
TYPICAL ORBITS OF QUADRATIC POLYNOMIALS WITH A NEUTRAL FIXED POINT: BRJUNO TYPE

by

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Abstract. — We describe the topological behavior of typical *orbits* of complex quadratic polynomials $P_\alpha(z) = e^{2\pi\alpha i}z + z^2$, with α of *high return* type. Here we prove that for such Brjuno values of α the closure of the *critical orbit*, which is the *measure theoretic attractor* of the map, has zero area. Then we show that the *limit set* of the orbit of a typical point in the *Julia set* of P_α is equal to the closure of the critical orbit. Our method is based on the near parabolic renormalization of Inou-Shishikura, and a uniform optimal estimate on the derivative of the Fatou coordinate that we prove here.

Introduction

The local, semi-local, and global dynamics of the maps

$$P_\alpha(z) := e^{2\pi\alpha i}z + z^2 : \mathbb{C} \rightarrow \mathbb{C},$$

for irrational values of α , have been extensively studied through various methods over the last decades. The aim of this work is to describe the topological behavior of the orbit of typical points under these maps. This is a step toward understanding the measurable dynamics of these maps.

The *post-critical* set of P_α is defined as the closure of the orbit of the critical value;

$$\mathcal{PC}(P_\alpha) := \overline{\cup_{j=1}^{\infty} P_\alpha^{\circ j}(-e^{2\pi\alpha i}/2)}.$$

It is well-known [Lyu83] that $\mathcal{PC}(P_\alpha)$ is the *measure theoretic attractor* of the dynamics of P_α on its *Julia set* $J(P_\alpha)$. That is, the orbit of Lebesgue almost every point in $J(P_\alpha)$ eventually stays in any given neighborhood of $\mathcal{PC}(P_\alpha)$. To understand the long term behavior of typical orbits in $J(P_\alpha)$, one needs to understand the structure of the set $\mathcal{PC}(P_\alpha)$ and the iterates of P_α near $\mathcal{PC}(P_\alpha)$.

Two different scenarios occur depending on the local dynamics of P_α at zero. Let $\alpha := [0; a_1, a_2, a_3, \dots]$ denote the continued fraction expansion of α with the

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convergents $p_n/q_n := [0; a_1, a_2, \dots, a_n]$. By classical theorems of Siegel and Brjuno [Sie42, Brj71], if the series $\sum_{n=1}^{\infty} \log q_{n+1}/q_n$ is finite, the map P_α is *linearizable* at zero, i.e. it is locally conformally conjugate to a rotation. When linearizable, the maximal domain of linearization is called the *Siegel disk* of P_α . The values of α for which the above sum is convergent are called *Brjuno numbers*. On the other hand, Yoccoz [Yoc95] proved that the convergence of this series is necessary for the linearizability of P_α . In [Mañ93], Mañé proves that for irrational α the orbit of the critical point of P_α is recurrent, and accumulates at the boundary of the Siegel disk (at zero) when P_α is linearizable (non-linearizable, respectively).

Petersen and Zakeri in [PZ04] describe the topology and geometry of the dynamics of the linearizable maps P_α for a.e. $\alpha \in [0, 1]$. For these values, they show that $J(P_\alpha)$ has zero area. However, a rather surprising result of Buff and Chéritat [BC12] states that there are parameters α , both of Brjuno and non-Brjuno type, for which $J(P_\alpha)$ has positive area. It is conjectured [Ché09] that for generic values of α , $J(P_\alpha)$ has positive area. Knowing all these, still there is not a single example of a quadratic map with a non-linearizable fixed point whose local dynamics is completely understood.

A major breakthrough in the field by Inou and Shishikura [IS06] has allowed further progress in the study of these maps. It is an essential part of [BC12] and is used in [Shi10] to show that the boundary of these Siegel disks are Jordan curves. Roughly speaking, Inou-Shishikura show that successive *renormalizations* of P_α (a sophisticated version of successive return maps depicted in Figure 1) are defined on “large enough” domains, and belong to a compact class of maps. This scheme requires the digits in the expansion of α to be larger than some constant N ,⁽¹⁾ i.e.

$$\alpha \in \text{Irrat}_{\geq N} := \{[0; a_1, a_2, \dots] \in (0, 1) \mid \inf_i a_i \geq N\}.$$

In [Che10] we started a systematic study of the measurable dynamics of these maps by quantifying the renormalization scheme of Inou-Shishikura as well as estimating the changes of coordinates between consecutive renormalization levels. This was mainly applied to the study of non-linearizable maps. In particular, we showed that for non-Brjuno values of α , $\mathcal{PC}(P_\alpha)$ is non-uniformly porous⁽²⁾ (and hence has zero area). Here, we prove the following counterpart.

Theorem A. — *For every Brjuno $\alpha \in \text{Irrat}_{\geq N}$, $\mathcal{PC}(P_\alpha)$ has zero area.*

When α is non-Brjuno, the arithmetic of α satisfies a particular property that we exploited in [Che10] to prove the existence of complementary holes at arbitrarily small scales on $\mathcal{PC}(P_\alpha) \setminus \{0\}$. But that arithmetic property does not necessarily hold for all Brjuno values of α , and hence, we need a different approach to prove the above theorem. In particular, the global estimates that appeared in [Che10] I are not enough and we need to prove a uniform infinitesimal estimate on the changes of coordinates.

⁽¹⁾However, they conjecture that $N = 1$.

⁽²⁾A set $E \subseteq \mathbb{C}$ is said to be non-uniformly porous, if there exists a $\lambda \in (0, 1)$ satisfying the following property. For every $z \in E$ there exists a sequence of real numbers $r_n \rightarrow 0$ such that the ball of radius r_n about z contains a ball of radius λr_n disjoint from E , for every n .

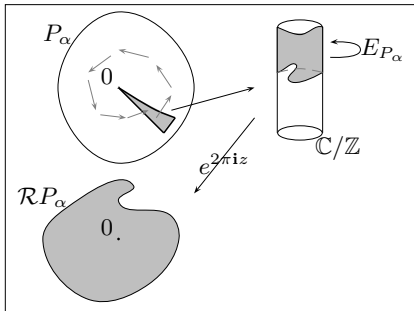


FIGURE 1. Identifying the sides of a sector landing at zero under P_α one obtains a half infinite cylinder which projects onto a neighborhood of zero under $e^{2\pi iz}$. The return map to this sector under P_α induces a map on that neighborhood, $\mathcal{R}P_\alpha$, called the renormalization of P_α .

Note that by Mane's Theorem, the boundary of the Siegel disk is contained in the post-critical set, and hence, must have zero area by the above result. The next statement is an immediate corollary of Theorem A.

Corollary B. — *For every Brjuno $\alpha \in \text{Irrat}_{\geq N}$, almost every point in $J(P_\alpha)$ is non-recurrent. In particular, there is no finite absolutely continuous invariant measure on $J(P_\alpha)$.*

Theorem A and its counterpart for non-Brjuno values of α in [Che10] enable us to prove the following statement here.

Theorem C. — *For all $\alpha \in \text{Irrat}_{\geq N}$, the limit set of the orbit of almost every point in $J(P_\alpha)$ is equal to $\mathcal{PC}(P_\alpha)$.*

Let $\overline{\Delta}_\alpha$ denote the closure of the Siegel disk of P_α . By [Her85] there are Brjuno values of α for which $\overline{\Delta}_\alpha$ does not contain the critical point, and therefore, $\mathcal{PC}(P_\alpha) \setminus \overline{\Delta}_\alpha$ is non-empty. Conjecturally, this set is homeomorphic to the Cantor bouquet minus its root. Our analysis of the post-critical set allows us to prove the following geometric property of these decorations.

Theorem D. — *For all $\alpha \in \text{Irrat}_{\geq N}$, $\mathcal{PC}(P_\alpha)$ is non-uniformly porous at every point in the set $\mathcal{PC}(P_\alpha) \setminus \overline{\Delta}_\alpha$.⁽³⁾*

Given an analytic germ f defined near zero with $f(0) = 0$ and $|f'(0)| = 1$, one blows up a small neighborhood of 0 under a suitable covering transformation and obtains a new map F defined near infinity that is close to the translation by one. The *Fatou coordinate* of F is a conformal change of coordinate that conjugates F to the translation by one above a horizontal line. Estimates on these changes of coordinates have been the core part of most of the results concerning the dynamics of near

⁽³⁾It follows from the proof of this theorem that under various arithmetic conditions on α , one can replace $\mathcal{PC}(P_\alpha) \setminus \overline{\Delta}_\alpha$ by $\mathcal{PC}(P_\alpha)$. However, we do not believe that one can do this for all Brjuno α .

parabolic maps, see [Shi98, ABC04, IS06, BC12, Che10], et al. However, the errors of the infinitesimal estimates proved so far, mainly in [Shi00, Shi98, IS06], have divergent integrals over infinite vertical strips of width one. In Section 5 we present a new estimate on the derivative of the Fatou coordinate with exponentially decaying (as a function of Imaginary part) error. Indeed, we show that given a compact class of germs f (in the compact open topology), the L^1 norm of the error (over infinite vertical strips) is uniformly bounded by a constant divided by the rotation of f at 0. This estimate allows us to prove the results under the sharp Brjuno condition on α here, and will be used to describe the statistical properties of the quadratic maps P_α of high type in [AC12]. Moreover, one may use these estimates to answer some major optimality questions, like Herman’s condition, in the class of quadratic maps of high type. Our approach to prove this estimate is different from the earlier ones, and is based on proposing an explicit C^2 change of coordinate that satisfies the optimal estimate and is “nearly conformal” and “nearly harmonic”. It is an application of Green’s Integral Formula (Hilbert transform) to compare the actual perturbed Fatou coordinate to this model.

For the parameters at the other end, i.e. when $\sup_i a_i < \infty$, the dynamic of P_α has been beautifully described in [McM98]. See [BC04, BC07, Yam08, BBCO10] and the references therein for some recent advancements in other aspects of the dynamics of these quadratic maps.

Frequently used notations. —

- $:=$ is used when a notation appears for the first time.
- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the integer, rational, real, and complex numbers, respectively.
- The bold face \mathbf{i} denotes the imaginary unit complex number.
- $\operatorname{Re} z, \operatorname{Im} z$, and $|z|$ denote the real part, the imaginary part, and the absolute value of a complex number z , respectively.
- $B(y, \delta) \subset \mathbb{C}$ denotes the ball of radius δ around y in the Euclidean metric. Given a set $X \subseteq \mathbb{C}$, $B(X, \delta) := \cup_{y \in X} B(y, \delta)$.
- $\operatorname{int}(S)$ denotes the interior of a set $S \subset \mathbb{C}$.
- $f^{\circ n}$ denotes the n times composition of a map f with itself, $f^{\circ 0} = \operatorname{id}$.
- $\operatorname{Dom} f, J(f)$, and $\mathcal{PC}(f)$ denote the domain of definition, the Julia set, and the post-critical set of a map f , respectively.
- Univalent map refers to a one to one holomorphic map.
- Given $g: \operatorname{Dom} g \rightarrow \mathbb{C}$, with only one critical point in its domain of definition, cp_g and cv_g denote the critical point and the critical value of g , respectively.
- For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x .

1. Preliminaries on renormalization

1.1. Inou-Shishikura class. — Consider the cubic polynomial $P(z) := z(1+z)^2$. This polynomial has a *parabolic* fixed point at 0, that is, a fixed point of multiplier $e^{2\pi\alpha\mathbf{i}}$ with $\alpha \in \mathbb{Q}$. It has a critical point at $\operatorname{cp}_P := -1/3$ with $P(\operatorname{cp}_P) := \operatorname{cv}_P = -4/27$, and another critical point at -1 which is mapped to 0 under P .

Consider the ellipse

$$E := \left\{ x + iy \in \mathbb{C} \mid \left(\frac{x + 0.18}{1.24} \right)^2 + \left(\frac{y}{1.04} \right)^2 \leq 1 \right\},$$

and let

$$(1) \quad U := g(\hat{\mathbb{C}} \setminus E), \text{ where } g(z) := -\frac{4z}{(1+z)^2}.$$

The domain U contains 0 and cp_P , but not -1 . Following [IS06], we define the classes of maps

$$\mathcal{IS} := \left\{ f := P \circ \varphi^{-1}: U_f \rightarrow \mathbb{C} \mid \begin{array}{l} \varphi: U \rightarrow U_f \text{ is univalent, } \varphi(0) = 0, \varphi'(0) = 1, \\ \text{and } \varphi \text{ has a quasi-conformal}^{(4)} \text{ extension to } \mathbb{C}. \end{array} \right\},$$

and,

$$\mathcal{IS}_A := \{ f(e^{2\pi\alpha i} z) \mid f \in \mathcal{IS}, \text{ and } \alpha \in A \}, \text{ where } A \subseteq \mathbb{R}.$$

Abusing the notation, \mathcal{IS}_β denotes the set $\mathcal{IS}_{\{\beta\}}$, for $\beta \in \mathbb{R}$.

Every map in \mathcal{IS} has a parabolic fixed point at 0 and a unique critical point at $\text{cp}_f := \varphi(-1/3) \in U_f$.

Consider a map $h: \text{Dom } h \rightarrow \mathbb{C}$, where $\text{Dom } h$ denotes the domain of definition (always assumed to be open) of h . Given a compact set $K \subset \text{Dom } h$ and an $\varepsilon > 0$, a neighborhood of h in the compact-open topology is defined as

$$\mathcal{N}(h; K, \varepsilon) := \{ g: \text{Dom } g \rightarrow \mathbb{C} \mid K \subset \text{Dom } g, \text{ and } \sup_{z \in K} |g(z) - h(z)| < \varepsilon \}.$$

In this topology, a sequence $h_n: \text{Dom } h_n \rightarrow \mathbb{C}$, $n = 1, 2, \dots$, converges to a map h if for any neighborhood of h defined as above, h_n is contained in that neighborhood for sufficiently large n . Note that the maps h_n need not be defined on the same domains.

The class \mathcal{IS}_A naturally embeds into the space of univalent maps on the unit disk with a neutral fixed point at 0. Therefore, it is a precompact class in the compact-open topology. In particular, it follows from the Koebe distortion Theorem that $\{h''(0) \mid h \in \mathcal{IS}_\mathbb{R}\}$ is relatively compact in $\mathbb{C} \setminus \{0\}$.

Any map $h = e^{2\pi\alpha i} f \in \mathcal{IS}_\alpha$ has a fixed point at 0 with multiplier $e^{2\pi\alpha i}$. Moreover, if α is small, h has another fixed point $\sigma_h \neq 0$ near 0 in U_h . The σ_h fixed point depends continuously on h and has asymptotic expansion $\sigma_h = -4\pi\alpha i / f''(0) + o(\alpha)$, when h converges to $f \in \mathcal{IS}$ in a fixed neighborhood of 0. Clearly $\sigma_h \rightarrow 0$ as $\alpha \rightarrow 0$.

The following theorem introduces a useful coordinate to study the local dynamics of maps in \mathcal{IS}_α . See Figure 2 for a geometric description of the following Theorem.

Theorem 1.1 (Inou–Shishikura [IS06]). — *There exists $\alpha_* \in (0, 1)$ such that for every $h: U_h \rightarrow \mathbb{C}$ in \mathcal{IS}_α (or $h = P_\alpha: \mathbb{C} \rightarrow \mathbb{C}$) with $\alpha \in (0, \alpha_*]$, there exist a domain $\mathcal{P}_h \subset U_h$ and a univalent map $\Phi_h: \mathcal{P}_h \rightarrow \mathbb{C}$ satisfying the following properties:*

- i. *The domain \mathcal{P}_h is bounded by piecewise smooth curves and is compactly contained in U_h . Moreover, it contains cp_h , 0, and σ_h on its boundary.*
- ii. *There exists a continuous branch of argument defined on \mathcal{P}_h such that*

$$\max_{w, w' \in \mathcal{P}_h} |\arg(w) - \arg(w')| \leq 2\pi\hat{\mathbf{k}}.$$

⁽⁴⁾See [Ahl06] for the definition of quasi-conformal mappings.

- iii. $\Phi_h(\mathcal{P}_h) = \{w \in \mathbb{C} \mid 0 < \operatorname{Re}(w) < [1/\alpha] - k\}$, $\operatorname{Im} \Phi_h(z) \rightarrow +\infty$ as $z \in \mathcal{P}_h \rightarrow 0$, and $\operatorname{Im} \Phi_h(z) \rightarrow -\infty$ as $z \in \mathcal{P}_h \rightarrow \sigma_h$.
- iv. Φ_h satisfies the Abel functional equation on \mathcal{P}_h , that is,

$$\Phi_h(h(z)) = \Phi_h(z) + 1, \text{ whenever } z \text{ and } h(z) \text{ belong to } \mathcal{P}_h.$$

Furthermore, Φ_h is unique once normalized by $\Phi_h(\operatorname{cp}_h) = 0$.

- v. The normalized map Φ_h depends continuously on h .

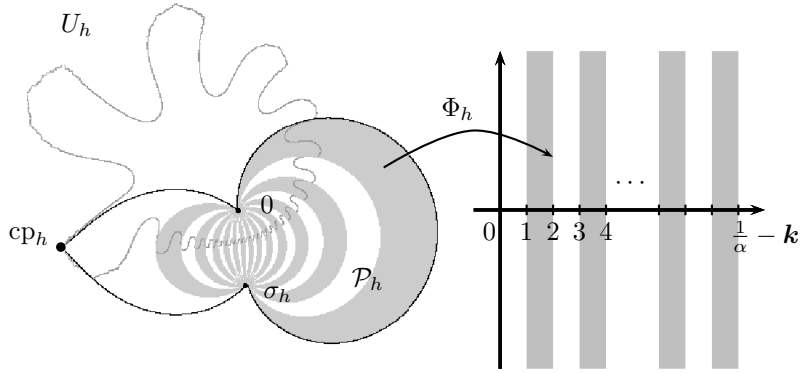


FIGURE 2. A perturbed Fatou coordinate Φ_h and its domain of definition \mathcal{P}_h . The first few iterates of cp_h under h appears in the Figure as the boundary of the amoeba containing 0.

