepsilon_{i,j}\rangle_{i=1}^\infty$, with integers $m_i \geq 1$, $b_{i,j} \geq 2$, and signs $\varepsilon_{i,j}$ for $i \geq 1$ and $1 \leq j \leq m_i$, such that $p_i/q_i = p_{i,m_i}/q_{i,m_i} = [\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i}]$, for $i \geq 1$. Then, we consider the integers

$$l_1 = 0,\ l_k = \sum_{i=1}^{k-1} m_i, k \geq 2. \tag{6.5}$$

Then, we define the map

$$\kappa : \big(\mathbb{Q} \cap \big((-1/2, 0) \cup (0, 1/2]\big)\big)^{\mathbb{N}} \to \mathcal{T}, \tag{6.6}$$

as follows. Given $\langle m_i : b_{i,j} : \varepsilon_{i,j}\rangle$ in $\big(\mathbb{Q} \cap \big((-1/2, 0) \cup (0, 1/2]\big)\big)^{\mathbb{N}}$ and $n \geq 1$, the n-th entry of $\kappa(\langle m_i : b_{i,j} : \varepsilon_{i,j}\rangle)$, denoted by $\kappa_n(\langle m_i : b_{i,j} : \varepsilon_{i,j}\rangle)$, is defined as

$$\kappa_n(\langle m_i : b_{i,j} : \varepsilon_{i,j}\rangle) = \begin{cases} b & \text{if } n \in \{l_k + 1 \mid k \geq 1\} \\ t & \text{otherwise.} \end{cases}$$

In other words, the first m_1 entries of κ are given by one "b" followed by $m_1 - 1$ times "t", the next m_2 entries of κ are given by one "b" followed by $m_2 - 1$ times "t", and so on. The map κ only depends on the sequence m_i (the lengths of the rational numbers) rather than the entries in each rational number. The individual entries come into play later.

In the following proposition r_3 denotes the constant introduced in Proposition 2.6.

Proposition 6.2. *Let $\langle p_i/q_i\rangle_{i=1}^{\infty} = \langle m_i : b_{i,j} : \varepsilon_{i,j}\rangle$ be a sequence of non-zero rational numbers in $(-1/2, 1/2]$ and consider the type $\kappa = \kappa(\langle m_i : b_{i,j} : \varepsilon_{i,j}\rangle) \in \mathcal{T}$. For every integer $k \geq 1$ and every α in the intersection of $M_\alpha(\langle p_i/q_i\rangle_{i=1}^{k})$ and $\Lambda^1_{r_3}(\kappa_1, \dots, \kappa_{l_k})$, if $\alpha_{l_k+1} \in A(r_3)$ then*

$$\beta_{l_k+1} = \frac{1}{2\pi\ \mathrm{i}} \log \rho_k.$$

Proof. Since $\alpha \in M_\alpha(p_1/q_1, \dots, p_k/q_k)$, for every integer j with $1 \leq j \leq k$ the dividing periodic point ν_j of Q_α is defined. In particular, the right hand side of the equality in the proposition is defined. On the other hand, as $\alpha \in \Lambda^1_{r_3}(\kappa_1, \dots, \kappa_{l_k})$, for every integer j with $2 \leq j \leq l_k$, $f_{j+1} = \mathcal{R}_{\text{NP-}\kappa_j} \circ \dots \circ \mathcal{R}_{\text{NP-}\kappa_1}(Q_\alpha)$ is defined. Moreover, by the definition of $\Lambda^1_{r_3}(\kappa_1, \dots, \kappa_{l_k})$ and the assumption in the proposition for every $1 \leq j \leq l_k + 1$, α_j is defined and belongs to $A(r_3)$. By Proposition 2.1, this implies that each f_j has a unique non-zero fixed point σ_j in the (fixed) neighbourhood W of 0. In particular, β_{l_k+1} in the left-hand side of the equation in the proposition is defined. It remains to relate these quantities.

Recall that for $\alpha \in M_\alpha(p_1/q_1, \dots, p_k/q_k)$, ν_1 is a fixed point, and in general for j with $2 \leq j \leq k$, ν_j is a periodic point of period $\prod_{i=1}^{j-1} q_i$. Moreover, as α varies in $M_\alpha(p_1/q_1, \dots, p_j/q_j)$, ν_j has holomorphic dependence on α. On the other hand, by the definition of the near-parabolic renormalisations, each σ_{l_j+1}, for $1 \leq j \leq k$, lifts to a periodic cycle of Q_α, which we denote by O_{l_j+1}. We claim that for each j with $1 \leq j \leq k$, O_{l_j+1} is equal to the cycle of ν_j. We prove this below by induction on j.