The map $\Phi_h : \mathcal{P}_h \rightarrow \mathbb{C}$ obtained in the above theorem is called the *perturbed Fatou coordinate*, or the *Fatou coordinate* for short, of h .

The class \mathcal{IS} is denoted by \mathcal{F}_1 in [IS06]. All parts in the above theorem, except the existence of uniform \hat{k} and k in ii. and iii., follow readily from Theorem 2.1, Main Theorems 1, 3, and Corollary 4.2 in [IS06]. Parts ii. and iii. also follow from those results but require some extra work. A detailed treatment of these statements are given in [BC12, Proposition 12].

1.2. Renormalization. — Let $h : U_h \rightarrow \mathbb{C}$ either be in \mathcal{IS}_α or be the quadratic polynomial P_α , with α in $(0, \alpha_*)$. Let $\Phi_h : \mathcal{P}_h \rightarrow \mathbb{C}$ denote the normalized Fatou coordinate of h . Define

$$(2) \quad \begin{aligned} \mathcal{C}_h &:= \{z \in \mathcal{P}_h : 1/2 \leq \operatorname{Re}(\Phi_h(z)) \leq 3/2, -2 < \operatorname{Im} \Phi_h(z) \leq 2\}, \text{ and} \\ \mathcal{C}_h^\sharp &:= \{z \in \mathcal{P}_h : 1/2 \leq \operatorname{Re}(\Phi_h(z)) \leq 3/2, 2 \leq \operatorname{Im} \Phi_h(z)\}. \end{aligned}$$

By definition, $\operatorname{cv}_h \in \operatorname{int}(\mathcal{C}_h)$ and $0 \in \partial(\mathcal{C}_h^\sharp)$.

Assume for a moment that there exists a positive integer k_h , depending on h , with the following properties:

- For every integer k , with $0 \leq k \leq k_h$, there exists a unique connected component of $h^{-k}(\mathcal{C}_h^\sharp)$ which is compactly contained in $\text{Dom } h$ and contains 0 on its boundary. We denote this component by $(\mathcal{C}_h^\sharp)^{-k}$.
- For every integer k , with $0 \leq k \leq k_h$, there exists a unique connected component of $h^{-k}(\mathcal{C}_h)$ which has non-empty intersection with $(\mathcal{C}_h^\sharp)^{-k}$, and is compactly contained in $\text{Dom } h$. This component is denoted by \mathcal{C}_h^{-k} .
- The sets $\mathcal{C}_h^{-k_h}$ and $(\mathcal{C}_h^\sharp)^{-k_h}$ are contained in

$$\left\{ z \in \mathcal{P}_h \mid \frac{1}{2} < \text{Re } \Phi_h(z) < \frac{1}{\alpha} - k - \frac{1}{2} \right\}.$$

- The maps $h : \mathcal{C}_h^{-k} \rightarrow \mathcal{C}_h^{-k+1}$, for $2 \leq k \leq k_h$, and $h : (\mathcal{C}_h^\sharp)^{-k} \rightarrow (\mathcal{C}_h^\sharp)^{-k+1}$, for $1 \leq k \leq k_h$, are univalent. The map $h : \mathcal{C}_h^{-1} \rightarrow \mathcal{C}_h$ is a degree two branched covering.

Let k_h be the smallest positive integer satisfying the above four properties, and define

$$S_h := \mathcal{C}_h^{-k_h} \cup (\mathcal{C}_h^\sharp)^{-k_h}.$$

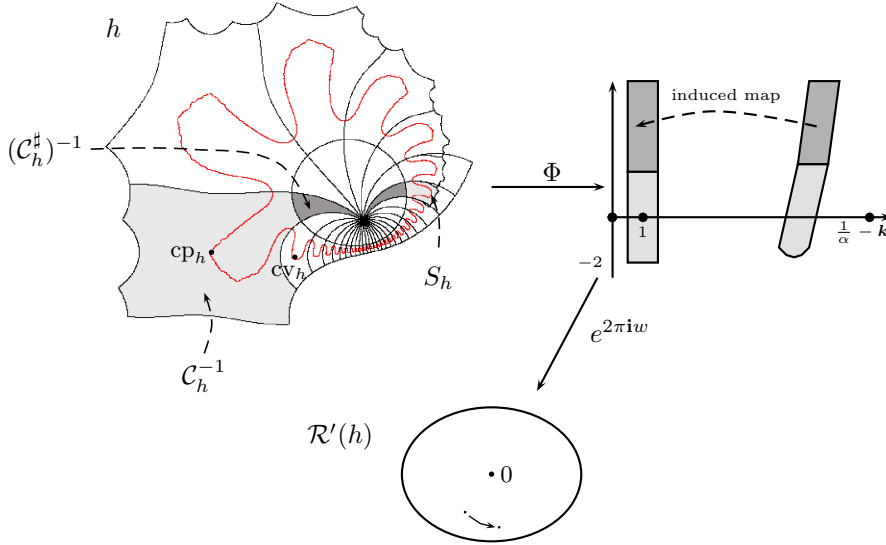


FIGURE 3. The figure shows the sets \mathcal{C}_h , $\mathcal{C}_h^\sharp, \dots, \mathcal{C}_h^{-k_h}$, and $(\mathcal{C}_h^\sharp)^{-k_h}$. The “induced map” projects via $e^{2\pi i w}$ to a map $\mathcal{R}(h)$ defined near 0.

Consider the map

$$(3) \quad \Phi_h \circ h^{\circ k_h} \circ \Phi_h^{-1} : \Phi_h(S_h) \rightarrow \mathbb{C}.$$

By the Abel functional equation, this map projects via $z = \frac{-4}{27}e^{2\pi iw}$ to a well-defined map $\mathcal{R}'(h)$, defined on a set containing 0 in its interior. One can see that $\mathcal{R}(h)$ has asymptotic expansion $e^{2\pi \frac{-1}{\alpha}i}z + O(z^2)$ near 0, see Figure 3.

The conjugate map $s \circ \mathcal{R}'(h) \circ s^{-1}$, where $s(z) := \bar{z}$ denotes the complex conjugation map, is of the form $z \mapsto e^{2\pi \frac{1}{\alpha}i}z + O(z^2)$ near 0. The map $\mathcal{R}(h) := s \circ \mathcal{R}'(h) \circ s^{-1}$, restricted to the interior of $s(\frac{-4}{27}e^{2\pi i(\Phi_h(S_h))})$, is called the *near-parabolic renormalization* of h by Inou and Shishikura. We simply refer to it as the *renormalization* of h . It is clear that one time iterating $\mathcal{R}(h)$ corresponds to several times iterating h , through the change of coordinates, see Lemma 2.1. For some applications of a closely related renormalization (Douady-Ghys renormalization) one may see [Dou87, Dou94, Yoc95, Shi98, ABC04] and the references therein.

The following theorem [IS06, Main theorem 3] states that this definition of renormalization \mathcal{R} can be carried out for perturbations of maps in \mathcal{IS} . In particular, this implies the existence of k_h satisfying the four properties listed in the definition of the renormalization. (See [BC12, Proposition 13] for a detailed argument on this, ⁽⁵⁾.)

Define

$$(4) \quad V := P^{-1}(B(0, \frac{4}{27}e^{4\pi})) \setminus ((-\infty, -1] \cup B)$$

where B is the component of $P^{-1}(B(0, \frac{4}{27}e^{-4\pi}))$ containing -1 (see Figure 4).

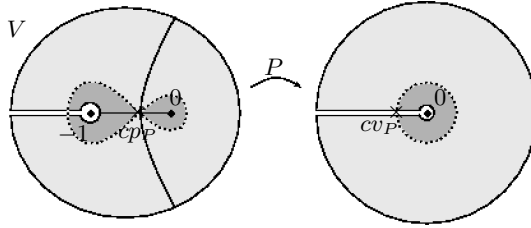


FIGURE 4. A schematic presentation of the polynomial P ; its domain, and its range. Similar colors and line styles are mapped on one another.

By an explicit calculation (see [IS06, Proposition 5.2]) one can see that the closure of U is contained in the interior of V .

Theorem 1.2 (Inou-Shishikura). — *There exists a constant $\alpha^* > 0$ such that if $h \in \mathcal{IS}_\alpha \cup \{P_\alpha\}$ with $\alpha \in (0, \alpha^*]$, then $\mathcal{R}(h)$ is well-defined and belongs to the class $\mathcal{IS}_{1/\alpha}$. Moreover, with the representation $\mathcal{R}(h) := e^{\frac{2\pi}{\alpha}i} \cdot P \circ \psi^{-1}$, the map $\psi : U \rightarrow \mathbb{C}$ extends to a univalent map on V .*

⁽⁵⁾The sets \mathcal{C}_h^{-k} and $(\mathcal{C}_h^\sharp)^{-k}$ defined here are (strictly) contained in the sets denoted by V^{-k} and W^{-k} in [BC12]. The set $\Phi_h(\mathcal{C}_h^{-k} \cup (\mathcal{C}_h^\sharp)^{-k})$ is contained in the union

$$D_{-k}^\sharp \cup D_{-k} \cup D'_{-k} \cup D'_{-k+1} \cup D_{-k+1} \cup D_{-k+1}^\sharp$$

in the notation used in [IS06, Section 5.A].

We need the constant \mathbf{k}'' defined by the next lemma.

Lemma 1.3. — *There exists a $\mathbf{k}'' \in \mathbb{Z}$ such that for every $h \in \mathcal{IS}_\alpha \cup \{P_\alpha\}$ with $\alpha \in (0, \alpha^*]$, $k_h \leq \mathbf{k}''$.*

Proof. — Comparing with the rotation of angle α , there exists an integer $j \in [\mathbf{k}, \mathbf{k}+3]$ such that for every $w \in \mathcal{C}_h \cup \mathcal{C}_h^\sharp$ close to zero, there exists a unique inverse orbit $w, h^{-1}(w), \dots, h^{-j}(w)$, contained in a neighborhood of zero, with $h^{-j}(w) \in \mathcal{P}_h$. As k_h is the smallest positive integer with $\mathcal{C}_h^{-k_h} \cup (\mathcal{C}_h^\sharp)^{-k_h} \subseteq \mathcal{P}_h$, we have $k_h \geq \mathbf{k}$.

Now let $\gamma : [0, \infty) \rightarrow \Phi_h(\mathcal{P}_h)$ be a continuous curve satisfying the following.

- $\Phi_h \circ h^{\circ k_h} \circ \Phi_h^{-1}(\gamma(t)) = s + ti$, for $t \in (-2, \infty)$, and a fixed $s \in [1/2, 3/2]$;
- $\text{diam} \{\text{Re } \gamma(t) \mid t \in [0, \infty)\} \geq k_h - \mathbf{k} - 4$.

The curve $\hat{\gamma} := \mathbb{E}x p \circ \gamma$ lands at zero, and part of it must spiral at least $k_h - \mathbf{k} - 4$ times around zero. By the definition of renormalization, $\mathcal{R}(h)$ maps $\hat{\gamma}$ to a straight ray landing at zero. On the other hand, by Theorem 1.2, $\mathcal{R}(h)$ is of the form $e^{2\pi\alpha i} \cdot P \circ \psi : U \rightarrow \mathbb{C}$ with ψ extending univalently over the larger domain V which compactly contains U . By the Koebe distortion Theorem, the total spiraling of the pull back of a straight ray landing at zero under $\mathcal{R}(h)$ must be uniformly bounded by some constant independent of h . That is, $k_h - \mathbf{k} - 4$ is uniformly bounded from above. \square

We also need the following lemma in the sequel.

Lemma 1.4. — *There exists a positive constant $\delta_1 \leq 1/8$ such that for every h in $\mathcal{IS}_\alpha \cup \{P_\alpha\}$ with $\alpha \in (0, \alpha^*]$ we have*

- $\forall j \in \mathbb{Z}, \mathbb{E}x p(B(j, \delta_1)) \subset \text{int}(\mathcal{C}_h)$,
- for every $\xi \in \mathbb{C}$ with $\mathbb{E}x p(\xi) \in \cup_{j=0}^{k_h + \lfloor 1/\alpha \rfloor - \mathbf{k} - 1} h^{\circ j}(S_h)$, $\mathbb{E}x p(B(\xi, \delta_1)) \subset \text{Dom } h$.

Proof. — The sets $\mathcal{C}_h^{-i} \cup (\mathcal{C}_h^\sharp)^{-i}$, for $i = 0, 1, 2, \dots, k_h$, are compactly contained in $\text{Dom } h$, therefore, $\cup_{j=0}^{k_h + \lfloor 1/\alpha \rfloor - \mathbf{k} - 1} h^{\circ j}(S_h)$ is compactly contained in $\text{Dom } h$. By the compactness of the class \mathcal{IS} , and the continuous dependence of the Fatou coordinates on the map, as well as Lemma 1.3, there exists $\delta > 0$ such that

$$B(-4/27, \delta) \subseteq \mathcal{C}_h, \text{ and } B(\cup_{j=0}^{k_h + \lfloor 1/\alpha \rfloor - \mathbf{k} - 1} h^{\circ j}(S_h), \delta) \subseteq \text{Dom } h.$$

Both statements in the lemma follow from the above inclusions. \square

Let $[0; a_1, a_2, \dots]$ denote the continued fraction expansion of α as

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

Define $\alpha_0 := \alpha$, and inductively for $i \geq 1$ define $\alpha_i \in (0, 1)$ as

$$\alpha_i := \frac{1}{\alpha_{i-1}} \pmod{1}.$$

For every $i \geq 0$, $a_{i+1} := \lfloor 1/\alpha_i \rfloor$ and $\alpha_i = [0; a_{i+1}, a_{i+2}, \dots]$.

If we fix a constant $N \geq 1/\alpha^*$, then $\alpha \in Irrat_{\geq N}$ implies that $\alpha_j \in (0, \alpha^*)$, for $j = 0, 1, 2, \dots$. We use this constant N in the rest of this paper.

Let $\alpha \in Irrat_{\geq N}$ and define $f_0 := P_\alpha$. Then, using Theorem 1.2, inductively define the sequence of maps

$$f_{n+1} := \mathcal{R}(f_n) : \text{Dom } f_{n+1} \rightarrow \mathbb{C}.$$

For every n ,

$$f_n : \text{Dom } f_n \rightarrow \mathbb{C}, \quad f_n(0) = 0, \quad \text{and } f_n'(0) = e^{2\pi\alpha_n i}.$$

2. The renormalization tower

2.1. Changes of coordinates and sectors around the fixed point. —

Remark. — To simplify the technical details of exposition, we assume that

$$(5) \quad N \geq \mathbf{k} + \hat{\mathbf{k}} + 1.$$

The reason for this is to make $\Phi_{f_n}(\mathcal{P}_{f_n})$ wide enough to contain a set that will be defined in a moment. However, one can avoid this condition by extending Φ_{f_n} and $\Phi_{f_n}^{-1}$ to larger domains, using the dynamics of f_n . See Subsection 3.1.

For $n \geq 0$, let $\Phi_n := \Phi_{f_n}$ denote the Fatou coordinate of $f_n : \text{Dom } f_n \rightarrow \mathbb{C}$ defined on the set $\mathcal{P}_n := \mathcal{P}_{f_n}$. For our convenience we define the covering map

$$\text{Exp}(\zeta) := \zeta \mapsto \frac{-4}{27} s(e^{2\pi i \zeta}) : \mathbb{C} \rightarrow \mathbb{C}^*, \quad \text{where } s(z) = \bar{z}.$$

By part ii of Theorem 1.1, Inequality (5), and that \mathcal{P}_n is simply connected, there is an inverse branch of $\mathbb{E}xp$ that maps \mathcal{P}_n into the range of Φ_{n-1} ;

$$\eta_n : \mathcal{P}_n \rightarrow \Phi_{n-1}(\mathcal{P}_{n-1}).$$

There may be several choices for this map but we choose one of them (for each n) with

$$\text{Re}(\eta_n(\mathcal{P}_n)) \subset [0, \hat{\mathbf{k}} + 1],$$

and fix this choice for the rest of this note. Now, define

$$\psi_n := \Phi_{n-1}^{-1} \circ \eta_n : \mathcal{P}_n \rightarrow \mathcal{P}_{n-1}.$$

Each ψ_n extends continuously to $0 \in \partial\mathcal{P}_n$ by mapping it to 0. Let $\Psi_1 := \psi_1$, and for $n \geq 2$ form the composition

$$\Psi_n := \psi_1 \circ \psi_2 \circ \dots \circ \psi_n : \mathcal{P}_n \rightarrow \mathcal{P}_0.$$

For every $n \geq 0$, let \mathcal{C}_n and \mathcal{C}_n^\sharp denote the corresponding sets for f_n defined in (2) (i.e., replace h by f_n). Denote by k_n the smallest positive integer with

$$S_n^0 := \mathcal{C}_n^{-k_n} \cup (\mathcal{C}_n^\sharp)^{-k_n} \subset \{z \in \mathcal{P}_n \mid 0 < \text{Re } \Phi_n(z) < \lfloor \frac{1}{\alpha_n} \rfloor - \mathbf{k} - 1\}.$$

For every $n \geq 0$ and $i \geq 2$, define the sectors

$$S_n^1 := \psi_{n+1}(S_{n+1}^0) \subset \mathcal{P}_n, \quad \text{and} \\ S_n^i := \psi_{n+1} \circ \dots \circ \psi_{n+i}(S_{n+i}^0) \subset \mathcal{P}_n.$$

By definition, the critical value of f_n is contained in $f_n^{\circ k_n}(S_n^0)$. Also, all these sectors contain 0 on their boundaries. We will mainly work with S_0^i , for $i \geq 0$.