For $j = 1$, $l_1 = 0$ and O_1 is equal to $\sigma_1 = \sigma(Q_\alpha) = \nu_1$. Assume that $O_{l_{j-1}+1}$ is equal to the cycle of ν_{j-1} for $j - 1 < l + 1$ and we want to prove that O_{l_j+1} is equal to the cycle of ν_j. By the definition of the types, $\kappa_{l_{j-1}+1} = b$ and for all integers i with $l_{j-1} + 1 < i \leq l_j$, we have $\kappa_i = t$. This implies that the zero fixed point of f_{l_j+1} lifts to $\sigma_{l_{j-1}+1}$ on the dynamic plane of $f_{l_{j-1}+1}$ (in all the intermediate levels i it lifts to 0). Thus, by the induction hypothesis, the zero fixed point of f_{l_j+1} lifts to the cycle of ν_{j-1}. On the other hand, as α_{l_j+1} tends to 0 in $A(r_3)$, σ_{l_j+1} tends to 0 and σ_{l_j+1} is the only fixed point of f_{l_j+1} within W-neighbourhood of 0. This implies that the lift of this fixed point, which is the cycle O_{l_j+1}, tends to the cycle of ν_{j-1}. However, as $\alpha \in M_\alpha(p_1/q_1, \dots, p_j/q_j)$, ν_j is the only periodic point of Q_α that bifurcates from ν_{j-1}. That is, for sufficiently small α_{l_j+1} (equivalently, for α sufficiently close to the root point of $M_\alpha(p_1/q_1, \dots, p_j/q_j)$), O_{l_j+1} is equal to the cycle of ν_j. By the holomorphic dependence of the cycles of ν_j and O_{l_j+1} on α, we conclude that these cycles are equal on the connected components of $\Lambda^1_{r_3}(\kappa_1, \dots, \kappa_{l_j})$.

By the above argument, σ_{l_k+1} lifts to the orbit of ν_k through the changes of the coordinates in the near-parabolic renormalisations. In particular, these cycles must have the same multipliers, as in the equation in the proposition. □

Corollary 6.3. *Let $\langle m_i : b_{i,j} : \varepsilon_{i,j}\rangle$ be a sequence of non-zero rational numbers in $(-1/2, 1/2]$. For every α in the intersection of $M_\alpha(\langle m_i : b_{i,j} : \varepsilon_{i,j}\rangle)$ and $\Lambda^1_{r_3}(\kappa(\langle m_i : b_{i,j} : \varepsilon_{i,j}\rangle))$, and every*

$k \geq 1$ *we have*

$$\beta_{l_k+1} = \frac{1}{2\pi \text{ i}} \log \rho_k.$$

6.4. **The quadratic growth condition and the complex rotations.** Here, our goal is to find a combinatorial condition, in terms of the combinatorial rotation numbers of the dividing periodic points, that guarantees an infinitely polynomial-like renormalisable map under that condition is infinitely near-parabolic renormalisable.

The algorithm defining the modified continued fraction expansion in Section 6.1 has a natural extension onto the complex plane which plays a crucial role in this section. It is defined as follows. Recall from Section 6.1 the closest integer function $[\cdot]$ defined on $\mathbb{R}$. We consider the map $\text{inv}(z) = -1/z$, for $z \in \mathbb{C} \setminus \{0\}$, and the map $\text{saw}(z) = z - [\text{Re}(z)]$, for $z \in \mathbb{C}$. We shall work with the composition of these two maps denote by

$$G(z) = \text{saw} \circ \text{inv}(z), \forall z \in \mathbb{C} \setminus \{0\}.$$

Then, G maps the interval $[-1/2, 0) \cup (0, 1/2]$ to $[-1/2, 1/2]$, and for a non-zero rational number x_1 in $[-1/2, 1/2]$,

$$x_i = G^{\circ i-1}(x_1), 1 \leq i \leq n.$$

where the numbers x_i are defined in Equation (6.1). In particular,

$$G([\langle b_j : \varepsilon_j \rangle_{j=i}^n]) = -\varepsilon_i \cdot [\langle b_j : \varepsilon_j \rangle_{j=i+1}^n].$$

That is, applying G to a continued fraction removes the first pair from the expansion, and then only modifies the first sign in the remaining expansion. In particular,

$$G^{\circ n-1}([\langle b_j : \varepsilon_j \rangle_{j=1}^n]) = \pm 1/b_n, \; G^{\circ n}([\langle b_j : \varepsilon_j \rangle_{j=1}^n]) = 0. \tag{6.7}$$

For each integer $b_1 \geq 2$ and $\varepsilon_1 \in \{+1, -1\}$, the image of the round ball of radius $1/2$ centred at $-\varepsilon_1 b_1$, $B(-\varepsilon_1 b_1, 1/2)$, under the map inv is a round ball containing ε_1/b_1. Note that this ball is not centred at ε_1/b_1. Let $\mathcal{F}_1$ denote the collection of all these balls for integers $b_1 \geq 2$ and $\varepsilon_1 \in \{+1, -1\}$. If we care to determine a specific ball in this collection, we use the notation $\mathcal{F}_1(\langle b_1 : \varepsilon_1 \rangle)$ to denote the one containing ε_1/b_1. It follows that $G : \mathcal{F}_1(\langle b_1 : \varepsilon_1 \rangle) \to B(0, 1/2)$ is a holomorphic bijection.

Similarly, for integers $n \geq 2$, we may define the collection $\mathcal{F}_n$ of round balls that are mapped onto $B(0, 1/2)$ by the iterate $G^{\circ n}$. The element of $\mathcal{F}_n$ containing $[\langle b_1 : \varepsilon_1 \rangle_{i=1}^n]$ is denoted by $\mathcal{F}_n(\langle b_i : \varepsilon_i \rangle_{i=1}^n)$ and we note that each such element is a disk that is symmetric with respect to the real line. Moreover,

$$G^{\circ n} : \mathcal{F}_n([\langle b_i : \varepsilon_i \rangle_{i=1}^n]) \to B(0, 1/2), n \geq 1,$$

is a holomorphic bijection given by a Möbius transformation that maps the real slice of the domain to the real slice of the image.

Lemma 6.4. *For every $n \geq 1$, every $[\langle b_j : \varepsilon_j \rangle_{j=1}^n] \in \mathbb{Q}$, every $z \in \mathcal{F}_n([\langle b_j : \varepsilon_j \rangle_{j=1}^n])$, and every k with $0 \leq k \leq n-1$, we have*

$$\frac{4}{5} \cdot \frac{1}{b_{k+1}} \leq |G^{\circ k}(z)| \leq \frac{4}{3} \cdot \frac{1}{b_{k+1}}, \quad \arg G^{\circ k}(z) \in \begin{cases} [\frac{-\pi}{4}, \frac{\pi}{4}] & \textit{if } \text{Re}(G^{\circ k}(z)) > 0 \\ [\frac{3\pi}{4}, \frac{5\pi}{4}] & \textit{if } \text{Re}(G^{\circ k}(z)) < 0 \end{cases}.$$

Proof. First note that

$$G^{\circ k}(\mathcal{F}_n([\langle b_j : \varepsilon_j\rangle_{j=1}^n])) = \mathcal{F}_{n-k}(G^{\circ k}([\langle b_j : \varepsilon_j\rangle_{j=1}^n])) \subseteq \mathcal{F}_1(G^{\circ k}([\langle b_j : \varepsilon_j\rangle_{j=1}^n])).$$

and the first pair in the expansion of $G^{\circ k}([\langle b_j : \varepsilon_j\rangle_{j=1}^n]$ has the form $(b_{k+1}, \pm 1)$. Hence, $G^{\circ k}(z)$ belongs to either $\mathcal{F}_1(1/b_{k+1})$ or $\mathcal{F}_1(-1/b_{k+1})$. Each of these sets is a round ball symmetric with respect to the real line passing through the pair of points $1/(b_{k+1}+1/2)$ and $1/(b_{k+1}-1/2)$ or the pair of points $-1/(b_{k+1}+1/2)$ and $-1/(b_{k+1}-1/2)$, respectively. In particular, for $G^{\circ k}(z) \in \mathcal{F}_1(\pm 1/b_{k+1})$,

$$\frac{4}{5}\cdot\frac{1}{b_{k+1}} \leq \frac{1}{b_{k+1}+1/2} \leq |z| \leq \frac{1}{(b_{k+1}-1/2)} \leq \frac{4}{3}\cdot\frac{1}{b_{k+1}}.$$

On the other hand, each of $\mathcal{F}_1(1/b_{k+1})$ and $\mathcal{F}_1(-1/b_{k+1})$ is a round disk of diameter

$$\frac{1}{b_{k+1}-1/2} - \frac{1}{b_{k+1}+1/2} \leq \frac{1}{\sqrt{2}b_{k+1}}.$$

Hence, $\mathcal{F}_1(\pm 1/b_{k+1})$ is contained in the round ball of radius $1/(\sqrt{2}b_{k+1})$ about $\pm 1/b_{k+1}$. This implies the bounds on $\arg(G^{\circ k}(z))$. □

Lemma 6.5. *There is a constant C_0 satisfying the following. Let $x = [\langle b_j : \varepsilon_j\rangle_{j=1}^n] \in \mathbb{Q}$, for some $n \geq 1$, and assume that $b_{j+1} \geq b_j^2$, for $1 \leq j \leq n-1$. Define the numbers*

$$x_0 = 1, x_i = G^{\circ(i-1)}(x), \quad 1 \leq i \leq n.$$

Then,

$$\frac{|x_n|}{|x_1|} \leq C_0 \cdot \prod_{i=0}^{n-1} |x_i|.$$

Proof. For each $1 \leq j \leq n$ we have

$$\big(1+\frac{1}{2b_j+1}\big)\frac{1}{b_j} = \frac{1}{b_j+1/2} \leq |x_j| \leq \frac{1}{b_j-1/2} = \frac{1}{b_j}\cdot\big(1+\frac{1}{2b_j-1}\big).$$

Then, since $b_{j+1} \geq b_j^2$, by the above equation we have

$$\begin{aligned}|x_{j+1}| &\leq \frac{1}{b_{j+1}}\big(1+\frac{1}{2b_{j+1}-1}\big)\\ &\leq \frac{1}{b_j^2}\big(1+\frac{1}{2b_{j+1}-1}\big) \leq |x_j|^2\big(1+\frac{1}{2b_{j+1}-1}\big)\big(1-\frac{1}{2b_j+2}\big)^2\\ &\leq |x_j|^2(1+\frac{C}{b_j}),\end{aligned}$$

for some uniform constant C. Then,

$$\frac{|x_n|}{\prod_{i=0}^{n-1}|x_i|} = \prod_{i=0}^{n-1}\big(\frac{|x_{i+1}|}{|x_i|^2}\big) \leq |x_1|\big(\prod_{i=1}^{n-1}(1+\frac{C}{b_i})\big) \leq |x_1|\exp(\sum_{i=1}^{\infty}\frac{C}{b_i}) \leq |x_1|e^C.$$