Lemma 2.1. — *Let $z \in \mathcal{P}_n$ be a point with $w := \mathbb{E}xp \circ \Phi_n(z) \in \text{Dom } f_{n+1}$. There exists an integer ℓ_z with $1 \leq \ell_z \leq \lfloor 1/\alpha_n \rfloor - \mathbf{k} + k_n - 1$, such that*

- *the finite orbit $z, f_n(z), f_n^{\circ 2}(z), \dots, f_n^{\circ \ell_z}(z)$ is defined, and $f_n^{\circ \ell_z}(z) \in \mathcal{P}_n$;*
- $\mathbb{E}xp \circ \Phi_n(f_n^{\circ \ell_z}(z)) = f_{n+1}(w)$.

Proof. — As $w \in \text{Dom } f_{n+1}$, by the definition of renormalization $\mathcal{R}(f_n) = f_{n+1}$, there are $\zeta \in \Phi_n(S_n^0)$ and $\zeta' \in \Phi_n(\mathcal{C}_n \cup \mathcal{C}_n^\#)$, such that

$$\mathbb{E}xp(\zeta) = w, \quad \mathbb{E}xp(\zeta') = f_{n+1}(w), \quad \text{and} \quad \zeta' = \Phi_n \circ f_n^{\circ k_n} \circ \Phi_n^{-1}(\zeta).$$

Since $\mathbb{E}xp(\Phi_n(z)) = w$, there exists an integer ℓ with

$$-k_n + 1 \leq \ell \leq \lfloor 1/\alpha_n \rfloor - \mathbf{k} - 1,$$

such that $\Phi_n(z) + \ell = \zeta$.

By the Abel functional equation for Φ_n , we have

$$\begin{aligned} \zeta' &= \Phi_n \circ f_n^{\circ k_n} \circ \Phi_n^{-1}(\zeta) \\ &= \Phi_n \circ f_n^{\circ k_n} \circ \Phi_n^{-1}(\Phi_n(z) + \ell) \\ &= \Phi_n \circ f_n^{\circ k_n + \ell}(z). \end{aligned}$$

Letting $\ell_z := k_n + \ell$, we have

$$1 \leq \ell_z \leq k_n + \lfloor 1/\alpha_n \rfloor - \mathbf{k} - 1, \quad f_n^{\circ \ell_z}(z) = \Phi_n^{-1}(\zeta') \in \mathcal{P}_n, \quad \text{and}$$

$$\begin{aligned} \mathbb{E}xp \circ \Phi_n(f_n^{\circ \ell_z}(z)) &= \mathbb{E}xp \circ \Phi_n(\Phi_n^{-1}(\zeta')) \\ &= \mathbb{E}xp(\zeta') \\ &= f_{n+1}(w). \end{aligned}$$

□

It is clear that in the above lemma there are many choices for ℓ_z . By making some specific choices for ℓ_z and inductively using the above lemma, the number of iterates on level n are related to the number of iterates on level 0

Define

$$\mathcal{P}'_n := \{w \in \mathcal{P}_n \mid 0 < \text{Re } \Phi_n(w) < \lfloor 1/\alpha_n \rfloor - \mathbf{k} - 1\}.$$

Lemma 2.2. — *For every $n \geq 1$ we have*

- i. *for every $w \in \mathcal{P}'_n$, $f_0^{\circ q_n} \circ \Psi_n(w) = \Psi_n \circ f_n(w)$,*
- ii. *for every $w \in S_n^0$, $f_0^{\circ(k_n q_n + q_{n-1})} \circ \Psi_n(w) = \Psi_n \circ f_n^{\circ k_n}(w)$.*

Proof. — By the previous lemma, an iterate of f_n on \mathcal{P}_n corresponds to some iterate of f_0 on $\Psi_n(\mathcal{P}_n)$. The correct numbers of iterates, q_n and $k_n q_n + q_{n-1}$, when $|w|$ is small, are determined by comparing the map to the rotation of angle α . Hence, the equalities hold at points near 0 and therefore, must hold over the domain of definitions by the analytic continuation. A more detailed proof may be found in [Che10]. □

2.2. The approximating neighborhoods. — For every $n \geq 0$, consider the union

$$(6) \quad \Omega_n^0 := \bigcup_{i=0}^{k_n + \lfloor 1/\alpha_n \rfloor - \mathbf{k} - 1} f_n^{\circ i}(S_n^0) \subset \text{Dom } f_n.$$

Using Lemma 2.2, we transfer the iterates in the above union to the dynamic plane of f_0 to form

$$\Omega_0^n := \bigcup_{i=0}^{q_n(k_n + \lfloor 1/\alpha_n \rfloor - \mathbf{k} - 1) + q_n - 1} f_0^{\circ i}(S_0^n) \bigcup \{0\}.$$

The upper bound in the above union is obtained as follows. The first k_n iterates in (6) correspond to $k_n q_n + q_n - 1$ number of iterates on level 0 by Lemma 2.2-ii, and the remaining $\lfloor 1/\alpha_n \rfloor - \mathbf{k} - 1$ iterates in (6) amounts to $q_n(\lfloor 1/\alpha_n \rfloor - \mathbf{k} - 1)$ number of iterates by Lemma 2.2-i. With these choices of the number of iterates, the unions satisfy the following property.

Proposition 2.3. — For every $n \geq 0$,

- i. Ω_0^{n+1} is compactly contained in the interior of Ω_0^n ,
- ii. $\mathcal{PC}(f_0)$ is contained in the interior of Ω_0^n .

Proof. — The proof is broken into several steps.

Step 1. $\forall n \geq 0, \Omega_0^{n+1} \subset \Omega_0^n$.

To prove this, it is enough to show the following claim. For every $z \in S_0^{n+1}$ there are points z_1, z_2, \dots, z_m in S_0^n as well as non-negative integers t_1, t_2, \dots, t_{m+1} , for some positive integer m , fulfilling the following properties:

- $f_0^{\circ t_1}(z_1) = z$ and $f_0^{\circ t_{m+1}}(z_m) = f_0^{\circ(q_{n+2} + (k_{n+1} - \mathbf{k} - 1)q_{n+1})}(z)$,
- $f_0^{\circ t_j}(z_{j-1}) = z_j$, for $j = 2, 3, \dots, t_{m-1}$,
- $t_j \leq q_{n+1} + (k_n - \mathbf{k} - 1)q_n$, for every $j = 1, 2, \dots, m + 1$.

Given $z \in S_0^{n+1}$, let $\zeta := \Psi_{n+1}^{-1}(z) \in S_{n+1}^0$. By the definition of S_{n+1}^0 , the iterates

$$\zeta, f_{n+1}(\zeta), f_{n+1}^{\circ 2}(\zeta), \dots, f_{n+1}^{\circ(k_{n+1} + \lfloor \frac{1}{\alpha_{n+1}} \rfloor - \mathbf{k} - 1)}(\zeta)$$

are defined and belong to the domain of f_{n+1} .

By Lemma 2.1 for $\psi_{n+1}(\zeta)$, there are two points $\xi_1 := \psi_{n+1}(\zeta)$ and ξ_2 in \mathcal{P}_n as well as a positive integer ℓ_0 with $\mathbb{E}x \circ \Phi_n(\xi_1) = \zeta$, $\mathbb{E}x \circ \Phi_n(\xi_2) = f_{n+1}(\zeta)$ and $f_n^{\circ \ell_0}(\xi_1) = \xi_2$. Let $\sigma_1 \in S_n^0$ and an integer ℓ_1 with $1 \leq \ell_1 \leq k_n + a_{n+1} - \mathbf{k} - 1$ be such that $f_n^{\circ \ell_1}(\sigma_1) = \xi_1$. By the same lemma, there is a point σ_2 in the orbit

$$\xi_1, f_n(\xi_1), f_n^{\circ 2}(\xi_1), \dots, f_n^{\circ(\ell_0 - 1)}(\xi_1), \xi_2$$

which belongs to S_n^0 . Let ℓ_2 be the positive integer with $1 \leq \ell_2 \leq k_n + \lfloor \frac{1}{\alpha_n} \rfloor - \mathbf{k} - 1$ such that $f_n^{\circ \ell_2}(\sigma_1) = \sigma_2$.

By the same argument for ξ_2 with $\mathbb{E}x \circ \Phi_n(\xi_2) = f_{n+1}(\zeta)$, we obtain points σ_3 in S_n^0 , ξ_3 in \mathcal{P}_n and a positive integer ℓ_3 with $1 \leq \ell_3 \leq k_n + a_{n+1} - \mathbf{k} - 1$, such that $f_n^{\circ \ell_3}(\sigma_2) = \sigma_3$.

Repeating this argument with $\xi_3, \xi_4, \dots, \xi_m$, where $m = k_{n+1} + \lfloor \frac{1}{\alpha_{n+1}} \rfloor - \mathbf{k} - 1$, one obtains a sequence $\sigma_1, \sigma_2, \dots, \sigma_m$ of points in S_n^0 and positive integers $\ell_1, \ell_2, \dots, \ell_{m+1}$, all bounded by $k_n + \lfloor \frac{1}{\alpha_n} \rfloor - \mathbf{k} - 1$, which satisfy

$$\begin{aligned} - f_n^{\circ \ell_{j+1}}(\sigma_j) &= \sigma_{j+1} \text{ for all } j = 2, 3, \dots, m-1 \\ - f_n^{\circ \ell_1}(\sigma_1) &= \xi_1 \text{ and } f_n^{\circ \ell_{m+1}}(\sigma_m) = \xi_m. \end{aligned}$$

Now we define $z_i := \Psi_n(\sigma_i) \in S_0^n$, for $i = 1, 2, \dots, m$. By definition, $\Psi_n(\xi_1) = z$. We claim that $\Psi_n(\xi_m) = f_0^{\circ(q_{n+2} + (k_{n+1} - \mathbf{k} - 1)q_{n+1})}(z)$. This is because

$$\mathbb{E}xp \circ \Phi_n(\xi_m) = f_{n+1}^{\circ(k_{n+1} + \lfloor \frac{1}{\alpha_{n+1}} \rfloor - \mathbf{k} - 1)}(\zeta),$$

which is mapped to $f_0^{\circ(q_{n+2} + (k_{n+1} - \mathbf{k} - 1)q_{n+1})}(z)$ under Ψ_{n+1} , using Lemma 2.2.

By Lemmas 2.2, ℓ_j times iterating f_n corresponds to t_j times iterating f_0 , for each $j = 1, 2, \dots, \ell_{m+1}$. With the same lemma, as ℓ_j is bounded by $k_n + \lfloor \frac{1}{\alpha_n} \rfloor - \mathbf{k} - 1$, each t_j is bounded by

$$k_n q_n + q_{n-1} + (\lfloor \frac{1}{\alpha_n} \rfloor - \mathbf{k} - 1)q_n \leq q_{n+1} + q_n(k_n - \mathbf{k} - 1).$$

Step 2. $\forall n \geq 0$, Ω_0^{n+1} is compactly contained in the interior of Ω_0^n .

To prove this we use the open mapping property of holomorphic and anti-holomorphic mappings. If z' is a point in the closure of Ω_0^{n+1} , as f_0 is a polynomial defined on the whole complex plane, there exists a point z in the closure of S_0^{n+1} with $f_0^{\circ t_0}(z) = z'$, for a non-negative integer t_0 less than $q_{n+2} + (k_n - \mathbf{k} - 1)q_{n+1}$. By the proof of Lemma 2.1, all the points σ_j in the above argument can be chosen to be in the interior of S_n^0 . Hence, all z_i are contained in the interior of S_0^n .

Step 3. $\forall n \geq 0$, the critical point of f_0 belongs to Ω_0^n and can be iterated at least $(a_{n+1} - \mathbf{k} - 1)q_n$ times within this set.

The map $f_n: S_n^0 \rightarrow f_n^{\circ k_n}(S_n^0)$ is a degree two branched covering. Thus, by Lemma 2.2, $f_0^{\circ(k_n q_n + q_{n+1})}: S_0^n \rightarrow \Psi_n(f_n^{\circ k_n}(S_n^0))$ is also a degree two map. That means that the critical point of f_0 is contained in the union $\cup_{i=0}^{k_n q_n + q_{n+1}} f_0^{\circ i}(S_0^n)$. Therefore, by the definition of Ω_0^n , the critical point can be iterated at least

$$q_{n+1} + (k_n - \mathbf{k} - 1)q_n - k_n q_n - q_{n-1} = (a_{n+1} - \mathbf{k} - 1)q_n$$

times within Ω_0^n .

As $a_{n+1} - \mathbf{k} - 1 \geq 1$ and q_n grows (exponentially) to infinity as n goes to infinity, the critical point of f_0 can be iterated infinite number of times within each Ω_0^n .

Finally, for every $n \geq 0$, Ω_0^n contains the closure of Ω_0^{n+1} in its interior, and Ω_0^{n+1} contains the orbit of the critical point. Therefore, the post-critical set must be contained in every Ω_0^n . \square

3. Area of the post-critical set

In this section we prove the following proposition which combined with Proposition 2.3 implies Theorem A. Let N be the constant introduced at the end of Section 1.

Proposition 3.1. — For all Brjuno α in $\text{Irrat}_{\geq N}$, $(\cap_{n=0}^{\infty} \Omega_n^0) \cap J(P_\alpha)$ has zero area.

3.1. Natural extensions. — For every $n \geq 1$, let $\text{Fil}(\Omega_n^0)$ denote the set obtained from adding the bounded components of $\mathbb{C} \setminus \Omega_n^0$ to Ω_n^0 , if there is any. For all integer $n \geq 1$ and $r \in [0, 1/\alpha_n - \mathbf{k}]$, let $I_{n,r}$ denote the connected component of

$$\text{Fil}(\Omega_n^0) \cap \Phi_n^{-1}\{r + ti : t \in \mathbb{R}\},$$

containing zero on its boundary. Each $I_{n,r}$ is a smooth curve in $\text{Fil}(\Omega_n^0)$ connecting the boundary of Ω_n^0 to 0. This implies that there is a continuous inverse branch of $\mathbb{E}\text{xp}$ defined on every $\Omega_n^0 \setminus I_{n,r}$.

By Theorem 1.1-ii, Lemma 1.3, and the precompactness of the class $\mathcal{IS}_{(0,\alpha^*)}$, there exists an integer \mathbf{k}' such that

$$(7) \quad \forall n \geq 1 \text{ and } \forall r \in [0, 1/\alpha_n - \mathbf{k}], \quad \sup_{z,w \in \Omega_n^0 \setminus I_{n,r}} |\arg(w) - \arg(z)| \leq 2\pi\mathbf{k}'$$

for any continuous branch of argument defined on $\Omega_n^0 \setminus I_{n,r}$.

Remark. — To avoid un-necessary technical details in the manuscript, from now on we assume that

$$(8) \quad N \geq 2\mathbf{k}' + \mathbf{k} + 1.$$

This guarantees that on every level n , $1/\alpha_n - \mathbf{k} \geq 2\mathbf{k}' + 1$ and hence one can embed a lift $\mathbb{E}\text{xp}^{-1}(\Omega_n^0 \setminus I_{n,r})$ into some $\Omega_{n-1} \setminus I_{n-1,r'}$, via the Fatou coordinates.

The Fatou coordinate Φ_h , and its inverse, have natural extensions onto some larger domains. In the forward direction it is achieved as follows. First define the set \mathcal{P}_h^l as follows:

– if $\mathbf{k}' \leq k_h$

$$\mathcal{P}_h^l := \bigcup_{j=0}^{2\mathbf{k}'} h^{\circ(k_h - \mathbf{k}' + j)}(S_h),$$

– if $\mathbf{k}' > k_h$

$$\mathcal{P}_h^l := \bigcup_{j=0}^{\mathbf{k}'} h^{\circ(k_h + j)}(S_h) \bigcup_{j=1}^{\mathbf{k}' - k_h} \Phi_h^{-1}(\Phi_h(S_h) - j).$$

Then $\Phi_h^l : \mathcal{P}_h^l \rightarrow \mathbb{C}$ is defined as

$$(9) \quad \Phi_h^l(z) := \Phi_h(h^{\circ j}(z)) - j,$$

where j is the smallest non-negative integer with $h^{\circ j}(z) \in \mathcal{P}_h$. The extended map is not univalent anymore, and has critical points at preimages of the critical point of h . These critical points of the extended map are sent to the points $0, -1, -2, \dots, -\mathbf{k}' + 1$.

The inverse map, which is denoted by ϕ_h^\dagger , is defined on the set

$$(10) \quad \mathcal{W}_h := \{w \in \mathbb{C} \mid \text{Re } w \in (\mathbf{k}', \alpha(h)^{-1} - \mathbf{k})\} \bigcup_{j=0}^{k_h + \mathbf{k}'} \Phi_h(S_h) + j.$$

If $\text{Re } w \in (\mathbf{k}', 1/\alpha - \mathbf{k})$, $\Phi_h^\dagger(w) := \Phi_h^{-1}(w)$, and when $w \in \Phi_h(S_h) + j$

$$(11) \quad \Phi_h^\dagger(w) := h^{\circ j} \circ \Phi_h^{-1}(z - j)$$

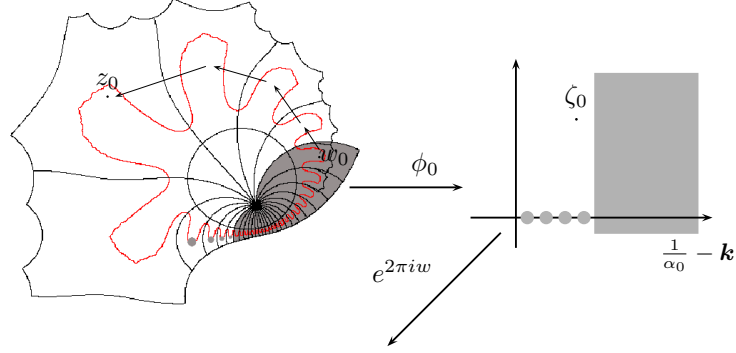


FIGURE 5. The two different colors correspond to the two different ways of going down the renormalization tower. The gray part corresponds to \mathcal{A} and the rest to \mathcal{B} .

By the equivariance property of the map Φ_h , one can see that this provides us with a well-defined map. The map Φ_h^\dagger has critical points that are mapped to

$$cv_h, h(cv_h), \dots, h^{\circ k_h - 1}(cv_h).$$

As usual, let $\Phi_n^\dagger := \Phi_{f_n}^\dagger$ and $\Phi_n^l := \Phi_{f_n}^l$, for $n = 0, 1, 2, \dots$.

3.2. The sequence ζ_n . — To each non-zero $z_0 \in \bigcap_{n=0}^\infty \Omega_0^n \cap J(P_\alpha)$, we associate a sequence of pairs

$$(12) \quad (z_i, \zeta_i), i = 0, 1, 2, \dots$$

(The sequence $\{z_i\}$ is the same as the first coordinates of the sequence of quadruples considered in [Che10]). For $n \geq 0$, define the sets

$$\begin{aligned} \mathcal{A}_n &:= \{z \in \mathcal{P}_n \mid \Phi_n(z) \in \bigcup_{j=1}^{k'} B(j, \delta_1), \text{ or } \operatorname{Re} \Phi_n(z) \in (k' + 1/2, 1/\alpha_n - k)\}, \\ \mathcal{B}_n &:= \Omega_n^0 \setminus \mathcal{A}_n, \end{aligned}$$

where δ_1 is obtained in Lemma 1.4. See Figure 3.2.

Sequence (12) is inductively defined as follows. To define ζ_0 ,

- if $z \in \mathcal{A}_0$, let $\zeta_0 := \Phi_h(z)$,
- if $z \in \mathcal{B}_0$, let ζ_0 be a preimage of z under Φ_h^\dagger .

The point ζ_0 satisfying this property is not necessarily unique. However, one can choose any of them. Indeed, when $|z_0|$ is small enough, ζ_0 is uniquely determined. Otherwise, there are at most two choices for it.

Now, define $z_1 := \mathbb{E}\exp(\zeta_0)$, and note that since $z_0 \in \Omega_0^1$, $z_1 \in \Omega_1^0$. Thus, we can repeat the above step to define the next pair (z_1, ζ_1) , and so on. In general, for $j \geq 0$,

$$(13) \quad \begin{aligned} z_{j+1} &= \mathbb{E}\exp(\zeta_j), \text{ and} \\ \Phi_j^l(z_j) &:= \zeta_j \text{ or } \Phi_j^\dagger(\zeta_j) = z_j. \end{aligned}$$

Note that by the definition of this sequence,

$$(14) \quad \forall l \geq 0, \mathbf{k}' + 1/2 \leq \operatorname{Re} \zeta_l \leq \alpha_l^{-1} - \mathbf{k} + k_l + \mathbf{k}' \quad \text{or} \quad \zeta_l \in \cup_{j=1}^{\mathbf{k}'} B(j, \delta_1).$$

By 1/4-theorem, every map $h \in \mathcal{IS}_{(0, \alpha^*)}$ is univalent on $B(0, 1/12)$. Let

$$D := 1 + \frac{1}{2\pi} \log \frac{16}{9},$$

and note that

$$(15) \quad D \geq \delta_1 \quad \text{and} \quad \mathbb{E}\exp(Di) \leq 1/12.$$

Since $\mathcal{R}h$ also belongs to $\mathcal{IS}_{(0, \alpha^*)}$, by the definition of renormalization, h^{k_h} is univalent on $\{z \in S_h \mid \operatorname{Im} \Phi_h(z) \geq D\}$. With this choice of D , we decompose the intersection into two sets

$$\begin{aligned} \Gamma &:= \{z \in \cap_{n=0}^{\infty} \Omega_0^n \cap J(P_\alpha) \mid \exists N \text{ such that for all } m \geq N, \operatorname{Im} \zeta_m \geq D/\alpha_m\}, \\ \Lambda &:= \{z \in \cap_{n=0}^{\infty} \Omega_0^n \cap J(P_\alpha) \mid \text{there are infinitely many } m \text{ with } \operatorname{Im} \zeta_m < D/\alpha_m\}. \end{aligned}$$

The areas of these two sets are treated in the following two subsections.

3.3. Area of Γ . — The set Γ is contained in the union of the sets

$$\Gamma^m := \{z \in \Gamma \mid \text{for all } n \geq m, \operatorname{Im} \zeta_n \geq D/\alpha_n\}, \quad \text{for } m = 0, 1, 2, \dots$$

Every Γ^m projects (under piecewise holomorphic or anti-holomorphic maps) onto the set Γ_m on the dynamic plane of level m ; defined as

$$\Gamma_m := \{z_m = \mathbb{E}\exp(\zeta_{m-1}) \mid z \in \Gamma^m\}.$$

Fix an $m \in \{0, 1, 2, \dots\}$, and for $j \geq m$ let

$$\Pi_j := \{\zeta_j \mid z \in \Gamma^m\}.$$

Note that here ζ_j is uniquely determined by z_j , by our choice of the constant D . Projecting the above set onto the round cylinder \mathbb{C}/\mathbb{Z} , we define

$$B_j := \Pi_j/\mathbb{Z}, \quad \text{for } j = m, m+1, m+2, \dots$$

That is, B_j is the set of all points $w \in \mathbb{C}/\mathbb{Z}$ for which there exists an integer i with $w + i \in \Pi_j$. Each B_j lies above the line $\operatorname{Im} w = D/\alpha_j$. We would like to show that these sets have zero area. To prove this, we first show that the sequence,

$$b_j := \sup_{\zeta, \zeta' \in B_j} \operatorname{Im}(\zeta - \zeta'), \quad \text{for } j = m, m+1, m+2, \dots,$$

satisfies the inequality in the following lemma.

Lemma 3.2. — *There exists a constant A such that for every $j > m$ we have*

$$b_{j-1} \leq \alpha_j b_j + A.$$

To prove the above lemma, we need an estimate on the derivative of the change of coordinates

$$(16) \quad \chi_h := \mathbb{L}\log \circ \Phi_h^\dagger,$$

where $\mathbb{L}\log$ is an arbitrary inverse branch of $\mathbb{E}\exp$ defined on a slit neighborhood of 0.

For every $r \in (0, 1/2]$ and $\alpha \in (0, 1)$ define the set

$$\Theta(r, \alpha) := \{w \in \mathbb{C} \mid \operatorname{Im} w \geq -2/\alpha\} \setminus B(\mathbb{Z}/\alpha, r/\alpha).$$

Proposition 3.3. — *There exists a constant C such that for every $r \in (0, 1/2]$, every $h \in \mathcal{IS}_{(0, \alpha^*)}$, and every $w \in \operatorname{Dom} \chi_h \cap \Theta(r, \alpha)$, we have*

$$|\chi'_h(w) - \alpha| \leq C \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im} w}.$$

This lemma is proved in Section 5.

Proof of Lemma 3.2. — Let $\chi_n := \chi_{f_n}$, for $n = 0, 1, 2, \dots$, denote the map defined by (16) when $h = f_n$. Since $\operatorname{Im} \zeta_j \geq D/\alpha_j \geq \delta_1$, one infers from the definition of the pairs (12) that $\Phi_j^\dagger(\zeta_j) := z_j$. Therefore, from (14), for $j = m, m+1, m+2, \dots$,

$$\begin{aligned} \Pi_j &\subseteq \{w \in \mathbb{C} \mid \operatorname{Im} w \geq D/\alpha_j, \mathbf{k}' + 1/2 \leq \operatorname{Re} w \leq \mathbf{k}' + 1/\alpha_j - \mathbf{k} + k_n\}, \\ \chi_j(\Pi_j)/\mathbb{Z} &= B_{j-1}. \end{aligned}$$

For arbitrary $\zeta, \zeta' \in \Pi_j$, let γ be a straight line segment connecting ζ' to ζ . By Proposition 3.3, with $r = 1/2$, we have

$$\begin{aligned} &|\chi_j(\zeta') - \chi_j(\zeta) - \alpha_j(\zeta' - \zeta)| \\ &= \left| \int_\gamma \chi'_j - \alpha_j dw \right| \\ &\leq \int_\gamma 2C\alpha_j e^{-2\pi\alpha_j \operatorname{Im} w} dw \\ &\leq 2C\alpha_j e^{-2\pi\alpha_j(D/\alpha_j)} \left(\frac{1}{\alpha_j} - \mathbf{k} + k_n \right) + 2 \int_{D/\alpha_j}^\infty C\alpha_j e^{-2\pi\alpha_j t} dt \\ &\leq 2C(1 + \mathbf{k}'') e^{-2\pi D} + \frac{e^{-2\pi D}}{\pi}. \end{aligned}$$

The result follows from taking the supremum over all $\chi_j(\zeta), \chi_j(\zeta')$, and then all ζ, ζ' . \square

Lemma 3.4. — $\limsup_{j \geq m} b_j < 4A$.

Proof. — For $j \geq 0$, recall the Brjuno sum,

$$\mathcal{B}(\alpha_j) := \log \frac{1}{\alpha_j} + \alpha_j \log \frac{1}{\alpha_{j+1}} + \alpha_j \alpha_{j+1} \log \frac{1}{\alpha_{j+2}} + \dots$$

By a result of Yoccoz [Yoc95] (this also follows from [Che10]), the Siegel disk of f_j contains the disk of radius $e^{-\mathcal{B}(\alpha_j)+M}$ about the origin, for some universal constant M . This implies that for every $j \geq m$, we have $b_j \leq \mathcal{B}(\alpha_{j+1})/2\pi$.

Fix $j \geq m$, and let l be an arbitrary integer bigger than j . Inductively using the inequality in Lemma 3.2, for $j+1, j+2, \dots, l$, we obtain

$$b_j \leq \alpha_{j+1} \alpha_{j+2} \cdots \alpha_l b_l + (1 + \alpha_{j+1} + \alpha_{j+1} \alpha_{j+2} + \cdots + \alpha_{j+1} \alpha_{j+2} \cdots \alpha_{l-1}) A.$$

Replacing b_l by $\mathcal{B}(\alpha_{l+1})/2\pi$, and that $\alpha_i\alpha_{i+1} \leq 1/2$, for all i , we come up with

$$b_j \leq \frac{1}{2\pi} \alpha_{j+1} \alpha_{j+2} \cdots \alpha_l \mathcal{B}(\alpha_{l+1}) + 4A.$$

As l tends to infinity, the first term in the above sum, which is the tail of the Brjuno series for α_{j+1} , tends to zero. \square

Lemma 3.5. — *There exists a constant $\varepsilon > 0$ such that if $\alpha_j < \varepsilon$ for some $j \geq m$, then $\text{area } B_{j-1} < \frac{1}{2} \text{area } B_j$.*

Proof. — Using Proposition 3.3 with $h = f_j$, $\chi_j := \chi_{f_j}$, and $r = 1/2$,

$$\begin{aligned} \text{area } B_{j-1} &= \text{area } \chi_j(\Pi_j) \\ &\leq \int_{\Pi_j} |\chi'_j|^2 \frac{dz \wedge d\bar{z}}{2} \\ &\leq \left(\frac{1}{\alpha_j} + \mathbf{k}''\right) \cdot \max_{\ell \in \mathbb{Z}} \int_{\{w \in \Pi_j \mid \ell \leq \text{Re } w \leq \ell+1\}} |\chi'_j|^2 \frac{dz \wedge d\bar{z}}{2} \\ &\leq \left(\frac{1}{\alpha_j} + \mathbf{k}''\right) (2C\alpha_n e^{-2\pi D})^2 \text{area } B_j \\ &\leq \frac{1}{2} \text{area } B_j, \end{aligned}$$

with the last inequality for small enough values of α_j . \square

Proposition 3.6. — *The set Γ has zero area.*

Proof. — As $\Gamma = \cup_{m \geq 0} \Gamma^m$, it is enough to show that each Γ^m has zero area. Clearly, Γ^m has zero area if and only if Γ_m has zero area. To prove this, we show that B_m has zero area in the two separate cases.

Case i: Eventually $\alpha_j \geq \varepsilon$, where ε is obtained in Lemma 3.5.

For these values of α , $\mathcal{B}(\alpha_j)$ is uniformly bounded, and hence, the set Γ is empty for large enough D . (In this case, α is of bounded type, and the boundary of the Siegel disk is known to be a quasi-circle passing through the critical point.)

Case ii: There are arbitrarily large j with $\alpha_j < \varepsilon$.

By Lemma 3.4, $\text{area } B_j \leq 4A$, for all j . On the other hand, there are infinitely many levels j with $\text{area } B_{j-1} < \frac{1}{2} \text{area } B_j$. \square

3.4. Area of Λ . —

Proposition 3.7. — *The set $\cap_{n=0}^{\infty} \Omega_0^n \cap J(P_\alpha)$ is non-uniformly porous at every point in Λ . In particular, Λ has zero area.*

Remark. — In [Che10] we prove that when α is a non-Brjuno number, for every non-zero point in $\cap_{n=0}^{\infty} \Omega_0^n$, there exists D such that infinitely often $\text{Im } \zeta_n \leq D/\alpha_n$. In other words, the intersection is contained in the union of all Λ , over all $D \in (0, +\infty)$. Then, this was used to show that such a point can not be the Lebesgue density point of $\mathcal{PC}(f_0)$. For the reader's convenience we present an argument, similar in principle but simpler in technical details, to produce the same result.

Lemma 3.8. — *There are $\delta_2, \delta_3 \in \mathbb{R}$ such that for every $n \geq 0$ with $\text{Im } \zeta_n < D/\alpha_n$, there exist a simply connected domain $V_n \subset \mathbb{C} \setminus \{0\}$ which equipped with its conformal metric $\rho_n|dz|$ of constant curvature -1 satisfies the following:*

- i. $V_n \subset \text{Dom } f_n \cap f_n(\text{Dom } f_n)$, and $z_n \in V_n$;
- ii. There exists $z'_n \in V_n$ with $d_{\rho_n}(z_n, z'_n) \leq \delta_2$ such that

$$B_{\rho_n}(z'_n, \delta_3) \cap \Omega_n^0 = \emptyset, \text{ and } f_n(B_{\rho_n}(z'_n, \delta_3)) \cap \Omega_n^0 = \emptyset;$$

- iii. $\sup_{z, w \in V_n} |\arg z - \arg w| \leq \mathbf{k}'$, for every branch of argument defined on V_n .

Proof. — Recall that ζ_n belongs to the set $\cup_{j=1}^{\mathbf{k}'} B(j, \delta_1) \cup W_{f_n}$, see Equation (10). Since the sectors $\mathcal{C}_n^{-j} \cup (\mathcal{C}_n^{\sharp})^{-j}$, for $j = 0, 1, 2, \dots, k_n$, are compactly contained in $\text{Dom } f_n$, k_n is uniformly bounded independent of n by Lemma 1.3, the continuous dependence of the Fatou coordinate on the map, and the compactness of the class \mathcal{IS} , there exists a constant ε_0 such that Φ_n^\dagger is defined on $B(W_{f_n}, \varepsilon_0)$ with

$$\Phi_n^\dagger(B(W_{f_n}, \varepsilon_0)) \subset \text{Dom } f_n.$$

By Theorem 1.2, the map $f_n : \text{Dom } f_n \rightarrow \mathbb{C}$ extends onto a domain V' with the modulus of $V' \setminus \text{Dom } f_n$ uniformly bounded from below, independent of n . By the Koebe distortion Theorem, this implies that $\text{Dom } f_n$, for all n , is contained in a fixed ball about 0. Similarly, all of f_n , for $n \geq 1$, are univalent on the ball $B(0, 1/12)$. By the definition of renormalization, this implies that there exists a constant $M_0 > 2$ such that each Φ_n^\dagger , for $n \geq 1$, is univalent above the line $\text{Im } w = M_0$, and that W_{f_n} lies above the line $\text{Im } w = -M_0$.

Let $U \subset \mathbb{C}$ be a domain with the hyperbolic metric $\rho|dz|$ of constant curvature equal to -1 . Using the 1/4-Theorem, one can show the following estimate on ρ at a given point $z \in U$

$$(17) \quad \frac{1}{2d(z, \partial U)} \leq \rho(z) \leq \frac{2}{d(z, \partial U)}.$$

The proof of the lemma is broken into two situations based on the position of ζ_n .