□

Lemma 6.6. *There exists $C_1 > 0$ such that for every $n \geq 1$ and every disk B_n in $\mathcal{F}_n$, the distortion of the map $G^{\circ n} : B_n \to B(0, 1/2)$ is bounded by C_1, that is,*

$$\forall z, w \in B_n, \frac{1}{C_1} \leq \Big|\frac{(G^{\circ n})'(z)}{(G^{\circ n})'(w)}\Big| \leq C_1.$$

Proof. Assume that B and B' are round disks that are symmetric with respect to the real line (invariant under complex conjugation), and $g : B \to B'$ be a Möbius map that sends $B \cap \mathbb{R}$ to $B' \cap \mathbb{R}$. Then, the distortion of g on B is realized at the two end points of the interval $B \cap \mathbb{R}$. This implies that the distortion of the map $G^{\circ n}$ on B_n is equal to its distortion on $B_n \cap \mathbb{R}$. Indeed, the latter statement is a classical result that follows from direct calculations. □

Let $N \geq 2$ be an integer and define the class of sequences of rational numbers[5]

$$\mathcal{QG}_N = \Big\{ \langle \frac{p_{i,m_i}}{q_{i,m_i}} \rangle_{i=1}^{\infty} = \langle m_i : b_{i,j} : \varepsilon_{i,j} \rangle \Big| \begin{array}{l} b_{1,1} \geq N; \forall i \geq 1, b_{i+1,1} \geq q_{i,m_i}^2 \\ \forall i \geq 1, 1 \leq j \leq m_i - 1, b_{i,j+1} \geq b_{i,j}^2. \end{array} \Big\}. \tag{6.8}$$

For example, if we let $m_i = 1$ for all $i \geq 1$, then a sequence $\langle p_{i,1}/q_{i,1} \rangle_{i=1}^{\infty}$ belongs to $\mathcal{QG}_N$ if and only if $q_1 \geq N$ and for all $i \geq 1$, $p_i \in \{+1, -1\}$ and $q_{i+1} \geq q_i^2$. However, choosing larger values of m_i at different stages allows us to cover more rational number at stage i. This imposes a stronger condition on the size of the next denominator through $q_{i+1} \geq b_{i,m_i}^2$.

Proposition 6.7. *For every $r \in (0, 1/2)$ there is a constant $N > 0$ satisfying the following property. Let $\langle p_i/q_i \rangle_{i=1}^{\infty}$ belong to $\mathcal{QG}_N$ and define the integers*

$$k_1 = 1, k_n = \prod_{i=1}^{n-1} q_{i,m_i}, \forall n \geq 2.$$

Then, for every $i \geq 1$ and every $z \in \mathbb{C}$ satisfying

$$\Big|z - (\frac{p_i}{q_i} - \mathrm{i}\, \frac{k_i}{q_i} \frac{\log 2}{2\pi})\Big| \leq \frac{\log 2}{2\pi} \frac{k_i}{q_i},$$

we have the following two properties.

a) For every n_i with $0 \leq n_i \leq m_i - 2$,

$$|G^{\circ n_i}(z)| \leq r, \text{ and } \arg(G^{\circ n_i}(z)) \in \begin{cases} [-\frac{\pi}{4}, \frac{\pi}{4}] & \text{if } \mathrm{Re}(G^{\circ n_i}(z)) > 0 \\ [\frac{3\pi}{4}, \frac{5\pi}{4}] & \text{if } \mathrm{Re}(G^{\circ n_i}(z)) < 0 \end{cases};$$

b)

$$|G^{\circ (m_i - 1)}(z)| \leq r.$$

Proof. Let us choose $\langle m_i : b_{i,j} : \varepsilon_{i,j} \rangle$ so that

$$\frac{p_i}{q_i} = \frac{p_{i,m_i}}{q_{i,m_i}} = [\langle b_{i,j} : \varepsilon_{i,j} \rangle_{j=1}^{m_i}].$$

[5] $\mathcal{QG}$ stands for **quadratic growth**.

Recall the constants C_0 and C_1 introduced in Lemmas 6.5 and 6.6. Then, choose $N \geq 2$ such that the following inequalities hold.

$$\frac{1}{N}\frac{4}{3} \leq \frac{r}{3}, \tag{6.9}$$

$$\frac{1}{N}C_1 \leq \frac{1}{C_1} \cdot \frac{\sqrt{C_1}-1}{\sqrt{C_1}+1} \cdot \frac{r}{3}, \tag{6.10}$$

$$\frac{1}{N}C_1 \frac{4}{3}\frac{\log 2}{2\pi} C_1 C_0 \leq \frac{1}{C_1} \cdot \frac{\sqrt{C_1}-1}{\sqrt{C_1}+1} \cdot \frac{r}{3} \tag{6.11}$$

We break the proof into several steps.