The proof when $\text{Im } \zeta_n < M + 1$:

If $\text{Re } \zeta_n \in (7/8, 1/\alpha_n - \mathbf{k} - 1)$ define

$$r'_n := \frac{1}{2} \min\{\text{Re } \zeta_n, \frac{1}{\alpha_n} - \mathbf{k} - \text{Re } \zeta_n\}, \quad \zeta'_n := \text{Re } \zeta_n - M\mathbf{i} - r'_n\mathbf{i}.$$

Let $V_n := \mathcal{P}_n$ and $z'_n := \Phi_n^{-1}(\zeta'_n)$. The ball $B(\zeta'_n, r'_n)$ and its translation by one are below the heights -2 and $-M$. Hence, by the choice of M , $\Phi_n^{-1}(B(\zeta'_n, r'_n))$ and $f_n(\Phi_n^{-1}(B(\zeta'_n, r'_n)))$ are outside of Ω_n^0 .

On the other hand, one can easily verify that the hyperbolic distance within $\Phi_n(V_n)$ from ζ_n to ζ'_n is uniformly bounded from above, and that $B(\zeta'_n, r'_n)$ contains a hyperbolic ball of definite size independent of n . As Φ_n is an isometry from V_n to $\Phi_n(V_n)$, with respect to the hyperbolic metrics, we conclude the second part of the lemma. The other parts are obvious here.

If $\operatorname{Re} \zeta_n \geq 1/\alpha_n - \mathbf{k} - 1$, define

$$\begin{aligned} V'_n &:= B(W_{f_n}, \varepsilon_0) \cup B(W_{f_n} + 1, \varepsilon_0) \cup \{\zeta \in \mathbb{C}; \operatorname{Re} \zeta \in (\frac{1}{\alpha_n} - \mathbf{k} - 2, \frac{1}{\alpha_n} - \mathbf{k})\}, \\ \zeta'_n &:= -2M\mathbf{i} + \frac{1}{\alpha_n} - \mathbf{k} - 1. \end{aligned}$$

Let $V_n := \Phi_n^\dagger(V'_n)$ and $z'_n := \Phi_n^\dagger(\zeta'_n)$. By the contraction of Φ_n^\dagger with respect to hyperbolic metrics, $d_{\rho_n}(z_n, z'_n)$ is uniformly bounded from above. By the Koebe distortion Theorem $\Phi_n^{-1}(B(\zeta'_n, 1))$ contains a hyperbolic ball of definite size about z'_n . The other properties stated in the lemma follow similarly.

When α_n is bigger than some constant ε , $\operatorname{Im} \zeta_n < D/\varepsilon$ and the problem is reduced to the above case. Thus, it remains to prove the lemma for small values of α_n .

The proof when α_n is small:

If $\operatorname{Re} \zeta_n \in (7/8, 1/\alpha_n - \mathbf{k} - 1)$ and $\operatorname{Im} \zeta_n > M + 1$ define

$$\begin{aligned} r'_n &:= \frac{1}{4} \min\{\operatorname{Re} \zeta_n, \frac{1}{\alpha_n} - \mathbf{k} - \operatorname{Re} \zeta_n\}, \\ \zeta'_n &:= \operatorname{Re} \zeta_n - M\mathbf{i} - r'_n\mathbf{i}, \\ V'_n &:= \{\zeta \in \mathbb{C}; |\operatorname{Re}(\zeta - \zeta_n)| < 3r'_n\} \cup \{\zeta \in \mathbb{C}; \operatorname{Im} \zeta > M, |\operatorname{Re}(\zeta - \zeta_n)| < \frac{1}{2\alpha_n}\}. \end{aligned}$$

Using the estimate in (17), one can verify that the hyperbolic distance within V'_n from ζ_n to ζ'_n is uniformly bounded independent of n . Similarly, $B(\zeta'_n, r'_n)$ contains a hyperbolic ball of definite size independent of n about ζ'_n .

The map Φ_n^{-1} extends to a map defined on V'_n , using the dynamics of f_n . Furthermore, it is univalent for sufficiently small values of α_n . Now, define $V_n := \Phi_n^{-1}(V'_n)$ and $z'_n := \Phi_n^{-1}(\zeta'_n)$. The isometry Φ_n^{-1} preserves the respective distance on V'_n and V_n , hence $d_{\rho_n}(z_n, z'_n)$ is uniformly bounded independent of n , and $\Phi_n^{-1}(B(\zeta'_n, r'_n))$ contains a hyperbolic ball of definite size about z'_n . The other properties in the lemma should be clear as well.

If $\zeta_n \in W_n$ and $\operatorname{Im} \zeta_n \geq M + 1$ define

$$\begin{aligned} r'_n &:= (\operatorname{Im} \zeta_n - M)/(4D), \\ \zeta'_n &:= (\frac{1}{\alpha_n} - \mathbf{k} - r'_n) - \mathbf{i}(M + r'_n), \\ V'_n &:= \{\zeta \in \mathbb{C}; \frac{1}{2\alpha_n} < \operatorname{Re} \zeta < \frac{1}{\alpha_n} - \mathbf{k}\} \cup \{\zeta \in \mathbb{C}; \operatorname{Im} \zeta > M, \frac{1}{\alpha_n} - \mathbf{k} \leq \operatorname{Re} \zeta < \frac{5}{4\alpha_n}\}. \end{aligned}$$

Similarly, extend Φ_n^\dagger to a univalent map on V'_n , using the dynamics of f_n , and let $V_n := \Phi_n^\dagger(V'_n)$ and $z'_n := \Phi_n^\dagger(\zeta'_n)$. By our choice of M , Φ_n^\dagger is an isometry from V'_n to V_n with respect to the hyperbolic metrics. Hence, one can verify that $d_{\rho_n}(z_n, z'_n)$ is uniformly bounded from above independent of n , and, $\Phi_n^{-1}(B(\zeta'_n, r'_n))$ contains a hyperbolic ball of definite size about z'_n . \square

Fix an n with $\operatorname{Im} \zeta_n < D/\alpha_n$. We inductively define a sequence of domains $V_n, V_{n-1}, V_{n-2}, \dots, V_0$, and maps g_n, g_{n-1}, \dots, g_1 , where each $V_i = \Omega_i^0 \setminus I_{i,r(i)}$, for some $r(i) \in [3/2, 1/\alpha_i - \mathbf{k}]$, $z_i \in V_i$, and each $g_i : V_i \rightarrow V_{i-1}$ is a holomorphic or anti-holomorphic map with $g_i(z_i) = z_{i-1}$.

Let V_n be the domain obtained in Lemma 3.8. Assuming V_l is defined for some l , the map g_l and the domain V_{l-1} are defined according to one of the following scenarios. Since V_l is a simply connected domain which spirals at most \mathbf{k}' times about 0, there

is an inverse branch of $\mathbb{E}xp$ defined on V_l , denoted by η_l , that maps z_l to ζ_{l-1} . As $\eta_l(V_l)$ contains ζ_l , by (14), one of the following must occur:

- (a) $\operatorname{Re} \eta_l(V_l) \subset (-\mathbf{k}' + 7/8, \mathbf{k}' + 1/2)$
- (b) $\operatorname{Re} \eta_l(V_l) \subset (1/2, 1/\alpha_l - \mathbf{k} - 1/2)$
- (c) $\operatorname{Re} \eta_l(V_l) \subset (1/\alpha_l - \mathbf{k} - 1/2, 1/\alpha_l - \mathbf{k} + k_n + 2\mathbf{k}')$

When (a) occurs, we must have $\zeta_{l-1} \in \cup_{j=1}^{\mathbf{k}'} B(j, \delta_1)$, and hence by Lemma 1.4, $\eta_l(V_l)$ contains one of these balls. This implies that $\eta_l(V_l)$ avoids δ_1 -neighborhoods of the points $0, -1, -2, \dots, -\mathbf{k}' + 1$. Now, we define $g_l := \Phi_l^{-1} \circ \eta_l$ on V_l . By definition, $g_l(z_l) = z_{l-1}$. By our assumption (8), $g_l(V_l)$ avoids the curve $I_{l-1, \mathbf{k}'+1}$, and therefore, g_l maps V_l into the set $V_{l-1} := \Omega_{l-1} \setminus I_{l-1, \mathbf{k}'+1}$. Indeed, by Lemma 1.4,

$$(18) \quad \Phi_l^{-1}(B(\eta_l(V_l), \delta_1)) \subseteq V_{l-1}.$$

When (b) occurs, let $g_l := \Phi_l^{-1} \circ \eta_l$, and $V_{l-1} := \Omega_{l-1} \setminus I_{l-1, \alpha_{l-1}^{-1} - \mathbf{k}}$. Here also, g_l maps z_l to z_{l-1} , and property (18) holds.

When (c) occurs, let $g_l := \Phi_l^\dagger \circ \eta_l$, and $V_{l-1} := \Omega_{l-1} \setminus I_{l-1, \alpha_{l-1}^{-1} - \mathbf{k}-1}$. Here, we have

$$(19) \quad \Phi_l^\dagger(B(\eta_l(V_l), \delta_1)) \subseteq V_{l-1}.$$

and $g_l(z_l) = z_{l-1}$, here as well.

The maps g_i enjoy the following properties.

Lemma 3.9. — *There are constants δ_4, δ_5 such that for every $i = 1, 2, \dots, n$*

- i. *the map $g_i : (V_i, \rho_i) \rightarrow (V_{i-1}, \rho_{i-1})$ is uniformly contracting, that is, for every $z \in V_i$*

$$\rho_{i-1}(g_i(z)) |g_i'(z)| \leq \delta_4 \rho_i(z);$$

- ii. *the map $g_i : V_i \rightarrow V_{i-1}$ is univalent on the hyperbolic ball*

$$\{z \in V_i \mid d_{\rho_i}(z, z_i) < \delta_5\};$$

- iii. *Each map g_i is either univalent or has a unique simple critical point in V_i .*

Proof. — *Part i.* By the definition of g_i , and Equations (18) and 19, each g_i can be decomposed as

$$(V_i, \rho_i) \xrightarrow{\eta_i} (\eta_i(V_i), \tilde{\rho}_i) \xrightarrow{inc.} (B_{\delta_1}(\eta_i(V_i)), \hat{\rho}_i) \xrightarrow{\Phi_i^{-1} \text{ or } \Phi_i^\dagger} (V_{i-1}, \rho_{i-1}),$$

where $\tilde{\rho}_i(z) |dz|$ and $\hat{\rho}_i(z) |dz|$ denote the hyperbolic metric of constant -1 curvature on $\eta_i(V_i)$ and $B_{\delta_1}(\eta_i(V_i))$, respectively.

By the Schwartz-Pick Lemma, the first map and the last map in the above decomposition are non-expanding, so it is enough to show that the inclusion map is uniformly contracting. To this end, fix an arbitrary point ζ_0 in $\eta_i(V_i)$, and define

$$H(\zeta) := \zeta + (\zeta - \zeta_0) \frac{\delta_1}{(\zeta - \zeta_0 + 2\mathbf{k}' + 1)} : \eta_i(V_i) \rightarrow \mathbb{C}.$$

As $\text{diam Re}(\eta_i(V_i)) \leq \mathbf{k}'$, for every $\zeta \in \eta_i(V_i)$ we have $|\text{Re}(\zeta - \zeta_0)| \leq \mathbf{k}'$. Consequently, $|\zeta - \zeta_0| < |\zeta - \zeta_0 + 2\mathbf{k}' + 1|$, and then

$$|H(\zeta) - \zeta| = \delta_1 \left| \frac{\zeta - \zeta_0}{\zeta - \zeta_0 + 2\mathbf{k}' + 1} \right| < \delta_1.$$

This implies that, $H(\zeta)$ is a holomorphic map from $\eta_i(V_i)$ into $B_{\delta_3}(\eta_i(V_i))$. By Schwartz-Pick Lemma, H is non-expanding. In particular, at $H(\zeta_0) = \zeta_0$ we have

$$\tilde{\rho}_i(\zeta_0)|H'(\zeta_0)| = \tilde{\rho}_i(\zeta_0)\left(1 + \frac{\delta_1}{2\mathbf{k}' + 1}\right) \leq \hat{\rho}_i(\zeta_0).$$

That is, $\tilde{\rho}_i(\zeta_0) \leq \delta_4 \cdot \hat{\rho}_i(\zeta_0)$ with $\delta_4 = (2\mathbf{k}' + 1)/(2\mathbf{k}' + 1 + \delta_1)$.

Part ii. In cases (a) and (b) of the definition of the map g_i both η_i and Φ_i^{-1} are univalent, hence g_i is univalent on the whole domain V_i .

In case (c) of the definition of the map g_i , η_i is univalent, and Φ_{i-1}^\dagger has \mathbf{k}' simple critical points that are mapped to

$$f_{i-1}(\text{cp}), f_{i-1}^{\circ 2}(\text{cp}), \dots, f_{i-1}^{\circ \mathbf{k}'}(\text{cp}).$$

However, in this case

$$z_{i-1} \notin \Phi_{i-1}^{-1}(B(j, \delta_1)), \text{ for } j = 1, 2, \dots, \mathbf{k}'.$$

The curve, $\Phi_{i-1}^{-1}(-2\mathbf{i} + t)$, for $t \in [0, \mathbf{k}']$, belongs to the boundary of V_{i-1} . By the continuous dependence of the Fatou coordinate on the map, there exists $\delta_5 > 0$ such that

$$B_{\rho_{i-1}}(f_{i-1}^{\circ j}(\text{cp}), \delta_5) \subseteq \Phi_{i-1}^{-1}(B(j, \delta_1)), \text{ for } j = 1, 2, \dots, \mathbf{k}'.$$

Hence, combining with the above argument, the branched covering Φ_{i-1}^\dagger has no critical value in $B_{\rho_{i-1}}(z_{i-1}, \delta_5)$. Let W be a connected component of $g_i^{-1}(B_{\rho_{i-1}}(z_{i-1}, \delta_5))$ containing z_i . As $B_{\rho_{i-1}}(z_{i-1}, \delta_5)$ is simply connected, g_i is univalent on W . In particular, by the contraction proved in Part i, g_i is univalent on $B_{\rho_i}(z_i, \delta_5) \subseteq W$.

Part iii. In both cases (a) and (b), the map g_i is a composition of the univalent map η_i and an inverse branch of Φ_i on $\eta_i(V_i)$. Hence, it is univalent.

In case (c), the lift $\eta_i(V_i)$ cannot contain any two points apart by an integer value. On the other hand, by the equivariance property of the Fatou coordinates, Φ_{i-1}^\dagger has \mathbf{k}' number of simple critical points apart by one. Therefore, no two of these critical points can belong to $\eta_i(V_i)$. \square

The composition

$$G_n := g_1 \circ \dots \circ g_{n-1} \circ g_n : V_n \rightarrow V_0,$$

maps z_n to z_0 .

Lemma 3.10. — *There are constants D_1 and $\delta_6 \in (0, 1)$ such that for every n with $\text{Im } \zeta_n < D/\alpha_n$, there is $r_n \in (0, 1)$ such that*

- i. $G_n(B_{\rho_n}(z'_n, \delta_3)) \cap \mathcal{PC}(f_0) = \emptyset$;
- ii. $B(G_n(z'_n), r_n) \subset G_n(B_{\rho_n}(z'_n, \delta_3))$, and $|G_n(z'_n) - z_0| \leq D_3 \cdot r_n$;

iii. $r_n \rightarrow 0$, as $n \rightarrow \infty$.

Proof. — *Part i.* First note that the conformal change of coordinate Ψ_n preserves the post-critical sets. That is,

$$\Psi_n(\mathcal{PC}(f_n) \cap \mathcal{P}_n) = \mathcal{PC}(f_0) \cap \Psi_n(\mathcal{P}_n).$$

The next step is to see that there exists an integer $l \leq q_n$ such that

$$G_n(B_{\rho_n}(z'_n, \delta_3)) = f_0^{ol}(\Psi_n(B_{\rho_n}(z'_n, \delta_3))).$$

The existence of such an l follows from the definition of the renormalization. It is less than q_n since when defining each g_i , $i = n-1, n-2, \dots, 1$, the map f_i is iterated less than $1/\alpha_i$.

Now,

$$\begin{aligned} & G_n(B_{\rho_n}(z'_n, \delta_3)) \cap \mathcal{PC}(f_0) \neq \emptyset \\ (\text{As } f_0(\mathcal{PC}(f_0)) = \mathcal{PC}(f_0)) & \Rightarrow f_0^{q_n-l} \circ G_n(B_{\rho_n}(z'_n, \delta_3)) \cap \mathcal{PC}(f_0) \neq \emptyset. \\ & \Rightarrow f_0^{q_n} \circ \Psi_n(B_{\rho_n}(z'_n, \delta_3)) \cap \mathcal{PC}(f_0) \neq \emptyset \\ (\text{By Lemma 2.2}) & \Rightarrow \Psi_n \circ f_n(B_{\rho_n}(z'_n, \delta_3)) \cap \mathcal{PC}(f_0) \neq \emptyset \\ & \Rightarrow f_n(B_{\rho_n}(z'_n, \delta_3)) \cap \mathcal{PC}(f_n) \neq \emptyset. \end{aligned}$$

The last line contradicts Lemma 3.8-ii, since Proposition 2.3 implies that $\mathcal{PC}(f_n)$ is contained in Ω_n^0 .⁽⁶⁾

Part ii. Let m be the smallest positive integer with $(\delta_3 + \delta_2) \cdot (\delta_4)^m \leq \delta_5$, and then decompose the map G_n into its first $m+1$ iterates and the rest as

$$G_n := G_n^u \circ G_n^l, \text{ where } G_n^l := g_{n-m} \circ \dots \circ g_{n-1} \circ g_n, \text{ and } G_n^u := g_1 \circ g_2 \circ \dots \circ g_{n-m-1}.$$

Let γ_n denote the hyperbolic geodesic in V_n connecting z_n to z'_n . By the above choice of m and Lemma 3.9-i, we have

$$G_n^l(B_{\rho_n}(z'_n, \delta_3) \cup \gamma_n) \subset B_{\rho_{n-m}}(z_{n-m}, \delta_4 \delta_5).$$

Since each g_i , for $i = n-m-1, n-m-2, \dots, 1$, is contacting and univalent on the ball $B_{\rho_i}(z_i, \delta_5)$, G_n^u is univalent on $B_{\rho_{n-m-1}}(z_{n-m-1}, \delta_5)$. By the Koebe distortion Theorem, G_n^u has uniformly bounded, independent of n , distortion on $G_n^l(B_{\rho_n}(z'_n, \delta_3) \cup \gamma_n)$.