Step 1. For $\langle m_i : b_{i,j} : \varepsilon_{i,j}\rangle \in \mathcal{QG}_N$, by Lemma 6.4, for every $i \geq 1$ and every n_i with $0 \leq n_i \leq m_i - 1$, by Equation (6.9), we have

$$\left|G^{\circ n_i}\big([\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i}]\big)\right| \leq \frac{4}{3} \cdot \frac{1}{b_{i,n_i+1}} \leq \frac{4}{3} \cdot \frac{1}{b_{i,1}} \leq \frac{4}{3} \cdot \frac{1}{b_{1,1}} \leq \frac{4}{3} \cdot \frac{1}{N} \leq \frac{r}{3}. \tag{6.12}$$

Step 2. By the definition of the class $\mathcal{QG}_N$ and Equation (6.2), for every $i \geq 1$,

$$q_{i+1,m_{i+1}} \geq q_{i+1,1} = b_{i+1,1} \geq q_{i,m_i}^2. \tag{6.13}$$

For $i = 1$, by Equation (6.9), we have

$$\frac{\log 2}{2\pi} \cdot \frac{k_1}{q_{1,m_1}} = \frac{\log 2}{2\pi} \cdot \frac{1}{q_{1,m_1}} \leq \frac{\log 2}{2\pi} \cdot \frac{1}{b_{1,1}} \leq \frac{\log 2}{2\pi} \cdot \frac{1}{N} \leq \frac{r}{3}. \tag{6.14}$$

Recall that $q_{i,0} = 1$, for $i \geq 1$. This is for the simplicity of the formulas in the following. For $i \geq 2$,

$$\begin{aligned} \frac{\log 2}{2\pi} \cdot \frac{k_i}{q_{i,m_i}} &= \frac{\log 2}{2\pi} \cdot \Big(\prod_{l=1}^{i-1} q_{l,m_l}\Big) \cdot \frac{1}{q_{i,m_i}} \\ &= \frac{\log 2}{2\pi} \prod_{l=0}^{i-1} \Big(\frac{q_{l,m_l}^2}{q_{l+1,m_{l+1}}}\Big) \\ &\leq \frac{\log 2}{2\pi} \cdot \frac{1}{q_{1,1}} \leq \frac{\log 2}{2\pi} \cdot \frac{1}{b_{1,1}} \qquad \text{(Eq. (6.13))} \\ &\leq \frac{\log 2}{2\pi} \cdot \frac{1}{N} \leq \frac{r}{2}. \end{aligned} \tag{6.15}$$

Step 3. Assume that $m_i = 1$ for some $i \geq 1$. For every z satisfying the hypothesis of the proposition, by Equations (6.12), (6.14), and (6.15), z belongs to a disk of radius bounded from above by $r/3$ attached to the real line at $p_{i,m_i}/q_{i,m_i}$ with $|p_{i,m_i}/q_{i,m_i}| \leq r/3$. Hence, $|z| \leq r/3 + 2r/3 = r$. This implies the inequality in part b), and there is nothing to prove for part a).

Let us fix an $i \geq 1$. From now on we assume that $m_i > 1$.

Step 4. Recall that

$$G^{\circ(m_i-1)} : \mathcal{F}_{m_i-1}([\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i-1}]) \to B(0,1/2)$$

is a bijection. Let us define the set

$$B_i^r \subseteq \mathcal{F}_{m_i-1}([\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i-1}]) \tag{6.16}$$

as the pre-images of $B(0,r)$ under the above restriction of $G^{\circ(m_i-1)}$. Then, B_i^r is a round ball containing the point $[\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i-1}]$.

Define the numbers $x_{i,l}$, for $i \geq 1$, as

$$x_{i,0} = 1,$$

as well as the numbers $x_{i,l}$, for $i \geq 1$ and $1 \leq l \leq m_i - 1$, as

$$x_{i,l} = \prod_{k=1}^{l} \left|[\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=k}^{m_i-1}]\right| = \prod_{k=0}^{l-1} \left|G^{\circ k}([\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i-1}])\right|.$$

For every l with $0 \leq l \leq m_i - 1$ we have

$$\left|(G^{\circ l})'([\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i-1}])\right| = (x_{i,l})^{-2}. \tag{6.17}$$

Hence, by the uniform bound on the distortion of $G^{\circ m_i-1}$ in Lemma 6.6, we have

$$r\frac{1}{C_1}x_{i,m_i-1}^2 \leq \operatorname{diam}(B_i^r) \leq rC_1 x_{i,m_i-1}^2. \tag{6.18}$$

Although the ball B_i^r is not centred at $[\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i-1}]$, the uniform bound on the distortion of $G^{\circ m_i-1}$ implies that the center of B_i^r is not too far from $[\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i-1}]$. One can verify that if h is a Möbius map of the unit disk with $1/C_1 \leq |h'(x)|/|h'(y)| \leq C_1$ for all x and y in the unit disk, then $|h(0)| \leq (\sqrt{C_1} - 1)/(\sqrt{C_1} + 1)$. By virtue of Lemma 6.6,

$$B\Big([\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i-1}], \frac{1}{C_1}\frac{\sqrt{C_1}-1}{\sqrt{C_1}+1}x_{i,m_i-1}^2 r\Big) \subseteq B_i^r. \tag{6.19}$$

That is, B_i^r contains a round ball of size comparable to rx_{i,m_i-1}^2 about $[\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i-1}]$. Let us define the constant

$$C_2 = \frac{1}{C_1}\frac{\sqrt{C_1}-1}{\sqrt{C_1}+1}.$$

Step 5. Define the numbers $y_{i,l}$ as

$$y_{i,0} = 1,\ i \geq 1,$$

and

$$y_{i,l} = \prod_{k=1}^{l} \left|[\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=k}^{m_i}]\right| = \prod_{k=0}^{l-1} \left|G^{\circ k}([\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i}])\right|, \quad i \geq 1 \text{ and } 1 \leq l \leq m_i. \tag{6.20}$$

By Equation (6.3), we have

$$\frac{1}{q_{i,m_i}} = y_{i,m_i}.$$

By Lemma 6.6, we have

$$\frac{1}{C_1}x_{i,m_i-1} \leq y_{i,m_i-1} \leq C_1 x_{i,m_i-1},$$

which implies

$$\frac{1}{C_1} x_{i,m_i-1} \frac{1}{b_{i,m_i}} \le \frac{1}{q_{i,m_i}} = y_{i,m_i} \le C_1 x_{i,m_i-1} \frac{1}{b_{i,m_i}}. \tag{6.21}$$

Step 6. Recall from Equation (6.7) that $G^{\circ m_i-1}(p_{i,m_i}/q_{i,m_i}) = \pm 1/b_{i,m_i}$ and $G^{\circ m_i-1}([\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i-1}]) = 0$. By the uniform bound on the distortion of $G^{\circ m_i-1}$ in Lemma 6.6 and the explicit value of its derivative at $[\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i-1}]$ in Equation (6.17), we obtain

$$\begin{aligned} \left|\frac{p_{i,m_i}}{q_{i,m_i}} - [\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i-1}]\right| &\le C_1 x_{i,m_i-1}^2 \cdot \frac{1}{b_{i,m_i}} \\ &\le C_1 x_{i,m_i-1}^2 \cdot \frac{1}{b_{i,1}} \le C_1 x_{i,m_i-1}^2 \cdot \frac{1}{b_{1,1}} \\ &\le C_1 x_{i,m_i-1}^2 \cdot \frac{1}{N} \le \frac{r}{3} \cdot C_2 \cdot x_{i,m_i-1}^2. \end{aligned} \tag{6.22}$$

In the last line of the above equation we have used Equations (6.10).

Step 7. We have,

$$\begin{aligned} \frac{k_i}{q_{i,m_i}} \cdot \frac{1}{x_{i,m_i-1}^2} &\le C_1 \frac{k_i}{1} \cdot \frac{x_{i,m_i-1}}{b_{i,m_i} x_{i,m_i-1}^2} && \text{(Eq. (6.21))} \\ &= C_1 \frac{k_i}{b_{i,1}} \cdot \frac{b_{i,1}}{b_{i,m_i} x_{i,m_i-1}} \\ &\le C_1 \frac{k_i}{q_{i-1}^2} \cdot \frac{b_{i,1}}{b_{i,m_i} x_{i,m_i-1}} && \text{(Eq. (6.13))} \\ &\le C_1 \frac{k_i}{q_{i-1}^2} \cdot \frac{4}{3} \frac{1}{y_{i,1} b_{i,m_i} x_{i,m_i-1}} && \text{(Lemma 6.4 with } k=0) \\ &\le C_1 \frac{1}{N} \cdot \frac{4}{3} C_1 \frac{1}{y_{i,1} b_{i,m_i}^2 y_{i,m_i}} && \text{(Eq. (6.15), Eq. (6.21))} \\ &= C_1 \frac{1}{N} \cdot \frac{4}{3} C_1 \frac{1}{y_{i,1} y_{i,m_i-1}} \cdot \frac{1}{b_{i,m_i}} && \text{(Eq. (6.20))} \\ &\le C_1 \frac{1}{N} \cdot \frac{4}{3} C_1 C_0 && \text{(Lem. 6.5)} \end{aligned}$$

In particular, by Equation (6.11), the above inequalities imply that

$$\frac{\log 2}{2\pi} \cdot \frac{k_i}{q_{i,m_i}} \le \frac{r}{3} C_2 x_{i,m_i-1}^2. \tag{6.23}$$

Step 8. Let $z \in \mathbb{C}$ be a point satisfying the hypothesis of the proposition. By Equation (6.23), z belongs to a ball of radius at most $\frac{r}{3} C_2 x_{i,m_i-1}^2$ that is tangent to the real line at $p_{i,m_i}/q_{i,m_i}$. On the other hand, by Equation (6.22), $p_{i,m_i}/q_{i,m_i}$ is within $\frac{r}{3} C_2 x_{i,m_i-1}^2$ distance from $[\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i-1}]$. Hence, $|z - [\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i-1}]| \le r C_2 x_{i,m_i-1}^2$. By Equations (6.19) and (6.16), the above inequality implies that

$$z \in B_i^r \subseteq \mathcal{F}_{m_i-1}([\langle b_{i,j} : \varepsilon_{i,j}\rangle_{j=1}^{m_i-1}]).$$

This finishes the proof of the proposition, by virtue of Lemma 6.4. □

6.5. **Rigidity of complex quadratic polynomials.** Recall the constant r_1 introduced in Proposition 2.1.