On the other hand, G_n^l is a uniformly bounded number of iterates of some g_i that belong to a compact class of maps in the compact open topology. More precisely, by definition, each g_i is a composition of η_i and either of Φ_i^{-1} or Φ_i^\dagger . The map η_i is univalent and, by Koebe's Theorem, has bounded distortion on hyperbolic sets of bounded diameter, and each of the other maps has extension over a larger domain by Equation (18).

Since near z_0 the hyperbolic and the Euclidean metrics are equivalent, combining the above arguments, one concludes that $G_n(B_{\rho_n}(z'_n, \delta_3))$ contains an round ball of Euclidean size proportional to its Euclidean diameter, and that the Euclidean diameter of $G_n(B_{\rho_n}(z'_n, \delta_3))$ is proportional to $|G_n(z'_n) - z_0|$.

⁽⁶⁾Indeed, one can show that $G_n(B_{\rho_n}(z'_n, \delta_3)) \cap \Omega_0^n = \emptyset$, see [Che10]-Lemma 4.7.

Part iii. The hyperbolic diameter of $G_n(B_{\rho_n}(z'_n, \delta_3))$ tends to zero exponentially fast, by Lemma 3.9-i. Hence, by equivalence of the hyperbolic and Euclidean metrics at local scales, r_n must tend to zero. \square

Proof of Proposition 3.7. — Given $z_0 \in \Lambda$, we have an increasing sequence n_i with $\text{Im } \zeta_{n_i} < D/\alpha_{n_i}$. Lemma 3.8 provides us with the balls $B_{\rho_{n_i}}(z'_n, \delta_3)$ in the complement of $\mathcal{PC}(f_{n_i})$. The maps \mathcal{G}_{n_i} , by Lemma 3.10, carry these balls to domains in the complement of $\mathcal{PC}(f_0)$ containing round balls $B(G_n(z'_{n_i}), r_{n_i})$ at arbitrarily small scales near z_0 . Hence, z_0 can not be a Lebesgue density point of Λ . \square

3.5. Non-uniform porosity of decorations. — For Brjuno values of α , let Δ_α denote the Siegel disk of P_α , and $\overline{\Delta}_\alpha$ denote its closure. Here we show that $\mathcal{PC}(P_\alpha)$ is non uniformly porous at every point in $\mathcal{PC}(P_\alpha) \setminus \overline{\Delta}_\alpha$.

Lemma 3.11. — *We have $\mathcal{PC}(P_\alpha) \setminus \overline{\Delta}_\alpha \subset \Lambda$.*

Proof. — Let $z \in \mathcal{PC}(P_\alpha) \setminus \overline{\Delta}_\alpha$ and assume on the contrary that there is an integer N with $\text{Im } \zeta_m \geq D/\alpha_m$, for all $m \geq N$.

By the first argument in the proof of Lemma 3.4, we have

$$d(\zeta_j, \Phi_j(\partial\Delta_j)) \leq \mathcal{B}(\alpha_{j+1})/2\pi,$$

for all j . The boundary of the Siegel disk of a map is sent to the boundary of the Siegel disk of its renormalization under the change of coordinates. Now it follows from Lemmas 3.2 and 3.4 that for all $j \geq N$ we have $d(\zeta_j, \Phi_j(\partial\Delta_j)) \leq 4A$. By a contraction argument for the changes of coordinates from a level of renormalization to the previous level with respect to appropriate hyperbolic metrics, see Lemma 3.9-i, one can see that all these distances must be zero. This implies that $z \in \overline{\Delta}_\alpha$. \square

Theorem D follows from the above lemma and Proposition 3.7.

4. Accumulation on the critical point

Let $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a rational mapping with $J(f) \neq \hat{\mathbb{C}}$, and U be an arbitrary neighborhood of its post-critical set $\mathcal{PC}(f)$. By a lemma of [Lyu83], the orbit of Lebesgue almost every point in $J(f)$ eventually stays in U . Therefore, by virtue of Proposition 2.3-ii, for every $n \geq 0$ and almost every $z \in J(P_\alpha)$, there is a positive integer k such that the orbit of $f^{o_k}(z)$ is contained in Ω^n . (Indeed, it is proved in [Che10] that while the orbit of a point stays in some Ω^n , it visits all the sectors involved in the union.)

Here we introduce a shrinking sequence of sets containing cp, denoted by Q_0^n , that are visited by the orbit of almost every $z \in J(P_\alpha)$.

Recall the sets $\mathcal{C}_n^{-k_n}$ and $(\mathcal{C}_n^\sharp)^{-k_n}$ obtained in the definition of $\mathcal{R}(f_n)$. For every $n \geq 0$, define the sets

$$W_n := \mathcal{C}_n^{-k_n} \subset S_n^0, \text{ and } W^n := \Psi_n(W_n) \subset S_0^n$$

Note that for every $n \geq 0$ there is a pre-critical point of f_n in W_n and hence, by Lemma 2.2-(2), there is a pre-critical point of f_0 in W^n . For every $n \geq 0$, define

$$Q^n := f_0^{\circ \tau(n)}(W^n),$$

where $\tau(n)$ is the smallest positive integer with $\text{cp} \in \text{int}(f_0^{\circ \tau(n)}(W^n))$.

Lemma 4.1. — For all $n \geq 0$ and all $z \in (\mathcal{C}_n^\sharp)^{-k_n}$, we have $\text{Im} \phi_n(z) \geq -2$.

Proof. — By the definition of the renormalization $\mathcal{R}f_n$, it is enough to show that

$$\forall w \in B(0, \frac{4}{27}e^{-4\pi}), \text{ we have } f_{n+1}^{-1}(w) \in B(0, \frac{4}{27}e^{4\pi}),$$

where $f_{n+1}^{-1}(w)$ is the unique pre-image of w . The map f_{n+1} has a unique simple critical point mapped to $-4/27$. Hence, there is an inverse branch of f_{n+1} defined on the ball $B(0, -4/27)$. By Koebe distortion Theorem, it follows that $f_{n+1}^{-1}(B(0, \frac{4}{27}e^{-4\pi}))$ is contained in $B(0, \frac{4}{27}e^{4\pi})$. \square

Lemma 4.2. — Assume that $z \in \Omega^n \setminus \Omega^{n+1}$ and $f_0(z) \in \Omega^{n+1}$, for some $n \geq 0$. Then, either $z \in Q^n$ or $f_0(z) \in W^{n+1}$ (hence, $f_0^{\circ(\tau(n+1)+1)}(z) \in Q^{n+1}$).

Proof. — There is a

$$j \in \{0, 1, \dots, q_{n+1}(k_{n+1} + \lfloor 1/\alpha_{n+1} \rfloor - k - 1) + q_n\},$$

such that $f_0(z) \in f_0^{\circ j}(S_0^{n+1})$, by the definition of Ω^{n+1} . If $j \neq 0$, there is a $w \in f_0^{j-1}(S_0^{n+1})$ with $f_0(w) = f_0(z)$. If $z \notin Q^n$, as $f_0 : \Omega^n \setminus Q^n \rightarrow \mathbb{C}$ is univalent, we must have $z = w \in \Omega^{n+1}$ which contradicts the assumption in the lemma. Therefore, either $j = 0$ or $z \in Q^n$.

If $j = 0$, then

$$\begin{aligned} f_0(z) \in S_0^{n+1} &= \Psi_{n+1}(S_{n+1}^0) \\ &= W^{n+1} \cup \Psi_{n+1}((\mathcal{C}_{n+1}^\sharp)^{-k_{n+1}}) \end{aligned}$$

However, if $f_0(z) \in \Psi_{n+1}((\mathcal{C}_{n+1}^\sharp)^{-k_{n+1}})$, by Lemma 4.1, $z \in \Omega^{n+1}$. This finishes the proof of the lemma. \square

Let us define the *eccentricity* of a domain Q at a point $q \in \text{int } Q$ as

$$\text{Ecc}(Q, q) := \frac{\inf\{r \mid Q \subset B(q, r)\}}{\sup\{r \mid B(q, r) \subset Q\}}.$$

Lemma 4.3. — We have

- $\sup_n \text{Ecc}(Q^n, \text{cp}) < \infty$,
- $\lim_{n \rightarrow \infty} \text{diam } Q^n = 0$.

Proof. — The proof of the first statement is broken into four steps.

Step 1. For every $n \geq 0$, there exists a simply connected domain \widehat{W}_n containing W_n with the following properties.

- $\widehat{W}_n \subset \mathcal{P}_n$, $f_n^{\circ k_n}(\widehat{W}_n) \subset \mathcal{P}_n$,
- $\inf_n \text{mod}(\widehat{W}_n \setminus W_n) > 0$,

– $f_n^{\circ k_n} : \widehat{W}_n \rightarrow f_n^{k_n}(\widehat{W}_n)$ is a proper branched covering of degree two.

Recall that $f_n^{\circ k_n} : W_n \rightarrow \mathcal{C}_n$ is a proper branched covering of degree two, k_n is uniformly bounded (Lemma 1.3), $B_{1/2}(W_n) \subset \mathcal{P}_n$, and $B_{1/2}(\mathcal{C}_n) \subset \mathcal{P}_n$. Now the above step follows from the compactness of the class \mathcal{IS} , and the continuous dependence of W_n on f_n .

Step 2. If $f_n^{-k_n}(\text{cv}_{f_n})$ is the unique preimage of cv_{f_n} contained in W_n then,

$$\sup_n \text{Ecc}(W_n, f_n^{-k_n}(\text{cv}_{f_n})) < \infty.$$

By Koebe distortion Theorem, $\sup_n \text{Ecc}(\mathcal{C}_n, \text{cv}_{f_n}) < \infty$. Indeed, Φ_n^{-1} is defined and univalent on the $1/2$ -neighborhood of $\Phi_n(\mathcal{C}_n)$. The map $f_n^{\circ k_n} : W_n \rightarrow \mathcal{C}_n$ belongs to a compact class by Step 1. This implies the above statement on eccentricity of W_n .

Step 3. $f_0^{\circ \tau(n)}$ is univalent on $\Psi_n(\widehat{W}_n)$, and $f_0^{\circ \tau(n)}(\Psi_n(f_n^{-k_n}(\text{cv}_{f_n}))) = \text{cp}$.

By Step 1 and Lemma 2.2, the map $f_0^{\circ(k_n q_n + q_{n-1})} : \Psi_n(\widehat{W}_n) \rightarrow \mathcal{P}_0$ is a proper branched covering of degree two mapping $\Psi_n(f_n^{-k_n}(\text{cv}_{f_n}))$ to cv . We must have

$$(20) \quad \tau(n) < k_n q_n + q_{n-1}.$$

On the other hand, $f_0^{\circ \tau(n)}(W^n)$ contains a critical point, and therefore, $f_0^{\circ \tau(n)}$ must be univalent on $\Psi_n(\widehat{W}_n)$.

Step 4. We have $\sup_n \text{Ecc}(Q^n, \text{cp}) < \infty$.

By definition $Q^n := f_0^{\circ \tau(n)} \circ \Psi_n(W_0^n)$, and by Step 3, $f_0^{\circ \tau(n)} \circ \Psi_n$ has univalent extension onto \widehat{W}_n with $\inf_n \text{mod}(\widehat{W}_n \setminus W_n) > 0$. Again by the Koebe distortion Theorem, there exists a constant $C(\varepsilon)$ such that

$$\text{Ecc}(Q^n, \text{cp}) \leq C(\varepsilon) \text{Ecc}(W_n, f_n^{-k_n}(\text{cv}_{f_n})).$$

Now Step 4 follows from Step 2.

To prove the second part, we use Montel's Normal Family Theorem. Since $Q^n \subset f_0^{\circ \tau(n)}(S_0^n)$, by definition of Ω^n , f_0 can be iterated at least

$$q_n(k_n + \lfloor 1/\alpha_n \rfloor - k - 1) + q_{n-1} - \tau(n)$$

times on Q^n with values in Ω^n . By (20), the above quantity goes to infinity as n goes to infinity.

Now if $\limsup_{n \rightarrow \infty} \text{diam}(Q^n) \neq 0$, by the first part, a ball of positive size about cp is contained in infinitely many Q^n 's. By Montel's Normal Family Theorem, this implies that the critical point belongs to the Fatou set of f_0 , which is not possible. \square

Indeed, one can show that $\text{diam} Q^n$ converges exponentially fast to zero by showing that the changes of coordinates between different levels of renormalization are uniformly contracting. See Subsection 4.3 in [Che10].

Proof of Theorem C. — By the argument after Proposition 2.3, the sets

$$E_n := \{z \in J(P_\alpha) \mid \mathcal{O}(z) \text{ eventually stays in } \Omega^n\},$$

for $n \geq 0$, have full Lebesgue measure in $J(P_\alpha)$. Therefore, their intersection has full Lebesgue measure in $J(P_\alpha)$.

Let $z \in \bigcap_{n=0}^{\infty} E_n \cap J(P_\alpha)$. As $\bigcap_{j=0}^{\infty} \Omega^j \cap J(P_\alpha)$ has zero area by Proposition 3.1 in the Introduction and Theorem C in [Che10], there are increasing sequences of positive integers $\{n_i\}_{i=1}^{\infty}$ and $\{t_i\}_{i=1}^{\infty}$ with the following property:

$$P_\alpha^{\circ t_i}(z) \in \Omega^{n_i} \setminus \Omega^{n_i+1}, \text{ and } P_\alpha^{\circ t_i+1}(z) \in \Omega^{n_i+1}.$$

Now by Lemma 4.2, there are positive integers $s_i \geq t_i$ with $P_\alpha^{\circ s_i}(z) \in Q_0^{n_i}$. Combining with Lemma 4.3, we conclude that the orbit of z gets arbitrarily close to the critical point of P_α . \square

5. Perturbed Fatou coordinate

5.1. The Pre-Fatou coordinate. — Let

$$h(z) = e^{2\pi\alpha i} \cdot P \circ \varphi^{-1}(z): \varphi(U) \rightarrow \mathbb{C}$$

be a map in the class \mathcal{IS}_α with $\alpha \neq 0$, and let σ_h denote the non-zero fixed point of h . For α small enough, $h(z)$ has only two fixed points 0 and σ_h in a fixed neighborhood of 0 (This property is guaranteed by assuming $\alpha \in Irrrat_{\geq N}$, [IS06]). By Koebe distortion Theorem one can see that $\text{cp}_h \in B(0, 2) \setminus B(0, 0.22)$.

Consider the universal covering

$$\tau_h(w) := \frac{\sigma_h}{1 - e^{-2\pi i \alpha w}} : \mathbb{C} \rightarrow \hat{\mathbb{C}} \setminus \{0, \sigma_h\}$$

that satisfies $\tau_h(w + \alpha^{-1}) = \tau_h(w)$, for all $w \in \mathbb{C}$, $\tau_h(w) \rightarrow 0$ as $\text{Im}(\alpha w) \rightarrow \infty$, and $\tau_h(w) \rightarrow \sigma_h$ as $\text{Im}(\alpha w) \rightarrow -\infty$.

The map h lifts under the covering τ_h to the map

$$F_h(w) := w + \frac{1}{2\pi\alpha i} \log \left(1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)} \right), \quad \text{with } z = \tau_\alpha(w), u_h(z) := \frac{h(z) - z}{z(z - \sigma_h)}$$

defined on the set of points w with $\tau_h(w) \in \text{Dom } h$. The branch of \log in the above formula is determined by $-\pi < \text{Im} \log(\cdot) < \pi$. We have

$$(21) \quad h \circ \tau_h = \tau_h \circ F_h, \quad \text{and} \quad F_h(w) + \alpha^{-1} = F_h(w + \alpha^{-1}).$$

The plan is to estimate the conformal mapping L_h that conjugates F_h to the translation by one. For every real number $R > 0$, let $\Theta(R)$ denote the set

$$\Theta(R) := \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} B(n/\alpha, R).$$

Lemma 5.1. — *There exist positive constants ε_0 , C_1 , C_2 , and C_3 such that for every map $h \in \mathcal{IS}_\alpha$ with $\alpha \in (0, \varepsilon_0)$ we have*

- i. *The induced map F_h is defined and is univalent on $\Theta(C_1)$. Moreover, on $\Theta(C_1)$*

$$|F_h(w) - (w + 1)| < 1/4, \quad \text{and} \quad |F_h'(w) - 1| < 1/4.$$

- ii. *For every $r \in (0, 1/2)$ and every $w \in \Theta(r/\alpha) \cap \Theta(C_1)$, we have*

$$|F_h(w) - (w + 1)| < C_2 \frac{\alpha}{r} e^{-2\pi\alpha \text{Im } w}.$$

- iii. *There exist a critical point $\text{cp}_{F_h} \in B(0, C_1)$ of F_h , and the smallest non-negative integer $i(h)$ with*

$$F_h^{\circ i(h)}(\text{cp}_{F_h}) \in \Theta(C_1), \quad \text{and} \quad \sup_h i(h) < \infty.$$

- iv. *For every positive integer $j \leq i(h) + \frac{2}{3\alpha}$, we have*

$$|F_h^{\circ j}(\text{cp}_{F_h}) - j| \leq C_3(1 + \log j).$$

Proof. — *Part i.* Recall that the model polynomial $P(z)$ has only two critical points at -1 and $-1/3$, so it is univalent on the disk of radius $1/3$. The $1/4$ -Theorem guarantees that $\varphi_h(B(0, 1/3))$ contains the disk of radius $1/12$. Hence, the composition $h(z) = \varphi_h^{-1} \circ P(e^{2\pi\alpha i} z)$ is univalent on the disk of radius $1/12$.