Lemma 6.8. *There is a constant $r_5 > 0$ satisfying the following. Let $f \in A(r_1) \ltimes \mathcal{F}_0$ or $f \in A(r_3) \ltimes \{Q_0\}$, with $|f'(0)| \geq 1$. If $\beta(f) \in A(r_5)$ then $\alpha(f) \in A(r_3)$.*

Proof. Recall the domain W from Lemma 2.1 and the constant B_4 from Lemma 3.24 such that for every $f \in A(r_3) \ltimes \mathcal{F}_0$ or $f \in A(r_3) \ltimes \{Q_0\}$ we have

$$\frac{1}{2\pi}\Big| \int_{\partial W} \frac{1}{z - f(z)}\, dz \Big| \leq B_4.$$

By the holomorphic index formula (2.2), this implies that

$$\Big|\frac{1}{1 - e^{2\pi i \alpha(f)}} + \frac{1}{1 - e^{2\pi i \beta(f)}}\Big| \leq B_4,$$

On the other hand, since $|f'(0)| \geq 1$ and $|f'(\sigma(f))| \geq 1$, we must have $\operatorname{Im} \alpha(f) \leq 0$ and $\operatorname{Im} \beta(f) \leq 0$. By an elementary calculation one can verify that there is $r_5 > 0$ such that if $\beta(f) \in A(r_5)$ then $\alpha(f) \in A(r_3)$. □

Remark 6.9. Indeed, the proof of the above lemma implies an stronger remarkable property on the relation between $\alpha(f)$ and $\beta(f)$. That is, the set of $\alpha(f)$ such that $\beta(f)$ is real and belongs to $(-r_5, r_5)$, is tangent to the real line at 0 with the order of the tangency being quadratic. One may use the pre-compactness of the $\mathcal{F}_0$ to prove stronger bounds on the location of this curve, which in turn may be used to give estimates on the location of the multipliers of the dividing periodic points of corresponding Q_α.

The following proposition is the main statement of this section. Recall the integers l_k introduced in Equation (6.5) and the map κ introduced in Equation (6.6).

Proposition 6.10. *Given $r_3 > 0$ as in Theorem 2.9 there is an integer $N > 0$ such that for every sequence of rational numbers $\langle m_i : b_{i,j} : \varepsilon_{i,j} \rangle \in \mathcal{QG}_N$ in the interval $(-1/2, 1/2]$ and every integer $k \geq 1$, $M_\alpha(\langle m_i : b_{i,j} : \varepsilon_{i,j} \rangle_{i=1}^k)$ is contained in $\Lambda^1_{r_3}(\langle \kappa_i \rangle_{i=1}^{l_k})$, where $\kappa = \kappa(\langle m_i : b_{i,j} : \varepsilon_{i,j} \rangle)$.*

The above proposition combined with Theorem 2.9 provides us with a constant N such that for every $\langle m_i : b_{i,j} : \varepsilon_{i,j} \rangle \in \mathcal{QG}_N$ and every $\alpha \in M_\alpha(\langle m_i : b_{i,j} : \varepsilon_{i,j} \rangle)$, Q_α is infinitely near-parabolic renormalisable of type $\kappa(\langle m_i : b_{i,j} : \varepsilon_{i,j} \rangle)$.

Proof. Let r_5 be the constant obtained in Lemma 6.8. Let N_1 be the constant obtained in Proposition 6.7 with $r = \min\{r_5, r_3\}$, and choose $N \geq N_1$ such that

$$\frac{\log 2}{2\pi} \cdot \frac{1}{N} \leq \frac{r_3}{2}, \; \frac{4}{3} \cdot \frac{1}{N} \leq \frac{r_3}{2}.$$

Fix $\langle m_i : b_{i,j} : \varepsilon_{i,j} \rangle$ in $\mathcal{QG}_N$, $k \geq 1$, and $\alpha \in M_\alpha(\langle m_i : b_{i,j} : \varepsilon_{i,j} \rangle_{i=1}^k)$. Define the type $\kappa = \kappa(\langle m_i : b_{i,j} : \varepsilon_{i,j} \rangle)$. We need to show that starting with $f_1 = Q_\alpha$, the sequence of maps f_i, for $1 \leq i \leq l_k$, in equation (5.2) is defined, and each $\alpha_i \in A(r_3)$. We prove this by an inductive argument.