Note that

$$\sigma_h = \frac{1 - e^{2\pi\alpha i}}{u_h(0)},$$

and

$$\{u_h(0) \mid h \in \mathcal{IS}_{[0, \alpha^*]}\}$$

is compactly contained in $\mathbb{C} \setminus \{0\}$. Now by explicit estimates on τ_h , there are constants A_1 and ε_0 such that for all $h \in \mathcal{IS}_\alpha$ with $\alpha \in (0, \varepsilon_0)$ and all $w \in \Theta(A_1)$, $|\tau_h(w)| \leq 1/12$. By definition, F_h must be univalent on $\Theta(A_1)$, for such maps h .

For the inequalities observe that

$$\begin{aligned} |F_h(w) - (w+1)| &= \left| \frac{1}{2\pi\alpha i} \log \left(1 - \frac{\sigma_h u_h(z)}{1 + zu_h(z)} \right) - 1 \right| \\ &= \left| \frac{1}{2\pi\alpha i} \log \left(1 - \frac{\sigma_h u_h(z)}{1 + zu_h(z)} \right) - \frac{\log e^{2\pi\alpha i}}{2\pi\alpha i} \right| \\ &= \frac{1}{2\pi\alpha} \left| \log \left(\left(1 - \frac{\sigma_h u_h(z)}{1 + zu_h(z)} \right) e^{-2\pi\alpha i} \right) \right|. \end{aligned}$$

Also, replacing σ_f by the above expression,

$$\begin{aligned} \left| \left(1 - \frac{\sigma_f u_f(z)}{1 + zu_f(z)} \right) e^{-2\pi\alpha i} - 1 \right| &= \left| \left(1 - \frac{\sigma_f u_f(z)}{1 + zu_f(z)} \right) - e^{2\pi\alpha i} \right| \\ &= |1 - e^{2\pi\alpha i}| \left| 1 - \frac{u_f(z)}{(1 + zu_f(z))u_f(0)} \right| \\ &\leq 2\pi\alpha \left| 1 - \frac{u_f(z)}{(1 + zu_f(z))u_f(0)} \right|. \end{aligned}$$

On the other hand, by the above argument, there exists a constant A'_1 (independent of h and α) such that the sets $\tau_h(\Theta(A'_1))$ are uniformly contained in a given neighborhood of zero. Since the set of maps u_h forms a compact set, the above estimates imply the first inequality. The second inequality on the set $\Theta(A'_1 + 1)$ follows from the first inequality and Cauchy Integral Formula.

Part ii. We further estimate the above expression as

$$\left| 1 - \frac{u_h(z)}{(1 + zu_h(z))u_h(0)} \right| \leq 2 \left| \frac{u_h'(0) - u_h^2(0)}{u_h(0)} \right| |z| \leq C_2' |\tau_h(w)| \leq C_2 \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im} w},$$

for some constants C_2', C_2 , by explicit estimates on the map τ_h .

Part iii. The unique critical point of h at $\varphi_h(-1/3)$ lifts under τ_h into a $1/\alpha$ -periodic set. By Part i, this set must contain an element in $B(0, C_2)$. Indeed, by a simple estimate one can see that $U = \operatorname{Dom} \varphi_h$ contains the ball $B(0, 2/3)$. Using the Koebe distortion Theorem, one can conclude that the critical point of h belongs to the annulus $B(0, 2) \setminus B(0, .22)$. This implies that the set of the critical points of F_h in $B(0, C_2)$ is indeed contained in a compact subset of $\mathbb{C} \setminus (\mathbb{Z}/\alpha)$. As $\alpha \rightarrow 0$, h converges to a map in \mathcal{IS} , and for the maps in \mathcal{IS} the orbit of the critical value is attracted by the parabolic fixed point at zero. By compactness of the class \mathcal{IS} , this implies that the chosen critical point of F_h leaves $B(0, C_2)$ in a uniformly bounded number of iterates.

Part iv. This follows from the estimates in Part i, and Part ii. One may refer to [Che10] for a detailed proof. \square

5.2. The linearizing coordinate. — Let $h \in \mathcal{IS}_{(0, \alpha^*]}$ with the perturbed Fatou coordinate $\Phi_h : \mathcal{P}_h \rightarrow \mathbb{C}$. The inverse image $\tau_h^{-1}(\mathcal{P}_h)$ has countably many components going from $-\mathbf{i}\infty$ to $+\mathbf{i}\infty$. Define

$$(22) \quad L_h := \Phi_h \circ \tau_h : \operatorname{Dom} L_h \rightarrow \mathbb{C},$$

where $\operatorname{Dom} L_h$ is, by definition, the component of $\tau_h^{-1}(\mathcal{P}_h)$ separating 0 and $1/\alpha$ in \mathbb{C} . It follows from Theorem 1.1 that the map $L_h : \operatorname{Dom} L_h \rightarrow \mathbb{C}$ is univalent, $L_h(\operatorname{cv}_{F_h}) = 1$, $L_h(\operatorname{Dom} L_h)$ contains the set

$$\{w \in \mathbb{C} : 0 < \operatorname{Re}(w) < \lfloor 1/\alpha \rfloor - \mathbf{k}\},$$

$L_h(w) \rightarrow +\mathbf{i}\infty$ as $\operatorname{Im}(w) \rightarrow +\infty$, and $L_h(w) \rightarrow -\mathbf{i}\infty$ as $\operatorname{Im}(w) \rightarrow -\infty$. Moreover, L_h satisfies the Abel functional equation

$$(23) \quad L_h(F_h(w)) = L_h(w) + 1,$$

whenever both sides are defined.

Using (23), if $\alpha < \varepsilon_0$, L_h extends to a univalent map on the set

$$\begin{aligned} \Sigma_{C_1} := & \{w \in \mathbb{C} : C_1 \leq \operatorname{Re}(w) \leq \alpha^{-1} - C_1\} \\ & \cup \{w \in \mathbb{C} : \operatorname{Re}(w) \geq \alpha^{-1} - C_1, \text{ and } |\operatorname{Im} w| \geq \operatorname{Re}(w) - \alpha^{-1} + 2C_1\} \\ & \cup \{w \in \mathbb{C} : \operatorname{Re}(w) \leq C_1, \text{ and } |\operatorname{Im} w| \geq -\operatorname{Re}(w) + 2C_1\}. \end{aligned}$$

More precisely, by Lemma 5.1-i, for every $w \in \Sigma_{C_1}$ there is an integer j with $F_h^j(w) \in \operatorname{Dom} L_h$. One defines $L_h(w) := L_h(F_h^j(w)) - j$. We denote the extended map by the same notation L_h .

Lemma 5.2. — *There exists a positive constant C_4 such that for every $h \in \mathcal{IS}_\alpha$, and every $\zeta \in L_h(\operatorname{Dom} L_h) \setminus B(0, 1/2)$ we have $1/C_4 \leq |(L_h^{-1})'(\zeta)| \leq C_4$. Moreover, $L_h'(z) \rightarrow 1$, as $\operatorname{Im} z \rightarrow \infty$.*

Proof. — This follows from the the Koebe distortion Theorem, and the compactness of the class \mathcal{IS} . \square

5.3. The model. — Given $A \in \mathbb{C}$, let \mathcal{K}_h denoted the domain (with asymptotic width one) bounded by the curves $A + t\mathbf{i}$, $F_h(A + t\mathbf{i})$, $(1-s)A + sF_h(A)$, for $t \in [0, \infty)$ and $s \in [0, 1]$. Moreover, assume that \mathcal{K}_h is contained in $\Theta(C_1 + 1) \cap \Theta(r/\alpha)$.

For $(s, t) \in [0, 1] \times [0, \infty)$ define the map

$$H(s, t) := A + \int_0^s X(\ell, t) d\ell + \mathbf{i}(t + \int_0^s Y(\ell, t) d\ell),$$

where X and Y are given as

$$X(s, t) := a_0(t) + a_1(t) \sin(\pi s) + a_2(t) \cos(\pi s) + a_3(t) \sin(2\pi s) + a_4(t) \cos(2\pi s),$$

$$Y(s, t) := b_0(t) + b_1(t) \sin(\pi s) + b_2(t) \cos(\pi s) + b_3(t) \sin(2\pi s) + b_4(t) \cos(2\pi s),$$

with

$$a_0(t) = \operatorname{Re}(F_h(A + \mathbf{i}t) - A) + \operatorname{Re} F_h''(A + \mathbf{i}t)/\pi,$$

$$b_0(t) = \operatorname{Im}(F_h(A + \mathbf{i}t) - A) - t + \operatorname{Im}(F_h''(A + \mathbf{i}t))/\pi,$$

$$a_1(t) = -\operatorname{Re} F_h''(A + \mathbf{i}t)/2,$$

$$b_1(t) = -\operatorname{Im} F_h''(A + \mathbf{i}t)/2,$$

$$a_2(t) = (1 - \operatorname{Re} F_h'(A + \mathbf{i}t))/2,$$

$$b_2(t) = -\operatorname{Im} F_h'(A + \mathbf{i}t)/2,$$

$$a_3(t) = \operatorname{Re} F_h''(A + \mathbf{i}t)/4,$$

$$b_3(t) = \operatorname{Im} F_h''(A + \mathbf{i}t)/4,$$

$$a_4(t) = 1 - a_0(t) - a_2(t),$$

$$b_4(t) = -a_0(t) - a_2(t).$$

(The above coefficients are obtained from solving a separable partial differential equation so that the next lemma holds.)

Instead of algebraically manipulating constants and frequently introducing new ones, we use the following convention from now on. Given real valued functions f and g defined on a set W , we write

$$f(x) \preceq g(x) \text{ on } W$$

to mean that there exists a constant M with $f(x) \leq Mg(x)$ for all $x \in W$. For example, by Lemma 5.1 and Cauchy Integral Estimate, on the set $\Theta(r/\alpha) \cap \Theta(C_1 + 1)$ we have

$$(24) \quad \max \{ |F_h(w) - w - 1|, |F_h'(w) - 1|, |F_h''(w)|, |F_h^{(3)}(w)|, |F_h^{(4)}(w)| \} \preceq \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im} w}.$$

Using the relation $H(s + 1, t) = F_h(H(s, t))$ we extend H onto a neighborhood of the form $(-\delta, 1 + \delta) \times (0, \infty)$, for some $\delta > 0$.

Lemma 5.3. — *The map H satisfies the following properties:*

- i. *For every $t \in (0, \infty)$ and $s \in (-\delta, \delta)$ we have*

$$F(H(s, t)) = H(s + 1, t).$$

- ii. *H is C^2 on $(-\delta, 1 + \delta) \times (0, \infty)$.*

iii. On $(-\delta, 1 + \delta) \times (0, \infty)$, we have

$$\max \{ |\partial_s H(s, t) - 1|, |\partial_t H(s, t) - \mathbf{i}| \} \preceq \frac{\alpha}{r} e^{-2\pi\alpha(t + \text{Im } A)},$$

and

$$\max \{ |\partial_{ss} H(s, t)|, |\partial_{tt} H(s, t)|, |\partial_{st} H(s, t)| \} \preceq \frac{\alpha}{r} e^{-2\pi\alpha(t + \text{Im } A)}.$$

Proof. — Clearly $H(0, t) = A + \mathbf{t}\mathbf{i}$. On the other hand

$$\begin{aligned} H(1, t) &= A + a_0(t) + 2a_1(t)/\pi + (t + b_0(t) + 2b_1(t)/\pi)\mathbf{i} \\ &= F(A + \mathbf{t}\mathbf{i}) \end{aligned}$$

For other values of s , the relation follows from the definition.

The map H is real analytic in the domain $(-\delta, 1 + \delta) \times (0, \infty) \setminus \{0, 1\} \times (0, \infty)$. A straightforward calculation shows that it is C^2 on the lines $\{0, 1\} \times (0, \infty)$.

From Inequality (24), on $[0, \infty)$ we have

$$\begin{aligned} |a_0(t) - 1| &\preceq \frac{\alpha}{r} e^{-2\pi\alpha(t + \text{Im } A)}, \\ |a_j(t)| &\preceq \frac{\alpha}{r} e^{-2\pi\alpha(t + \text{Im } A)}, \text{ for } j = 1, 2, 3, 4, \\ |b_j(t)| &\preceq \frac{\alpha}{r} e^{-2\pi\alpha(t + \text{Im } A)}, \text{ for } j = 0, 1, 2, 3, 4. \end{aligned}$$

Similar estimates hold for the first and second derivatives of these functions. This implies that on $[0, 1] \times (0, \infty)$

$$\max \{ |X - 1|, |\partial_s X|, |\partial_t X|, |\partial_{tt} X|, |Y|, |\partial_t Y|, |\partial_{tt} Y| \}(s, t) \preceq \frac{\alpha}{r} e^{-2\pi\alpha(t + \text{Im } A)}.$$

One infers the inequalities in Part iii from these estimates. \square

To simplify the calculations, from now on we use the complex notation $z = s + \mathbf{i}t$, $dz = ds + \mathbf{i}dt$, $d\bar{z} = ds - \mathbf{i}dt$, $\partial_z = (\partial_s - \mathbf{i}\partial_t)/2$, $\partial_{\bar{z}} = (\partial_s + \mathbf{i}\partial_t)/2$.

To compare L_h^{-1} to H we work on the map

$$G := L_h \circ H : (-\delta, 1 + \delta) + \mathbf{i}(0, \infty) \rightarrow \mathbb{C}.$$

In particular, we would like to prove that

$$(25) \quad |\partial_z G(z) - 1| \preceq \frac{1}{r} e^{-2\pi\alpha \text{Im } z}.$$

Lemma 5.4. — *The map G has a C^2 extension onto \mathbb{C} which satisfies*

$$(26) \quad G(z + 1) = G(z) + 1, \quad \forall z \in \mathbb{C},$$

and is the translation by $G(0, 0)$ on $\{z \mid \text{Im } z \leq -1\}$.

i. On $\{z \mid \text{Im } z \geq 0\}$,

$$\max \{ |\partial_{\bar{z}} G(z)|, |\partial_{z\bar{z}} G(z)| \} \preceq \frac{\alpha}{r} e^{-2\pi\alpha \text{Im}(z+A)},$$

and on $\{z \mid \text{Im } z \in [-1, 0]\}$,

$$\max \{ |\partial_{\bar{z}} G(z)|, |\partial_{z\bar{z}} G(z)| \} \preceq 1.$$

ii. *There exists a constant z_0 such that as $\text{Im } z \rightarrow \infty$, $G(z) - z \rightarrow z_0$.*

iii For every constant $\delta \in \mathbb{R}$ there is a constant C_7 independent of r such that if $\mathcal{K}_h \subseteq \Theta(r/\alpha)$ and $\liminf_{w \in \mathcal{K}_h} \operatorname{Im} w \geq \delta/\alpha$ then for every $z_1, z_2 \in \operatorname{Dom} G$,

$$|G(z_2) - G(z_1) - (z_2 - z_1)| \leq C_7/r.$$

Proof. — From Lemma 5.3-i and functional equation (23), the Relation (26) holds for every z with $\operatorname{Re} z \in (-\delta, \delta)$ and $\operatorname{Im} z > 0$. Moreover, it is C^2 as H is C^2 smooth and L_h is analytic. We can use (26) to extend G onto the upper half plane.

Below the line $\operatorname{Im} z = -1$ we define $G(z) := z + G(0, 0)$. Then, there exists a C^2 smooth interpolation of G on the strip $\operatorname{Im} z \in [-1, 0]$ that satisfies (26), and has uniformly bounded first and second order partial derivatives. (For example, one may obtain this interpolation by first C^1 gluing the vector field $\partial G/\partial s$ over the upper half plane to the constant vector field 1 below the line $\operatorname{Im} z = -1$, and then integrate this vector field.)

Part i: By complex chain rule, with $w = H(z)$, we have

$$\begin{aligned} |\partial_z G(z)| &= |\partial_w L_h(H(z)) \cdot \partial_z H(z)| \\ &\leq \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im}(z+A)}, \end{aligned}$$

by Lemmas 5.2 and 5.3-iii.

On the other hand, Lemma 5.2 and the Cauchy Integral Formula imply that $|\partial_{ww} L_h|$ is uniformly bounded well inside the domain of L_h . Therefore, from the estimates in Lemma 5.3, and the complex chain rule

$$\begin{aligned} |\partial_{z\bar{z}} G(z)| &= |\partial_{ww} L_h(H(z)) \cdot \partial_z H(z) \cdot \partial_{\bar{z}} H(z) + \partial_w L_h(H(z)) \cdot \partial_{z\bar{z}} H(z)| \\ &\leq \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im}(z+A)}. \end{aligned}$$

Part ii: Since G is periodic of period one, it is enough to prove the asymptotic behavior at z with $\operatorname{Re} z \in [0, 1]$. In this region, it is enough to prove that for every $\varepsilon > 0$ there is $R \in \mathbb{R}$ such that if $\operatorname{Im} z_1$ and $\operatorname{Im} z_2$ are larger than R then $|(G(z_2) - z_2) - (G(z_1) - z_1)|$ is less than ε .

Given real constants $t_2 > t_1$, define $\mathcal{D} := \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z \leq 1, t_1 \leq \operatorname{Im} z \leq t_2\}$. By Green's Integral Formula, we have

$$(27) \quad \int_{\partial \mathcal{D}} G(z) dz = \iint_{\mathcal{D}} \partial_{\bar{z}} G(z) d\bar{z} dz.$$

The right-hand integral is bounded by

$$\begin{aligned} \left| \iint_{\mathcal{D}} \partial_{\bar{z}} G(z) d\bar{z} dz \right| &\leq 2 \int_{t_1}^{t_2} \int_0^1 |\partial_{\bar{z}} G(s, t)| ds dt \\ &\leq \int_{t_1}^{t_2} \frac{\alpha}{r} e^{-2\pi\alpha(t+\operatorname{Im} A)} ds dt && \text{(Lemma 5.4-ii)} \\ &\leq \frac{e^{-2\pi\alpha \operatorname{Im} A}}{2\pi r} e^{-2\pi\alpha t_1}. \end{aligned}$$

This bound tends to zero, as t_1 goes to infinity.

By periodicity of G , the left hand side of (27) is equal to

$$-\mathbf{i}(t_2 - t_1) + \int_{\gamma_1} G(z) ds + \int_{\gamma_2} G(z) ds,$$

where γ_1 and γ_2 are the top and bottom boundaries of \mathcal{D} with appropriate orientations. By Lemmas 5.2 and 5.3-iii, as $\text{Im } z \rightarrow \infty$, $\partial_z G(z) \rightarrow 1$. Hence

$$\int_{\gamma_1} G(z) ds \rightarrow - \int_0^1 G(t_1 \mathbf{i}) + 1 \cdot s ds = -G(t_1 \mathbf{i}) - 1/2$$

and

$$\int_{\gamma_2} G(z) ds \rightarrow \int_0^1 G(t_2 \mathbf{i}) + 1 \cdot s ds = G(t_2 \mathbf{i}) + 1/2.$$

Putting these together, we conclude that as $\text{Im } z_1, \text{Im } z_2 \rightarrow \infty$,

$$\begin{aligned} |(G(z_2) - z_2) - (G(z_1) - z_1)| \\ \rightarrow |(G(\mathbf{i} \text{Im } z_2) - \mathbf{i} \text{Im } z_2) - (G(\mathbf{i} \text{Im } z_1) - \mathbf{i} \text{Im } z_1)| \rightarrow 0. \end{aligned}$$

This finishes the proof of this part.

Part iii: This follows from the above relations and inequalities once we have a lower bound on $\text{Im } z_1$ and $\text{Im } z_2$. (See [Che10], proof of Lemma 5.4 for further details.) \square

The map G projects via $e^{2\pi \mathbf{i} z}$ to a well-defined map on \mathbb{C} . More precisely, let

$$\phi(z) := e^{2\pi \mathbf{i} z}, \psi(\zeta) := \phi^{-1}(\zeta) = \frac{1}{2\pi \mathbf{i}} \log \zeta, \text{ with } \text{Im } \log(\cdot) \in [0, 2\pi),$$

and define

$$K(\zeta) := \phi \circ G \circ \psi(\zeta).$$

The map K has continuous extension to 0; $K(0) = 0$.

Lemma 5.5. — *The map $K : \mathbb{C} \rightarrow \mathbb{C}$ is complex differentiable at 0, and*

$$|\partial_\zeta K(\zeta) - \partial_\zeta K(0)| \leq \frac{1}{r} |\zeta|^\alpha.$$

Proof. — By the definition of derivative and Lemma 5.4-iii,

$$\begin{aligned} \partial_\zeta K(0) &= \lim_{\zeta \rightarrow 0} \frac{e^{2\pi \mathbf{i} G(\frac{1}{2\pi \mathbf{i}} \log \zeta)}}{\zeta} \\ (28) \quad &= \lim_{\zeta \rightarrow 0} e^{2\pi \mathbf{i} (G(\frac{1}{2\pi \mathbf{i}} \log \zeta) - \frac{1}{2\pi \mathbf{i}} \log \zeta)} \\ &= e^{2\pi \mathbf{i} z_0}. \end{aligned}$$

To prove the inequality, first we estimate the Laplacian of K on $\mathbb{C} \setminus \{0\}$. Using the complex chain rule, with $\zeta = \phi(z)$ and $\psi(\zeta) = z$, at $\zeta \neq 0$

$$\begin{aligned} \partial_\zeta K &= \partial_\zeta(\phi \circ G \circ \psi) = \partial_z \phi \circ (G \circ \psi) \cdot \partial_\zeta(G \circ \psi) + \partial_{\bar{z}} \phi \circ (G \circ \psi) \cdot \partial_{\bar{\zeta}}(\overline{G \circ \psi}) \\ (29) \quad &= \partial_z \phi \circ G \circ \psi \cdot \partial_\zeta(G \circ \psi) \\ &= \partial_z \phi \circ G \circ \psi \cdot \partial_z G \circ \psi \cdot \partial_\zeta \psi, \end{aligned}$$

therefore,

$$\begin{aligned}
\partial_{\bar{\zeta}}\partial_{\zeta}K &= \partial_{\bar{\zeta}}(\partial_z\phi \circ (G \circ \psi)) \cdot \partial_z G \circ \psi \cdot \partial_{\zeta}\psi \\
&\quad + \partial_z\phi \circ (G \circ \psi) \cdot \partial_{\bar{\zeta}}(\partial_z G \circ \psi) \cdot \partial_{\zeta}\psi \\
&= (\partial_{zz}\phi \circ G \circ \psi \cdot \partial_{\bar{z}}G \circ \psi \cdot \partial_{\bar{\zeta}}\bar{\psi} \cdot \partial_z G \circ \psi) \cdot \partial_z\psi \\
&\quad + \partial_z\phi \circ G \circ \psi \cdot ((\partial_{zz}G \circ \psi) \cdot \partial_{\bar{\zeta}}\bar{\psi} + \partial_{z\bar{z}}G \circ \psi \cdot \partial_{\bar{\zeta}}\bar{\psi}) \cdot \partial_z\psi \\
&= \partial_{zz}\phi \circ G \circ \psi \cdot \partial_{\bar{z}}G \circ \psi \cdot \overline{\partial_{\zeta}\bar{\psi}} \cdot \partial_z G \circ \psi \cdot \partial_z\psi \\
&\quad + \partial_z\phi \circ G \circ \psi \cdot \partial_{z\bar{z}}G \circ \psi \cdot \overline{\partial_{\zeta}\bar{\psi}} \cdot \partial_z\psi \\
&= 2\pi\mathbf{i}K|\partial_{\zeta}\psi|^2(\partial_{z\bar{z}}G \circ \psi + 2\pi\mathbf{i}\partial_{\bar{z}}G \circ \psi \cdot \partial_z G \circ \psi).
\end{aligned}$$

(To get the above expression, one could also find the coefficient of $(\zeta - \zeta_0)(\bar{\zeta} - \bar{\zeta}_0)$ in the expansion of $K(\zeta)$ near ζ_0 , as K is real analytic away from the positive real axis.)

Since G is analytic below the horizontal line $\text{Im } z = -1$, $\partial_{\bar{\zeta}}\partial_{\zeta}K(\zeta) = 0$ outside of the disk of radius $e^{2\pi}$. Above this line, we have the estimates in Lemma 5.4-i that provide us with

$$|\partial_{\zeta\bar{\zeta}}K(\zeta)| \leq \frac{\alpha}{r}|\zeta|^{\alpha-1}.$$

Fix $\zeta_0 \in \mathbb{C}$ and choose a disk $B(0, R)$ of radius $R > e^{2\pi}$ containing ζ_0 . The general form of Cauchy Integral Formula states that for the continuous function $\partial_{\zeta}K$

$$\partial_{\zeta}K(\zeta_0) = \frac{1}{2\pi\mathbf{i}} \int_{\partial B(0, R)} \frac{\partial_{\zeta}K(\zeta)}{\zeta - \zeta_0} d\zeta + \frac{1}{2\pi\mathbf{i}} \iint_{B(0, R)} \frac{\partial_{\bar{\zeta}}\partial_{\zeta}K(\zeta)}{\zeta - \zeta_0} d\zeta d\bar{\zeta}$$

Outside of the disk $B(0, e^{2\pi})$, $\partial_{\zeta}K = e^{2\pi\mathbf{i}G(0,0)}$ and $\partial_{\zeta\bar{\zeta}}K = 0$ by Lemma 5.4, thus the above formula reduces to

$$\partial_{\zeta}K(\zeta_0) = 1 + \frac{1}{2\pi\mathbf{i}} \iint_{B(0, e^{2\pi})} \frac{\partial_{\bar{\zeta}}\partial_{\zeta}K(\zeta)}{\zeta - \zeta_0} d\zeta d\bar{\zeta}.$$

We may now estimate the difference at $\zeta_0 \in B(0, 1)$ as

$$\begin{aligned}
|\partial_{\zeta}K(\zeta_0) - \partial_{\zeta}K(0)| &\leq \frac{1}{2\pi} \iint_{B(0, e^{2\pi})} |\partial_{\bar{\zeta}}\partial_{\zeta}K(\zeta)| \frac{|\zeta_0|}{|\zeta - \zeta_0| \cdot |\zeta|} d\zeta d\bar{\zeta} \\
&\leq \frac{\alpha}{r} |\zeta_0|^{\alpha} \iint_{B(0, e^{2\pi})} \frac{|\zeta_0|^{1-\alpha}}{|\zeta - \zeta_0| \cdot |\zeta|^{2-\alpha}} d\zeta d\bar{\zeta} \\
&\leq \frac{1}{r} |\zeta_0|^{\alpha}.
\end{aligned}$$

The last inequality is obtained by virtue of the following calculations.

Define $B_1 := B(0, |\zeta_0|/2)$, $B_2 := B(\zeta_0, |\zeta_0|/2)$, and $B_3 := B(0, e^{2\pi}) \setminus (B_1 \cup B_2)$. Note that on B_1 , we have $|\zeta - \zeta_0| \geq |\zeta_0|/2$, on B_2 we have $|\zeta| \geq |\zeta_0|/2$, and on B_3 we have $|\zeta - \zeta_0| \geq |\zeta|/2$. Hence

$$\begin{aligned}
\iint_{B_1} \frac{|\zeta_0|^{1-\alpha}}{|\zeta - \zeta_0| \cdot |\zeta|^{2-\alpha}} d\zeta d\bar{\zeta} &\leq \frac{2}{|\zeta_0|^{\alpha}} \iint_{B_1} \frac{1}{|\zeta|^{2-\alpha}} d\zeta d\bar{\zeta} = \frac{2^{3-\alpha}\pi}{\alpha}, \\
\iint_{B_2} \frac{|\zeta_0|^{1-\alpha}}{|\zeta - \zeta_0| \cdot |\zeta|^{2-\alpha}} d\zeta d\bar{\zeta} &\leq \frac{2^{2-\alpha}}{|\zeta_0|} \iint_{B_2} \frac{1}{|\zeta - \zeta_0|} d\zeta d\bar{\zeta} = 2^{3-\alpha}\pi,
\end{aligned}$$

and

$$\begin{aligned}
 \iint_{B_3} \frac{|\zeta_0|^{1-\alpha}}{|\zeta - \zeta_0| \cdot |\zeta|^{2-\alpha}} d\zeta d\bar{\zeta} &\leq 2|\zeta_0|^{1-\alpha} \iint_{B_3} \frac{1}{|\zeta|^{3-\alpha}} d\zeta d\bar{\zeta} \\
 &\leq 2|\zeta_0|^{1-\alpha} \iint_{B(0, e^{2\pi}) \setminus B_1} \frac{1}{|\zeta|^{3-\alpha}} d\zeta d\bar{\zeta} \\
 &= \frac{8\pi|\zeta_0|^{1-\alpha}}{\alpha-1} (e^{2\pi(\alpha-1)} - (|\zeta_0|/2)^{\alpha-1}) \\
 &\leq \frac{2^{4-\alpha}\pi}{1-\alpha}.
 \end{aligned}$$

□

Lemma 5.6. — *We have Inequality (25) on the upper half plane.*

Proof. — This results from Lemmas 5.5 and 5.4-ii, using Equations (28) and (29). □

Remark. — An alternative approach to get Inequality (25) from the model map H is using the Beltrami differential equation. It follows from the properties of H that the complex dilatation of H , $\mu := \partial_{\bar{z}}H/\partial_zH$, can be extended to a $C^{1+\epsilon}$ map onto the upper half plane, using the relation $\mu(z+1) = \mu(z)$. The function μ has absolute value strictly less than 1 at points with large imaginary part. The Beltrami equation $\mu\partial_zG = \partial_{\bar{z}}G$ has a C^2 periodic solution with Fourier expansion. Indeed, one can find the coefficients of this expansion in terms of a_j 's and b_j 's, by comparing the coefficients in the Beltrami equation term by term.

5.4. Main Estimate. —

Proposition 5.7 (main estimate). — *There exists a constant M such that for every $r \in (0, 1/2]$, and every $w \in \text{Dom } L_h \cap \Theta(r/\alpha) \cap \Theta(C_1 + 1)$, we have*

$$(30) \quad |L'_h(w) - 1| \leq \frac{M}{r} e^{-2\pi\alpha \text{Im } w}.$$

Proof. — Since $|L'_h(w)|$ is uniformly bounded by Lemma 5.2, it is enough to prove the proposition for w with $\text{Im } w \geq 0$. Given such w , we choose a $\mathcal{K}_h \subset \Theta(r/\alpha) \cap \Theta(C_1 + 1)$ containing w , and consider the corresponding map H and G . (Here $\text{Im } A \geq 0$.)

By Lemmas 5.3 and 5.4, at $z := H^{-1}(w)$ we have

$$|\partial_z H(z) - 1| \leq M_1 \frac{\alpha}{r} e^{-2\pi\alpha \text{Im } w}, \quad \text{and} \quad |\partial_z G(z) - 1| \leq M_2 \frac{1}{r} e^{-2\pi\alpha \text{Im } w},$$

for some constants M_1 and M_2 independent of r and w (and G, H).

Applying ∂_z to $G = L_h \circ H$ at $w = H(z)$, we obtain

$$\partial_z G(z) = \partial_w L_h(w) \cdot \partial_z H(z).$$

This implies the desired inequality. □

5.5. Proof of the main technical lemmas. —

Lemma 5.8. — For every $h \in \mathcal{IS}_{(0, \alpha^*]}$ and $z \in \mathbb{C}$, let $\mathbb{L}og$ denote an arbitrary inverse branch of $\mathbb{E}xp$ containing $\tau_h(z)$ in the interior of its domain. We have

$$|(\mathbb{L}og \circ \tau_h)'(z) - \alpha| \leq C_8 \alpha e^{-2\pi \alpha \operatorname{Im} z},$$

where C_8 is a constant independent of h and w .

Proof. — The proof follows from explicit calculations. \square

Proof of Proposition 3.3. — By definition, $\chi_h := \mathbb{L}og \circ (\Phi_h^\dagger)^{-1}$, and $(\Phi_h^\dagger)^{-1} = \tau_h \circ L_h^{-1}$. To estimate the derivative of χ_h , first we estimate the derivative of the inverse map L_h^{-1} at w using Lemma 5.7. To this end, we need to locate $L_h^{-1}(w)$.

Given $w \in \Theta(r, \alpha) \cap \operatorname{Dom} \chi_h$, by pre-compactness of the class $\mathcal{IS}_{(0, \alpha^*]}$, and Lemma 5.2, there exists a r' , with r'/r uniformly bounded, such that $L_h^{-1}(w)$ belongs to $\Theta(r'/\alpha)$.

On the other hand, from Lemma 5.1-iv, we know that $\operatorname{Im} L_h^{-1}(1/2\alpha)$ is bigger than $-C_3 \log(1 + 1/2\alpha)$. Now, from the first part of the same lemma and the relation $L_h(F_h(w)) = L_h(w) + 1$ one concludes that $\operatorname{Im} L_h^{-1}(w)$ must be at least

$$-C_3 \log(1 + 1/2\alpha) - 1/4\alpha.$$

Now choose $\delta \in \mathbb{R}$ such that

$$-C_3 \log(1 + 1/2\alpha) - 1/4\alpha \geq \delta/\alpha$$

holds for every $\alpha \in (0, 1)$. The above argument implies that

$$\operatorname{Im} L_h^{-1}(w) \geq -C_3 \log(1 + 1/2\alpha) - 1/4\alpha + \operatorname{Im} L_h^{-1}(i \operatorname{Im} w + 1/2\alpha),$$

which is at least $\delta/\alpha + \operatorname{Im} w - 4(C_7 + 8)$, by the second statement in Lemma 5.4-iii, and a similar property for H .

Now we may use Lemma 5.7, at $w' := L_h^{-1}(w)$, to obtain

$$|L_h'(w') - 1| \leq \frac{M'}{r} e^{-2\pi \alpha \operatorname{Im} w},$$

for an appropriate constant M' .

With $z = L_h^{-1}(w)$,

$$\begin{aligned} |\chi_h'(w) - \alpha| &\leq |(\mathbb{L}og \circ \tau_h)'(z) \cdot (L_h^{-1})'(z) - (\mathbb{L}og \circ \tau_h)'(w) + (\mathbb{L}og \circ \tau_h)'(z) - \alpha| \\ &\leq |(\mathbb{L}og \circ \tau_h)'(z)| |(L_h^{-1})'(z) - 1| + |(\mathbb{L}og \circ \tau_h)'(z) - \alpha| \\ &\leq C \frac{\alpha}{r} e^{-2\pi \alpha \operatorname{Im} w}, \end{aligned}$$

for some constant C . \square

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