Recall that $\beta_1 = \frac{1}{2\pi i} \log \rho_1$, Equation (6.4). By the PLY inequality (4.2), and the above condition on N, α is contained in a disk of radius bounded by $r_3/2$ attached to the real line at $[\langle b_{1,j} : \varepsilon_{1,j} \rangle_{j=1}^{m_1}]$. Moreover, by Equation (6.12),

$$\left|[\langle b_{1,j} : \varepsilon_{1,j} \rangle_{j=1}^{m_1}]\right| \leq \frac{4}{3} \cdot \frac{1}{N} \leq \frac{r_3}{2}.$$

Hence, α is contained in $A(r_3)$. Therefore, by Theorem 2.9 and Definition 2.11, f_1 is near-parabolic renormalisable of type $\kappa_1 = b$. That is, $\mathcal{R}_{\text{NP-b}}(Q_\alpha)$ is defined.

By the definitions, $l_2 = m_1$ and $\alpha_2 = -1/\beta_1$. Also, for all i with $2 \leq i \leq l_2$ (if there is any), we have $\kappa_i = t$. Thus, for all such i, we have $\alpha_{i+1} = -1/\alpha_i$. Proposition 6.7, combined with the PLY inequality, implies that, for every i with $2 \leq i \leq l_2$, $\alpha_i \in A(r_3)$. In particular, this implies that for every i with $2 \leq i \leq l_2+1$, f_i is defined. But we still don't know whether α_{l_2+1} belongs to $A(r_3)$ or not.

Let j be an integer with $1 \leq j \leq k-1$. For $\alpha \in M_\alpha(\langle m_i : b_{i,j} : \varepsilon_{i,j} \rangle_{i=1}^{k})$, we want to show that if $\alpha \in \Lambda^1_{r_3}(\langle \kappa_i \rangle_{i=1}^{l_j})$ then $\alpha \in \Lambda^1_{r_3}(\langle \kappa_i \rangle_{i=1}^{l_{j+1}})$. Since $\alpha \in \Lambda^1_{r_3}(\langle \kappa_i \rangle_{i=1}^{l_j})$, by definition, α_{l_j} belongs to $A(r_3)$ and hence, f_{l_j+1} is defined. However, since α belongs to $M_\alpha(\langle m_i : b_{i,j} : \varepsilon_{i,j} \rangle_{i=1}^{j+1})$, PLY inequality, Corollary 6.3, and Proposition 6.7 with $n_j = 0$, imply that $\beta_{l_j+1} = \frac{1}{2\pi i} \log \rho_j$ belongs to $A(r_5)$. Then, by Lemma 6.8, α_{l_j+1} belongs to $A(r_3)$, and therefore, $\mathcal{R}_{\text{NP-b}}(f_{l_j+1})$ is defined. Note the choice of N and N_1 at the beginning of the proof. By the definition, $\kappa_{l_j+1} = b$ and for all l with $l_j + 2 \leq l \leq l_{j+1}$ (if there is any) $\kappa_l = t$. That is, $\alpha_{l_j+2} = -1/\beta_{l_j+1}$ and $\alpha_{l+1} = -1/\alpha_l$ for all l with $l_j + 2 \leq l \leq l_{j+1}$. Now we use Proposition 6.7 to conclude that for every l with $l_j \leq l \leq l_{j+1}$, $\alpha_l \in A(r_3)$ and f_{l+1} is defined.

By an inductive argument, the proposition follows from the above paragraphs. □

Proposition 6.11. *Let p_i/q_i, $i \geq 1$, be a sequence of non-zero rational numbers in $(-1/2, 1/2]$ such that for every c in $M(\langle p_i/q_i \rangle_{i=1}^{\infty})$, $Q_{\alpha(c)}$ is infinitely near parabolic renormalisable and for every $n \geq 1$ the rotation α_n belongs to $A(r_3)$. Then, the nest of Mandelbrot copies $M(\langle p_i/q_i \rangle_{i=1}^{n})$ shrinks to a single point.*

Proof. By the hypothesis, $M_\alpha(p_1/q_1, p_2/q_2, \dots)$ is contained in $\Lambda^1_{r_3}(\kappa)$, where

$$\kappa = \kappa(\langle p_i/q_i \rangle_{i=1}^{\infty})$$

is defined in Equation (6.6). By Theorem 5.1, the connected set $M_\alpha(\langle p_i/q_i \rangle_{i=1}^{\infty})$ must be a single point. □

We will not use the following proposition in this paper, but it is stated for future purposes.

Proposition 6.12. *For every sequence of rational numbers $\langle m_i : a_{i,j} : \varepsilon_{i,j} \rangle$ with $a_{i,j} \geq N$, there is $\alpha \in M_\alpha(\langle m_i : a_{i,j} : \varepsilon_{i,j} \rangle)$ such that Q_α is infinitely near-parabolic renormalisable of type $\kappa(\langle m_i : a_{i,j} : \varepsilon_{i,j} \rangle)$.*

Proof. By the continuity of the relations between α_n and β_n as well as α_{n-1} in terms of α_n or β_n, there is $\alpha \in A(r_3)$ such that Q_α is infinitely near parabolic renormalisation of type κ. On the other hand, if Q_α is infinitely near-parabolic renormalisable, then the orbit of the critical point remains uniformly bounded in $\mathbb{C}$. This implies that α belongs to M_α. It is not difficult to see that Q_α has the correct combinatorial rotations of the dividing periodic points. More details shall be added later. □

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